Supplement to "Continuous Time Random Matching"

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For the convenience of the reader, we provide a brief introduction of nonstandard analysis in Appendix D to set up the basic concepts and notations. Theorem A.2 is then proved in Appendix E. Proofs of the results in Section 2 are then completed in Appendix F.

D A Brief Introduction to Nonstandard Analysis

The proof of Theorem A.2 makes extensive use of some basic results in nonstandard analysis. In order to make the proof more readable, this section follows Appendix D in Duffie, Qiao and Sun (2018) by setting up notation and summarizing background knowledge, adopting some material from Loeb and Wolff (2015) and presenting several other related results (including Lemmas D.1 and D.2). First, a simple construction of the nonstandard number system that extends the usual ordered field of real numbers is given in Subsection D.1. A more general framework of nonstandard analysis is then presented in Subsection D.2. The key constructions of Loeb measure spaces and Loeb transition probabilities are introduced in Subsections D.3 and D.4 respectively. The crucial relevant result is the so-called Fubini property for Loeb product and transition probabilities.

D.1 Non-standard number system

First, we extend the ordered field of real numbers $\mathbb{R}$ to an ordered field ${}^*\mathbb{R}$ that contains infinitesimals. To this end, we introduce the concept of a free ultrafilter.

Definition D.1. *A free ultrafilter on the set* $\mathbb{N}$ *of positive integers is a collection* $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N}) = \{A : A \subseteq \mathbb{N}\}$ *such that*

1. $\emptyset \notin \mathcal{U}$.
2. $A \in \mathcal{U}$ *and* $B \in \mathcal{U} \Longrightarrow A \cap B \in \mathcal{U}$.

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3. *$A \subseteq \mathbb{N}$ and $A \notin \mathcal{U} \Longrightarrow \mathbb{N} \backslash A \in \mathcal{U}$.*

4. *A is a finite subset of $\mathbb{N} \Longrightarrow \mathbb{N} \backslash A \in \mathcal{U}$.*

Fix a free ultrafilter $\mathcal{U}$. One can define a set function ι on the power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$ such that $\iota(A) = 1$ if $A \in \mathcal{U}$, and $\iota(A) = 0$ if $A \notin \mathcal{U}$. It is easy to check that ι is a finitely additive probability measure on $\mathcal{P}(\mathbb{N})$. If a property holds on some set $A \in \mathcal{U}$, then the property holds with ι-probability one; we can simply say that the property holds almost everywhere, expressed for brevity as "a.e.".

Two sequences $\langle r_i \rangle$ and $\langle s_i \rangle$ of real numbers are said to be equivalent if $r_i = s_i$ a.e. (i.e., $\{i \in \mathbb{N} : r_i = s_i\} \in \mathcal{U}$). We write $[\langle r_i \rangle]$ for the equivalence class containing the sequence $\langle r_i \rangle$, and we use ${}^*\mathbb{R}$ to denote the collection of such equivalence classes. The set ${}^*\mathbb{R}$ is called the set of nonstandard real numbers, or the "hyperreal" numbers. Such a construction using an ultrafilter is called an ultrapower construction.[1] We note that the set $\mathbb{R}$ of real numbers is embedded in the set of nonstandard real numbers ${}^*\mathbb{R}$ via the map $c \to [\langle c \rangle]$, where $\langle c \rangle$ is the constant sequence with term $c \in \mathbb{R}$. We write ${}^*c$ for $[\langle c \rangle]$, but later drop the star for convenience. In contrast to hyperreal numbers in ${}^*\mathbb{R}$, the numbers in $\mathbb{R}$ are also called standard real numbers.

The summation and multiplication operations $+$, $\centerdot$ and the absolute value function $|\cdot|$ together with the "less than" order relation $<$ for ${}^*\mathbb{R}$ are defined as follows.

Definition D.2. *Given real sequences $\langle r_i \rangle$ and $\langle s_i \rangle$, we set*

1. $[\langle r_i \rangle] + [\langle s_i \rangle] = [\langle r_i + s_i \rangle]$.

2. $[\langle r_i \rangle] \centerdot [\langle s_i \rangle] = [\langle r_i \centerdot s_i \rangle]$.

3. $|[\langle r_i \rangle]| = [\langle |r_i| \rangle]$.

4. $[\langle r_i \rangle] < [\langle s_i \rangle]$ *if $r_i < s_i$ a.e.*

It is easy to check that the operations $+$ and $\centerdot$, as well as $|\cdot|$ and the ordering $<$, are independent of the choices of the representing sequences. The structure $({}^*\mathbb{R}, +, \cdot, <)$ forms an ordered field that extends the ordered field $(\mathbb{R}, +, \cdot, <)$.

For any $r \in {}^*\mathbb{R}$, r is **infinite** (or **unlimited**) if $|r| > n$ for every standard positive integer $n \in \mathbb{N}$; r is **finite** (or **limited**) if $|r| < n$ for some $n \in \mathbb{N}$; and r is **infinitesimal** if $|r| < \frac{1}{n}$ for every $n \in \mathbb{N}$. Recall that for $r = [\langle r_i \rangle] \in {}^*\mathbb{R}$ and $c \in \mathbb{R}$, $|r| < c$ ($|r| > c$) means that $|r_i| < c$ ($|r_i| > c$) holds a.e.

[1]Though the set ${}^*\mathbb{R}$ of nonstandard real numbers depends on the underlying ultrafilter, the particular choice of such an ultrafilter is not an issue. When we consider applications of nonstandard analysis, what we use are some general properties of nonstandard models, such as the Transfer Principle in Proposition D.1 and the Countable Saturation Principle in Proposition D.3 below.

For $x, y \in {}^*\mathbb{R}$, we say that x and y are infinitesimally close or infinitely close if $x - y$ is infinitesimal and in that case we write $x \simeq y$. The equivalence class for $\simeq$ containing x is called the monad of x, written as $monad(x)$. That is, $monad(x) = \{y \in {}^*\mathbb{R} : y \simeq x\}$. For $x, z \in {}^*\mathbb{R}$, we use $x \lesssim z$ ($x \gtrsim z$) to denote that there exists $y \in {}^*\mathbb{R}$ with $y \simeq x$ such that $y \leq z$ ($y \geq z$).

If $\rho \in {}^*\mathbb{R}$ is finite, then the unique real number c with $\rho \simeq c$ is called the standard part of ρ. We write $c = \mathrm{st}(\rho)$ or $c = {}^\circ\rho$.

Let ${}^*\mathbb{N} = \{[\langle r_i \rangle] : r_i \in \mathbb{N} \text{ a.e.}\} \subseteq {}^*\mathbb{R}$ be the set of positive hyperfinite integers, and ${}^*\mathbb{N}_\infty$ the set of unlimited hyperfinite integers.

D.2 General framework of nonstandard analysis

To develop the general framework of nonstandard analysis, we need to work with the concept of superstructure. Fix a set X containing $\mathbb{R}$. Let $V_0(X) = X$, and for each positive integer $n \in \mathbb{N}$, let $V_n(X) = V_{n-1}(X) \cup \mathcal{P}(V_{n-1}(X))$, where $\mathcal{P}(V_{n-1}(X))$ is the power set of $V_{n-1}(X)$. The superstructure over X is the set $V(X) = \cup_{n=0}^{\infty} V_n(X)$. Entities in X are said to be of rank 0, and for $n \geq 1$, entities in $V_n(X) \backslash V_{n-1}(X)$ are said to be of rank n.

For $a, b \in V_n(X)$, one can define an ordered pair (a, b) as the set $\{\{a\}, \{a, b\}\}$, which is an element in $V_{n+2}(X)$. With the definition of ordered pairs, one can then define the Cartesian product of two sets in $V(X)$, as well as relations and functions in $V(X)$. For $k \geq 3$, one can define ordered k-tuples $(a_1, a_2, \ldots, a_k)$ as $\{(1, a_1), (2, a_2), \ldots, (k, a_k)\}$. The k-tuple versions of Cartesian products, relations and functions in $V(X)$ can be similarly defined. In fact, the superstructure can be used to cover basically all of the relevant mathematical structures that are useful for applications.

We now describe the construction of formal statements, or "formulas," in a formal language $\mathcal{L}_X$ about the superstructure $V(X)$. Given X, the language $\mathcal{L}_X$ for the superstructure $V(X)$ over X has the following symbols:

1. Connectives: $\urcorner$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$.
2. Quantifiers: $\forall$, $\exists$.
3. Parentheses: $[\ ]$, $(\ )$, $\langle\ \rangle$.
4. Constant Symbols: One name for each entity in $V(X)$.
5. Variable Symbols: A fixed collection of symbols representing variables.
6. Equality Symbol: Denotes equality for elements of X, and set equality otherwise.
7. Set membership: $\in$.

The above symbols serve as the "alphabet" of the language $\mathcal{L}_X$. A fixed set of variable symbols together with other symbols in $\mathcal{L}_X$ will lead to a well-defined collection of formal syntactical statements as defined recursively in the following definition.

Definition D.3. *A formula of $\mathcal{L}_X$ is built up inductively with the following rules:*

(a) If $x_1, \cdots, x_n$, x, and y are either constants or variables, then the following are called atomic formulas: $x \in y$, $x = y$; $(x_1, \cdots, x_n) \in y$; $(x_1, \cdots, x_n) = y$; $((x_1, \cdots, x_n), x) \in y$; $((x_1, \cdots, x_n), x) = y$.

(b) If Φ and Θ are formulas, so are $(\neg\Phi)$, $(\Phi \wedge \Theta)$, $(\Phi \rightarrow \Theta)$, $(\Phi \vee \Theta)$, and $(\Phi \leftrightarrow \Theta)$.

(c) If x is a variable symbol and y is either a variable symbol or a constant symbol and Φ is a formula, then $(\forall x \ \in \ y)\Phi$ and $(\exists x \in y)\Phi$ are formulas.

The logical connectives $\urcorner$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$ have the usual meanings in terms of the satisfiability of formulas connected by them. For example, $(\neg\Phi)$ means that Φ is not satisfied while $\vee$, $\wedge$ mean "or", "and" respectively. For the formulas $(\forall x \ \in \ y)\Phi$ and $(\exists x \in y)\Phi$, the scope of the quantifiers $\forall$ and $\exists$ is Φ. One can define the scope of a quantifier within a formula inductively.

Definition D.4. *A variable x is free in a formula Φ if it is not within the scope of any quantifier for x. A closed formula in $\mathcal{L}_X$ is a formula without free variables.*

Fix a free ultrafilter $\mathcal{U}$. Given $\langle a_i \rangle$ and $\langle b_i \rangle$, both in the space X^∞ of sequences in X, are said to be equivalent if $a_i = b_i$ a.e. For any $c \in X$, let ${}^*c = [\langle c, c, \ldots \rangle]$ be the equivalence class of sequences in X^∞ that contains the constant sequence $\langle c, c, \ldots \rangle$. For any sequence $\{A_i\}_{i=1}^\infty$ of sets in $V_n(X) \backslash X$ for some $n \geq 1$, define the set $[\langle A_i \rangle] = \{[\langle x_i \rangle] : x_i \in A_i \text{ a.e.}\}$. For $A \in V(X)$, let ${}^*A = [\langle A, A, A, \ldots \rangle]$. In particular, ${}^*X$ is the set of equivalent classes of sequences in X^∞.

Definition D.5. *If Φ is a formula in $\mathcal{L}_X$, the $*$-transform of Φ, denoted ${}^*\Phi$, is the formula in $\mathcal{L}_{{}^*X}$ that is obtained by replacing each constant c in Φ with ${}^*c$.*

The following result is a basic tool in nonstandard analysis.[2]

Proposition D.1 (Transfer Principle)**.** *If Φ is a closed formula in $\mathcal{L}_X$ that is true for $V(X)$, then ${}^*\Phi$ is true for $V({}^*X)$.*

All entities in $V(X)$ and entities in $V({}^*X)$ of the form ${}^*b$, for some $b \in V(X)$, are called **standard**. An entity a in $V({}^*X)$ is called **internal** if for some set $b \in V(X)$, $a \in {}^*b$. All other

[2]For a detailed proof, readers are referred to Sections 2.2-2.5 of Loeb and Wolff (2015).

entities in $V({}^*X)$ are called **external**. In particular, a set A in $V({}^*X)$ is internal if it is an element of ${}^*\mathcal{A}$ for some set $\mathcal{A}$ (of sets) in $V(X)$. In other words, a set A in $V({}^*X)$ is internal if one can find a sequence $\{A_i\}_{i=1}^{\infty}$ of sets in $V_n(X)\backslash X$ for some $n \geq 1$ such that A is the set of equivalence classes $\{[\langle a_i \rangle] : a_i \in A_i \text{ a.e.}\}$. If any kind of internal operations are applied to internal sets, one still obtains internal sets; see, for example, Theorem 2.8.4 in Loeb and Wolff (2015). In particular, if A and B are internal, then so are $A \cup B$, $A \cap B$, $A \backslash B$, and $A \times B$. An internal function is a function whose graph is internal.

The following result about internal sets is important for applications.

Proposition D.2 (Spillover Principle)**.** *Let A be an internal subset of ${}^*\mathbb{R}$.*

(1) If A contains all standard natural numbers, then A contains all elements of ${}^*\mathbb{N}$ less than some unlimited natural number.

(2) If A contains all unlimited positive hyperfinite integers, then A contains all elements of ${}^*\mathbb{N}$ greater than some unlimited natural number.

(3) If A contains the positive infinitesimals, then A contains all elements of ${}^*\mathbb{R}_+$ smaller than some standard positive real number.

(4) Assume that for each unlimited positive hyperfinite integer H there exists an unlimited natural number $K < H$ such that $K \in A$. Then A contains a standard natural number.

For $B \in V(X)\backslash X$, let $\mathcal{P}_F(B)$ denote the finite subsets of B. An element $A \in {}^*\mathcal{P}_F(B)$ will be called a **hyperfinite** set. In particular, A is the set of equivalence classes $\{[\langle b_i \rangle] : b_i \in B_i \text{ a.e.}\}$ for some sequence $\langle B_i \rangle$ of finite subsets of B. The internal cardinality of A is simply the hyperinteger $[\langle |B_i| \rangle]$, where $|B_i|$ is the cardinality of the finite set B_i. When the internal cardinality of A is an unlimited hyperfinite integer, then the external cardinality of A is the cardinality of the continuum (see Proposition 6 in Duffie, Qiao and Sun (2018)).

By the Spillover Principle as stated in Proposition D.2, it is easy to prove the following lemma.

Lemma D.1. *Let $\mathbb{R}_{++}$ be the set of positive standard real numbers, c a standard real number in $\mathbb{R}$, and M an unlimited hyperfinite integer. Let f be an internal function from the internal set $B = \{1, 2, \ldots, M^2\}$ to ${}^*\mathbb{R}$ such that for any $n \in B$, $f(n) \simeq c$ when $\frac{n}{M}$ is infinitesimal. Then, for any $\epsilon \in \mathbb{R}_{++}$, there exists $\delta \in \mathbb{R}_{++}$ such that for any $n \in B$ with $\mathrm{st}(\frac{n}{M}) < \delta$, $|\mathrm{st}(f(n)) - c| < \epsilon$.*

Proof. Suppose not. Then, there exists $\epsilon \in \mathbb{R}_{++}$ and a sequence $\{n_m\}_{m \in \mathbb{N}}$ in B such that for any $m \in \mathbb{N}$, $\frac{n_m}{M} < 2^{-m}$ and $|f(n_m) - c| > \epsilon$. Define an internal set

$$A = \{m \in {}^*\mathbb{N} : \exists n \in B \text{ such that } \frac{n}{M} < 2^{-m} \text{ and } |f(n) - c| > \epsilon\}.$$

Then $\mathbb{N}$ is a subset of A. Since A is internal, by the Spillover Principle, there exists $m_0 \in A \cap {}^*\mathbb{N}_\infty$. Hence, there exists $n_0 \in B$ such that $\frac{n_0}{M} < 2^{-m_0}$ and $|f(n_0) - c| > \epsilon$. It is clear that $\frac{n_0}{M}$ is infinitesimal but $f(n_0) \not\simeq c$. This is a contradiction. ∎

The following is an important uniformity principle that transforms a local property expressed by finite intersections to a global property described by the intersection of all the sets in the sequence.

Proposition D.3 (Countable Saturation Principle). *For a sequence of nonempty internal sets, $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \ldots$, we have $\cap_{n\in\mathbb{N}}^{\infty} A_n \neq \emptyset$.*

D.3 Hyperfinite internal probability spaces and their Loeb spaces

Fix any unlimited hyperfinite integer M. Let $\Lambda = \{1, 2, \ldots, M\}$, and $\mathcal{C}$ be the internal power set of all the internal subsets of Λ. Let $w : \Lambda \to {}^*\mathbb{R}_+$ be an internal function such that $\sum_{i\in\Lambda} w(i) = 1$.

Define an internal finitely-additive measure from $\mathcal{C}$ to ${}^*[0,1]$ such that $\mu(A) = \sum_{i\in A} w(i)$ for any $A \in \mathcal{C}$. Then $(\Lambda, \mathcal{C}, \mu)$ is called a hyperfinite internal probability space. If $w(i) \equiv \frac{1}{M}$ for all $i \in \Lambda$, then $(\Lambda, \mathcal{C}, \mu)$ is called a hyperfinite counting probability space. For an internal function h from Λ to ${}^*\mathbb{R}$, the (internal) integral $\int_\Lambda h\, d\mu$ of h over a hyperfinite internal probability space $(\Lambda, \mathcal{C}, \mu)$ is simply the weighted sum $\sum_{i\in\Lambda} h(i)w(i)$. The integral for an internal vector-valued function can be defined componentwise.

Let f be an internal function from Λ to ${}^*\mathbb{R}^N$, where N is a standard positive integer. The expectation $\mathbb{E}f$ and variance $\mathrm{Var}(f)$ of f over a hyperfinite internal probability space $(\Lambda, \mathcal{C}, \mu)$ are defined to be $\int_\Lambda f\, d\mu$ and $\int_\Lambda ||f - \mathbb{E}f||_\infty^2\, d\mu$ respectively, where $||\cdot||_\infty$ is the sup norm on ${}^*\mathbb{R}^N$.

The following lemma is an internal version of the elementary Chebyshev's Inequality.

Lemma D.2. *Let $(\Lambda, \mathcal{C}, \mu)$ be a hyperfinite internal probability space and f an internal function from Λ to ${}^*\mathbb{R}^N$, where N is a standard positive integer. Then, for any $a \in {}^*\mathbb{R}_{++}$, we have*

$$\mu\left(||f - \mathbb{E}f||_\infty \geq a\right) \leq \frac{\mathrm{Var}(f)}{a^2}.$$

Proof. For a subset A of Λ, $\mathbf{1}_A$ denotes its indicator function, which has value one on A and zero on $\Lambda \setminus A$. For any $a \in {}^*\mathbb{R}_{++}$, we have

$$\begin{aligned}\mu\left(||f - \mathbb{E}f||_\infty \geq a\right) &= \int_\Lambda \mathbf{1}_{\{||f-\mathbb{E}f||_\infty \geq a\}}\, d\mu \\ &= \int_\Lambda \mathbf{1}_{\{||f-\mathbb{E}f||_\infty^2 \geq a^2\}} d\mu \leq \int_\Lambda \frac{||f-\mathbb{E}f||_\infty^2}{a^2}\, d\mu.\end{aligned}$$

Since $\mathrm{Var}(f) = \int_\Lambda ||f - \mathbb{E}f||_\infty^2 \, d\mu$, we have $\mu\left(||f - \mathbb{E}f||_\infty \geq a\right) \leq \frac{\mathrm{Var}(f)}{a^2}$. ■

We let $\mathrm{st}(\mu)$ be the function from $\mathcal{C}$ into $\mathbb{R}_+$ defined by $\mathrm{st}(\mu)(A) = \mathrm{st}(\mu(A))$ for any $A \in \mathcal{C}$. It is clear that $\mathrm{st}(\mu)$ is a finitely-additive measure on the algebra $\mathcal{C}$. The important point is that $\mathrm{st}(\mu)$ is a countably-additive measure on the algebra $\mathcal{C}$. To see this, consider a sequence $A_1 \supseteq A_2 \supseteq \ldots$ of internal sets in $\mathcal{C}$ such that $\cap_{n\in\mathbb{N}} A_n = \emptyset$. The countable saturation principle implies the existence of $m \in \mathbb{N}$ such that $A_m = \emptyset$. It is thus clear that $\lim_{n\to\infty} \mathrm{st}(\mu)(A_n) = 0$. By the well-known Caratheodory's extension theorem (see, for example, page 181 in Loeb (2016)), $\mathrm{st}(\mu)$ can be extended to a measure μ_L on the σ-algebra $\sigma(\mathcal{C})$ that is generated by $\mathcal{C}$. By including all μ_L-null subsets, we obtain a standard complete probability space $(\Lambda, L_\mu(\mathcal{C}), \mu_L)$, which is called a Loeb measure space.

D.4 Transition probabilities

Let $(I, \mathcal{I}_0, \lambda_0)$ be a hyperfinite internal probability space for which $\mathcal{I}_0$ is the internal power set on some hyperfinite set I. Let Ω be a hyperfinite internal set with $\mathcal{F}_0$ its internal power set. Let P_0 be an internal function from I to the space of hyperfinite internal probability measures on $(\Omega, \mathcal{F}_0)$, which is called an internal transition probability. For $i \in I$, denote the hyperfinite internal probability measure $P_0(i)$ on $(\Omega, \mathcal{F}_0)$ by P_{0i}.

It is clear that the Cartesian product $I \times \Omega$ is a hyperfinite set. Let $\mathcal{I}_0 \otimes \mathcal{F}_0$ be the internal power set on $I \times \Omega$. Define a hyperfinite internal probability measure τ_0 on $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0)$ by letting $\tau_0(\{(i,\omega)\}) = \lambda_0(\{i\})P_{0i}(\{\omega\})$ for $(i,\omega) \in I \times \Omega$. The measure τ_0 will be called the product transition probability of the measure λ_0 and the transition probability P_0. Let $(I, \mathcal{I}, \lambda)$, $(\Omega, \mathcal{F}_i, P_i)$, and $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \tau)$ be the Loeb spaces corresponding respectively to $(I, \mathcal{I}_0, \lambda_0)$, $(\Omega, \mathcal{F}_0, P_{0i})$, and $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \tau_0)$. The collection $\{P_i : i \in I\}$ of Loeb measures will be called a Loeb transition probability, and denoted by P. The measure τ will be called the Loeb product transition probability of the measure λ and the Loeb transition probability P. We shall also denote τ_0 by $\lambda_0 \otimes P_0$ and τ by $\lambda \boxtimes P$.

The following result presents a generalized Fubini theorem for a Loeb transition probability, which is proved in Section 5 of Duffie and Sun (2007).[3]

Proposition D.4. *Let f be a real-valued integrable function on $(I \times \Omega, \sigma(\mathcal{I}_0 \otimes \mathcal{F}_0), \tau)$. Then, (1) $f_i = f(i,\cdot)$ is $\sigma(\mathcal{F}_0)$-measurable for each $i \in I$ and integrable on $(\Omega, \sigma(\mathcal{F}_0), P_i)$ for λ-almost all $i \in I$; (2) $\int_\Omega f_i(\omega)\, dP_i(\omega)$ is integrable on $(I, \sigma(\mathcal{I}_0), \lambda)$; (3) $\int_I \int_\Omega f_i(\omega)\, dP_i(\omega)\, d\lambda(i) = \int_{I\times\Omega} f(i,\omega)\, d\tau(i,\omega)$.*

[3] For simplicity, we only state the result in terms of the σ-algebra $\sigma(\mathcal{I}_0 \otimes \mathcal{F}_0)$. One can also re-state the result to the case when the underlying measure space is the completion of $(I \times \Omega, \sigma(\mathcal{I}_0 \otimes \mathcal{F}_0), \tau)$.

If P_{0i} does not depend on i, then $\tau = \lambda \boxtimes P$ is called the Loeb product measure. The corresponding measure space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is called the Loeb product space. In this case, the symmetric position of the two probability spaces respectively on I and Ω implies that the properties as stated in Proposition D.4 also hold when the iterated integral is taken in different order. It is clear that $\mathcal{I} \boxtimes \mathcal{F}$ contains the usual product σ-algebra $\sigma(\mathcal{I}_0) \otimes \sigma(\mathcal{F}_0)$. Thus, the Loeb product space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a Fubini extension.[4]

E Proof of Theorem A.2

This section is organized as follows. Lemma E.1 in Subsection E.1 presents a static model for internal random matching as well as some estimations on the relevant matching probabilities. Such a static matching model will be used in the construction of a hyperfinite dynamic matching model in Subsection E.2. To make the proof of Theorem A.2 more accessible, we first state some properties of the hyperfinite dynamic matching model (Lemmas E.3 – E.13 in Subsection E.3) that are needed for proving Theorem A.2 in Subsection E.4. The proofs of the technical results in Lemmas E.1 through E.13 are postponed to Subsection E.5. In particular, Lemmas E.1 and E.2 are proved in Subsections E.5.1 and E.5.2 respectively. In order to prove Lemmas E.3 – E.13, some additional technical results are presented as Lemma E.17 through E.24 in Subsection E.5.3. Then, Lemmas E.3 through E.13 are shown in Subsections E.5.4 through E.5.14, respectively.

E.1 Static internal matching model

Let $I = \{1, \ldots, \hat{M}\}$ be a hyperfinite set with $\hat{M}$ an unlimited hyperfinite even integer in ${}^*\mathbb{N}_\infty$, $\mathcal{I}_0$ the internal power set on I, and λ_0 the hyperfinite counting probability measure on $\mathcal{I}_0$ with $\lambda_0(A) = |A|/|I|$ for any $A \in \mathcal{I}_0$, where $|A|$ is the internal cardinality of $|A|$. An internal partial matching ψ on I is an internal mapping from I to I such that $\psi(\psi(i)) = i$ for any $i \in I$. When $\psi(i) \neq i$ ($\psi(i) = i$), agent i is matched with agent $\psi(i)$ (agent i is not matched). When $\psi(i) \neq i$ for each $i \in I$, ψ is said to be an internal full matching on I. For a given hyperfinite internal probability space $(\Omega, \mathcal{F}_0, P_0)$, an internal random (partial) matching π on I is an internal mapping from $I \times \Omega$ to I such that $\pi_\omega = \pi(\,\cdot\,, \omega)$ is an internal partial matching on I for each $\omega \in \Omega$.

The following result is essential to the construction of hyperfinite dynamic matching model in Subsection E.2.

[4]For the development of Loeb (product) spaces, see Loeb (1975), Anderson (1976), Kiesler (1977), Sun (1998), and Section 6.2 of Loeb and Wolff (2015) (by Horst Osswald). When both λ and P have atomless parts, Proposition 8.4.5 in Chapter 8 of Loeb and Wolff (2015) (by the third author of this paper) indicates that $\mathcal{I} \boxtimes \mathcal{F}$ is always a strict extension of $\sigma(\mathcal{I}_0) \otimes \sigma(\mathcal{F}_0)$.

Lemma E.1. *Let $(I, \mathcal{I}_0, \lambda_0)$ be the hyperfinite internal counting probability space as above. Then, there exists a hyperfinite internal set Ω with its internal power set $\mathcal{F}_0$ such that for any internal type function α^0 from I to S and internal partial matching π^0 on I with*

$$g^0(i) = \begin{cases} \alpha^0(\pi^0(i)) & \text{if } \pi^0(i) \neq i \\ J & \text{if } \pi^0(i) = i, \end{cases}$$

and for any internal function q from $S \times S$ to ${}^*\mathbb{R}_+$ with $\sum_{r \in S} q_{kr} \leq 1$ and $\hat{\rho}_{kJ} q_{kl} = \hat{\rho}_{lJ} q_{lk}$ for any $k, l \in S$, where $\hat{\rho} = \lambda_0 \left(\alpha^0, g^0\right)^{-1}$ is the internal extended type distribution induced by $\left(\alpha^0, g^0\right)$ on $\hat{S}$,[5] *there exists an internal random matching π from $I \times \Omega$ to I and an internal probability measure P_0 on $(\Omega, \mathcal{F}_0)$ with the following properties.*

(i) Let $H = \{i \in I : \pi^0(i) \neq i\}$. Then $P_0 \left(\{\omega \in \Omega : \pi_\omega(i) = \pi^0(i) \text{ for any } i \in H\}\right) = 1$.

(ii) Let g be the internal mapping from $I \times \Omega$ to $S \cup \{J\}$, defined by

$$g(i, \omega) = \begin{cases} \alpha^0(\pi(i, \omega)) & \text{if } \pi(i, \omega) \neq i \\ J & \text{if } \pi(i, \omega) = i \end{cases}$$

for any $(i, \omega) \in I \times \Omega$.

Fix any $i, j \in I$ with $i \neq j$, $\pi^0(i) = i$ and $\pi^0(j) = j$; denote $\alpha^0(i)$ and $\alpha^0(j)$ by k_1 and k_2 respectively. For any $l_1, l_2 \in S$, the random matching π and the associated type process g satisfy the following inequalities:

$$P_0 \left(\pi_i = j\right) \leq \frac{2}{\hat{M} \hat{\rho}_{k_1 J}},$$

$$q_{k_1 l_1} - \frac{2}{\hat{M}^{\frac{2}{3}}} \leq P_0(g_i = l_1) \leq q_{k_1 l_1} \text{ if } \hat{\rho}_{k_1 J} \geq \frac{1}{\hat{M}^{\frac{1}{3}}},$$

$$q_{k_1 l_1} q_{k_2 l_2} - \frac{5}{\hat{M}^{\frac{2}{3}}} \leq P_0(g_i = l_1, g_j = l_2) \leq q_{k_1 l_1} q_{k_2 l_2} + \frac{1}{\hat{M}^{\frac{2}{3}}} \text{ if } \hat{\rho}_{k_1 J} \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \text{ and } \hat{\rho}_{k_2 J} \geq \frac{1}{\hat{M}^{\frac{1}{3}}}.$$

(iii) For any $k, l \in S$ and $\omega \in \Omega$,

$$\left|\lambda_0 \left(\{i \in I : \alpha^0(i) = k, g^0(i) = J, g(i, \omega) = l\}\right) - \hat{\rho}_{kJ} q_{kl}\right| \leq \frac{2}{\hat{M}}.$$

To reflect their dependence on (α^0, π^0, q), π and P_0 will also be denoted by $\pi_{(\alpha^0, \pi^0, q)}$ and $P_{(\alpha^0, \pi^0, q)}$ respectively.[6]

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[5] That is, for any subset C of $\hat{S}$, $\hat{\rho}(C) = \lambda_0 \left(\left(\alpha^0, g^0\right)^{-1}(C)\right)$.

[6] The above equation shows that $g_i, i \in I$ are approximately pairwise independent. In fact, we can use similar techniques to prove that $g_i, i \in I$ are approximately mutually independent. For simplicity, we only demonstrate the case for approximate pairwise independence.

Part (i) means that initially matched agents are not rematched. The three inequalities in Part (ii) provide respectively (1) an upper bound on the probability for two single agents to be matched, (2) an estimation on the distribution of the partner's type of an agent, (3) the approximate pairwise independence of the random types of agents' partners.

E.2 Hyperfinite dynamic matching model

What we need to do is to construct a hyperfinite sequence of internal transition probabilities and a hyperfinite sequence of internal extended type functions. Since we need to consider random mutation, random matching, random type changing and break-up at each infinitesimal time period, three internal measurable spaces with internal transition probabilities will be constructed at each time period.

Before the formal construction, we briefly describe the timeline. In each period, there are three steps. The first step is the mutation step, agents (single or matched) change their types independently. The second step is the matching step, only single agents take part in a static random matching described in Lemma E.1. The third step is the type changing with break-up step, at which agents who were just matched in the last step either enter into a long-term partnership or do not, and then experience a change in their types according to the specified type-changing probabilities. At this step, agents who have been matched for more than one step break up with some probability, and change their types according to the specified type-changing probabilities if they indeed break up.

Let M and $\hat{M}$ be fixed unlimited hyperfinite integers in ${}^*\mathbb{N}_\infty$, with $\hat{M}$ sufficiently larger than M^{M^M} (an explicit expression for $\hat{M}$ will be given after Lemma E.3). As in Subsection E.1, let $I_0 = \{1, 2, \ldots, \hat{M}\}$, $\mathcal{I}_0$ the internal power set on I, and λ_0 the internal counting probability measure on $\mathcal{I}_0$. Let $\mathbb{T}_0$ be the hyperfinite set $\{n\}_{n=0}^{M^2}$. The corresponding hyperfinite discrete time line is $\{n/M\}_{n=0}^{M^2}$.

We define the parameters for the hyperfinite dynamical system as follows. For any $k, k', l, l' \in S$, and $\hat{\rho} \in {}^*\hat{\Delta}$, let

$$\hat{\eta}_{kl} = \begin{cases} \frac{1}{M}\eta_{kl} + \frac{1}{M^2} & \text{if } k \neq l \\ 1 - \sum_{r \in (S \setminus \{k\})} \hat{\eta}_{kr} & \text{if } k = l, \end{cases}$$

$$\hat{q}_{kl}(\hat{\rho}) = \frac{1}{M}({}^*\theta_{kl})(\hat{\rho}) \text{ and } \hat{q}_k(\hat{\rho}) = 1 - \sum_{l \in S} \hat{q}_{kl}(\hat{\rho}),$$

$$\hat{\xi}_{kl} = \min\{\xi_{kl}, 1 - \frac{1}{M^2}\},$$

$$\hat{\sigma}_{kl} = \sigma_{kl},$$

$$\hat{\varsigma}_{kl} = \varsigma_{kl},$$

$$\hat{\vartheta}_{kl} = \frac{1}{M}\vartheta_{kl} + \frac{1}{M^2}.$$

In this way, we have defined $\hat{\eta}_{kl}$, $\hat{\xi}_{kl}$ and $\hat{\vartheta}_{kl}$ so that $\hat{\eta}_{kl}$ and $\hat{\vartheta}_{kl}$ have lower bound $\frac{1}{M^2}$, and $\hat{\xi}_{kl}$ has upper bound $1-\frac{1}{M^2}$. Such bounds will be used in the proof of Lemma E.18.

Denote

$$\begin{aligned}
\bar{\eta} &= \max\{\eta_{kl} + \frac{1}{M} : k,l \in S, k \neq l\},\\
\bar{q} &= \max\{({}^*\theta_{kl})(\hat{\rho}) : k,l \in S, \hat{\rho} \in {}^*\hat{\Delta}\} = \max\{\theta_{kl}(\hat{p}) : k,l \in S, \hat{p} \in \hat{\Delta}\},\\
\bar{\vartheta} &= \max\{\vartheta_{kl} + \frac{1}{M} : k,l \in S\},\\
\bar{a} &= \max\{\bar{\eta}, \bar{q}, \bar{\vartheta}\}.
\end{aligned}$$

For the initial stage at time $t=0$, we need a hyperfinite internal sample probability space $(\Omega_0, \mathcal{E}_0, Q_0)$, an internal type process $\hat{\alpha}^0$ from $I \times \Omega_0$ to S, and an internal random matching $\hat{\pi}^0$ from $I \times \Omega_0$ to I. Let $\hat{g}^0$ be the internal mapping from $I \times \Omega_0$ to $S \cup \{J\}$ defined by

$$\hat{g}^0(i,\omega_0) = \begin{cases} \hat{\alpha}^0(\hat{\pi}^0(i,\omega_0),\omega_0) & \text{if } \hat{\pi}^0(i,\omega_0) \neq i \\ J & \text{if } \hat{\pi}^0(i,\omega_0) = i, \end{cases}$$

for any $(i,\omega_0) \in I \times \Omega_0$. Let $\hat{\rho}^0_{\omega_0} = \lambda_0 \left(\hat{\alpha}^0_{\omega_0}, \hat{g}^0_{\omega_0}\right)^{-1}$ be the initial internal cross-sectional extended type distribution on $\hat{S}$. We require that $\mathbb{E}(\hat{\rho}^0) \simeq \hat{p}^0$,[7] $\mathbb{E}(\hat{\rho}^0_{kJ}) \geq \frac{1}{M^2}$ for any $k \in S$,

$$Q_0\left(||\hat{\rho}^0 - \mathbb{E}(\hat{\rho}^0)||_\infty \geq \frac{1}{\hat{M}^{\frac{1}{10}}}\right) \leq \frac{1}{\hat{M}^{\frac{1}{10}}}, \tag{E.1}$$

and for any $(k_1,l_1),(k_2,l_2) \in \hat{S}$,

$$\left|Q_0\left(A_{i1} \cap A_{j2}\right) - Q_0\left(A_{i1}\right) Q_0\left(A_{j2}\right)\right| \leq \frac{1}{\hat{M}^{\frac{1}{10}}}, \tag{E.2}$$

where

$$A_{i1} = \{\omega_0 \in \Omega_0 : (\hat{\alpha}^0_i(\omega_0), \hat{g}^0_i(\omega_0)) = (k_1,l_1)\}, \quad A_{j2} = \{\omega_0 \in \Omega_0 : (\hat{\alpha}^0_j(\omega_0), \hat{g}^0_j(\omega_0)) = (k_2,l_2)\}.$$

The following lemma shows the existence of such initial sample spaces and initial processes.

Lemma E.2. *The above requirement can be achieved by an internal extended type process $(\hat{\alpha}^0, \hat{g}^0)$ that is deterministic, or has identical distributions in the sense that for each $i \in I$, the internal extended type distribution of agent i $Q_0\left(\hat{\alpha}^0_i, \hat{g}^0_i\right)^{-1} \simeq \hat{p}^0$. In the identical distribution case, we can also require that $Q_0\left(\hat{\pi}^0_i = j\right) \leq \frac{1}{\hat{M}^{\frac{1}{5}}}$ holds for any $i,j \in I$ with $i \neq j$.*

Suppose that the construction for the dynamical system $\mathbb{D}$ has been done up to time period $n-1$. Thus, $\{(\Omega_m, \mathcal{E}_m, Q_m)\}_{m=0}^{3n-3}$ and $\{\hat{\alpha}^m, \hat{\pi}^m, \hat{g}^m\}_{m=0}^{3n-3}$ have been constructed, where

[7] Two vectors a and b are infinitely close to each other (denoted by $a \simeq b$) if their distance is infinitesimal.

each Ω_m is a hyperfinite internal set with its internal power set $\mathcal{E}_m$, Q_m an internal transition probability from Ω^{m-1} to $(\Omega_m, \mathcal{E}_m)$, $\hat{\alpha}^m$ an internal type function from $I \times \Omega^{m-1}$ to the type space S, and $\hat{\pi}^m$ an internal random matching from $I \times \Omega^{m-1}$ to I. Here, $\Omega^m = \prod_{j=0}^m \Omega_j$, and $\{\omega_j\}_{j=1}^m$ will also be denoted by ω^m when there is no confusion. Denote the internal product transition probability $Q_1 \otimes Q_2 \otimes \cdots \otimes Q_m$ by Q^m, and $\otimes_{j=1}^m \mathcal{E}_j$ by $\mathcal{E}^m$ (which is simply the internal power set on Ω^m). Then, Q^m is the internal product of the internal transition probability Q_m with the internal probability measure Q^{m-1}.

We shall now consider the constructions for period n. We first work with the random mutation step. Let $\Omega_{3n-2} = S^I$ (the space of all internal functions from I to S) with its internal power set $\mathcal{E}_{3n-2}$. For each $i \in I$, $\omega^{3n-3} \in \Omega^{3n-3}$, if $\hat{\alpha}^{3n-3}(i, \omega^{3n-3}) = k$, define a probability measure $\gamma_i^{\omega^{3n-3}}$ on S by letting $\gamma_i^{\omega^{3n-3}}(l) = \hat{\eta}_{kl}$ for each $l \in S$. Define an internal probability measure $Q_{3n-2}^{\omega^{3n-3}}$ on $(S^I, \mathcal{E}_{3n-2})$ to be the internal product measure $\prod_{i \in I} \gamma_i^{\omega^{3n-3}}$. Let $\hat{\alpha}^{3n-2} : \left(I \times \prod_{m=0}^{3n-2} \Omega_m\right) \to S$ be such that $\hat{\alpha}^{3n-2}\left(i, \omega^{3n-2}\right) = \omega_{3n-2}(i)$. Let $\hat{\pi}^{3n-2} : \left(I \times \prod_{m=0}^{3n-2} \Omega_m\right) \to I$ be such that $\hat{\pi}^{3n-2}\left(i, \omega^{3n-2}\right) = \hat{\pi}^{3n-3}\left(i, \omega^{3n-3}\right)$. Let $\hat{g}^{3n-2} : \left(I \times \prod_{m=0}^{3n-2} \Omega_m\right) \to S \cup \{J\}$ be such that

$$\hat{g}^{3n-2}\left(i, \omega^{3n-2}\right) = \begin{cases} \hat{\alpha}^{3n-2}(\hat{\pi}^{3n-2}(i, \omega^{3n-2}), \omega^{3n-2}) & \text{if } \hat{\pi}^{3n-2}(i, \omega^{3n-2}) \neq i \\ J & \text{if } \hat{\pi}^{3n-2}(i, \omega^{3n-2}) = i. \end{cases}$$

Let $\hat{\rho}_{\omega^{3n-2}}^{3n-2} = \lambda_0 \left(\hat{\alpha}_{\omega^{3n-2}}^{3n-2}, \hat{g}_{\omega^{3n-2}}^{3n-2}\right)^{-1}$ be the internal cross-sectional extended type distribution after the random mutation step.

Next, we consider the step of internal random matching. Let $(\Omega_{3n-1}, \mathcal{E}_{3n-1}) = (\bar{\Omega}, \bar{\mathcal{E}})$ be the measurable space constructed in Lemma E.1. For any given $\omega^{3n-2} \in \Omega^{3n-2}$, the type function is $\hat{\alpha}_{\omega^{3n-2}}^{3n-2}(\cdot)$, while the partial matching function is $\hat{\pi}_{\omega^{3n-3}}^{3n-3}(\cdot)$. We can construct an internal probability measure $Q_{3n-1}^{\omega^{3n-2}} = P_{\hat{\alpha}_{\omega^{3n-2}}^{3n-2}, \hat{\pi}_{\omega^{3n-3}}^{3n-3}, \hat{q}(\hat{\rho}_{\omega^{3n-2}}^{3n-2})}$ and an internal random matching $\pi_{\hat{\alpha}_{\omega^{3n-2}}^{3n-2}, \hat{\pi}_{\omega^{3n-3}}^{3n-3}, \hat{q}(\hat{\rho}_{\omega^{3n-2}}^{3n-2})}$ by Lemma E.1. Let $\hat{\alpha}^{3n-1} : \left(I \times \prod_{m=0}^{3n-1} \Omega_m\right) \to S$, $\hat{\pi}^{3n-1} : \left(I \times \prod_{m=0}^{3n-1} \Omega_m\right) \to I$ and $\hat{g}^{3n-1} : \left(I \times \prod_{m=0}^{3n-1} \Omega_m\right) \to S \cup \{J\}$ be such that

$$\begin{aligned} \hat{\alpha}^{3n-1}\left(i, \omega^{3n-1}\right) &= \hat{\alpha}^{3n-2}\left(i, \omega^{3n-2}\right), \\ \hat{\pi}^{3n-1}\left(i, \omega^{3n-1}\right) &= \pi_{\hat{\alpha}_{\omega^{3n-2}}^{3n-2}, \hat{\pi}_{\omega^{3n-3}}^{3n-3}, \hat{q}(\hat{\rho}_{\omega^{3n-2}}^{3n-2})}\left(i, \omega_{3n-1}\right), \end{aligned}$$

$$\hat{g}^{3n-1}\left(i, \omega^{3n-1}\right) = \begin{cases} \hat{\alpha}^{3n-2}(\hat{\pi}^{3n-1}(i, \omega^{3n-1}), \omega^{3n-2}) & \text{if } \hat{\pi}^{3n-1}(i, \omega^{3n-1}) \neq i \\ J & \text{if } \hat{\pi}^{3n-1}(i, \omega^{3n-1}) = i. \end{cases}$$

Let $\hat{\rho}_{\omega^{3n-1}}^{3n-1} = \lambda_0 \left(\hat{\alpha}_{\omega^{3n-1}}^{3n-1}, \hat{g}_{\omega^{3n-1}}^{3n-1}\right)^{-1}$ be the internal cross-sectional extended type distribution after the random matching step.

Now, we consider the final step of random type changing with break-up for matched agents. Let $\Omega_{3n} = (S \times \{0,1\})^I$ with its internal power set $\mathcal{E}_{3n}$, where 0 represents "unmatched" and 1 represents "paired". Each point $\omega_{3n} = (\omega_{3n}^1, \omega_{3n}^2) \in \Omega_{3n}$ is an internal function from I to $S \times \{0,1\}$. Define a new type function $\hat{\alpha}^{3n} : (I \times \Omega^{3n}) \to S$ by letting $\hat{\alpha}^{3n}(i, \omega^{3n}) = \omega_{3n}^1(i)$. Fix $\omega^{3n-1} \in \Omega^{3n-1}$. For each $i \in I$,

1. if $\hat{\pi}^{3n-1}(i, \omega^{3n-1}) = i$ (i is not paired after the matching step at period n), let $\tau_i^{\omega^{3n-1}}$ be the probability measure on the space $S \times \{0,1\}$ that gives probability one to $\left(\hat{\alpha}^{3n-2}(i, \omega^{3n-2}), 0\right)$ and zero for the rest.

2. if $\hat{\pi}^{3n-1}(i, \omega^{3n-1}) \neq i$ and $\hat{\pi}^{3n-3}(i, \omega^{3n-3}) = i$ (i is newly paired after the matching step at period n), $\hat{\alpha}^{3n-2}(i, \omega^{3n-2}) = k$, $\hat{\pi}^{3n-1}(i, \omega^{3n-1}) = j$ and $\hat{\alpha}^{3n-2}(j, \omega^{3n-2}) = l$, define a probability measure $\tau_{ij}^{\omega^{3n-1}}$ on $(S \times \{0,1\}) \times (S \times \{0,1\})$ such that
$$\tau_{ij}^{\omega^{3n-1}} \left((k',1),(l',1)\right) = \hat{\xi}_{kl} \hat{\sigma}_{kl}(k', l')$$
and
$$\tau_{ij}^{\omega^{3n-1}} \left((k',0),(l',0)\right) = \left(1 - \hat{\xi}_{kl}\right) \hat{\varsigma}_{kl}(k') \hat{\varsigma}_{lk}(l')$$
for $k', l' \in S$, and zero for the rest.

3. if $\hat{\pi}^{3n-1}(i, \omega^{3n-1}) \neq i$ and $\hat{\pi}^{3n-3}(i, \omega^{3n-3}) \neq i$ (i is already paired at time $n-1$), $\hat{\alpha}^{3n-2}(i, \omega^{3n-2}) = k$, $\hat{\pi}^{3n-1}(i, \omega^{3n-1}) = j$ and $\hat{\alpha}^{3n-2}(j, \omega^{3n-2}) = l$, define a probability measure $\tau_{ij}^{\omega^{3n-1}}$ on $(S \times \{0,1\}) \times (S \times \{0,1\})$ such that
$$\tau_{ij}^{\omega^{3n-1}} \left((k',1),(l',1)\right) = \left(1 - \hat{\vartheta}_{kl}\right) \delta_k(k') \delta_l(l')$$
and
$$\tau_{ij}^{\omega^{3n-1}} \left((k',0),(l',0)\right) = \hat{\vartheta}_{kl} \hat{\varsigma}_{kl}(k') \hat{\varsigma}_{lk}(l')$$
for $k', l' \in S$, and zero for the rest.

Let
$$\begin{aligned} A^n_{\omega^{3n-1}} &= \{(i,j) \in I \times I : i < j,\ \hat{\pi}^{3n-1}(i, \omega^{3n-1}) = j\} \\ B^n_{\omega^{3n-1}} &= \{i \in I : \hat{\pi}^{3n-1}(i, \omega^{3n-1}) = i\}. \end{aligned}$$

Define an internal probability measure $Q_{3n}^{\omega^{3n-1}}$ on $(S \times \{0,1\})^I$ to be the internal product measure
$$\prod_{i \in B^n_{\omega^{3n-1}}} \tau_i^{\omega^{3n-1}} \otimes \prod_{(i,j) \in A^n_{\omega^{3n-1}}} \tau_{ij}^{\omega^{3n-1}}.$$

We define $\hat{\pi}^{3n}$ and $\hat{g}^{3n}$ such that for any $(i,\omega^{3n}) \in I \times \Omega^{3n}$,

$$\hat{\pi}^{3n}(i,\omega^{3n}) = \begin{cases} i & \text{if } \hat{\pi}^{3n-1}(i,\omega^{3n-1}) = i \text{ or } \omega^2_{3n}(i) = 0 \text{ or } \omega^2_{3n}(\hat{\pi}^n(i,\omega^{3n-1})) = 0 \\ \hat{\pi}^{3n-1}(i,\omega^{3n-1}) & \text{otherwise,} \end{cases}$$

$$\hat{g}^{3n}\left(i,\omega^{3n}\right) = \begin{cases} \hat{\alpha}^{3n}(\hat{\pi}^{3n}(i,\omega^{3n}),\omega^{3n}) & \text{if } \hat{\pi}^{3n}(i,\omega^{3n}) \neq i \\ J & \text{if } \hat{\pi}^{3n}(i,\omega^{3n}) = i. \end{cases}$$

It is to check that for each $\omega^{3n} \in \Omega^{3n}$, $\hat{\pi}^{3n}_{\omega^{3n}}(\cdot)$ is indeed an internal matching on I.[8] Let $\hat{\rho}^{3n}_{\omega^{3n}} = \lambda_0 \left(\hat{\alpha}^{3n}_{\omega^{3n}}, \hat{g}^{3n}_{\omega^{3n}}\right)^{-1}$ be the internal cross-sectional extended type distribution after the step of random type changing with break-up for matched agents.

Repeating this construction, we can construct a hyperfinite sequence of internal transition probabilities $\{(\Omega_m, \mathcal{E}_m, Q_m)\}_{m=0}^{3M^2}$ and a hyperfinite sequence of internal functions $\{(\hat{\alpha}^m, \hat{\pi}^m, \hat{g}\}_{m=0}^{3M^2}$.

Let $(I \times \Omega^{3M^2}, \mathcal{I}_0 \otimes \mathcal{E}^{3M^2}, \lambda_0 \otimes Q^{3M^2})$ be the internal product probability space of $(I, \mathcal{I}_0, \lambda_0)$ and $(\Omega^{3M^2}, \mathcal{E}^{3M^2}, Q^{3M^2})$. Let $(I \times \Omega^{3M^2}, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be the Loeb space of the internal product probability space. For simplicity, we denote Ω^{3M^2} by Ω and Q^{3M^2} by P_0. For a standard natural number N, any internal function f from $(\Omega^{m+1}, \mathcal{E}^{m+1}, Q^{m+1})$ to ${}^*\mathbb{R}^N$ and $\omega^m \in \Omega^m$, $\mathbb{E}^{\omega^m}(f)$ and $\mathrm{Var}^{\omega^m}(f)$ are defined to be $\int_{\Omega_{m+1}} f(\omega^{m+1})\, dQ^{\omega^m}_{m+1}$ and $\int_{\Omega_{m+1}} ||f(\omega^{m+1}) - \mathbb{E}^{\omega^m} f||^2_\infty \, dQ^{\omega^m}_{m+1}$, respectively.

In the following, we will often work with functions or sets that are measurable in $(\Omega^m, \mathcal{E}^m, Q^m)$ or its Loeb space for some $m \leq 3M^2$, which may be viewed as functions or sets based on $(\Omega^{3M^2}, \mathcal{E}^{3M^2}, Q^{3M^2})$ or its Loeb space by allowing for dummy components for the tail part. We can thus continue to use P to denote the Loeb measure generated by Q^m for convenience. Since all the type functions, random matchings, and partners' type functions are internal in the relevant hyperfinite settings, they are all $\mathcal{I} \boxtimes \mathcal{F}$-measurable when viewed as functions on $I \times \Omega$.

E.3 Properties of the hyperfinite dynamic matching model

In this subsection, we first introduce an internal process $\tilde{\beta}^m$ to capture the types of the agents and their partners, and whether the agents are newly matched. For $1 \leq m \leq 3M^2$ and $i \in I$, let $\tilde{\beta}^m_i = (\hat{\alpha}^m_i, \hat{g}^m_i, \hat{h}^m_i)$, where

$$\hat{h}^m_i = \begin{cases} 0 & \text{if } \hat{g}^m_i \neq J \text{ and } \hat{g}^{m-1}_i \neq J \\ 1 & \text{otherwise.} \end{cases}$$

[8]For any given $\omega^{3n} \in \Omega^{3n}$, let $C^n_{\omega^{3n}} = \{i \in I : \hat{\pi}^{3n-1}(i,\omega^{3n-1}) = i \text{ or } \omega^2_{3n}(i) \cdot \omega^2_{3n}(j) = 0\}$. Then, for any $i \in C^n_{\omega^{3n}}$, we have $\hat{\pi}^{3n}_{\omega^{3n}}(i) = i$ by the definition of $\hat{\pi}^{3n}$. For any $i \notin C^n_{\omega^{3n}}$, we know that $\hat{\pi}^{3n-1}(i,\omega^{3n-1}) = j \neq i$, and $\omega^2_{3n}(i) \cdot \omega^2_{3n}(j) = 1$. The definition of $\hat{\pi}^{3n}$ indicates that $\hat{\pi}^{3n}_{\omega^{3n}}(i) = \hat{\pi}^{3n-1}(i,\omega^{3n-1}) = j$. Since $\hat{\pi}^{3n-1}_{\omega^{3n-1}}(\cdot)$ is an internal matching, we know that $\hat{\pi}^{3n-1}(j,\omega^{3n-1}) = i \neq j$. It is also clear that $\omega^2_{3n}(j) \cdot \omega^2_{3n}(i) = 1$, which implies that $j \notin C^n_{\omega^{3n}}$. It follows from the definition of $\hat{\pi}^{3n}$ that $\hat{\pi}^{3n}_{\omega^{3n}}(j) = \hat{\pi}^{3n-1}(j,\omega^{3n-1}) = i$. Therefore, $\hat{\pi}^{3n}_{\omega^{3n}}(\cdot)$ is an internal matching on I.

It is clear that $\hat{h}_i^m = 0$ if and only if agent i has been matched with another agent for at least two steps. Note that in the third step of each time period, agents who have been matched for more than one step break up with some probability; agents who have just been matched in the previous step (the matching step) form a longterm partnership with some probability. That is why we need $\hat{h}$ to identify agents who have been matched for more than one step. By the construction of the model, if an agent has a partner at the end of the mutation step, he or she must have the same partner in the previous step. It is easy to verify that for any $n \in \{1, \ldots M\}$,

$$\hat{h}_i^{3n-2} = \begin{cases} 0 & \text{if } \hat{g}_i^{3n-2} \neq J \\ 1 & \text{if } \hat{g}_i^{3n-2} = J. \end{cases} \tag{E.3}$$

Similarly, for the type changing with break-up step,

$$\hat{h}_i^{3n} = \begin{cases} 0 & \text{if } \hat{g}_i^{3n} \neq J \\ 1 & \text{if } \hat{g}_i^{3n} = J. \end{cases} \tag{E.4}$$

Let $\tilde{S} = S \times (S \cup \{J\}) \times \{0,1\}$. Any $(k,l,r) \in \tilde{S}$ is called an expanded type. Let $\tilde{\Delta}$ be the space whose elements consist of any probability measure $\tilde{p}$ on $\tilde{S} = S \times (S \cup \{J\}) \times \{0,1\}$ satisfying $\tilde{p}_{klr} = \tilde{p}_{lkr}$ and $\tilde{p}_{kJ0} = 0$ (which means that $\tilde{p}_{kJ1} = \hat{p}_{kJ}$) for any $k,l \in S$ and $r \in \{0,1\}$, which can be viewed as a subset of the simplex in a Euclidean space. We will work with the sup norm $||\cdot||_\infty$ on $\tilde{\Delta}$. For each $k,l \in S$, we use the same notation $\hat{q}_{kl}$ to denote the matching probability from ${}^*\tilde{\Delta} \to {}^*\mathbb{R}$ that is defined by letting $\hat{q}_{kl}(\tilde{\rho}) = \hat{q}_{kl}(\hat{\rho})$, where $\hat{\rho}_{kl} = \tilde{\rho}_{kl0} + \tilde{\rho}_{kl1}$.

Let $\tilde{\rho}^m$ be the internal cross-sectional expanded type distribution $\lambda_0 \left(\tilde{\beta}^m\right)^{-1}$. For $k,l \in S$, $\tilde{\rho}_{kl0}^m$ is the fraction of agents who are of type k, matched with type-l agents at the m-th step and paired at the $(m-1)$-th step as well, while $\tilde{\rho}_{kl1}^m$ is the fraction of agents who are of type k, matched with type-l agents at the m-th step and single at the $(m-1)$-th step. Note that $\hat{\rho}_{kl}^m$ is the proportion of type-k agents matched with type-l agents at the m-th step, which implies $\hat{\rho}_{kl}^m = \tilde{\rho}_{kl0}^m + \tilde{\rho}_{kl1}^m$.

Next, we define three mappings T_1, T_2, T_3 on ${}^*\tilde{\Delta}$ to represent the transformation of the expanded type distribution after each step of random mutation, random matching, and random type changing and break-up.[9] For any $\rho \in {}^*\tilde{\Delta}$, let

$$[T_1(\tilde{\rho})]_{kl0} = \begin{cases} \sum_{k',\,l' \in S} \tilde{\rho}_{k'l'0}\, \hat{\eta}_{k'k}\, \hat{\eta}_{l'l} & \text{if } l \neq J \\ 0 & \text{if } l = J, \end{cases}$$

$$[T_1(\tilde{\rho})]_{kl1} = \begin{cases} 0 & \text{if } l \neq J \\ \sum_{k' \in S} \tilde{\rho}_{k'J1}\, \hat{\eta}_{k'k} & \text{if } l = J, \end{cases}$$

$$[T_2(\tilde{\rho})]_{kl0} = \begin{cases} \tilde{\rho}_{kl0} & \text{if } l \neq J \\ 0 & \text{if } l = J, \end{cases}$$

$$[T_2(\tilde{\rho})]_{kl1} = \begin{cases} \tilde{\rho}_{kJ1}\hat{q}_{kl}(\tilde{\rho}) & \text{if } l \neq J \\ \tilde{\rho}_{kJ1}\hat{q}_{k}(\tilde{\rho}) & \text{if } l = J, \end{cases}$$

$$[T_3(\tilde{\rho})]_{kl0} = \begin{cases} \tilde{\rho}_{kl0}\left(1 - \hat{\vartheta}_{kl}\right) + \sum_{k',l' \in S} \tilde{\rho}_{k'l'1}\, \hat{\xi}_{k'l'}\, \hat{\sigma}_{k'l'}(k,l) & \text{if } l \neq J \\ 0 & \text{if } l = J, \end{cases}$$

$$[T_3(\tilde{\rho})]_{kl1} = \begin{cases} 0 & \text{if } l \neq J \\ \sum_{k',l' \in S} \tilde{\rho}_{k'l'1}\left(1 - \hat{\xi}_{k'l'}\right) \hat{\varsigma}_{k'l'}(k) + \sum_{k',l' \in S} \tilde{\rho}_{k'l'0}\, \hat{\vartheta}_{k'l'}\, \hat{\varsigma}_{k'l'}(k) + \tilde{\rho}_{kJ1} & \text{if } l = J. \end{cases}$$

The following lemma shows the equicontinuity of T_1, T_2, T_3 and $\hat{q}$.

Lemma E.3. *There exists a sequence of positive hyperreal numbers $\{\xi_m\}_{m=-1}^{3M^2+1}$ with $\xi_{-1} = \frac{1}{M^{M^M}}$ and $(3M^2+1)\xi_m \leq \xi_0 \leq \xi_{-1}$ for any $m \in \{1, \ldots, 3M^2+1\}$ such that for any $m \in \{-1, 0, \ldots, 3M^2\}$, $r \in \{1,2,3\}$, $\tilde{\rho}, \tilde{\rho}' \in {}^*\tilde{\Delta}$, if $\|\tilde{\rho} - \tilde{\rho}'\|_\infty \leq \xi_{m+1}$, then*

$$\|T_r(\tilde{\rho}) - T_r(\tilde{\rho}')\|_\infty \leq \xi_m,$$

$$\|\hat{q}(\tilde{\rho}) - \hat{q}(\tilde{\rho}')\|_\infty \leq \xi_m.$$

In the rest of this paper, we shall take $\hat{M}$ to be $\left[\left(\frac{1}{\xi_{3M^2+1}}\right)^{10}\right] + 1$. Here, for a hyperreal number x, $[x]$ represents the smallest hyperinteger less than or equal to x. Since $\xi_{3M^2+1} \leq \frac{1}{M^{M^M}}$, we know that $\hat{M} > M^{M^M}$, which is compatible with the condition imposed at the beginning of Subsection E.2.

Let $e(m) = [\frac{m+2}{3}]$ and $f(m) = m - 3e(m) + 3$. Then for any $m \in \{1, \ldots, 3M^2\}$, the m-th step in the hyperfinite dynamical system is also the $f(m)$-th step in the $e(m)$-th period. For hyperintegers $1 \leq m_1 \leq m_2 \leq 3M^2$, we use $U_{m_1}^{m_2}$ to represent $T_{f(m_2)} \circ T_{f(m_2-1)} \circ \cdots \circ T_{f(m_1)}$. For convenience, when $1 \leq m_2 < m_1 \leq 3M^2$, $U_{m_1}^{m_2}$ is defined to be the identity mapping on ${}^*\tilde{\Delta}$.

[9] If the expanded type distribution at the beginning of step $3n-2$ is $\tilde{\rho}$, Lemma E.14 indicates that the expected expanded type distribution at the end of step $3n-2$ is $T_1(\tilde{\rho})$. Similarly, Lemma E.16 says that the expected expanded type distribution at the end of step $3n$ is $T_3(\tilde{\rho})$ if the expanded type distribution at the beginning of step $3n$ is $\tilde{\rho}$. However, $T_2(\tilde{\rho})$ is not the expected expanded type distribution at the end of step $3n-1$ if the type distribution at the beginning of step $3n-1$ is $\tilde{\rho}$. Nevertheless, by Lemma E.15, $T_2(\tilde{\rho})$ is a good approximation of the expected expanded type distribution at the end of step $3n-1$.

The following lemma shows that the expected expanded type distribution at the m-th step $\mathbb{E}\tilde{\rho}^m$ is infinitesimally close to the repeated applications of the transformation T_1, T_2, T_3 to the initial expected expanded type distribution.

Lemma E.4. *For any $m \in \{1, 2, \ldots, 3M^2\}$, we have $\mathbb{E}(\tilde{\rho}^m) \simeq U_1^m(\mathbb{E}(\tilde{\rho}^0))$.*

Let $\mathcal{F}^m = \{F \in \mathcal{E}^{3M^2} : F = F^m \times \Pi_{m'=m+1}^{3M^2} \Omega_{m'} \text{ and } F^m \in \mathcal{E}^m\}$. Any set F in $\mathcal{F}^m$ represents an event that "happens" by step m. For example, we use $\left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}$ to represent some event that happens by step $3n-2$ in which $\tilde{\beta}_i^{3n-2} = (k, J, 1)$. The following two lemmas consider conditional probabilities[10] of the form of $P_0\left(\tilde{\beta}_i^{m+1} = b \,\middle|\, \left(\tilde{\beta}_i^m = a\right) \cap F^m\right)$ for $F^m \in \mathcal{F}^m$, which will be used in Subsection E.4 below.

The following lemma provides an upper bound on the difference between $\hat{q}_{kl}\left(U_1^{3n-2}(\mathbb{E}(\tilde{\rho}^0))\right)$ and $P_0\left(\tilde{\beta}_i^{3n-1} = (k, l, 1) \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right)$.

Lemma E.5. *For any $i \in I$, $n \in \{1, 2, \ldots, M^2\}$, $k, l \in S$, and $F^{3n-2} \in \mathcal{F}^{3n-2}$ with $P_0\left(\left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right) > 0$, we have*

$$\begin{aligned} &\left| P_0\left(\tilde{\beta}_i^{3n-1} = (k, l, 1) \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right) - \hat{q}_{kl}\left(U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)\right) \right| \\ &\qquad \leq \frac{1}{M^3 P_0\left(\left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right)} + \frac{1}{M^2}. \end{aligned}$$

The next lemma shows the relationship between $P_0\left(\tilde{\beta}_i^{m+1} = b \,\middle|\, \tilde{\beta}_i^m = a\right)$ and $P_0\left(\tilde{\beta}_i^{m+1} = b \,\middle|\, \left(\tilde{\beta}_i^m = a\right) \cap F^m\right)$.

Lemma E.6. *Fix any $i \in I$, $a, b \in \tilde{S}$, and $n \in \{1, 2, \ldots, M^2\}$.*

(i) For any $F^{3n-3} \in \mathcal{F}^{3n-3}$ with $P_0\left(\left(\tilde{\beta}_i^{3n-3} = a\right) \cap F^{3n-3}\right) > 0$, the following identity holds:

$$P_0\left(\tilde{\beta}_i^{3n-2} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-3} = a\right) \cap F^{3n-3}\right) = P_0\left(\tilde{\beta}_i^{3n-2} = b \,\middle|\, \tilde{\beta}_i^{3n-3} = a\right).$$

(ii) For any $F^{3n-2} \in \mathcal{F}^{3n-2}$ with $P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) > 0$, we have the following inequality

$$\begin{aligned} &\left| P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) - P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \tilde{\beta}_i^{3n-2} = a\right) \right| \\ &\qquad \leq \frac{1}{M^3 P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right)} + \frac{1}{M^2}. \end{aligned}$$

(iii) For any $F^{3n-1} \in \mathcal{F}^{3n-1}$ with $P_0\left(\left(\tilde{\beta}_i^{3n-1} = a\right) \cap F^{3n-1}\right) > 0$, we have

$$P_0\left(\tilde{\beta}_i^{3n} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-1} = a\right) \cap F^{3n-1}\right) = P_0\left(\tilde{\beta}_i^{3n} = b \,\middle|\, \tilde{\beta}_i^{3n-1} = a\right).$$

[10] For given events A and B with $P_0(A) = 0$, we can define the value of the conditional probability $P_0(B|A)$ to be any number in $[0, 1]$ that suits a particular context.

For any $i \in I$ and $m \in \{0, 1, \ldots, 3M^2\}$, let $\mathcal{F}_i^m$ be the internal algebra generated by $\{\tilde{\beta}_i^{m'}\}_{m'=0}^m$. Any set in $\mathcal{F}_i^m$ represents an event for agent i that happens by step m.

A Markov property for the expanded type process is presented below.

Lemma E.7. *Fix any $i \in I$. $\tilde{\beta}_i$ satisfies the Markov property in the sense that for any $m, m' \in \{0, 1, \ldots, 3M^2\}$ with $m > m'$, $a, a' \in \tilde{S}$, and $F_i^{m'-1} \in \mathcal{F}_i^{m'-1}$,*

$$P\left(\left(\tilde{\beta}_i^m = a, \tilde{\beta}_i^{m'} = a'\right) \cap F_i^{m'-1}\right) P\left(\tilde{\beta}_i^{m'} = a'\right) = P\left(\tilde{\beta}_i^m = a, \tilde{\beta}_i^{m'} = a'\right) P\left(\left(\tilde{\beta}_i^{m'} = a'\right) \cap F_i^{m'-1}\right).$$

The following lemma shows that the expanded type process satisfies a pairwise independence condition.

Lemma E.8. *For any $i, j \in I$ with $i \neq j$ and $P_0\left(\hat{\pi}_i^0 = j\right) \leq \frac{1}{\hat{M}^{\frac{1}{5}}}$, we have*

$$P\left(F_i^m \cap F_j^m\right) = P\left(F_i^m\right) P\left(F_j^m\right)$$

for any $m \in \{0, 1, \ldots, 3M^2\}$, $F_i^m \in \mathcal{F}_i^m$, and $F_j^m \in \mathcal{F}_j^m$.

In the rest of this subsection, we consider an estimation of the number of mutations, matchings and break-ups that can happen within a finite time interval, and consider the expected cross-sectional expanded type distribution. For any $\omega \in \Omega$, let

$$\begin{aligned}
\hat{H}_i^m(\omega) &= \left|\{n \in \mathbb{T}_0 : \hat{\alpha}_i^{3n-2}(\omega) \neq \hat{\alpha}_i^{3n-3}(\omega) \text{ or } \hat{g}_i^{3n-2}(\omega) \neq \hat{g}_i^{3n-3}(\omega), 3n-2 \leq m\}\right|, \\
\hat{N}_i^m(\omega) &= \left|\{n \in \mathbb{T}_0 : \hat{g}_i^{3n-1}(\omega) \neq \hat{g}_i^{3n-2}(\omega), 3n-1 \leq m\}\right|, \\
\hat{R}_i^m(\omega) &= \left|\{n \in \mathbb{T}_0 : \hat{g}_i^{3n}(\omega) = J \text{ and } \hat{h}_i^{3n-1}(\omega) = 0, 3n \leq m\}\right|.
\end{aligned}$$

Here, $\hat{H}_i^m$ is the number of mutations of agent i and of the partner of agent i, by the m-th step, while $\hat{N}_i^m$ and $\hat{R}_i^m$ are the numbers of matchings and breakups of agent i by the m-th step. Let $\hat{X}_i^m = \hat{H}_i^m + \hat{N}_i^m + \hat{R}_i^m$.

The following lemma provides a lower bound for the probability that there is no jump for the counting process $\hat{X}_i$ between two different steps.

Lemma E.9. *For any $m, \Delta m \in \{0, \ldots, 3M^2\}$ and $F^m \in \mathcal{F}^m$ such that $m + \Delta m \leq 3M^2$, $\frac{\Delta m}{M}$ is finite and $P_0(F^m) > 0$, we have*

$$P_0(\hat{X}_i^{m+\Delta m} = \hat{X}_i^m | F^m) \geq \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m} \simeq e^{-\frac{2K\bar{a}\Delta m}{M}}.$$

An estimation on the probability of changing type twice in a given time period is presented below.

Lemma E.10. *For any $m, \Delta m \in \{0, \ldots, 3M^2\}$ such that $m + \Delta m \leq 3M^2$, $\frac{\Delta m}{M}$ is finite and $P_0(F^m) > 0$, we have*

$$P_0\left(\hat{X}_i^{m+\Delta m} - \hat{X}_i^m \geq 2 \,\middle|\, F^m\right) \lesssim \left(1 - e^{-\frac{2K\bar{a}\Delta m}{M}}\right)^2 .$$[11]

The next lemma shows that up to any finite time, any agent can only change their types finitely many times with probability one.

Lemma E.11. *For any $i \in I$, let $A_i = \{\omega \in \Omega : \hat{X}_i^m(\omega)$ is finite for any m such that $\frac{m}{M}$ is finite$\}$. Let $A = \{(j, \omega) \in I \times \Omega : \hat{X}_j^m(\omega)$ is finite for any m such that $\frac{m}{M}$ is finite$\}$. Then $P(A_i) = 1$ for any $i \in I$, and $\lambda \boxtimes P(A) = 1$.*

An upper bound is provided below for $\|\mathbb{E}\left(\tilde{\rho}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}^m\right)\|_\infty$.

Lemma E.12. *For any $m, \Delta m \in \{0, \ldots, 3M^2\}$ such that $m + \Delta m \leq 3M^2$, $\frac{\Delta m}{M}$ is finite, we have*

$$\|\mathbb{E}\left(\tilde{\rho}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}^m\right)\|_\infty \lesssim 1 - e^{-\frac{2K\bar{a}\Delta m}{M}}.$$

By Lemma E.12, it is easy to prove the following lemma.

Lemma E.13. *Let f be a standard real-valued continuous function on $\tilde{\Delta}$. For any $m, \Delta m \in \{0, \ldots, 3M^2\}$ such that $m + \Delta m \leq 3M^2$, $\frac{\Delta m}{M}$ is infinitesimal, we have $\mathbb{E}\left({}^*f\left(\tilde{\rho}^{m+\Delta m}\right)\right) - \mathbb{E}\left({}^*f\left(\tilde{\rho}^m\right)\right)$ is infinitesimal.*

E.4 Existence of continuous-time random matching

In this subsection, we prove that there exist $\alpha : I \times \Omega \times \mathbb{R}_+ \to S$, $\pi : I \times \Omega \times \mathbb{R}_+ \to I$, and $g : I \times \Omega \times \mathbb{R}_+ \to S \cup \{J\}$ satisfying all the properties described in Appendix A.1. Towards this end, we divide the proof into six parts. In Part 1, we define these processes, and discuss their basic properties. In Part 2, we prove that (α, g) is Markovian and independent. We then check that the transition-intensity matrix of the relevant Markov chains at time t is $Q(\check{p}(t))$. In particular, we verify that the transition intensities for agents' expanded types at time t satisfy Equations (A.1), (A.2), (A.3), and (A.4) in Parts 3, 4, 5, and 6 respectively.

Part 1: Fix any $t \in \mathbb{R}_+$, and denote the integer part of tM by $\bar{n}$. Based on the hyperfinite dynamic system defined in Appendix E.2, let $\alpha'(t) = \hat{\alpha}^{3\bar{n}}$, $\pi(t) = \hat{\pi}^{3\bar{n}}$, $g'(t) = \hat{g}^{3\bar{n}}$. Since $\hat{\alpha}^{3\bar{n}}$, $\hat{\pi}^{3\bar{n}}$, and $\hat{g}^{3\bar{n}}$ are internal, it is clear that $\alpha'(t)$, $\pi(t)$, and $g'(t)$ are measurable on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

[11] The notation $\lesssim$ is defined in Subsection D.1, which means "$<$" or "$\simeq$".

Fix any $i \in I$. The stochastic processes α_i' and g_i' may not be RCLL. Lemma E.11 shows that the following set

$$A_i = \{\omega \in \Omega : \hat{X}_i^m(\omega) \text{ is finite for any } m \text{ such that } \frac{m}{M} \text{ is finite}\}$$

is measurable and has probability one under P. We define

$$\alpha_i(\omega, t) = \begin{cases} \lim_{t' \to t^+} \alpha_i'(\omega, t') & \text{if } \omega \in A_i \\ 1 & \text{otherwise,} \end{cases}$$

$$g_i(\omega, t) = \begin{cases} \lim_{t' \to t^+} g_i'(\omega, t') & \text{if } \omega \in A_i \\ 1 & \text{otherwise.} \end{cases}$$

Now we prove that α and g are well defined and measurable on $(I \times \Omega \times \mathbb{R}_+, (\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}(\mathbb{R}_+))$. For any $\omega \in A_i$, $\alpha_i'(\omega, t')$ can only change finitely many times in the time interval $[0, t+1]$. Then there exists $\epsilon > 0$ such that $\alpha_i'(\omega, t')$ are constant on $(t, t+\epsilon)$. Then $\lim_{t' \to t^+} \alpha_i'(\omega, t')$ is well defined for any $\omega \in A_i$. We can prove that g is well defined in the same way. By the definition of α and g, and the fact that A_i is measurable, it is clear that $\alpha_i(\omega, t)$ and $g_i(\omega, t)$ are measurable on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, and for any $(i, \omega) \in I \times \Omega$, the stochastic processes $\alpha_i(\omega, t)$ and $g_i(\omega, t)$ are RCLL in $t \in \mathbb{R}_+$. By Proposition 1.13 in Karatzas and Shreve (1991), α and g are measurable on $(I \times \Omega \times \mathbb{R}_+, (\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}(\mathbb{R}_+))$.

For each $n \in \mathbb{T}_0 = \{n\}_{n=0}^{M^2}$ and $\omega \in \Omega$, since $\hat{\pi}_\omega^{3n}$ is an internal involution on I and λ_0 is the hyperfinite counting probability measure on $\mathcal{I}_0$, it is obvious that the particular case $\hat{\pi}_\omega^{3\bar{n}}$ is measure-preserving from the Loeb space $(I, \mathcal{I}, \lambda)$ to itself. Hence, for any $t \in \mathbb{R}_+$ and $\omega \in \Omega$, $\pi_{\omega t}(\cdot)$ is an internal involution on I and is measure-preserving.

Part 2: First, we need to find out the relationship between α and $\hat{\alpha}$. Fix any $i \in I$ and $t \in \mathbb{R}_+$. Letting $E_t = \{n \in {}^*\mathbb{N} : \frac{n}{M} \in monad(t)\}$, it is obvious that $\bar{n} \in E_t$. Fix any $\omega \in A_i$. If $\hat{\alpha}_i^{3n}(\omega) \equiv C$ for any $n \in E_t$, then the Spillover Principle implies that there exists $n_1, n_2 \in \mathbb{T}_0$ such that $\mathrm{st}(\frac{n_1}{M}) < t < \mathrm{st}(\frac{n_2}{M})$ for $t > 0$, $n_1 = 0$, $\mathrm{st}(\frac{n_2}{M}) > 0$ for $t = 0$, and $\hat{\alpha}_i^{3n}(\omega) \equiv C$ for any $n \in \{n_1, n_1+1, \ldots, n_2\}$. Hence for any t' in the time interval $\left(\mathrm{st}(\frac{n_1}{M}), \mathrm{st}(\frac{n_2}{M})\right)$, $\alpha_i'(t') = C$. Therefore,

$$\alpha_i(\omega, t) = \lim_{t' \to t^+} \alpha_i'(\omega, t') = C.$$

Fix any $n_0 \in E_t$. For any $\omega \in A_i$, if $\hat{\alpha}_i^{3n_0}(\omega) \neq \alpha_i(\omega, t)$, by the argument above, $\hat{\alpha}_i^{3n}(\omega)$ can not be constant for $n \in E_t$. Hence there is a mutation, matching, or break up at some period in E_t. Therefore, by Lemma E.9, for any $n_1, n_2 \in \mathbb{T}_0$ such that $\mathrm{st}(\frac{n_1}{M}) < t < \mathrm{st}(\frac{n_2}{M})$ for $t > 0$, $n_1 = 0$, and $\mathrm{st}(\frac{n_2}{M}) > 0$ for $t = 0$, we have

$$\begin{aligned} P\left(\hat{\alpha}_i^{3n_0} \neq \alpha_{it}\right) &\leq P\left(\hat{X}_i^{3n_1} \neq \hat{X}_i^{3n_2}\right) \\ &\leq 1 - \mathrm{st}\left(e^{-\frac{6K\bar{a}(n_2 - n_1)}{M}}\right). \end{aligned}$$

If $\mathrm{st}\left(\frac{n_2-n_1}{M}\right) \to 0$, then $\mathrm{st}\left(e^{-\frac{6K\bar{a}(n_2-n_1)}{M}}\right) \to 1$. Hence, we have $P\left(\hat{\alpha}_i^{3n_0} \neq \alpha_{it}\right) = 0$, which implies that $P\left(\hat{\alpha}_i^{3n_0} = \alpha_{it}\right) = 1$. Similarly, we can prove that $P\left(\hat{g}_i^{3n_0} = g_{it}\right) = 1$. Denote $\beta_i(t) = (\alpha_i(t), g_i(t))$ and $\hat{\beta}_i^m = (\hat{\alpha}_i^m, \hat{g}_i^m)$ for any $0 \leq m \leq 3M^2$. Then we have

$$P\left(\omega \in \Omega : \hat{\beta}_i^{3n_0}(\omega) = \beta_i(\omega, t)\right) = 1. \tag{E.5}$$

Then, we can derive that for P-almost all $\omega \in \Omega$,

$$g_i(\omega, t) = \hat{g}_i^{3\bar{n}}(\omega) = \begin{cases} \hat{\alpha}_i^{3\bar{n}}(\omega) & \text{if } \hat{\pi}_i^{3\bar{n}}(\omega) \neq i \\ J & \text{if } \hat{\pi}_i^{3\bar{n}}(\omega) = i \end{cases} = \begin{cases} \alpha(\pi(i,\omega,t)) & \text{if } \pi(i,\omega,t) \neq i \\ J & \text{if } \pi(i,\omega,t) = i. \end{cases}$$

By Lemma E.2, $(\hat{\alpha}^0, \hat{g}^0)$ can be deterministic or identically distributed, and $\mathbb{E}(\hat{\rho}^0) \simeq \hat{p}^0$ for both cases. Equation (E.5) implies that for any $i \in I$,

$$P\left((\hat{\alpha}_i^0(\omega), \hat{g}_i^0(\omega)) = (\alpha_i(\omega, 0), g_i(\omega, 0))\right) = 1.$$

Hence, (α_0, g_0) can be deterministic or identically distributed, and $\mathbb{E}(\hat{p}(0)) \simeq \mathbb{E}(\hat{\rho}^0) \simeq \hat{p}^0$ for both cases, where $\hat{p}(0)$ is the cross-sectional extended type distribution of (α_0, g_0).

For any $n \in \mathbb{T}_0$, Equation (E.4) indicates that

$$\hat{h}_i^{3n} = \begin{cases} 0 & \text{if } \hat{g}_i^{3n} \neq J \\ 1 & \text{if } \hat{g}_i^{3n} = J \end{cases} = \mathbf{1}_{\{J\}}\left(\hat{g}_i^{3n}\right),$$

which means that the value of $\hat{h}_i^{3n}$ is completely determined by $\hat{g}_i^{3n}$. It follows from Lemma E.7 that for any $r \in \mathbb{N}$, $m_1 = 3n_1, m_2 = 3n_2, \ldots, m_r = 3n_r$ with $n_1 > n_2 > \cdots > n_r$ in $\mathbb{T}_0$, and any expanded types $a_1, a_2, \ldots, a_r$ in $\tilde{S}$,

$$\begin{aligned} & P\left(\tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{m_2} = a_2, \ldots, \tilde{\beta}_i^{m_r} = a_r\right) P\left(\tilde{\beta}_i^{m_2} = a_2\right) \\ = \; & P\left(\tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{m_2} = a_2\right) P\left(\tilde{\beta}_i^{m_2} = a_2, \ldots, \tilde{\beta}_i^{m_r} = a_r\right). \end{aligned}$$

Hence, we obtain that for any $n_1 > n_2 > \cdots > n_r$ in $\mathbb{T}_0$, and any extended types $b_1, b_2, \ldots, b_r$ in $\hat{S}$,

$$\begin{aligned} & P\left(\hat{\beta}_i^{3n_1} = b_1, \hat{\beta}_i^{3n_2} = b_2, \ldots, \hat{\beta}_i^{3n_r} = b_r\right) P\left(\hat{\beta}_i^{3n_2} = b_2\right) \\ = \; & P\left(\hat{\beta}_i^{3n_1} = b_1, \hat{\beta}_i^{3n_2} = b_2\right) P\left(\hat{\beta}_i^{3n_2} = b_2, \ldots, \hat{\beta}_i^{3n_r} = b_r\right). \end{aligned} \tag{E.6}$$

For any $r \in \mathbb{N}$, and real time sequence $t_1 > t_2 > \cdots > t_r$ in $\mathbb{R}_+$, choose $n_k \in \mathbb{T}_0$ such that $\frac{n_k}{M} \simeq t_k$ for $1 \leq k \leq r$. Then, it follows from Equations (E.5) and (E.6) that for any extended types $b_1, b_2, \ldots, b_r$ in $\hat{S}$

$$\begin{aligned} & P\left(\beta_i(t_1) = b_1, \beta_i(t_2) = b_2, \ldots, \beta_i(t_r) = b_r\right) P\left(\beta_i(t_2) = b_2\right) \\ = \; & P\left(\beta_i(t_1) = b_1, \beta_i(t_2) = b_2\right) P\left(\beta_i(t_2) = b_2, \ldots, \beta_i(t_r) = b_r\right), \end{aligned}$$

which implies that the stochastic process $\beta_i = (\alpha_i, g_i)$ has the Markov property.

For any $j \in I$ with $j \neq i$ and $P_0\left(\hat{\pi}_i^0 = j\right) \leq \frac{1}{\hat{M}^{\frac{1}{5}}}$, Lemma E.8 implies that for any $n_1 > n_2 > \cdots > n_r$ in $\mathbb{T}_0$, and any extended types $b_1, c_1, b_2, c_2, \ldots, b_r, c_r$ in $\hat{S}$,

$$\begin{aligned} & P\left(\hat{\beta}_i^{3n_1} = b_1, \hat{\beta}_j^{3n_1} = c_1, \ldots, \hat{\beta}_i^{3n_r} = b_r, \hat{\beta}_j^{3n_r} = c_r\right) \\ = \; & P\left(\hat{\beta}_i^{3n_1} = b_1, \ldots, \hat{\beta}_i^{3n_r} = b_r\right) P\left(\hat{\beta}_j^{3n_1} = c_1, \ldots, \hat{\beta}_j^{3n_r} = c_r\right). \end{aligned} \tag{E.7}$$

Assume that $(\hat{\alpha}^0, \hat{g}^0)$ is deterministic as in Lemma E.2. Fix any agent $j \in I$ $(j \neq i)$ who is not matched with agent i at time zero. Then, we know that $P_0\left(\hat{\pi}_i^0 = j\right) = 0 < \frac{1}{\hat{M}^{\frac{1}{5}}}$. For any $r \in \mathbb{N}$, and real time sequence $t_1 > t_2 > \cdots > t_r$ in $\mathbb{R}_+$, choose $n_k \in \mathbb{T}_0$ such that $\frac{n_k}{M} \simeq t_k$ for $1 \leq k \leq r$. We can obtain from Equations (E.5) and (E.7) that for any extended types $b_1, c_1, b_2, c_2, \ldots, b_r, c_r$ in $\hat{S}$,

$$\begin{aligned} & P\left(\beta_i(t_1) = b_1, \beta_j(t_1) = c_1, \ldots, \beta_i(t_r) = b_r, \beta_j(t_r) = c_r\right) \\ = \; & P\left(\beta_i(t_1) = b_1, \ldots, \beta_i(t_r) = b_r\right) P\left(\beta_j(t_1) = c_1, \ldots, \beta_j(t_r) = c_r\right), \end{aligned} \tag{E.8}$$

which implies that the stochastic processes (α_i, g_i) and (α_j, g_j) are independent.

Next, consider the case that $(\hat{\alpha}^0, \hat{g}^0)$ is identically distributed as in Lemma E.2. Fix any agent $j \in I$ with $j \neq i$. Lemma E.2 indicates that $P_0\left(\hat{\pi}_i^0 = j\right) \leq \frac{1}{\hat{M}^{\frac{1}{5}}}$, which implies that Equation (E.7) holds. By Equations (E.5) and (E.7), we know that Equation (E.8) also holds in this case. Hence, the stochastic processes (α_i, g_i) and (α_j, g_j) are independent.

Part 3: Fix any $i \in I$, $t \in \mathbb{R}_+$, $k, l, k', l' \in S$ with $(k, l) \neq (k', l')$ and $P\left(\beta_i(t) = (k, l)\right) > 0$. The purpose of this part is to verify that the transition intensity agent i from expanded type (k, l) to expanded type (k', l') at time t is given by Equation (A.1).

For any $\Delta t \in \mathbb{R}_{++}$, let $n, \Delta n \in {}^*\mathbb{N}$ such that $\frac{n}{M} \in monad(t)$ and $\frac{\Delta n}{M} \in monad(\Delta t)$. By Equation (E.5),

$$P\left(\beta_i(t + \Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k, l)\right) \simeq P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', l') \,\middle|\, \hat{\beta}_i^{3n} = (k, l)\right).$$

Lemma E.10 indicates that the probability for agent i to change her extended type twice in the time interval $[t, t + \Delta t]$ is of order Δt^2. Hence,

$$\begin{aligned} & P\left(\beta_i(t + \Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k, l)\right) \\ & \quad = P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', l'), \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, l)\right) + O(\Delta t^2). \end{aligned} \tag{E.9}$$

For any $k_1, l_1 \in S$ and $m, m' \in \{3n, 3n+1, \ldots, 3M^2\}$ with $m > m'$, let

$$B_{k_1 l_1}^m = \{\omega \in \Omega : \hat{\beta}_i^m(\omega) = (k_1, l_1), \hat{X}_i^{3n}(\omega) = \hat{X}_i^{m-1}(\omega), \hat{\beta}_i^{3n}(\omega) = (k, l)\},$$

which is the event that $\hat{\beta}_i^m = (k_1, l_1)$, $\hat{\beta}_i^{3n} = (k, l)$, and there is neither mutation, nor matching, nor break-up for agent i between $3n$-th step and $(m-1)$-th step. Further,

$$C_{m'}^m = \{\omega \in \Omega : \hat{X}_i^{m'}(\omega) = \hat{X}_i^m(\omega)\}$$

is the event that there is neither mutation, nor matching, nor break-up for agent i between m'-th step and m-th step. In particular, when the event $C_{m'}^m$ happens, agent i does not change her extended type between m'-th step and m-th step.

If the events $\left(\hat{\beta}_i^{3n} = (k, l)\right)$ and $\left(\hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1\right)$ happen, then mutation is the only way for agent i to change her extended type to (k', l') by the end of step $3n + 3\Delta n$ (since the other two steps must involve single agents). Based on the definition of conditional probabilities, Equation (E.9) can be expanded as follows:

$$\begin{aligned}
&P\left(\beta_i(t+\Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k, l)\right) \\
&\quad = P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', l'), \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, l)\right) + O(\Delta t^2) \\
&\quad = \sum_{r=n}^{n+\Delta n-1} P_0\left(B_{k'l'}^{3r+1} \cap C_{3r+1}^{3n+3\Delta n} \,\middle|\, \hat{\beta}_i^{3n} = (k, l)\right) + O(\Delta t^2) \\
&\quad = \sum_{r=n}^{n+\Delta n-1} \left[P_0\left(B_{k'l'}^{3r+1} \,\middle|\, \hat{\beta}_i^{3n} = (k, l)\right) P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'l'}^{3r+1}\right)\right] + O(\Delta t^2) \\
&\quad = \sum_{r=n}^{n+\Delta n-1} \left[P_0\left(\hat{\beta}_i^{3r+1} = (k', l') \,\middle|\, C_{3n}^{3r} \cap \left(\hat{\beta}_i^{3n} = (k, l)\right)\right) P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n} = (k, l)\right)\right. \\
&\qquad\qquad \left. P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'l'}^{3r+1}\right)\right] + O(\Delta t^2).
\end{aligned}$$

By Equation (E.3) and Lemma E.6, we obtain that

$$\begin{aligned}
&P_0\left(\hat{\beta}_i^{3r+1} = (k', l') \,\middle|\, C_{3n}^{3r} \cap \left(\hat{\beta}_i^{3n} = (k, l)\right)\right) \\
&\quad = P_0\left(\hat{\beta}_i^{3r+1} = (k', l') \,\middle|\, C_{3n}^{3r} \cap \left(\hat{\beta}_i^{3r} = (k, l)\right)\right) \\
&\quad = P_0\left(\tilde{\beta}_i^{3r+1} = (k', l', 0) \,\middle|\, C_{3n}^{3r} \cap \left(\tilde{\beta}_i^{3r} = (k, l, 0)\right)\right) \\
&\quad = P_0\left(\tilde{\beta}_i^{3r+1} = (k', l', 0) \,\middle|\, \tilde{\beta}_i^{3r} = (k, l, 0)\right) \\
&\quad = P_0\left(\hat{\beta}_i^{3r+1} = (k', l') \,\middle|\, \hat{\beta}_i^{3r} = (k, l)\right) \\
&\quad = \hat{\eta}_{kk'}\hat{\eta}_{ll'},
\end{aligned}$$

where the last identity follows from the step of random mutation for matched agents in the construction of the hyperfinite dynamic matching model. Then, the above identities imply that

$$\begin{aligned}
&P\left(\beta_i(t+\Delta t)=(k',l') \,\middle|\, \beta_i(t)=(k,l)\right) \\
&= \sum_{r=n}^{n+\Delta n-1}\Big[P_0\left(\hat{\beta}_i^{3r+1}=(k',l') \,\middle|\, \hat{\beta}_i^{3n}=(k,l)\right) P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,l)\right) \\
&\qquad P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'l'}^{3r+1}\right)\Big]+O(\Delta t^2). \qquad \text{(E.10)}
\end{aligned}$$

When $k \neq k'$ and $l = l'$, $P_0\left(\hat{\beta}_i^{3r+1}=(k',l) \,\middle|\, \hat{\beta}_i^{3r}=(k,l)\right)=\hat{\eta}_{kk'}\hat{\eta}_{ll}$, which implies that

$$\begin{aligned}
&\left|P_0\left(\hat{\beta}_i^{3r+1}=(k',l) \,\middle|\, \hat{\beta}_i^{3r}=(k,l)\right)-\hat{\eta}_{kk'}\right| \\
&= |\hat{\eta}_{kk'}\hat{\eta}_{ll}-\hat{\eta}_{kk'}| = \hat{\eta}_{kk'}(1-\hat{\eta}_{ll}) = \hat{\eta}_{kk'}\sum_{l''\in S\setminus\{l\}}\hat{\eta}_{ll''} \le K\left(\frac{\bar{a}}{M}\right)^2.
\end{aligned}$$

Now, we estimate the difference

$$\begin{aligned}
&\left|P\left(\beta_i(t+\Delta t)=(k',l) \,\middle|\, \beta_i(t)=(k,l)\right)-\sum_{r=n}^{n+\Delta n-1}\hat{\eta}_{kk'}\right| \\
&\le \sum_{r=n}^{n+\Delta n-1}\Big|\left(P_0\left(\hat{\beta}_i^{3r+1}=(k',l) \,\middle|\, \hat{\beta}_i^{3n}=(k,l)\right)-\hat{\eta}_{kk'}\right)P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,l)\right) \\
&\qquad P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'l'}^{3r+1}\right)\Big| \\
&\quad + \sum_{r=n}^{n+\Delta n-1}\hat{\eta}_{kk'}\left|P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,l)\right)P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'l'}^{3r+1}\right)-1\right|+O(\Delta t^2) \\
&\le \sum_{r=n}^{n+\Delta n-1}K\left(\frac{\bar{a}}{M}\right)^2 \\
&\quad + \sum_{r=n}^{n+\Delta n-1}\hat{\eta}_{kk'}\left|P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,l)\right)P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'l'}^{3r+1}\right)-1\right|+O(\Delta t^2).
\end{aligned}$$

Since $\bar{a}$ is finite, we know that $\sum_{r=n}^{n+\Delta n-1}K\left(\frac{\bar{a}}{M}\right)^2=\frac{\bar{a}^2}{M}\frac{\Delta n}{M}$ is infinitesimal and can be absorbed into $O(\Delta t^2)$. By Lemma E.9,

$$P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,l)\right)P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'l'}^{3r+1}\right) \gtrsim e^{-\frac{6K(r-n)\bar{a}}{M}}e^{-\frac{6K(n+\Delta n-r)\bar{a}}{M}}.$$

Then, it follows from the above inequalities that

$$\begin{aligned}
&\left|P\left(\beta_i(t+\Delta t)=(k',l) \,\middle|\, \beta_i(t)=(k,l)\right)-\sum_{r=n}^{n+\Delta n-1}\hat{\eta}_{kk'}\right| \\
&\le \sum_{r=n}^{n+\Delta n-1}\frac{\bar{a}}{M}\left(1-e^{-\frac{6K(r-n)\bar{a}}{M}}e^{-\frac{6K(n+\Delta n-r)\bar{a}}{M}}\right)+O(\Delta t^2) \\
&= \bar{a}\Delta t\left(1-e^{-6K\bar{a}\Delta t}\right)+O(\Delta t^2) \\
&= O(\Delta t^2).
\end{aligned}$$

Therefore, we obtain the estimation

$$
\begin{aligned}
P\left(\beta_i(t+\Delta t)=(k',l)\,\middle|\,\beta_i(t)=(k,l)\right) &= \sum_{r=n}^{n+\Delta n-1} \hat{\eta}_{kk'} + O(\Delta t^2) \\
&= \eta_{kk'}\Delta t + O(\Delta t^2). \qquad \text{(E.11)}
\end{aligned}
$$

When $k=k'$ and $l\neq l'$, we can also prove in the same way as above that

$$
P\left(\beta_i(t+\Delta t)=(k,l')\,\middle|\,\beta_i(t)=(k,l)\right) = \eta_{ll'}\Delta t + O(\Delta t^2). \qquad \text{(E.12)}
$$

It remains to consider the case that $k\neq k'$ and $l\neq l'$. It is clear that

$$
P_0\left(\hat{\beta}_i^{3r+1}=(k',l')\,\middle|\,\hat{\beta}_i^{3r}=(k,l)\right) = \hat{\eta}_{kk'}\hat{\eta}_{ll'} \leq \left(\frac{\bar{a}}{M}\right)^2.
$$

Therefore, Equation (E.10) implies that

$$
\begin{aligned}
&P\left(\beta_i(t+\Delta t)=(k',l')\,\middle|\,\beta_i(t)=(k,l)\right) \\
&= \sum_{r=n}^{n+\Delta n-1} \Big[P_0\left(\hat{\beta}_i^{3r+1}=(k',l')\,\middle|\,\hat{\beta}_i^{3n}=(k,l)\right) P_0\left(C_{3n}^{3r}\,\middle|\,\hat{\beta}_i^{3n}=(k,l)\right) \\
&\qquad P_0\left(C_{3r+1}^{3n+3\Delta n}\,\middle|\,B_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,l)\right)\right)\Big] + O(\Delta t^2) \\
&\leq \sum_{r=n}^{n+\Delta n-1} \left(\frac{\bar{a}}{M}\right)^2 + O(\Delta t^2) \\
&= O(\Delta t^2). \qquad \text{(E.13)}
\end{aligned}
$$

By combining Equations (E.11), (E.12), (E.13), we obtain that

$$
P\left(\beta_i(t+\Delta t)=(k',l')\,\middle|\,\beta_i(t)=(k,l)\right) = \left(\eta_{kk'}\delta_l(l')+\eta_{ll'}\delta_k(k')\right)\Delta t + O(\Delta t^2).
$$

Hence, agent i's transition intensity for her expanded types from (k,l) to (k',l') at time t is indeed $Q_{(k,l)(k',l')}(\check{p}(t))$, as given in Equation (A.1).

Part 4: Fix any $i\in I$, $t\in\mathbb{R}_+$, $k,l,k'\in S$ with $P(\beta_i(t)=(k,l))>0$. The purpose of this part is to verify that agent i's transition intensity for her expanded types from (k,l) to (k',J) at time t is given in Equation (A.2).

For any $\Delta t\in\mathbb{R}_{++}$, let $n,\Delta n\in{}^*\mathbb{N}$ such that $\frac{n}{M}\in monad(t)$ and $\frac{\Delta n}{M}\in monad(\Delta t)$. By Equation (E.5), we have

$$
P\left(\beta_i(t+\Delta t)=(k',J)\,\middle|\,\beta_i(t)=(k,l)\right) \simeq P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J)\,\middle|\,\hat{\beta}_i^{3n}=(k,l)\right).
$$

By Lemma E.10, the probability for agent i to change her extended type twice in the time interval $[t,t+\Delta t]$ is at level of Δt^2. Hence, we have

$$
\begin{aligned}
&P\left(\beta_i(t+\Delta t)=(k',J)\,\middle|\,\beta_i(t)=(k,l)\right) \\
&= P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J), \hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1\,\middle|\,\hat{\beta}_i^{3n}=(k,l)\right) + O(\Delta t^2). \qquad \text{(E.14)}
\end{aligned}
$$

For any $k_1 \in S$ and $m, m' \in \{3n, 3n+1, \ldots, 3M^2\}$ with $m > m'$, let

$$B^m_{k_1 J} = \{\omega \in \Omega : \hat{\beta}^m_i(\omega) = (k_1, J), \hat{X}^{3n}_i(\omega) = \hat{X}^{m-1}_i(\omega), \hat{\beta}^{3n}_i(\omega) = (k, l)\}$$

and $C^m_{m'} = \{\omega \in \Omega : \hat{X}^{m'}_i(\omega) = \hat{X}^m_i(\omega)\}$. Then, $B^m_{k_1 l_1}$ is the event that $\hat{\beta}^m_i = (k_1, J)$, $\hat{\beta}^{3n}_i = (k, l)$, and there is neither mutation, nor matching, nor break-up for agent i between $3n$-th step and $(m-1)$-th step; $C^m_{m'}$ is the event that there is neither mutation, nor matching, nor break-up for agent i between m'-th step and m-th step. In particular, when the event $C^m_{m'}$ happens, agent i does not change her extended type between m'-th step and m-th step.

If the events $\left(\hat{\beta}^{3n}_i = (k, l)\right)$ and $\left(\hat{X}^{3n+3\Delta n}_i - \hat{X}^{3n}_i = 1\right)$ happen, break-up is the only way for agent i to change her extended type to (k', J) by the end of step $3n + 3\Delta n$ (since, in the other two steps, paired agents must stay paired). Based on the definition of conditional probabilities, Equation (E.14) can be expanded as follows:

$$\begin{aligned}
& P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k, l)\right) \\
= \;& P_0\left(\hat{\beta}^{3n+3\Delta n}_i = (k', J), \hat{X}^{3n+3\Delta n}_i - \hat{X}^{3n}_i = 1 \,\middle|\, \hat{\beta}^{3n}_i = (k, l)\right) + O(\Delta t^2) \\
= \;& \sum_{r=n}^{n+\Delta n - 1} P_0\left(B^{3r+3}_{k'J} \cap C^{3n+3\Delta n}_{3r+3} \,\middle|\, \hat{\beta}^{3n}_i = (k, l)\right) + O(\Delta t^2) \\
= \;& \sum_{r=n}^{n+\Delta n - 1} \left[P_0\left(B^{3r+3}_{k'J} \,\middle|\, \hat{\beta}^{3n}_i = (k, l)\right) P_0\left(C^{3n+3\Delta n}_{3r+3} \,\middle|\, B^{3r+3}_{k'J}\right)\right] + O(\Delta t^2) \\
= \;& \sum_{r=n}^{n+\Delta n - 1} \left[P_0\left(\hat{\beta}^{3r+3}_i = (k', J) \,\middle|\, C^{3r+2}_{3n} \cap \left(\hat{\beta}^{3n}_i = (k, l)\right)\right) P_0\left(C^{3r+2}_{3n} \,\middle|\, \hat{\beta}^{3n}_i = (k, l)\right)\right. \\
& \qquad \left. P_0\left(C^{3n+3\Delta n}_{3r+3} \,\middle|\, B^{3r+3}_{k'J}\right)\right] + O(\Delta t^2).
\end{aligned}$$

It follows from Equation (E.4) and Lemma E.6 that

$$\begin{aligned}
& P_0\left(\hat{\beta}^{3r+3}_i = (k', J) \,\middle|\, C^{3r+2}_{3n} \cap \left(\hat{\beta}^{3n}_i = (k, l)\right)\right) \\
= \;& P_0\left(\hat{\beta}^{3r+3}_i = (k', J) \,\middle|\, C^{3r+2}_{3n} \cap \left(\hat{\beta}^{3r+2}_i = (k, l)\right)\right) \\
= \;& P_0\left(\tilde{\beta}^{3r+3}_i = (k', J, 1) \,\middle|\, C^{3r+2}_{3n} \cap \left(\tilde{\beta}^{3r+2}_i = (k, l, 0)\right)\right) \\
= \;& P_0\left(\tilde{\beta}^{3r+3}_i = (k', J, 1) \,\middle|\, \tilde{\beta}^{3r+2}_i = (k, l, 0)\right) \\
= \;& \hat{\vartheta}_{kl} \hat{\varsigma}_{kl}(k'),
\end{aligned}$$

where the last identity follows from the step of random type changing with break-up for agents (who are not newly matched, but break up the partnership) in the construction of the hyperfinite dynamic matching model. Then, the above identities imply that

$$\begin{aligned}
& P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k, l)\right) \\
= \;& \sum_{r=n}^{n+\Delta n - 1} \hat{\vartheta}_{kl} \hat{\varsigma}_{kl}(k') P_0\left(C^{3r+2}_{3n} \,\middle|\, \hat{\beta}^{3n}_i = (k, l)\right) P_0\left(C^{3n+3\Delta n}_{3r+3} \,\middle|\, B^{3r+3}_{k'J}\right) + O(\Delta t^2). \quad \text{(E.15)}
\end{aligned}$$

Next, we estimate the difference

$$
\begin{aligned}
& \left| P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k,l)\right) - \sum_{r=n}^{n+\Delta n-1} \hat{\vartheta}_{kl}\hat{\varsigma}_{kl}(k') \right| \\
\leq & \sum_{r=n}^{n+\Delta n-1} \hat{\vartheta}_{kl}\hat{\varsigma}_{kl}(k') \left| P_0\left(C_{3n}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n} = (k,l)\right) P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, B_{k'J}^{3r+3}\right) - 1 \right| \\
+ & \; O(\Delta t^2).
\end{aligned}
$$

By Lemma E.9, we obtain that

$$
\begin{aligned}
& P_0\left(C_{3n}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n} = (k,l)\right) P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, B_{k'J}^{3r+3}\right) \\
\gtrsim & \; e^{-\frac{2K\bar{a}(3r-3n+2)}{M}} e^{-\frac{2K\bar{a}(3n+3\Delta n-3r-3)}{M}} \\
\simeq & \; e^{-6K\bar{a}\Delta t}.
\end{aligned}
$$

Then, it follows from the above inequalities that

$$
\begin{aligned}
& \left| P\left(\beta_i(t+\Delta t) = (k', l) \,\middle|\, \beta_i(t) = (k,l)\right) - \sum_{r=n}^{n+\Delta n-1} \hat{\vartheta}_{kl}\hat{\varsigma}_{kl}(k') \right| \\
\leq & \sum_{r=n}^{n+\Delta n-1} \frac{\bar{a}}{M}\left(1 - e^{-6K\bar{a}\Delta t}\right) + O(\Delta t^2) \\
= & \; \bar{a}\Delta t\left(1 - e^{-6K\bar{a}\Delta t}\right) + O(\Delta t^2) \\
= & \; O(\Delta t^2).
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k,l)\right) &= \sum_{r=n}^{n+\Delta n-1} \hat{\vartheta}_{kl}\hat{\varsigma}_{kl}(k') + O(\Delta t^2) \\
&= \vartheta_{kl}\varsigma_{kl}(k')\Delta t + O(\Delta t^2), \qquad \text{(E.16)}
\end{aligned}
$$

which implies that agent i's transition intensity for her expanded types from (k,l) to (k',J) at time t is $Q_{(k,l)(k',J)}(\check{p}(t))$, as given in Equation (A.2).

Part 5: Fix any $i \in I$, $t \in \mathbb{R}_+$, $k, k', l' \in S$ with $P\left(\beta_i(t) = (k,J)\right) > 0$. The purpose of this part is to verify that agent i's transition intensity for her expanded types from (k,J) to (k',l') at time t is given in Equation (A.3).

For any $\Delta t \in \mathbb{R}_{++}$, let $n, \Delta n \in {}^*\mathbb{N}$ such that $\frac{n}{M} \in monad(t)$ and $\frac{\Delta n}{M} \in monad(\Delta t)$. By Equation (E.5), we have

$$
P\left(\beta_i(t+\Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k,J)\right) \simeq P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', l') \,\middle|\, \hat{\beta}_i^{3n} = (k,J)\right).
$$

Lemma E.10 says that the probability for agent i to change her extended type twice in the time interval $[t, t+\Delta t]$ is at level of Δt^2. Hence, we have

$$\begin{aligned} & P\left(\beta_i(t+\Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k, J)\right) \\ = \; & P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', l'), \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) + O(\Delta t^2). \end{aligned} \tag{E.17}$$

For any $k_1, l_1 \in S$ and $m, m' \in \{3n, 3n+1, \ldots, 3M^2\}$ with $m > m'$, let

$$B_{k_1 l_1}^m = \{\omega \in \Omega : \hat{\beta}_i^m(\omega) = (k_1, l_1), \hat{X}_i^{3n}(\omega) = \hat{X}_i^{m-1}(\omega), \hat{\beta}_i^{3n}(\omega) = (k, J)\}$$

and $C_{m'}^m = \{\omega \in \Omega : \hat{X}_i^{m'}(\omega) = \hat{X}_i^m(\omega)\}$. Then $B_{k_1 l_1}^m$ is the event that $\hat{\beta}_i^m = (k_1, l_1)$, $\hat{\beta}_i^{3n} = (k, J)$ and there is neither mutation, nor matching, nor break-up for agent i between $3n$-th step and $(m-1)$-th step; $C_{m'}^m$ is the event that there is neither mutation, nor matching, nor break-up for agent i between m'-th step and m-th step. In particular, when the event $C_{m'}^m$ happens, agent i does not change her extended type between m'-th step and m-th step. It is clear that

$$B_{k_1 l_1}^m = \{\omega \in \Omega : \hat{\beta}_i^m(\omega) = (k_1, l_1), \hat{\beta}_i^{3n}(\omega) = (k, J)\} \cap C_{3n}^{m-1}. \tag{E.18}$$

If the events $\left(\hat{\beta}_i^{3n} = (k, J)\right)$ and $\left(\hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1\right)$ happen, then matching is the only way for agent i to change her extended type to (k', l') by the end of step $3n + 3\Delta n$ (since, in the other two steps, single agents must stay single). Equation (E.17) can be expanded as follows:

$$\begin{aligned} & P\left(\beta_i(t+\Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k, J)\right) \\ = \; & P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', l'), \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) + O(\Delta t^2) \\ = \; & \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} P_0\left(\left(\hat{\beta}_i^{3r+3} = (k', l')\right) \cap B_{kl}^{3r+2} \cap C_{3r+3}^{3n+3\Delta n} \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) + O(\Delta t^2) \\ = \; & \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} \Big[P_0\left(\left(\hat{\beta}_i^{3r+3} = (k', l')\right) \cap B_{kl}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\ & \qquad P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left(\hat{\beta}_i^{3r+3} = (k', l')\right) \cap B_{kl}^{3r+2}\right)\Big] \\ & + O(\Delta t^2) \\ = \; & \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} \Big[P_0\left(\hat{\beta}_i^{3r+3} = (k', l') \,\middle|\, B_{kl}^{3r+2}\right) P_0\left(B_{kl}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\ & \qquad P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left(\hat{\beta}_i^{3r+3} = (k', l')\right) \cap B_{kl}^{3r+2}\right)\Big] \\ & + O(\Delta t^2). \end{aligned}$$

By Equations (E.4), (E.18) and Lemma E.6,

$$
\begin{aligned}
& P_0\left(\hat{\beta}_i^{3r+3}=(k',l') \,\middle|\, B_{kl}^{3r+2}\right) \\
= \;& P_0\left(\hat{\beta}_i^{3r+3}=(k',l') \,\middle|\, \left(\hat{\beta}_i^{3r+2}=(k,l)\right)\cap C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right) \\
= \;& P_0\left(\tilde{\beta}_i^{3r+3}=(k',l',0) \,\middle|\, \left(\tilde{\beta}_i^{3r+2}=(k,l,1)\right)\cap C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right) \\
= \;& P_0\left(\tilde{\beta}_i^{3r+3}=(k',l',0) \,\middle|\, \tilde{\beta}_i^{3r+2}=(k,l,1)\right) \\
= \;& \hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l'),
\end{aligned}
$$

where the last identity follows from the step of random type changing with break-up for agents (who are newly matched with an enduring relationship) in the construction of the hyperfinite dynamic matching model. Then, the above identities and Equation (E.18) imply that

$$
\begin{aligned}
& P\left(\beta_i(t+\Delta t)=(k',l') \,\middle|\, \beta_i(t)=(k,J)\right) \\
= \;& \sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')P_0\left(B_{kl}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right) \\
& \qquad P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left(\hat{\beta}_i^{3r+3}=(k',l')\right)\cap B_{kl}^{3r+2}\right) \\
& +O(\Delta t^2) \\
= \;& \sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\left[\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')P_0\left(\hat{\beta}_i^{3r+2}=(k,l) \,\middle|\, C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\right. \\
& \qquad \left. P_0\left(C_{3n}^{3r+1} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left(\hat{\beta}_i^{3r+3}=(k',l')\right)\cap B_{kl}^{3r+2}\right)\right] \\
& +O(\Delta t^2).
\end{aligned} \tag{E.19}
$$

Fix any sample realization $\omega^{3r+1}\in\Omega^{3r+1}$ such that $\hat{\beta}_i^{3r+1}\left(\omega^{3r+1}\right)=(k,J)$. By the definition of $\hat{h}_i^m$, we know that $\hat{h}_i^{3r+1}\left(\omega^{3r+1}\right)=1$, and $\hat{h}_i^{3r+2}\left(\omega^{3r+1},\omega_{3r+2}\right)=1$ for any $\omega_{3r+2}\in\Omega_{3r+2}$. Hence, these facts together with Lemma E.5 imply that

$$
\begin{aligned}
& \left|\hat{q}_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right)-P_0\left(\hat{\beta}_i^{3r+2}=(k,l) \,\middle|\, C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\right| \\
= \;& \left|\hat{q}_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right)-P_0\left(\hat{\beta}_i^{3r+2}=(k,l) \,\middle|\, C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3r+1}=(k,J)\right)\right)\right| \\
= \;& \left|\hat{q}_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right)-P_0\left(\tilde{\beta}_i^{3r+2}=(k,l,1) \,\middle|\, C_{3n}^{3r+1}\cap\left(\tilde{\beta}_i^{3r+1}=(k,J,1)\right)\right)\right| \\
\le \;& \frac{1}{M^3P_0\left(C_{3n}^{3r+1}\cap\left(\tilde{\beta}_i^{3r+1}=(k,J,1)\right)\right)}+\frac{1}{M^2} \\
= \;& \frac{1}{M^3P_0\left(C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)}+\frac{1}{M^2}.
\end{aligned}
$$

By Lemma E.9, $P\left(C_{3n}^{3r+1} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)>0$, which implies that $P\left(C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)>$

0. Then $P_0\left(C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)$ is not infinitesimal. It is then clear that

$$\frac{1}{M^3P_0\left(C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)}<\frac{1}{M^2}.$$

Therefore, we obtain the following estimation

$$\left|\hat{q}_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right)-P_0\left(\hat{\beta}_i^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\right|\le\frac{2}{M^2},\tag{E.20}$$

which implies that

$$\begin{aligned}&P_0\left(\hat{\beta}_i^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\\ \le\;&\hat{q}_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right)+\frac{2}{M^2}\\ \le\;&\frac{\bar{a}}{M}+\frac{2}{M^2}.\end{aligned}\tag{E.21}$$

It follows from the above inequality that

$$\begin{aligned}&\Bigg|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\Bigg[\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')P_0\left(\hat{\beta}_i^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\\ &\qquad P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',l')\right)\cap B_{kl}^{3r+2}\right)\Bigg]\\ &\quad-\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')P_0\left(\hat{\beta}_i^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\Bigg|\\ =\;&\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\Bigg[\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')P_0\left(\hat{\beta}_i^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\\ &\left(1-P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',l')\right)\cap B_{kl}^{3r+2}\right)\right)\Bigg]\\ \le\;&\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\left(\frac{\bar{a}}{M}+\frac{2}{M^2}\right)\\ &\left(1-P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',l')\right)\cap B_{kl}^{3r+2}\right)\right).\end{aligned}\tag{E.22}$$

By Lemma E.9, we obtain that

$$\begin{aligned}&P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',l')\right)\cap B_{kl}^{3r+2}\right)\\ \gtrsim\;&e^{-\frac{2K\bar{a}(3r-3n+1)}{M}}e^{-\frac{2K\bar{a}(3n+3\Delta n-3r-3)}{M}}\\ \simeq\;&e^{-6K\bar{a}\Delta t}.\end{aligned}\tag{E.23}$$

Then, Equations (E.22) and (E.23) imply that

$$\begin{aligned}
&\Bigg|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\Big[\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')P_0\left(\hat{\beta}_i^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\\
&\qquad P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',l')\right)\cap B_{kl}^{3r+2}\right)\Big]\\
-&\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')P_0\left(\hat{\beta}_i^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\Bigg|\\
\lesssim&\sum_{r=n}^{n+\Delta n-1}K\left(\frac{\bar{a}}{M}+\frac{2}{M^2}\right)\left(1-e^{-6K\bar{a}\Delta t}\right)\\
\lesssim&\;K\bar{a}\left(1-e^{-6K\bar{a}\Delta t}\right)\Delta t\\
=&\;O(\Delta t^2).
\end{aligned}\tag{E.24}$$

Therefore, Equations (E.19) and (E.24) lead to the following estimation

$$\begin{aligned}
&P\left(\beta_i(t+\Delta t)=(k',l')\,\middle|\,\beta_i(t)=(k,J)\right)\\
=&\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')P_0\left(\hat{\beta}_i^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\\
&+O(\Delta t^2).
\end{aligned}\tag{E.25}$$

We can use Equation (E.20) to deduce that

$$\begin{aligned}
&\Bigg|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\hat{q}_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right)\\
&\quad-\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')P_0\left(\hat{\beta}_i^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\Bigg|\\
\leq&\left|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\frac{2}{M^2}\right|\\
\leq&\;\Delta nK\frac{2}{M^2},
\end{aligned}\tag{E.26}$$

which is an infinitesimal and can be absorbed into the term $O(\Delta t^2)$. Therefore, Equations (E.25) and (E.26) imply that

$$\begin{aligned}
&P\left(\beta_i(t+\Delta t)=(k',l')\,\middle|\,\beta_i(t)=(k,J)\right)\\
=&\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\hat{q}_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right)+O(\Delta t^2).
\end{aligned}\tag{E.27}$$

Equation (E.5) implies that $\check{p}_t=\mathbb{E}\left(\hat{p}_t\right)\simeq\mathbb{E}\left(\hat{\rho}^{3n}\right)$. By Lemma E.4, $U_1^{3r+1}(\tilde{\rho}^0)\simeq$

$\mathbb{E}\left(\tilde{\rho}^{3r+1}\right)$. By the continuity of θ_{kl}, we obtain the following estimation

$$
\begin{aligned}
& \frac{1}{\Delta t}\left|\sum_{r=n}^{n+\Delta n-1} \sum_{l\in S} \hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\hat{q}_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right) - \sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\theta_{kl}\left(\check{p}_t\right)\Delta t\right| \\
\lesssim\ & \frac{1}{\Delta t}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\left|\sum_{r=n}^{n+\Delta n-1}\frac{1}{M}{}^*\theta_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right) - {}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3n}\right)\right)\frac{\Delta n}{M}\right| \\
\lesssim\ & \frac{1}{M\Delta t}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\sum_{r=n}^{n+\Delta n-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3r+1}\right)\right) - {}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3n}\right)\right)\right| \\
\lesssim\ & \frac{K}{\Delta n}\sum_{r=n}^{n+\Delta n-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3r+1}\right)\right) - {}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3n}\right)\right)\right|.
\end{aligned}
$$

Fix any $\Delta n' \in \mathbb{T}_0$ such that $\frac{\Delta n'}{M}$ is infinitesimal. Lemma E.12 implies that $\|\mathbb{E}\left(\tilde{\rho}^{3r+1}\right) - \mathbb{E}\left(\tilde{\rho}^{3n}\right)\|_\infty$ is infinitesimal for any r between n and $n+\Delta n'$. By the continuity of θ_{kl}, $\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3r+1}\right)\right) - {}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3n}\right)\right)\right|$ is also infinitesimal. Then, we obtain that

$$
\frac{K}{\Delta n'}\sum_{r=n}^{n+\Delta n'-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3r+1}\right)\right) - {}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3n}\right)\right)\right| \simeq 0.
$$

By Lemma D.1, we know that for any $\epsilon \in \mathbb{R}_{++}$, there exists $\delta \in \mathbb{R}_{++}$ such that for any $\Delta n \in \mathbb{T}_0$ with $\mathrm{st}\left(\frac{\Delta n}{M}\right) < \delta$, the standard part of

$$
\frac{K}{\Delta n}\sum_{r=n}^{n+\Delta n-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3r+1}\right)\right) - {}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3n}\right)\right)\right|
$$

is less than ϵ. In other words, we know that

$$
\frac{K\Delta t}{\Delta n}\sum_{r=n}^{n+\Delta n-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3r+1}\right)\right) - {}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3n}\right)\right)\right| = o(\Delta t), \tag{E.28}
$$

which implies that

$$
\left|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\hat{q}_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right) - \sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\theta_{kl}\left(\check{p}_t\right)\Delta t\right| = o(\Delta t).
$$

Hence, Equation (E.27) implies that

$$
P\left(\beta_i(t+\Delta t) = (k',l') \,\middle|\, \beta_i(t) = (k,J)\right) = \sum_{l\in S}\xi_{kl}\sigma_{kl}(k',l')\theta_{kl}\left(\check{p}_t\right)\Delta t + o(\Delta t),
$$

which implies agent i's transition intensity for her expanded types from (k,J) to (k',l') at time t to be $Q_{(k,J)(k',l')}(\check{p}(t))$ as in Equation (A.3).

Part 6: Fix any $i \in I$, $t \in \mathbb{R}_+$, $k,k' \in S$ with $k \neq k'$ and $P\left(\beta_i(t) = (k,J)\right) > 0$. The purpose of this part is to verify that agent i's transition intensity for her expanded types from (k,J) to (k',J) at time t is given in Equation (A.4).

For any $\Delta t \in \mathbb{R}_{++}$, let $n, \Delta n \in {}^*\mathbb{N}$ such that $\frac{n}{M} \in monad(t)$ and $\frac{\Delta n}{M} \in monad(\Delta t)$. By Equation (E.5), we have

$$P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k, J)\right) \simeq P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J) \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right).$$

Lemma E.10 says that the probability for agent i to change her extended type twice in the time interval $[t, t+\Delta t]$ is at level of Δt^2. Hence, we have

$$\begin{aligned} & P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k, J)\right) \\ = \;& P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) + O(\Delta t^2). \end{aligned} \tag{E.29}$$

For any $k_1 \in S$ and $m, m' \in \{3n, 3n+1, \ldots, 3M^2\}$ with $m > m'$, let

$$B_{k_1 J}^m = \{\omega \in \Omega : \hat{\beta}_i^m(\omega) = (k_1, J), \hat{X}_i^{3n}(\omega) = \hat{X}_i^{m-1}(\omega), \hat{\beta}_i^{3n}(\omega) = (k, J)\}$$

and $C_{m'}^m = \{\omega \in \Omega : \hat{X}_i^{m'}(\omega) = \hat{X}_i^m(\omega)\}$. Then $B_{k_1 J}^m$ is the event that $\hat{\beta}_i^m = (k_1, J)$, $\hat{\beta}_i^{3n} = (k, J)$ and there is neither mutation, nor matching, nor break-up for agent i between $3n$-th step and $(m-1)$-th step; $C_{m'}^m$ is the event that there is neither mutation, nor matching, nor break-up for agent i between m'-th step and m-th step. In particular, when the event $C_{m'}^m$ happens, agent i does not change her extended type between m'-th step and m-th step. It is clear that

$$B_{k_1 J}^m = \{\omega \in \Omega : \hat{\beta}_i^m(\omega) = (k_1, J), \hat{\beta}_i^{3n}(\omega) = (k, J)\} \cap C_{3n}^{m-1}. \tag{E.30}$$

If the events $\left(\hat{\beta}_i^{3n} = (k, J)\right)$ and $\left(\hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1\right)$ happen, agent i can become an agent with extended type (k', J) via mutation, or matching (without entering an enduring partnership) by the end of step $3n + 3\Delta n$ (since a single agent does not involve in the break-up of a long-term relationship). Equation (E.29) can be expanded as follows:

$$\begin{aligned} & P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k, J)\right) \\ = \;& P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{H}_i^{3n+3\Delta n} - \hat{H}_i^{3n} = 1, \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\ + \;& P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{N}_i^{3n+3\Delta n} - \hat{N}_i^{3n} = 1, \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\ + \;& O(\Delta t^2). \end{aligned} \tag{E.31}$$

The first term on the right hand side can be expanded as follows:

$$
\begin{aligned}
& P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J),\hat{H}_i^{3n+3\Delta n}-\hat{H}_i^{3n}=1,\hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1 \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)\\
= & \sum_{r=n}^{n+\Delta n-1} P_0\left(B_{k'J}^{3r+1}\cap C_{3r+1}^{3n+3\Delta n} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)\\
= & \sum_{r=n}^{n+\Delta n-1}\left[P_0\left(B_{k'J}^{3r+1} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'J}^{3r+1}\right)\right]\\
= & \sum_{r=n}^{n+\Delta n-1}\left[P_0\left(\hat{\beta}_i^{3r+1}=(k',J) \,\middle|\, C_{3n}^{3r}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)\right.\\
& \left. P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'J}^{3r+1}\right)\right].
\end{aligned}
$$

By Equation (E.3) and Lemma E.6 (i), we obtain that

$$
\begin{aligned}
& P_0\left(\hat{\beta}_i^{3r+1}=(k',J) \,\middle|\, C_{3n}^{3r}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\\
= & P_0\left(\hat{\beta}_i^{3r+1}=(k',J) \,\middle|\, C_{3n}^{3r}\cap\left(\hat{\beta}_i^{3r}=(k,J)\right)\right)\\
= & P_0\left(\tilde{\beta}_i^{3r+1}=(k',J,1) \,\middle|\, C_{3n}^{3r}\cap\left(\tilde{\beta}_i^{3r}=(k,J,1)\right)\right)\\
= & P_0\left(\tilde{\beta}_i^{3r+1}=(k',J,1) \,\middle|\, \tilde{\beta}_i^{3r}=(k,J,1)\right)\\
= & P_0\left(\hat{\beta}_i^{3r+1}=(k',J) \,\middle|\, \hat{\beta}_i^{3r}=(k,J)\right)\\
= & \hat{\eta}_{kk'},
\end{aligned}
$$

where the last identity follows from the step of random mutation for matched agents in the construction of the hyperfinite dynamic matching model. Then, the above identities imply that

$$
\begin{aligned}
& P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J),\hat{H}_i^{3n+3\Delta n}-\hat{H}_i^{3n}=1,\hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1 \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)\\
= & \sum_{r=n}^{n+\Delta n-1}\left[\hat{\eta}_{kk'}P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'J}^{3r+1}\right)\right].
\end{aligned}
$$

Now, we estimate the difference

$$
\begin{aligned}
& \left|P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J),\hat{H}_i^{3n+3\Delta n}-\hat{H}_i^{3n}=1,\hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1 \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)-\sum_{r=n}^{n+\Delta n-1}\hat{\eta}_{kk'}\right|\\
\leq & \sum_{r=n}^{n+\Delta n-1}\hat{\eta}_{kk'}\left|P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'J}^{3r+1}\right)-1\right|.
\end{aligned}
$$

We can obtain from Lemma E.9 that

$$
P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'J}^{3r+1}\right)\gtrsim e^{-\frac{6K(r-n)\bar{a}}{M}}e^{-\frac{2K(3n+3\Delta n-3r-1)\bar{a}}{M}}.
$$

Then, it follows from the above inequalities that

$$\begin{aligned}
& \left| P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J), \hat{H}_i^{3n+3\Delta n}-\hat{H}_i^{3n}=1, \hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1 \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right) - \sum_{r=n}^{n+\Delta n-1} \hat{\eta}_{kk'} \right| \\
\leq\ & \sum_{r=n}^{n+\Delta n-1} \frac{\bar{a}}{M}\left(1-e^{-\frac{6K(r-n)\bar{a}}{M}} e^{-\frac{2K(3n+3\Delta n-3r-1)\bar{a}}{M}}\right) \\
\simeq\ & \bar{a}\Delta t\left(1-e^{-6K\bar{a}\Delta t}\right) \\
=\ & O(\Delta t^2).
\end{aligned}$$

Therefore, we obtain the following estimation

$$\begin{aligned}
& P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J), \hat{H}_i^{3n+3\Delta n}-\hat{H}_i^{3n}=1, \hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1 \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right) \\
=\ & \sum_{r=n}^{n+\Delta n-1} \hat{\eta}_{kk'} + O(\Delta t^2) \\
=\ & \eta_{kk'}\Delta t + O(\Delta t^2). \qquad \text{(E.32)}
\end{aligned}$$

Next, we need to estimate the second term on the right hand side of Equation (E.31). The proof for such an estimation is very close to the proof in Part 5. For the sake of completeness and readability, we present the detailed proof below.

The second term on the right hand side of Equation (E.31) can be expanded as follows:

$$\begin{aligned}
& P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J), \hat{N}_i^{3n+3\Delta n}-\hat{N}_i^{3n}=1, \hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1 \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right) \\
=\ & \sum_{r=n}^{n+\Delta n-1} \sum_{l\in S} P_0\left(\left(\hat{\beta}_i^{3r+3}=(k',J)\right) \cap B_{kl}^{3r+2} \cap C_{3r+3}^{3n+3\Delta n} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right) \\
=\ & \sum_{r=n}^{n+\Delta n-1} \sum_{l\in S} \Big[P_0\left(\left(\hat{\beta}_i^{3r+3}=(k',J)\right) \cap B_{kl}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right) \\
& \qquad P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left(\hat{\beta}_i^{3r+3}=(k',J)\right) \cap B_{kl}^{3r+2}\right)\Big] \\
=\ & \sum_{r=n}^{n+\Delta n-1} \sum_{l\in S} \Big[P_0\left(\hat{\beta}_i^{3r+3}=(k',J) \,\middle|\, B_{kl}^{3r+2}\right) P_0\left(B_{kl}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right) \\
& \qquad P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left(\hat{\beta}_i^{3r+3}=(k',J)\right) \cap B_{kl}^{3r+2}\right)\Big].
\end{aligned}$$

It follows from Equations (E.4) and (E.30), and Lemma E.6 that

$$\begin{aligned}
& P_0\left(\hat{\beta}_i^{3r+3}=(k',J) \,\middle|\, B_{kl}^{3r+2}\right) \\
=\ & P_0\left(\hat{\beta}_i^{3r+3}=(k',J) \,\middle|\, \left(\hat{\beta}_i^{3r+2}=(k,l)\right) \cap C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n}=(k,J)\right)\right) \\
=\ & P_0\left(\tilde{\beta}_i^{3r+3}=(k',J,1) \,\middle|\, \left(\tilde{\beta}_i^{3r+2}=(k,l,1)\right) \cap C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n}=(k,J)\right)\right) \\
=\ & P_0\left(\tilde{\beta}_i^{3r+3}=(k',J,1) \,\middle|\, \tilde{\beta}_i^{3r+2}=(k,l,1)\right) \\
=\ & (1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k'),
\end{aligned}$$

where the last identity follows from the step of random type changing with break-up for matched agents (without entering an enduing partnership) in the construction of the hyperfinite dynamic matching model. Then, the above identities and Equation (E.30) imply that

$$\begin{aligned}
& P_0\left(\hat\beta_i^{3n+3\Delta n}=(k',J),\hat N_i^{3n+3\Delta n}-\hat N_i^{3n}=1,\hat X_i^{3n+3\Delta n}-\hat X_i^{3n}=1 \,\middle|\, \hat\beta_i^{3n}=(k,J)\right) \\
= & \sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}(1-\hat\xi_{kl})\hat\varsigma_{kl}(k')P_0\left(B_{kl}^{3r+2}\,\middle|\,\hat\beta_i^{3n}=(k,J)\right) \\
& \qquad P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat\beta_i^{3r+3}=(k',J)\right)\cap B_{kl}^{3r+2}\right) \\
= & \sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\Big[(1-\hat\xi_{kl})\hat\varsigma_{kl}(k')P_0\left(\hat\beta_i^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat\beta_i^{3n}=(k,J)\right)\right) \\
& \qquad P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat\beta_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat\beta_i^{3r+3}=(k',J)\right)\cap B_{kl}^{3r+2}\right)\Big]. \quad \text{(E.33)}
\end{aligned}$$

It follows from Equation (E.21) that

$$\begin{aligned}
& \Bigg|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\Big[(1-\hat\xi_{kl})\hat\varsigma_{kl}(k')P_0\left(\hat\beta_i^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat\beta_i^{3n}=(k,J)\right)\right) \\
& \qquad P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat\beta_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat\beta_i^{3r+3}=(k',J)\right)\cap B_{kl}^{3r+2}\right)\Big] \\
& \quad -\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}(1-\hat\xi_{kl})\hat\varsigma_{kl}(k')P_0\left(\hat\beta_i^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat\beta_i^{3n}=(k,J)\right)\right)\Bigg| \\
= & \sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\Big[(1-\hat\xi_{kl})\hat\varsigma_{kl}(k')P_0\left(\hat\beta_i^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat\beta_i^{3n}=(k,J)\right)\right) \\
& \left(1-P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat\beta_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat\beta_i^{3r+3}=(k',J)\right)\cap B_{kl}^{3r+2}\right)\right)\Big] \\
\le & \sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\left(\frac{\bar a}{M}+\frac{2}{M^2}\right) \\
& \left(1-P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat\beta_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat\beta_i^{3r+3}=(k',J)\right)\cap B_{kl}^{3r+2}\right)\right). \quad \text{(E.34)}
\end{aligned}$$

Then, Equations (E.23) and (E.34) imply that

$$
\begin{aligned}
& \left| \sum_{r=n}^{n+\Delta n-1} \sum_{l\in S} \Big[(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')P_0\left(\hat{\beta}_i^{3r+2}=(k,l)\,\middle|\, C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right) \right. \\
& \qquad P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',J)\right)\cap B_{kl}^{3r+2}\right)\Big] \\
- & \left. \sum_{r=n}^{n+\Delta n-1} \sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')P_0\left(\hat{\beta}_i^{3r+2}=(k,l)\,\middle|\, C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\right| \\
\lesssim & \sum_{r=n}^{n+\Delta n-1} K\left(\frac{\bar{a}}{M}+\frac{2}{M^2}\right)\left(1-e^{-6K\bar{a}\Delta t}\right) \\
\lesssim & \; K\bar{a}\left(1-e^{-6K\bar{a}\Delta t}\right)\Delta t \\
= & \; O(\Delta t^2).
\end{aligned}
\tag{E.35}
$$

By Equations (E.33) and (E.35), we have the following estimation

$$
\begin{aligned}
& P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J),\hat{N}_i^{3n+3\Delta n}-\hat{N}_i^{3n}=1,\hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right) \\
= & \sum_{r=n}^{n+\Delta n-1} \sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')P_0\left(\hat{\beta}_i^{3r+2}=(k,l)\,\middle|\, C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right) \\
& +O(\Delta t^2).
\end{aligned}
\tag{E.36}
$$

It follows from Equation (E.23) that

$$
\begin{aligned}
& \left| \sum_{r=n}^{n+\Delta n-1} \sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\hat{q}_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right) \right. \\
& \qquad \left. - \sum_{r=n}^{n+\Delta n-1} \sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')P_0\left(\hat{\beta}_i^{3r+2}=(k,l)\,\middle|\, C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\right| \\
\leq & \left| \sum_{r=n}^{n+\Delta n-1} \sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\frac{2}{M^2}\right| \\
\leq & \; \Delta n K\frac{2}{M^2},
\end{aligned}
\tag{E.37}
$$

which is an infinitesimal and can be absorbed into the term $O(\Delta t^2)$. Therefore, Equations (E.36) and (E.37) imply that

$$
\begin{aligned}
& P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J),\hat{N}_i^{3n+3\Delta n}-\hat{N}_i^{3n}=1,\hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right) \\
= & \sum_{r=n}^{n+\Delta n-1} \sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\hat{q}_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right)+O(\Delta t^2).
\end{aligned}
\tag{E.38}
$$

Equation (E.5) implies that $\check{p}_t = \mathbb{E}\left(\hat{p}_t\right) \simeq \mathbb{E}\left(\hat{\rho}^{3n}\right)$. By Lemma E.4, $U_1^{3r+1}(\tilde{\rho}^0) \simeq$

$\mathbb{E}\left(\tilde{\rho}^{3r+1}\right)$. By the continuity of θ_{kl}, we obtain the following estimation

$$
\begin{aligned}
& \frac{1}{\Delta t}\left|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\hat{q}_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right)-\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\theta_{kl}\left(\check{p}_t\right)\Delta t\right| \\
\lesssim\; & \frac{1}{\Delta t}\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\left|\sum_{r=n}^{n+\Delta n-1}\frac{1}{M}{}^*\theta_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right)-{}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3n}\right)\right)\frac{\Delta n}{M}\right| \\
\lesssim\; & \frac{1}{M\Delta t}\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\sum_{r=n}^{n+\Delta n-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3r+1}\right)\right)-{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3n}\right)\right)\right| \\
\lesssim\; & \frac{K}{\Delta n}\sum_{r=n}^{n+\Delta n-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3r+1}\right)\right)-{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3n}\right)\right)\right|.
\end{aligned}
$$

Then, the above inequality and Equation (E.28) allow us to claim that

$$
\left|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\hat{q}_{kl}\left(U_1^{3r+1}\left(\mathbb{E}\tilde{\rho}^0\right)\right)-\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\theta_{kl}\left(\check{p}_t\right)\Delta t\right|=o(\Delta t).
$$

Hence, Equation (E.38) implies that

$$
\begin{aligned}
& P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J),\hat{N}_i^{3n+3\Delta n}-\hat{N}_i^{3n}=1,\hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right) \\
=\; & \sum_{l\in S}(1-\xi_{kl})\varsigma_{kl}(k')\theta_{kl}\left(\check{p}_t\right)\Delta t+o(\Delta t). \qquad \text{(E.39)}
\end{aligned}
$$

By Equations (E.32) and (E.39), we can obtain that

$$
\begin{aligned}
& P\left(\beta_i(t+\Delta t)=(k',J)\,\middle|\,\beta_i(t)=(k,J)\right) \\
=\; & P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J),\hat{H}_i^{3n+3\Delta n}-\hat{H}_i^{3n}=1,\hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right) \\
& +P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J),\hat{N}_i^{3n+3\Delta n}-\hat{N}_i^{3n}=1,\hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)+O(\Delta t^2) \\
=\; & \eta_{kk'}\Delta t+\sum_{l\in S}(1-\xi_{kl})\varsigma_{kl}(k')\theta_{kl}\left(\check{p}_t\right)\Delta t+o(\Delta t),
\end{aligned}
$$

which implies that agent i's transition intensity for her expanded types from (k,J) to (k',J) at time t is indeed $Q_{(k,J)(k',J)}(\check{p}(t))$, as given in Equation (A.4).

E.5 Proofs of Lemmas E.1 – E.13

The proofs of Lemmas E.1 and E.2 are given in Subsections E.5.1 and E.5.2 respectively. In order to prove Lemmas E.3 – E.13, some additional lemmas are presented in Subsection E.5.3. Lemmas E.3 – E.13 are then proved in Subsections E.5.4 – E.5.14 respectively.

E.5.1 Proof of Lemma E.1

The proof consists of three steps. In the first step, we (randomly) choose a set A_{kl} of agents among the type-k single agents, which is to be matched with type-l agents. We require that the internal cardinality $|A_{kl}|$ of A_{kl} is even and $|A_{kl}| = |A_{lk}|$, which allow the agents in A_{kl} and A_{lk} to be matched. The second step is to randomly match the agents in A_{kl} and A_{lk}. In the third step, the random matching obtained by combining the match of agents in those groups is shown to satisfy Lemma E.1 (i) and (ii).

Step 1: For each $k \in S$, let $I_k = \{i \in I : \alpha^0(i) = k, \pi^0(i) = i\}$ be the set of type-k agents who are initially unmatched. Let

$$\begin{aligned}\Omega_0 &= \big\{(A_{kl})_{k,l\in S} : \forall k,l,l' \in S, A_{kl} \subseteq I_k,\ A_{kl} \text{ is internal; } |A_{kl}| \text{ is the largest even integer} \\ &\quad \text{less than or equal to } |I_k|q_{kl}, A_{kl} \text{ and } A_{kl'} \text{ are disjoint for different } l \text{ and } l'.\big\}\end{aligned}$$

Note that $\hat{\rho}_{kJ}$ is the proportion of agents of type k who are initially unmatched, which implies that $|I_k| = \hat{M}\hat{\rho}_{kJ}$. Hence, we have $|I_k|q_{kl} = \hat{M}\hat{\rho}_{kJ}q_{kl} = \hat{M}\hat{\rho}_{lJ}q_{lk} = |I_l|q_{lk}$. Then for any $(A_{kl})_{k,l\in S} \in \Omega_0$, $|A_{kl}| = |A_{lk}|$ for any $k,l \in S$. Let μ_0 be the internal counting probability measure on $(\Omega_0, \mathcal{A}_0)$, where $\mathcal{A}_0$ is the internal power set of Ω_0.

Step 2: For any fixed $\omega_0 = (A_{kl})_{k,l\in S} \in \Omega_0$, we consider internal partial matchings on I that match agents from A_{kl} to A_{lk}. We only need to consider those sets A_{kl} which are nonempty. For each $k \in S$, let $\Omega_{kk}^{\omega_0}$ be the internal set of all the internal full matchings on A_{kk}, and $\mu_{kk}^{\omega_0}$ the internal counting probability measure on $\Omega_{kk}^{\omega_0}$. For $k,l \in S$ with $k < l$, let $\Omega_{kl}^{\omega_0}$ be the internal set of all the internal bijections from A_{kl} to A_{lk}, and $\mu_{kl}^{\omega_0}$ the internal counting probability measure on A_{kl}. Let Ω_1 be the internal set of all the internal partial matchings from I to I. Define $\Omega_1^{\omega_0}$ to be the set of $\phi \in \Omega_1$, with

(i) the restriction $\phi|_H = \pi^0|_H$, where H is the set $\{i : \pi^0(i) \neq i\}$ of initially matched agents;

(ii) $\{i \in I_k : \phi(i) = i\} = I_k \backslash \left(\cup_{l=1}^K A_{kl}\right)$ for each $k \in S$;

(iii) the restriction $\phi|_{A_{kk}} \in \Omega_{kk}^{\omega_0}$ for $k \in S$;

(iv) for $k,l \in S$ with $k < l$, $\phi|_{A_{kl}} \in \Omega_{kl}^{\omega_0}$.

(i) means that initially matched agents remain matched with the same partners. The rest is clear.

Define an internal probability measure $\mu_1^{\omega_0}$ on Ω_1 such that such that

(i) for $\phi \in \Omega_1^{\omega_0}$,

$$\mu_1^{\omega_0}(\phi) = \prod_{1 \le k \le l \le K, A_{kl} \ne \emptyset} \mu_{kl}^{\omega_0}(\phi|_{A_{kl}});$$

(ii) $\phi \notin \Omega_1^{\omega_0}$, $\mu_1^{\omega_0}(\phi) = 0$.

The purpose of introducing the space $\Omega_1^{\omega_0}$ and the internal probability measure $\mu_1^{\omega_0}$ is to match the agents in A_{kl} to the agents in A_{lk} randomly. The probability measure $\mu_1^{\omega_0}$ is trivially extended to the common sample space Ω_1.

Define an internal probability measure P_0 on $\Omega = \Omega_0 \times \Omega_1$ with the internal power set $\mathcal{F}_0$ by letting

$$P_0\left((\omega_0, \omega_1)\right) = \mu_0(\omega_0) \times \mu_1^{\omega_0}(\omega_1).$$

For $(i, \omega) \in I \times \Omega$, let $\pi(i, (\omega_0, \omega_1)) = \omega_1(i)$, and $g(i, \omega) = \begin{cases} \alpha^0(\pi(i,\omega)) & \text{if } \pi(i,\omega) \ne i \\ J & \text{if } \pi(i,\omega) = i. \end{cases}$

Denote the internal set $\{(\omega_0, \omega_1) \in \Omega : \omega_0 \in \Omega_0,\ \omega_1 \in \Omega_1^{\omega_0,}\}$ by $\hat{\Omega}$. The definition of P_0 indicates that $P_0\left(\hat{\Omega}\right) = 1$.

Step 3: It is clear that π is an internal random matching and satisfies part (i) of the lemma. For any $k, l \in S$ and $\omega \in \Omega$, we have $\lambda_0\left(\{i \in I : \alpha^0(i) = k, g^0(i) = J, g(i,\omega) = l\}\right) = \frac{|A_{kl}|}{\hat{M}}$. Since $|A_{kl}|$ is the largest even integer less than or equal to $|I_k| q_{kl}$, we have $||A_{kl}| - |I_k| q_{kl}| \le 2$. Hence,

$$\begin{aligned} & \left|\lambda_0\left(\{i \in I : \alpha^0(i) = k, g^0(i) = J, g(i,\omega) = l\}\right) - \hat{\rho}_{kJ} q_{kl}\right| \\ = & \left|\frac{|A_{kl}|}{\hat{M}} - \frac{|I_k|}{\hat{M}} q_{kl}\right| \le \frac{2}{\hat{M}}, \end{aligned}$$

which implies part (iii) of the lemma.

It remains to prove part (ii). Fix any $i, j \in I$ with $i \ne j$, $\pi^0(i) = i$ and $\pi^0(j) = j$; denote $\alpha^0(i)$ and $\alpha^0(j)$ by k_1 and k_2 respectively.

We start with the first inequality in part (ii). By the construction above, we have

$$P_0\left(\pi_i = j\right) = P_0\left(\{((A_{kl})_{k,l \in S}, \omega_1) : i \in A_{k_1 k_2},\ j \in A_{k_2 k_1},\ \omega_1(i) = j\}\right).$$

Let $\bar{A} = \{(A_{kl})_{k,l \in S} : i \in A_{k_1 k_2}, j \in A_{k_2 k_1}\}$. Then, the definition of P_0 implies that

$$P_0\left(\pi_i = j\right) = \sum_{(A_{kl})_{k,l \in S} \in \bar{A}} \mu_0\left((A_{kl})_{k,l \in S}\right) \mu_1^{(A_{kl})_{k,l \in S}}(\omega_1(i) = j).$$

When $k_1 \ne k_2$, for any $(A_{kl})_{k,l \in S} \in \bar{A}$, we know that

$$\mu_1^{(A_{kl})_{k,l \in S}}(\omega_1(i) = j) = \frac{1}{|A_{k_1 k_2}|}.$$

When $k_1 = k_2$, for any $(A_{kl})_{k,l\in S} \in \bar{A}$, we have

$$\mu_1^{(A_{kl})_{k,l\in S}}(\omega_1(i) = j) = \frac{1}{|A_{k_1k_2}| - 1} \leq \frac{2}{|A_{k_1k_2}|},$$

since $|A_{k_1k_2}| \geq 2$ for any $(A_{kl})_{k,l\in S} \in \bar{A}$. Then, it is clear that $\mu_1^{(A_{kl})_{k,l\in S}}(\omega_1(i) = j) \leq \frac{2}{|A_{k_1k_2}|}$ always holds for any $(A_{kl})_{k,l\in S} \in \bar{A}$. Therefore, we can obtain that

$$\begin{aligned} P_0(\pi_i = j) &\leq \sum_{(A_{kl})_{k,l\in S}\in\bar{A}} \mu_0\left((A_{kl})_{k,l\in S}\right) \frac{2}{|A_{k_1k_2}|} \\ &= \frac{2}{|A_{k_1k_2}|}\mu_0\left(\{(A_{kl})_{k,l\in S} : i \in A_{k_1k_2},\, j \in A_{k_2k_1}\}\right) \\ &\leq \frac{2}{|A_{k_1k_2}|}\mu_0\left(\{(A_{kl})_{k,l\in S} : i \in A_{k_1k_2}\}\right). \end{aligned}$$

Let M_k and m_{kl} be the internal cardinality of I_k and A_{kl} respectively. Let $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ denote the binomial coeffiecient. Then we have

$$P_0(\pi_i = j) \leq \frac{2}{m_{k_1k_2}} \frac{\binom{M_{k_1}-1}{m_{k_1k_2}-1}}{\binom{M_{k_1}}{m_{k_1k_2}}} = \frac{2}{m_{k_1k_2}} \frac{m_{k_1k_2}}{M_{k_1}} = \frac{2}{M_{k_1}} = \frac{2}{\hat{M}\hat{\rho}_{k_1J}},$$

where the last identity follows from the fact that $\hat{M}\hat{\rho}_{k_1J} = |I_k| = M_{k_1}$.

Next, we prove the second inequality in part (ii). Assume that $\hat{\rho}_{k_1J} \geq \hat{M}^{\frac{-1}{3}}$. We have

$$P_0(g(i) = l_1) = \mu_0\left(\{(A_{kl})_{k,l\in S} : i \in A_{k_1l_1}\}\right) = \frac{\binom{M_{k_1}-1}{m_{k_1l_1}-1}}{\binom{M_{k_1}}{m_{k_1l_1}}} = \frac{m_{k_1l_1}}{M_{k_1}}.$$

It is clear that $P_0(g(i) = l_1) \leq \frac{M_{k_1}q_{k_1l_1}}{M_{k_1}} = q_{k_1l_1}$. Note that

$$\begin{aligned} P_0(g(i) = l_1) &\geq \frac{M_{k_1}q_{k_1l_1} - 2}{M_{k_1}} = q_{k_1l_1} - \frac{2}{M_{k_1}} \\ &= q_{k_1l_1} - \frac{2}{\hat{M}\hat{\rho}_{k_1J}} \geq q_{k_1l_1} - \frac{2}{\hat{M}^{\frac{2}{3}}}. \end{aligned}$$

Then, we have

$$q_{k_1l_1} - \frac{2}{\hat{M}^{\frac{2}{3}}} \leq P_0(g(i) = l_1) \leq q_{k_1l_1}. \tag{E.40}$$

Then we prove the third inequality in part (ii). We make the further assumption that $\hat{\rho}_{k_2J} \geq \hat{M}^{\frac{-1}{3}}$. When $k_1 \neq k_2$, we can obtain that

$$\begin{aligned} P_0(g(i) = l_1, g(j) = l_2) &= \mu_0(\{(A_{kl})_{k,l\in S} : i \in A_{k_1l_1}, j \in A_{k_2l_2}\}) \\ &= \frac{\binom{M_{k_1}-1}{m_{k_1l_1}-1}}{\binom{M_{k_1}}{m_{k_1l_1}}} \frac{\binom{M_{k_2}-1}{m_{k_2l_2}-1}}{\binom{M_{k_2}}{m_{k_2l_2}}} = P_0(g(i) = l_1)P_0(g(j) = l_2). \end{aligned}$$

Equation (E.40) implies the following inequalities:

$$q_{k_1l_1}q_{k_2l_2} \geq P_0(g(i)=l_1, g(j)=l_2) \geq (q_{k_1l_1} - \frac{2}{\hat{M}^{\frac{2}{3}}})(q_{k_2l_2} - \frac{2}{\hat{M}^{\frac{2}{3}}}) \geq q_{k_1l_1}q_{k_2l_2} - \frac{4}{\hat{M}^{\frac{2}{3}}}. \quad \text{(E.41)}$$

When $k_1 = k_2$ but $l_1 \neq l_2$, we have

$$P_0(g(i)=l_1, g(j)=l_2) = \mu_0(\{(A_{kl})_{k,l\in S} : i \in A_{k_1l_1}, j \in A_{k_1l_2}\}) = \frac{\binom{M_{k_1}-2}{m_{k_1l_1}-1, m_{k_1l_2}-1}}{\binom{M_{k_1}}{m_{k_1l_1}, m_{k_1l_2}}},$$

where $\binom{a}{b,c} = \frac{a!}{b!c!(a-b-c)!}$ is the multinomial coefficient. It is clear that

$$\begin{aligned} P_0(g(i)=l_1, g(j)=l_2) &= \frac{m_{k_1l_1}m_{k_1l_2}}{M_{k_1}(M_{k_1}-1)} \leq \frac{m_{k_1l_1}(m_{k_1l_2}+1)}{M_{k_1}^2} \\ &\leq q_{k_1l_1}q_{k_1l_2} + q_{k_1l_1}\frac{1}{M_{k_1}} \leq q_{k_1l_1}q_{k_1l_2} + \frac{1}{M_{k_1}} \\ &= q_{k_1l_1}q_{k_1l_2} + \frac{1}{\hat{M}\hat{\rho}_{k_1J}} \leq q_{k_1l_1}q_{k_1l_2} + \frac{1}{\hat{M}^{\frac{2}{3}}}. \end{aligned}$$

On the other hand, we can obtain that

$$\begin{aligned} & \frac{m_{k_1l_1}m_{k_1l_2}}{M_{k_1}(M_{k_1}-1)} \\ \geq\ & \frac{(M_{k_1}q_{k_1l_1}-2)}{M_{k_1}}\frac{(M_{k_1}q_{k_1l_2}-2)}{M_{k_1}} \\ \geq\ & q_{k_1l_1}q_{k_1l_2} - \frac{2}{M_{k_1}}q_{k_1l_1} - \frac{2}{M_{k_1}}q_{k_1l_2} \\ \geq\ & q_{k_1l_1}q_{k_1l_2} - \frac{4}{M_{k_1}} \\ =\ & q_{k_1l_1}q_{k_1l_2} - \frac{4}{\hat{M}\hat{\rho}_{k_1J}} \\ \geq\ & q_{k_1l_1}q_{k_1l_2} - \frac{4}{\hat{M}^{\frac{2}{3}}}. \end{aligned}$$

By combining the above inequalities, we have

$$q_{k_1l_1}q_{k_1l_2} - \frac{4}{\hat{M}^{\frac{2}{3}}} \leq P_0(g(i)=l_1, g(j)=l_2) \leq q_{k_1l_1}q_{k_1l_2} + \frac{1}{\hat{M}^{\frac{2}{3}}}. \quad \text{(E.42)}$$

When $k_1 = k_2$ and $l_1 = l_2$, we can obtain that

$$P_0(g(i)=l_1, g(j)=l_1) = \mu_0(\{(A_{kl})_{k,l\in S} : i, j \in A_{k_1l_1}\}) = \frac{\binom{M_{k_1}-2}{m_{k_1l_1}-2}}{\binom{M_{k_1}}{m_{k_1l_1}}}.$$

It is clear that

$$P_0(g(i)=l_1, g(j)=l_1) = \frac{(m_{k_1l_1})(m_{k_1l_1}-1)}{M_{k_1}(M_{k_1}-1)} \leq \frac{m_{k_1l_1}^2}{M_{k_1}^2} \leq q_{k_1l_1}^2.$$

On the other hand,

$$\frac{(m_{k_1l_1})(m_{k_1l_1}-1)}{M_{k_1}(M_{k_1}-1)} \geq \frac{(M_{k_1}q_{k_1l_1}-2)}{M_{k_1}} \frac{(M_{k_1}q_{k_1l_1}-3)}{M_{k_1}}$$
$$\geq q_{k_1l_1}^2 - \frac{5}{M_{k_1}} q_{k_1l_1} \geq q_{k_1l_1}^2 - \frac{5}{\hat{M}^{\frac{2}{3}}}.$$

Therefore, we obtain that

$$q_{k_1l_1}^2 - \frac{5}{\hat{M}^{\frac{2}{3}}} \leq P_0(g(i)=l_1, g(j)=l_2) \leq q_{k_1l_1}^2. \tag{E.43}$$

By combining Equations (E.41), (E.42) and (E.43), we know that for any $(k_1,l_1),(k_2,l_2) \in S^2$,

$$q_{k_1l_1}q_{k_2l_2} - \frac{5}{\hat{M}^{\frac{2}{3}}} \leq P_0(g(i)=l_1, g(j)=l_2) \leq q_{k_1l_1}q_{k_2l_2} + \frac{1}{\hat{M}^{\frac{2}{3}}}.$$

E.5.2 Proof of Lemma E.2

The Deterministic Case

Recall that $\hat{p}^0$ is the initial extended type distribution. Let $\{A_{kl}\}_{(k,l)\in\hat{S}}$ be an internal partition of I such that $\frac{|A_{kl}|}{\hat{M}} \simeq \hat{p}_{kl}^0$ for any $k \in S$ and $l \in S \cup \{J\}$, $\frac{|A_{kJ}|}{\hat{M}} \geq \frac{1}{M^2}$ for any $k \in S$, and $|A_{kl}| = |A_{lk}|$ for any $k,l \in S$, and $|A_{kk}|$ is even for any $k \in S$. Let $\hat{\alpha}^0$ be an internal function from $(I, \mathcal{I}_0, \lambda_0)$ to S such that $\hat{\alpha}^0(i) = k$ if $i \in \bigcup_{l\in S\cup\{J\}} A_{kl}$. Let $\hat{\pi}^0$ be an internal partial matching on I such that $\hat{\pi}^0(i) = i$ on $\bigcup_{k\in S} A_{kJ}$, and the restriction $\hat{\pi}^0|_{A_{kl}}$ is an internal bijection from A_{kl} to A_{lk} for any $k,l \in S$. Let $\hat{g}^0(i) = \hat{\alpha}^0(\hat{\pi}^0(i))$. It is clear that $\lambda_0(\{i : \hat{\alpha}^0(i) = k, \hat{g}^0(i) = l\}) \simeq \hat{p}_{kl}^0$ for any $k \in S$ and $l \in S \cup \{J\}$. Since $\hat{\alpha}^0, \hat{\pi}^0, \hat{g}^0$ are deterministic, the other requirements are trivially satisfied.

The Case of Identical Distributions

We divide the proof of this case into three steps. In the first step, we generate an internal type process by choosing a random type for each agent based an internal distribution μ^0 modified from a marginal distribution of $\hat{p}^0$. The second step applies Lemma E.1 to each realization of agents' type profile to obtain an internal random matching $\hat{\pi}^0$. The third step verifies that $\hat{\pi}^0$ satisfies the required properties.

Step 1: Pick an element $\ddot{\rho}^0$ from ${}^*\hat{\Delta}$ such that for any $(k,l) \in \hat{S}$, $\ddot{\rho}_{kl}^0 \simeq \hat{p}_{kl}^0$ and $\ddot{\rho}_{kJ}^0 \geq \frac{1}{M}$. Let μ^0 be an element in ${}^*\Delta$ such that $\mu_k^0 = \sum_{r\in S\cup\{J\}} \ddot{\rho}_{kr}^0$ for any $k \in S$. It is clear that $\mu_k^0 \geq \frac{1}{M}$ for any $k \in S$. Let $\Omega_{-2} = S^I$ be the internal set of all the internal functions from I to S, and Q_{-2} the internal product probability measure $\Pi_{i\in I}\mu^0$ on $(\Omega_{-2}, \mathcal{A}_{-2})$, where $\mathcal{A}_{-2}$ is the internal power set of Ω_{-2}. For any realization $\omega_{-2} \in \Omega_{-2}$, agent i's type is $\omega_{-2}(i)$ for any $i \in I$, while the realized cross-sectional extended type distribution is $\lambda\left(\omega_{-2}, J\right)^{-1}$, which is denoted by $\hat{\gamma}(\omega_{-2})$ (and also by $\hat{\gamma}_{\omega_{-2}}$) for simplicity.

Step 2: Let $(\Omega_{-1}, \mathcal{A}_{-1})$ be the measurable space constructed in Lemma E.1. In order to apply Lemma E.1, we first specify the internal matching probability function q^0 as follows. For any $\hat{\rho} \in {}^*\hat{\Delta}$, let

$$q^0_{kl}(\hat{\rho}) = \begin{cases} \frac{1}{\hat{\rho}_{kJ}} \min\left(\frac{\ddot{\rho}^0_{kl}\hat{\rho}_{kJ}}{\mu^0_k}, \frac{\ddot{\rho}^0_{kl}\hat{\rho}_{lJ}}{\mu^0_l}\right) & \text{if } \hat{\rho}_{kJ} > 0 \\ 0 & \text{if } \hat{\rho}_{kJ} = 0. \end{cases}$$

It is clear that $\sum_{r\in S} q^0_{kr}(\hat{\rho}) \leq 1$ and $\hat{\rho}_{kJ} q^0_{kl}(\hat{\rho}) = \hat{\rho}_{lJ} q^0_{lk}(\hat{\rho})$ for any $k, l \in S$ and $\hat{\rho} \in {}^*\Delta$.

Fix any realization $\omega_{-2} \in \Omega_{-2}$. The type function for all the agents is ω_{-2}. Since no agents are matched after Step 1, the internal partial matching is simply the identity mapping id_I on I. By Lemma E.1, we can construct an internal probability measure $Q^{\omega_{-2}}_{-1} = P_{\omega_{-2}, id_I, q^0(\hat{\gamma}_{\omega_{-2}})}$ and an internal random matching $\pi_{\omega_{-2}, id_I, q^0(\hat{\gamma}_{\omega_{-2}})}$.

Define an internal probability measure Q_0 on $\Omega_0 = \Omega_{-2} \times \Omega_{-1}$ with the internal power set $\mathcal{E}_0$ by letting

$$Q_0\left((\omega_{-2}, \omega_{-1})\right) = Q_{-2}(\omega_{-2}) \times Q^{\omega_{-2}}_{-1}(\omega_{-1}).$$

Let $\hat{\alpha}^0 : (I \times \Omega_0) \to S$, $\hat{\pi}^0 : (I \times \Omega_0) \to I$ and $\hat{g}^0 : (I \times \Omega_0) \to S \cup \{J\}$ be such that for any $i \in I$, $\omega_0 = (\omega_{-2}, \omega_{-1}) \in \Omega_0$,

$$\hat{\alpha}^0\,(i, \omega_0) = \omega_{-2}(i),$$

$$\hat{\pi}^0\,(i, \omega_0) = \pi_{\omega_{-2}, id_I, q^0(\hat{\gamma}_{\omega_{-2}})}(i, \omega_{-1}),$$

$$\hat{g}^0\,(i, \omega_0) = \begin{cases} \hat{\alpha}^0\left(\hat{\pi}^0\,(i, \omega_0)\,, \omega_0\right) & \text{if } \hat{\pi}^0\,(i, \omega_0) \neq i \\ J & \text{if } \hat{\pi}^0\,(i, \omega_0) = i. \end{cases}$$

For any $i \in I$, let $\hat{\beta}^0_i = (\hat{\alpha}^0_i, \hat{g}^0_i)$.

Step 3: By the construction of $(\Omega_{-2}, \mathcal{A}_{-2}, Q_{-2})$, the internal type distribution of each agent is μ^0. Since no agent is matched at this stage, the expected cross-sectional extended type distribution $\mathbb{E}\hat{\gamma}$ concentrates on $S \times \{J\}$ with $\mathbb{E}\hat{\gamma}_{kJ} = \mu^0_k$ and $\mathbb{E}\hat{\gamma}_{kl} = 0$ for any $k, l \in S$. We use $\mathbf{1}_k$ to denote the indicator function of the singleton set $\{k\}$. For any $i, j \in I$ with $i \neq j$, it is clear that $\mathbf{1}_k(\hat{\alpha}^0_i)$ and $\mathbf{1}_k(\hat{\alpha}^0_j)$ are independent on $(\Omega_{-2}, \mathcal{A}_{-2}, Q_{-2})$. Since the variance of any zero-one valued random variable is less than or equal to $1/4$, we have for any $k \in S$,

$$\begin{aligned} \text{Var}\hat{\gamma}_{kJ} &= \text{Var}\frac{1}{\hat{M}}\sum_{i\in I}\mathbf{1}_k(\hat{\alpha}^0_i) = \frac{1}{\hat{M}^2}\sum_{i\in I}\text{Var}\mathbf{1}_k(\hat{\alpha}^0_i) \\ &\leq \frac{1}{\hat{M}^2}\hat{M}\frac{1}{4} = \frac{1}{4\hat{M}}. \end{aligned}$$

Lemma D.2 implies that

$$\begin{aligned}
& Q_{-2}\left(\|\hat{\gamma}-\mathbb{E}\hat{\gamma}\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right) \\
\leq\ & \sum_{k\in S} Q_{-2}\left(|\hat{\gamma}_{kJ}-\mathbb{E}\hat{\gamma}_{kJ}| \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right) \\
\leq\ & K\frac{\frac{1}{4\hat{M}}}{\frac{1}{\hat{M}^{\frac{2}{3}}}} = \frac{K}{4\hat{M}^{\frac{1}{3}}}.
\end{aligned}$$

Let $B=\{\omega_{-2}\in\Omega_{-2} : \|\hat{\gamma}(\omega_{-2})-\mathbb{E}\hat{\gamma}\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\}$. It is clear that $Q_{-2}(B) \leq \frac{K}{4\hat{M}^{\frac{1}{3}}}$.

Fix any $i,j\in I$ with $i\neq j$, $\omega_{-2}\notin B$ and $k_1,l_1,k_2,l_2\in S$. We have $|\hat{\gamma}_{k_1J}(\omega_{-2})-\mathbb{E}\hat{\gamma}_{k_1J}| \leq \frac{1}{\hat{M}^{\frac{1}{3}}}$, which implies that

$$\hat{\gamma}_{k_1J}(\omega_{-2}) \geq \mathbb{E}\hat{\gamma}_{k_1J} - \frac{1}{\hat{M}^{\frac{1}{3}}} = \mu^0_{k_1} - \frac{1}{\hat{M}^{\frac{1}{3}}} \geq \frac{1}{M} - \frac{1}{\hat{M}^{\frac{1}{3}}} \geq \frac{1}{2M}. \tag{E.44}$$

Since k_1 is a general type in S, we also have $|\hat{\gamma}_{k_2J}(\omega_{-2})-\mathbb{E}\hat{\gamma}_{k_2J}| \leq \frac{1}{\hat{M}^{\frac{1}{3}}}$ and $\hat{\gamma}_{k_2J}(\omega_{-2}) \geq \frac{1}{2M}$. We shall also use $\hat{\gamma}^{\omega_{-2}}$ to represent $\hat{\gamma}(\omega_{-2})$ for notational simplicity. Since $\frac{1}{2M} \geq \frac{1}{\hat{M}^{\frac{1}{3}}}$, Lemma E.1 (ii) indicates that ($\hat{\rho}$ and q in the lemma correspond to $\hat{\gamma}^{\omega_{-2}}$ and $q^0\left(\hat{\gamma}^{\omega_{-2}}\right)$ respectively)

$$\left|Q^{\omega_{-2}}_{-1}\left(\hat{g}^0_i = l_1\right) - q^0_{k_1l_1}\left(\hat{\gamma}^{\omega_{-2}}\right)\right| \leq \frac{2}{\hat{M}^{\frac{2}{3}}} \tag{E.45}$$

and

$$\left|Q^{\omega_{-2}}_{-1}\left(\hat{g}^0_i = l_1, \hat{g}^0_j = l_2\right) - q^0_{k_1l_1}\left(\hat{\gamma}^{\omega_{-2}}\right) q^0_{k_2l_2}\left(\hat{\gamma}^{\omega_{-2}}\right)\right| \leq \frac{5}{\hat{M}^{\frac{2}{3}}}. \tag{E.46}$$

Next, we shall consider the relationship between $Q^{\omega_{-2}}_{-1}$ and $q^0\left(\mathbb{E}\hat{\gamma}\right)$. When $\hat{p}^0_{k_1} > 0$ and $\hat{p}^0_{l_1} > 0$, the definition of q^0 implies that

$$\begin{aligned}
& \left|q^0_{k_1l_1}\left(\hat{\gamma}^{\omega_{-2}}\right) - q^0_{k_1l_1}\left(\mathbb{E}\hat{\gamma}\right)\right| \\
=\ & \left|\min\left(\frac{\hat{p}^0_{k_1l_1}}{\hat{p}^0_{k_1}}, \frac{\hat{p}^0_{k_1l_1}\hat{\gamma}^{\omega_{-2}}_{l_1J}}{\hat{p}^0_{l_1}\hat{\gamma}^{\omega_{-2}}_{k_1J}}\right) - \min\left(\frac{\hat{p}^0_{k_1l_1}}{\hat{p}^0_{k_1}}, \frac{\hat{p}^0_{k_1l_1}\mathbb{E}\hat{\gamma}_{l_1J}}{\hat{p}^0_{l_1}\mathbb{E}\hat{\gamma}_{k_1J}}\right)\right| \\
\leq\ & \frac{\hat{p}^0_{k_1l_1}}{\hat{p}^0_{l_1}}\left|\frac{\hat{\gamma}^{\omega_{-2}}_{l_1J}}{\hat{\gamma}^{\omega_{-2}}_{k_1J}} - \frac{\mathbb{E}\hat{\gamma}_{l_1J}}{\mathbb{E}\hat{\gamma}_{k_1J}}\right| \\
\leq\ & \frac{\left|\hat{\gamma}^{\omega_{-2}}_{l_1J}\mathbb{E}\hat{\gamma}_{k_1J} - \hat{\gamma}^{\omega_{-2}}_{k_1J}\mathbb{E}\hat{\gamma}_{l_1J}\right|}{\hat{\gamma}^{\omega_{-2}}_{k_1J}\mathbb{E}\hat{\gamma}_{k_1J}} \\
\leq\ & 2M^2\left(\left|\hat{\gamma}^{\omega_{-2}}_{l_1J}\mathbb{E}\hat{\gamma}_{k_1J} - \hat{\gamma}^{\omega_{-2}}_{l_1J}\hat{\gamma}^{\omega_{-2}}_{k_1J}\right| + \left|\hat{\gamma}^{\omega_{-2}}_{l_1J}\hat{\gamma}^{\omega_{-2}}_{k_1J} - \hat{\gamma}^{\omega_{-2}}_{k_1J}\mathbb{E}\hat{\gamma}_{l_1J}\right|\right) \\
\leq\ & 2M^2\left(\left|\mathbb{E}\hat{\gamma}_{k_1J} - \hat{\gamma}^{\omega_{-2}}_{k_1J}\right| + \left|\hat{\gamma}^{\omega_{-2}}_{l_1J} - \mathbb{E}\hat{\gamma}_{l_1J}\right|\right) \\
\leq\ & \frac{4M^2}{\hat{M}^{\frac{1}{3}}}.
\end{aligned}$$

The first inequality above follows from the fact that $|\min\{a,b\}-\min\{a,c\}| \leq |b-c|$ for any real numbers a,b,c. The third inequality is implied by $\hat{\gamma}_{k_1 J}^{\omega_{-2}} \geq \frac{1}{2M}$ and $\mathbb{E}\hat{\gamma}_{k_1 J} = \mu_{k_1}^0 \geq \frac{1}{M}$. The last inequality holds because $\omega_{-2} \notin B$. The rest is clear. When $\hat{p}_{k_1}^0 = 0$ or $\hat{p}_{l_1}^0 = 0$, we have $\left|q_{k_1 l_1}^0\left(\hat{\gamma}^{\omega_{-2}}\right) - q_{k_1 l_1}^0\left(\mathbb{E}\hat{\gamma}\right)\right| = |0-0| = 0$. Hence the inequality

$$\left|q_{k_1 l_1}^0\left(\hat{\gamma}^{\omega_{-2}}\right) - q_{k_1 l_1}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \leq \frac{4M^2}{\hat{M}^{\frac{1}{3}}} \tag{E.47}$$

always holds. Since k_1 and l_1 are general types in S, we also have $\left|q_{k_2 l_2}^0\left(\hat{\gamma}^{\omega_{-2}}\right) - q_{k_2 l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \leq \frac{4M^2}{\hat{M}^{\frac{1}{3}}}$. Therefore, the last two inequalities imply that

$$\begin{aligned}
& \left|q_{k_1 l_1}^0\left(\hat{\gamma}^{\omega_{-2}}\right) q_{k_2 l_2}^0\left(\hat{\gamma}^{\omega_{-2}}\right) - q_{k_1 l_1}^0\left(\mathbb{E}\hat{\gamma}\right) q_{k_2 l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \\
\leq\ & \left|q_{k_1 l_1}^0\left(\hat{\gamma}^{\omega_{-2}}\right) q_{k_2 l_2}^0\left(\hat{\gamma}^{\omega_{-2}}\right) - q_{k_1 l_1}^0\left(\hat{\gamma}^{\omega_{-2}}\right) q_{k_2 l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \\
& + \left|q_{k_1 l_1}^0\left(\hat{\gamma}^{\omega_{-2}}\right) q_{k_2 l_2}^0\left(\mathbb{E}\hat{\gamma}\right) - q_{k_1 l_1}^0\left(\mathbb{E}\hat{\gamma}\right) q_{k_2 l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \\
\leq\ & \left|q_{k_2 l_2}^0\left(\hat{\gamma}^{\omega_{-2}}\right) - q_{k_2 l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right| + \left|q_{k_1 l_1}^0\left(\hat{\gamma}^{\omega_{-2}}\right) - q_{k_1 l_1}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \\
\leq\ & \frac{8M^2}{\hat{M}^{\frac{1}{3}}}.
\end{aligned} \tag{E.48}$$

It follows from Equations (E.45) and (E.47) that

$$\begin{aligned}
& \left|Q_{-1}^{\omega_{-2}}\left(\hat{g}_i^0 = l_1\right) - q_{k_1 l_1}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \\
\leq\ & \left|Q_{-1}^{\omega_{-2}}\left(\hat{g}_i^0 = l_1\right) - q_{k_1 l_1}^0\left(\hat{\gamma}^{\omega_{-2}}\right)\right| + \left|q_{k_1 l_1}^0\left(\hat{\gamma}^{\omega_{-2}}\right) - q_{k_1 l_1}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \\
\leq\ & \frac{2}{\hat{M}^{\frac{2}{3}}} + \frac{4M^2}{\hat{M}^{\frac{1}{3}}} \leq \frac{5M^2}{\hat{M}^{\frac{1}{3}}}.
\end{aligned} \tag{E.49}$$

By Equations (E.46) and (E.48), we obtain that

$$\begin{aligned}
& \left|Q_{-1}^{\omega_{-2}}\left(\hat{g}_i^0 = l_1, \hat{g}_j^0 = l_2\right) - q_{k_1 l_1}^0\left(\mathbb{E}\hat{\gamma}\right) q_{k_2 l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \\
\leq\ & \left|Q_{-1}^{\omega_{-2}}\left(\hat{g}_i^0 = l_1, \hat{g}_j^0 = l_2\right) - q_{k_1 l_1}^0\left(\hat{\gamma}^{\omega_{-2}}\right) q_{k_2 l_2}^0\left(\hat{\gamma}^{\omega_{-2}}\right)\right| \\
& + \left|q_{k_1 l_1}^0\left(\hat{\gamma}^{\omega_{-2}}\right) q_{k_2 l_2}^0\left(\hat{\gamma}^{\omega_{-2}}\right) - q_{k_1 l_1}^0\left(\mathbb{E}\hat{\gamma}\right) q_{k_2 l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \\
\leq\ & \frac{5}{\hat{M}^{\frac{2}{3}}} + \frac{8M^2}{\hat{M}^{\frac{1}{3}}} \leq \frac{9M^2}{\hat{M}^{\frac{1}{3}}}.
\end{aligned} \tag{E.50}$$

We shall now consider the approximate pairwise independence of $\hat{\beta}^0 = \left(\hat{\alpha}^0, \hat{g}^0\right)$ as stated in Equation (E.2). For this purpose, fix any $i, j \in I$ with $i \neq j$, and $k_1, l_1, k_2, l_2 \in S$. Let $C_i = \{\omega_{-2} \in \Omega_{-2} : \omega_{-2}(i) = k_1\}$, and $C_{ij} = \{\omega_{-2} \in \Omega_{-2} : \omega_{-2}(i) = k_1, \omega_{-2}(j) = k_2\}$. Based on the construction of Q_{-2}, it is clear that $Q_{-2}\left(C_i\right) = \mu_{k_1}^0$ and $Q_{-2}\left(C_{ij}\right) = \mu_{k_1}^0 \mu_{k_2}^0$. It follows from Equation (E.49) and the definition of set B that

$$\begin{aligned}
& \left|Q_0\left(\hat{\beta}_i^0=(k_1,l_1)\right)-\mu_{k_1}^0\,q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \\
= & \left|\int_{C_i}\left[Q_{-1}^{\omega_{-2}}\left(\hat{g}_i^0=l_1\right)-q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)\right]dQ_{-2}\right| \\
\leq & \int_{C_i\cap B}\left|Q_{-1}^{\omega_{-2}}\left(\hat{g}_i^0=l_1\right)-q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)\right|dQ_{-2} \\
& \quad +\int_{C_i\setminus B}\left|Q_{-1}^{\omega_{-2}}\left(\hat{g}_i^0=l_1\right)-q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)\right|dQ_{-2} \\
\leq & Q_{-2}(B)+\int_{C_{ij}\setminus B}\frac{5M^2}{\hat{M}^{\frac{1}{3}}}dQ_{-2} \\
\leq & \frac{K}{4\hat{M}^{\frac{1}{3}}}+\frac{5M^2}{\hat{M}^{\frac{1}{3}}}\leq\frac{6M^2}{\hat{M}^{\frac{1}{3}}}.
\end{aligned} \tag{E.51}$$

One can derive from the definition of set B and Equation (E.50) to obtain that

$$\begin{aligned}
& \left|Q_0\left(\hat{\beta}_i^0=(k_1,l_1),\hat{\beta}_j^0=(k_2,l_2)\right)-\mu_{k_1}^0\,\mu_{k_2}^0\,q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)q_{k_2l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \\
= & \left|\int_{C_{ij}}\left[Q_{-1}^{\omega_{-2}}\left(\hat{g}_i^0=l_1,\hat{g}_j^0=l_2\right)-q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)q_{k_2l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right]dQ_{-2}\right| \\
\leq & \int_{C_{ij}\cap B}\left|Q_{-1}^{\omega_{-2}}\left(\hat{g}_i^0=l_1,\hat{g}_j^0=l_2\right)-q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)q_{k_2l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right|dQ_{-2} \\
& \quad +\int_{C_{ij}\setminus B}\left|Q_{-1}^{\omega_{-2}}\left(\hat{g}_i^0=l_1,\hat{g}_j^0=l_2)\right)-q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)q_{k_2l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right|dQ_{-2} \\
\leq & Q_{-2}(B)+\int_{C_{ij}\setminus B}\frac{9M^2}{\hat{M}^{\frac{1}{3}}}dQ_{-2} \\
\leq & \frac{K}{4\hat{M}^{\frac{1}{3}}}+\frac{9M^2}{\hat{M}^{\frac{1}{3}}}\leq\frac{10M^2}{\hat{M}^{\frac{1}{3}}}.
\end{aligned} \tag{E.52}$$

Equation (E.51) states an inequality in terms of the general indices i,k_1,l_1, which can be restated for the corresponding indices j,k_2,l_2. Based on those two inequalities, we can obtain that

$$\begin{aligned}
& \left|Q_0\left(\hat{\beta}_i^0=(k_1,l_1)\right)Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right)-\mu_{k_1}^0\,\mu_{k_2}^0\,q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)q_{k_2l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \\
\leq & \left|Q_0\left(\hat{\beta}_i^0=(k_1,l_1)\right)Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right)-\mu_{k_1}^0\,q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right)\right| \\
+ & \left|\mu_{k_1}^0\,q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right)-\mu_{k_1}^0\,\mu_{k_2}^0\,q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)q_{k_2l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \\
\leq & \left|Q_0\left(\hat{\beta}_i^0=(k_1,l_1)\right)-\mu_{k_1}^0\,q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)\right|+\left|Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right)-\mu_{k_2}^0\,q_{k_2l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \\
\leq & \frac{12M^2}{\hat{M}^{\frac{1}{3}}}.
\end{aligned} \tag{E.53}$$

Equations (E.52) and (E.53) imply that

$$\begin{aligned}
&\left|Q_0\left(\hat{\beta}_i^0=(k_1,l_1),\hat{\beta}_j^0=(k_2,l_2)\right)-Q_0\left(\hat{\beta}_i^0=(k_1,l_1)\right)Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right)\right| \\
\leq\ &\left|Q_0\left(\hat{\beta}_i^0=(k_1,l_1),\hat{\beta}_j^0=(k_2,l_2)\right)-\mu_{k_1}^0\,\mu_{k_2}^0\,q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)q_{k_2l_2}^0\left(\mathbb{E}\hat{\gamma}\right)\right| \\
&+\left|\mu_{k_1}^0\,\mu_{k_2}^0\,q_{k_1l_1}^0\left(\mathbb{E}\hat{\gamma}\right)q_{k_2l_2}^0\left(\mathbb{E}\hat{\gamma}\right)-Q_0\left(\hat{\beta}_i^0=(k_1,l_1)\right)Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right)\right| \\
\leq\ &\frac{22M^2}{\hat{M}^{\frac{1}{3}}}. \qquad\qquad (\text{E}.54)
\end{aligned}$$

Next we consider the case that either agent i or agent j is not matched. It is clear that

$$Q_0\left(\hat{\beta}_i^0=(k_1,J),\hat{\beta}_j^0=(k_2,l_2)\right)=Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right)-\sum_{l'\in S}Q_0\left(\hat{\beta}_i^0=(k_1,l'),\hat{\beta}_j^0=(k_2,l_2)\right),$$

$$Q_0\left(\hat{\beta}_i^0=(k_1,J)\right)Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right)=Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right)-\sum_{l'\in S}Q_0\left(\hat{\beta}_i^0=(k_1,l')\right)Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right).$$

Then, it follows from Equation (E.54) that

$$\begin{aligned}
&\left|Q_0\left(\hat{\beta}_i^0=(k_1,J),\hat{\beta}_j^0=(k_2,l_2)\right)-Q_0\left(\hat{\beta}_i^0=(k_1,J)\right)Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right)\right| \\
=\ &\left|\sum_{l'\in S}\left[Q_0\left(\hat{\beta}_i^0=(k_1,l'),\hat{\beta}_j^0=(k_2,l_2)\right)-Q_0\left(\hat{\beta}_i^0=(k_1,l')\right)Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right)\right]\right| \\
\leq\ &\sum_{l'\in S}\left|Q_0\left(\hat{\beta}_i^0=(k_1,l'),\hat{\beta}_j^0=(k_2,l_2)\right)-Q_0\left(\hat{\beta}_i^0=(k_1,l')\right)Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right)\right| \\
\leq\ &\frac{22KM^2}{\hat{M}^{\frac{1}{3}}}. \qquad\qquad (\text{E}.55)
\end{aligned}$$

The following identities are obvious:

$$Q_0\left(\hat{\beta}_i^0=(k_1,J),\hat{\beta}_j^0=(k_2,J)\right)=Q_0\left(\hat{\beta}_j^0=(k_2,J)\right)-\sum_{l'\in S}Q_0\left(\hat{\beta}_i^0=(k_1,l'),\hat{\beta}_j^0=(k_2,J)\right),$$

$$Q_0\left(\hat{\beta}_i^0=(k_1,J)\right)Q_0\left(\hat{\beta}_j^0=(k_2,J)\right)=Q_0\left(\hat{\beta}_j^0=(k_2,J)\right)-\sum_{l'\in S}Q_0\left(\hat{\beta}_i^0=(k_1,l')\right)Q_0\left(\hat{\beta}_j^0=(k_2,J)\right).$$

By Equation (E.55), we obtain that

$$\begin{aligned}
&\left|Q_0\left(\hat{\beta}_i^0=(k_1,J),\hat{\beta}_j^0=(k_2,J)\right)-Q_0\left(\hat{\beta}_i^0=(k_1,J)\right)Q_0\left(\hat{\beta}_j^0=(k_2,J)\right)\right| \\
=\ &\left|\sum_{l'\in S}\left[Q_0\left(\hat{\beta}_i^0=(k_1,l'),\hat{\beta}_j^0=(k_2,J)\right)-Q_0\left(\hat{\beta}_i^0=(k_1,l')\right)Q_0\left(\hat{\beta}_j^0=(k_2,J)\right)\right]\right| \\
\leq\ &\sum_{l'\in S}\left|Q_0\left(\hat{\beta}_i^0=(k_1,l'),\hat{\beta}_j^0=(k_2,J)\right)-Q_0\left(\hat{\beta}_i^0=(k_1,l')\right)Q_0\left(\hat{\beta}_j^0=(k_2,J)\right)\right| \\
\leq\ &\frac{22K^2M^2}{\hat{M}^{\frac{1}{3}}}. \qquad\qquad (\text{E}.56)
\end{aligned}$$

Therefore, Equations (E.54), (E.55) and (E.56) imply that for any $(k,l),(k',l') \in \hat{S}$,

$$\begin{aligned} & \left| Q_0\left(\hat{\beta}_i^0=(k,l),\hat{\beta}_j^0=(k',l')\right) - Q_0\left(\hat{\beta}_i^0=(k,l)\right) Q_0\left(\hat{\beta}_j^0=(k',l')\right) \right| \\ \leq \quad & \frac{22K^2M^2}{\hat{M}^{\frac{1}{3}}} \leq \frac{1}{\hat{M}^{\frac{1}{10}}}, \end{aligned} \tag{E.57}$$

which means that Equation (E.2) holds.

Next, we consider Equation (E.1); namely, $Q_0\left(||\hat{\rho}^0 - \mathbb{E}\hat{\rho}^0||_\infty \geq \frac{1}{\hat{M}^{\frac{1}{10}}}\right) \leq \frac{1}{\hat{M}^{\frac{1}{10}}}$. Recall that $\hat{\rho}^0_{\omega_0} = \lambda_0\left(\hat{\alpha}^0_{\omega_0}, \hat{g}^0_{\omega_0}\right)^{-1}$ is the internal cross-sectional extended type distribution. For any $(k,l) \in \hat{S}$, Equation (E.57) implies that

$$\begin{aligned} & \mathrm{Var}\hat{\rho}^0_{kl} = \mathrm{Var}\frac{1}{\hat{M}}\sum_{i\in I}\mathbf{1}_{kl}(\hat{\beta}_i^0) \\ = \quad & \frac{1}{\hat{M}^2}\sum_{i\in I}\mathrm{Var}\,\mathbf{1}_{kl}(\hat{\beta}_i^0) + \frac{1}{\hat{M}^2}\sum_{(i,j)\in I^2,\, i\neq j}\mathrm{Cov}\left(\mathbf{1}_{kl}(\hat{\beta}_i^0), \mathbf{1}_{kl}(\hat{\beta}_j^0)\right) \\ = \quad & \frac{1}{\hat{M}^2}\sum_{i\in I}\mathrm{Var}\,\mathbf{1}_{kl}(\hat{\beta}_i^0) \\ & +\frac{1}{\hat{M}^2}\sum_{(i,j)\in I^2,\, i\neq j}\left(Q_0\left(\hat{\beta}_i^0=(k_1,l_1),\hat{\beta}_j^0=(k_2,l_2)\right) - Q_0\left(\hat{\beta}_i^0=(k_1,l_1)\right)Q_0\left(\hat{\beta}_j^0=(k_2,l_2)\right)\right) \\ \leq \quad & \frac{1}{\hat{M}^2}\hat{M}\frac{1}{4} + \frac{1}{\hat{M}^2}\hat{M}(\hat{M}-1)\frac{22K^2M^2}{\hat{M}^{\frac{1}{3}}} \\ \leq \quad & \frac{1}{4\hat{M}} + \frac{22K^2M^2}{\hat{M}^{\frac{1}{3}}} \\ \leq \quad & \frac{23K^2M^2}{\hat{M}^{\frac{1}{3}}}. \end{aligned}$$

Then, Equation (E.1) follows from the above inequality and Lemma D.2 ($a = \frac{1}{\hat{M}^{\frac{1}{10}}}$):

$$\begin{aligned} & Q_0\left(||\hat{\rho}^0 - \mathbb{E}\hat{\rho}^0||_\infty \geq \frac{1}{\hat{M}^{\frac{1}{10}}}\right) \\ \leq \quad & \sum_{(k,l)\in\hat{S}} Q_0\left(|\hat{\rho}^0_{kl} - \mathbb{E}\hat{\rho}^0_{kl}| \geq \frac{1}{\hat{M}^{\frac{1}{10}}}\right) \\ \leq \quad & K(K+1)\frac{\frac{23K^2M^2}{\hat{M}^{\frac{1}{3}}}}{\frac{1}{\hat{M}^{\frac{2}{10}}}} = K(K+1)\frac{23K^2M^2}{\hat{M}^{\frac{4}{30}}} \leq \frac{1}{\hat{M}^{\frac{1}{10}}}. \end{aligned}$$

In this paragraph, we prove that $\mathbb{E}\hat{\rho}^0 \simeq \hat{p}^0$. Fix any $k,l \in S$. Since $\mathbb{E}\hat{\gamma}_{kJ} = \mu^0_k$ and $\mathbb{E}\hat{\gamma}_{lJ} = \mu^0_l$, it follows from the definition of q^0 that

$$q^0_{kl}\left(\mathbb{E}\hat{\gamma}\right) = \frac{1}{\mathbb{E}\hat{\gamma}_{kJ}}\min\left(\frac{\ddot{\rho}^0_{kl}\mathbb{E}\hat{\gamma}_{kJ}}{\mu^0_k}, \frac{\ddot{\rho}^0_{kl}\mathbb{E}\hat{\gamma}_{lJ}}{\mu^0_l}\right),$$

which implies that $\mu_k^0 \, q_{kl}^0 \left(\mathbb{E}\hat{\gamma}\right) = \ddot{\rho}_{kl}^0 \simeq \hat{p}_{kl}^0$. Equation (E.51) implies that

$$Q_0 \left(\hat{\beta}_i^0 = (k,l)\right) \simeq \mu_k^0 \, q_{kl}^0 \left(\mathbb{E}\hat{\gamma}\right) \simeq \hat{p}_{kl}^0.$$

Next, we consider the case that agent i is not matched. We have

$$\begin{aligned} Q_0 \left(\hat{\beta}_i^0 = (k,J)\right) &= Q_0 \left(\hat{\alpha}_i^0 = k\right) - \sum_{l \in S} Q_0 \left(\hat{\beta}_i^0 = (k,l)\right) \\ &\simeq \mu_k^0 - \sum_{l \in S} \hat{p}_{kl}^0 = \sum_{l \in S \cup \{J\}} \ddot{\rho}_{kl}^0 - \sum_{l \in S} \hat{p}_{kl}^0 \\ &\simeq \sum_{l \in S \cup \{J\}} \hat{p}_{kl}^0 - \sum_{l \in S} \hat{p}_{kl}^0 = \hat{p}_{kJ}^0. \end{aligned}$$

Therefore, for any $(k,l) \in \hat{S}$, $Q_0 \left(\hat{\beta}_i^0 = (k,l)\right) \simeq \hat{p}_{kl}^0$, which implies that

$$\mathbb{E}\hat{\rho}_{kl}^0 = \frac{1}{\hat{M}} \sum_{i \in I} Q_0 \left(\hat{\beta}_i^0 = (k,l)\right) \simeq \hat{p}_{kl}^0.$$

Now, we prove that $\mathbb{E}\hat{\rho}_{kJ}^0 \geq \frac{1}{M^2}$ for any $k \in S$. Fix any $k \in S$. By Equation (E.51), we have

$$Q_0 \left(\hat{\beta}_i^0 = (k,l)\right) \leq \mu_{k_1}^0 \, q_{kl}^0 \left(\mathbb{E}\hat{\gamma}\right) + \frac{6M^2}{\hat{M}^{\frac{1}{3}}} = \ddot{\rho}_{kl}^0 + \frac{6M^2}{\hat{M}^{\frac{1}{3}}},$$

for any $l \in S$. Then we can obtain the following estimation

$$\begin{aligned} Q_0 \left(\hat{\beta}_i^0 = (k,J)\right) &= Q_0 \left(\hat{\alpha}_i^0 = k\right) - \sum_{l \in S} Q_0 \left(\hat{\beta}_i^0 = (k,l)\right) \\ &\geq \mu_k^0 - \sum_{l \in S} \left(\ddot{\rho}_{kl}^0 + \frac{6M^2}{\hat{M}^{\frac{1}{3}}}\right) \\ &= \ddot{\rho}_{kJ}^0 - \frac{6KM^2}{\hat{M}^{\frac{1}{3}}} \geq \frac{1}{M} - \frac{6KM^2}{\hat{M}^{\frac{1}{3}}} \geq \frac{1}{M^2}, \end{aligned}$$

which implies that

$$\mathbb{E}\hat{\rho}_{kJ}^0 = \frac{1}{\hat{M}} \sum_{i \in I} Q_0 \left(\hat{\beta}_i^0 = (k,J)\right) \geq \frac{1}{M^2}.$$

It remains to prove that for any $i,j \in I$ with $i \neq j$, $Q_0 \left(\hat{\pi}_i^0 = j\right) \leq \frac{1}{\hat{M}^{\frac{1}{5}}}$. It is clear that

$$\begin{aligned} Q_0 \left(\hat{\pi}_i^0 = j\right) &= \int_{\Omega_{-2}} Q_{-1}^{\omega_{-2}} \left(\hat{\pi}_i^0 = j)\right) dQ_{-2} \\ &\leq \int_B Q_{-1}^{\omega_{-2}} \left(\hat{\pi}_i^0 = j)\right) dQ_{-2} + \int_{B^c} Q_{-1}^{\omega_{-2}} \left(\hat{\pi}_i^0 = j)\right) dQ_{-2}. \end{aligned}$$

When $\omega_{-2} \in B^c$ (the complement of B) and k is agent i's type ($\omega_{-2}(i) = k$), Equation (E.44) indicates that $\hat{\gamma}_{kJ}(\omega_{-2}) \geq \frac{1}{2M}$; by Lemma E.1 (ii),

$$Q_{-1}^{\omega_{-2}} \left(\hat{\pi}_i^0 = j)\right) \leq \frac{2}{\hat{M} \, \hat{\gamma}_{kJ}(\omega_{-2})} \leq \frac{2}{\hat{M} \frac{1}{2M}} = \frac{4M}{\hat{M}}.$$

Therefore, the above inequalities and the definition of B imply that

$$\begin{aligned} Q_0\left(\hat{\pi}_i^0 = j\right) &\leq Q_{-2}(B) + \int_{B^c} \frac{4M}{\hat{M}} dQ_{-2} \\ &\leq \frac{K}{4\hat{M}^{\frac{1}{3}}} + \frac{4M}{\hat{M}} \leq \frac{1}{\hat{M}^{\frac{1}{5}}}. \end{aligned}$$

E.5.3 Some additional lemmas

The following lemma demonstrates the identity of $\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}^{3n-2}$ and $T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)$.

Lemma E.14. *For any $n \in \mathbb{T}_0$, $\omega^{3n-3} \in \Omega^{3n-3}$, we have*

$$\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}^{3n-2} = T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right).$$

Proof. Fix any $n \in \mathbb{T}_0$, $\omega^{3n-3} \in \Omega^{3n-3}$ and $(k, l, r) \in \tilde{S}$. For any $(k', l', r') \in \tilde{S}$, let

$$B_{k'l'r'}^{\omega^{3n-3}} = \{i \in I : \tilde{\beta}_i^{3n-3}(\omega^{3n-3}) = (k', l', r')\}.$$

It follows from the definition of $\tilde{\rho}^{3n-2}$ that

$$\begin{aligned} \mathbb{E}^{\omega^{3n-3}}\tilde{\rho}_{klr}^{3n-2} &= \int_{\Omega_{3n-2}} \tilde{\rho}_{klr}^{3n-2}(\omega^{3n-2}) dQ_{3n-2}^{\omega^{3n-3}} \\ &= \int_{\Omega_{3n-2}} \frac{1}{\hat{M}} \sum_{i \in I} \mathbf{1}_{klr}(\tilde{\beta}_i^{3n-2}(\omega^{3n-2})) dQ_{3n-2}^{\omega^{3n-3}} \\ &= \frac{1}{\hat{M}} \sum_{(k',l',r') \in \tilde{S}} \sum_{i \in B_{k'l'r'}^{\omega^{3n-3}}} \int_{\Omega_{3n-2}} \mathbf{1}_{klr}(\tilde{\beta}_i^{3n-2}(\omega^{3n-2})) dQ_{3n-2}^{\omega^{3n-3}} \\ &= \frac{1}{\hat{M}} \sum_{(k',l',r') \in \tilde{S}} \sum_{i \in B_{k'l'r'}^{\omega^{3n-3}}} Q_{3n-2}^{\omega^{3n-3}} \left(\tilde{\beta}_i^{3n-2}(\omega^{3n-2}) = (k, l, r)\right). \end{aligned}$$

When $l \in S$ and $r = 0$, we have

$$\begin{aligned} \mathbb{E}^{\omega^{3n-3}}\tilde{\rho}_{kl0}^{3n-2} &= \frac{1}{\hat{M}} \sum_{k',l' \in S} \sum_{i \in B_{k'l'0}^{\omega^{3n-3}}} Q_{3n-2}^{\omega^{3n-3}} \left(\tilde{\beta}_i^{3n-2}(\omega^{3n-2}) = (k, l, 0)\right) \\ &= \frac{1}{\hat{M}} \sum_{k',l' \in S} \sum_{i \in B_{k'l'0}^{\omega^{3n-3}}} \hat{\eta}_{kk'}\hat{\eta}_{ll'} \\ &= \sum_{k',l' \in S} \tilde{\rho}_{k'l'0}^{3n-3}(\omega^{3n-3})\hat{\eta}_{kk'}\hat{\eta}_{ll'} \\ &= \left[T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)\right]_{kl0}. \end{aligned} \tag{E.58}$$

When $l = J$ and $r = 1$, we have

$$\begin{aligned}\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}_{kJ1}^{3n-2} &= \frac{1}{\hat{M}}\sum_{k'\in S}\sum_{i\in B_{k'J1}^{\omega^{3n-3}}} Q_{3n-2}^{\omega^{3n-3}}\left(\tilde{\beta}_i^{3n-2}(\omega^{3n-2}) = (k, J, 1)\right)\\ &= \frac{1}{\hat{M}}\sum_{k'\in S}\sum_{i\in B_{k'J1}^{\omega^{3n-3}}} \hat{\eta}_{kk'}\\ &= \sum_{k'\in S}\tilde{\rho}_{k'J1}^{3n-3}(\omega^{3n-3})\hat{\eta}_{kk'}\\ &= \left[T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)\right]_{kJ1}. \end{aligned} \tag{E.59}$$

By the construction of the mutation step and the definition of $\tilde{\beta}^{3n-2}$, it is clear that

$$\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}_{kl1}^{3n-2} = 0 = \left[T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)\right]_{kl1}, \tag{E.60}$$

$$\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}_{kJ0}^{3n-2} = 0 = \left[T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)\right]_{kJ0}. \tag{E.61}$$

The identity $\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}^{3n-2} = T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)$ then follows from Equations (E.58), (E.59), (E.60) and (E.61) ■

The following lemma shows the relationship between $\mathbb{E}^{\omega^{3n-2}}\tilde{\rho}^{3n-1}$ and $T_2\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)$.

Lemma E.15. *For any $n \in \mathbb{T}_0$, $\omega^{3n-1} \in \Omega^{3n-1}$, we have*

$$||\tilde{\rho}^{3n-1}(\omega^{3n-1}) - T_2\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)||_\infty \leq \frac{2K}{\hat{M}}.$$

Proof. Fix any $n \in \mathbb{T}_0$, $\omega^{3n-2} \in \Omega^{3n-2}$ and $(k, l, r) \in \tilde{S}$. When $l \in S$ and $r = 0$, it is clear that for any $\omega^{3n-1} \in \Omega^{3n-1}$ with $\omega^{3n-1} = \left(\omega^{3n-2}, \omega_{3n-1}\right)$,

$$\tilde{\rho}_{kl0}^{3n-1}(\omega^{3n-1}) = \tilde{\rho}_{kl0}^{3n-2}(\omega^{3n-2}) = \left[T_2\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)\right]_{kl0}. \tag{E.62}$$

When $l \in S$ and $r = 1$, it follows from Lemma E.1 (iii) that for any $\omega_{3n-1} \in \Omega_{3n-1}$,

$$\begin{aligned} &\left|\tilde{\rho}_{kl1}^{3n-1}(\omega^{3n-1}) - \left[T_2(\tilde{\rho}^{3n-2}(\omega^{3n-2}))\right]_{kl1}\right|\\ =& \left|\tilde{\rho}_{kl1}^{3n-1}(\omega^{3n-1}) - \tilde{\rho}_{kJ1}^{3n-2}(\omega^{3n-2})\hat{q}_{kl}\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)\right|\\ \leq& \frac{2}{\hat{M}}. \end{aligned} \tag{E.63}$$

When $l = J$ and $r = 1$, we have for any $\omega_{3n-1} \in \Omega_{3n-1}$,

$$\begin{aligned} &\left|\tilde{\rho}_{kJ1}^{3n-1}(\omega^{3n-1}) - \left[T_2(\tilde{\rho}^{3n-2}(\omega^{3n-2}))\right]_{kJ1}\right|\\ =& \left|\sum_{l'\in S}\tilde{\rho}_{kl'1}^{3n-1}(\omega^{3n-1}) - \sum_{l'\in S}\left[T_2(\tilde{\rho}^{3n-2}(\omega^{3n-2}))\right]_{kl'1}\right|\\ \leq& \frac{2K}{\hat{M}}. \end{aligned} \tag{E.64}$$

When $l \in S$ and $r = 0$, by the construction of the matching step and the definition of $\tilde{\beta}^{3n-1}$, it is clear that for any $\omega_{3n-1} \in \Omega_{3n-1}$,

$$\tilde{\rho}^{3n-1}_{kJ0}(\omega_{3n-1}) = 0 = \left[T_2\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)\right]_{kJ0}. \tag{E.65}$$

By Equations (E.62), (E.63), (E.64) and (E.65), we have,

$$||\tilde{\rho}^{3n-1}(\omega^{3n-1}) - T_2\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)||_\infty \leq \frac{2K}{\hat{M}}$$

for any $\omega^{3n-1} \in \Omega^{3n-1}$. ■

The identity of $\mathbb{E}^{\omega^{3n-1}}\tilde{\rho}^{3n}$ and $T_1\left(\tilde{\rho}^{3n-1}(\omega^{3n-1})\right)$ is proved in the next lemma.

Lemma E.16. *For any $n \in \mathbb{T}_0$, $\omega^{3n-1} \in \Omega^{3n-1}$, we have*

$$\mathbb{E}^{\omega^{3n-1}}\tilde{\rho}^{3n} = T_3\left(\tilde{\rho}^{3n-1}(\omega^{3n-1})\right).$$

Proof. Fix any $n \in \mathbb{T}_0$, $\omega^{3n-1} \in \Omega^{3n-1}$ and $(k, l, r) \in \tilde{S}$. For any $(k', l', r') \in \tilde{S}$, let

$$B^{\omega^{3n-1}}_{k'l'r'} = \{i \in I : \tilde{\beta}^{3n-1}_i(\omega^{3n-1}) = (k', l', r')\}.$$

It follows from the definition of $\tilde{\rho}^{3n}$ that

$$\begin{aligned}
\mathbb{E}^{\omega^{3n-1}}\tilde{\rho}^{3n}_{klr} &= \int_{\Omega_{3n}} \tilde{\rho}^{3n}_{klr}(\omega^{3n}) dQ^{\omega^{3n-1}}_{3n} \\
&= \int_{\Omega_{3n}} \frac{1}{\hat{M}} \sum_{i \in I} \mathbf{1}_{klr}(\tilde{\beta}^{3n}_i) dQ^{\omega^{3n-1}}_{3n} \\
&= \frac{1}{\hat{M}} \sum_{(k',l',r') \in \tilde{S}} \sum_{i \in B^{\omega^{3n-1}}_{k'l'r'}} \int_{\Omega_{3n}} \mathbf{1}_{klr}(\tilde{\beta}^{3n}_i) dQ^{\omega^{3n-1}}_{3n} \\
&= \frac{1}{\hat{M}} \sum_{(k',l',r') \in \tilde{S}} \sum_{i \in B^{\omega^{3n-1}}_{k'l'r'}} Q^{\omega^{3n-1}}_{3n}\left(\tilde{\beta}^{3n}_i = (k, l, r)\right).
\end{aligned}$$

When $l \in S$ and $r = 0$, we have

$$\begin{aligned}
\mathbb{E}^{\omega^{3n-1}}\tilde{\rho}^{3n}_{kl0} &= \frac{1}{\hat{M}} \sum_{i \in B^{\omega^{3n-1}}_{kl0}} Q^{\omega^{3n-1}}_{3n}\left(\tilde{\beta}^{3n}_i = (k, l, 0)\right) \\
&\quad + \frac{1}{\hat{M}} \sum_{k',l' \in S} \sum_{i \in B^{\omega^{3n-1}}_{k'l'1}} Q^{\omega^{3n-1}}_{3n}\left(\tilde{\beta}^{3n}_i = (k, l, 0)\right) \\
&= \tilde{\rho}^{3n-1}_{kl0}(\omega^{3n-1})(1 - \hat{\vartheta}_{kl}) + \sum_{k',l' \in S} \tilde{\rho}^{3n-1}_{k'l'1}(\omega^{3n-1})\hat{\xi}_{k'l'}\hat{\sigma}_{k'l'}(k, l) \\
&= \left[T_3\left(\tilde{\rho}^{3n-1}(\omega^{3n-1})\right)\right]_{kl0}.
\end{aligned} \tag{E.66}$$

When $l = J$ and $r = 1$, we obtain that

$$\begin{aligned}
\mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n}_{kJ1} &= \frac{1}{\hat{M}} \sum_{i \in B^{\omega^{3n-1}}_{kJ1}} Q^{\omega^{3n-1}}_{3n} \left(\tilde{\beta}^{3n}_i = (k, J, 1) \right) \\
&\quad + \frac{1}{\hat{M}} \sum_{k',l' \in S} \sum_{i \in B^{\omega^{3n-1}}_{k'l'0}} Q^{\omega^{3n-1}}_{3n} \left(\tilde{\beta}^{3n}_i = (k, J, 1) \right) \\
&\quad + \frac{1}{\hat{M}} \sum_{k',l' \in S} \sum_{i \in B^{\omega^{3n-1}}_{k'l'1}} Q^{\omega^{3n-1}}_{3n} \left(\tilde{\beta}^{3n}_i = (k, J, 1) \right) \\
&= \tilde{\rho}^{3n-1}_{kJ1}(\omega^{3n-1}) + \tilde{\rho}^{3n-1}_{kl0}(\omega^{3n-1})(1 - \hat{\vartheta}_{kl}) + \sum_{k',l' \in S} \tilde{\rho}^{3n-1}_{k'l'1}(\omega^{3n-1}) \hat{\xi}_{k'l'} \hat{\sigma}_{k'l'}(k, l) \\
&= \left[T_3 \left(\tilde{\rho}^{3n-1}(\omega^{3n-1}) \right) \right]_{kJ1}. \qquad \text{(E.67)}
\end{aligned}$$

By the construction of the type changing with break-up step, and the definition of $\tilde{\beta}^{3n}$, it is clear that

$$\mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n}_{kl1} = 0 = \left[T_3 \left(\tilde{\rho}^{3n-1}(\omega^{3n}) \right) \right]_{kl1}, \qquad \text{(E.68)}$$

$$\mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n}_{kJ0} = 0 = \left[T_3 \left(\tilde{\rho}^{3n-1}(\omega^{3n}) \right) \right]_{kJ0}. \qquad \text{(E.69)}$$

The identity $\mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n} = T_3 \left(\tilde{\rho}^{3n-1}(\omega^{3n-1}) \right)$ is then implied by Equations (E.66), (E.67), (E.68) and (E.69). ∎

The following lemma shows that the cross-sectional expanded type distribution $\tilde{\rho}^m$ at the end of step m can be approximated by $U_1^m(\mathbb{E}\tilde{\rho}^0)$.

Lemma E.17. *Let $\epsilon_0 = \frac{2}{\hat{M}^{\frac{1}{10}}}$. For any $m \in \{1, 2, \ldots, 3M^2\}$, let*

$$V^m = \{\omega^m \in \Omega^m : \|\tilde{\rho}^m(\omega^m) - U_1^m(\mathbb{E}\tilde{\rho}^0)\|_\infty > \xi_0\},$$

where ξ_0 is defined in Lemma E.3. Then, for any $m \in \{1, 2, \ldots, 3M^2\}$, we have $Q^m(V^m) \leq \epsilon_0$.

Proof. In period 0, let $W^0 = \{\omega_0 \in \Omega_0 : \|\tilde{\rho}^0(\omega_0) - \mathbb{E}\tilde{\rho}^0\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{10}}}\}$. By Equation (E.1), $Q_0(W^0) \leq \frac{1}{\hat{M}^{\frac{1}{10}}}$.

Fix any $n \in \mathbb{T}_0$. For the mutation step in period n, fix any $\omega^{3n-3} \in \Omega^{3n-3}$. Let

$$C^{\omega^{3n-3}} = \{(i, j) \in I \times I : i < j, \hat{\pi}^{3n-3}(i, \omega^{3n-3}) = j\}.$$

For any $i, j \in I$ such that $i \neq j$ and $\hat{\pi}^{3n-3}(i, \omega^{3n-3}) \neq j$, and any $(k, l, r) \in \tilde{S}$, it is clear that $\mathbf{1}_{klr}(\tilde{\beta}^{3n-2}_i)$ and $\mathbf{1}_{klr}(\tilde{\beta}^{3n-2}_j)$ are independent on $(\Omega_{3n-2}, \mathcal{E}_{3n-2}, Q^{\omega^{3n-3}}_{3n-2})$. Therefore, such independence and the definition of $C^{\omega^{3n-3}}$ imply that

$$\begin{aligned}
\mathrm{Var}^{\omega^{3n-3}} \tilde{\rho}^{3n-2}_{klr} &= \mathrm{Var}^{\omega^{3n-3}} \frac{1}{\hat{M}} \sum_{i \in I} \mathbf{1}_{klr}(\tilde{\beta}^{3n-2}_i) \\
&= \frac{1}{\hat{M}^2} \sum_{i \in I} \mathrm{Var}^{\omega^{3n-3}} \mathbf{1}_{klr}(\tilde{\beta}^{3n-2}_i) + \frac{2}{\hat{M}^2} \sum_{(i,j) \in C^{\omega^{3n-3}}} \mathrm{Cov}\left(\mathbf{1}_{klr}(\tilde{\beta}^{3n-2}_i), \mathbf{1}_{klr}(\tilde{\beta}^{3n-2}_j)\right) \\
&\leq \frac{1}{\hat{M}^2} \hat{M} \frac{1}{4} + \frac{2}{\hat{M}^2} \frac{\hat{M}}{2} \frac{1}{4} \\
&= \frac{1}{2\hat{M}}.
\end{aligned}$$

It follows from Lemmas D.2 and E.14 that

$$\begin{aligned}
& Q^{\omega^{3n-3}}_{3n-2} \left(\|\tilde{\rho}^{3n-2} - T_1\left(\tilde{\rho}^{3n-3}\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \right) \\
= \;& Q^{\omega^{3n-3}}_{3n-2} \left(\|\tilde{\rho}^{3n-2} - \mathbb{E}^{\omega^{3n-3}} \tilde{\rho}^{3n-2}\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \right) \\
\leq \;& \sum_{(k,l,r) \in \tilde{S}} Q^{\omega^{3n-3}}_{3n-2} \left(\left| \tilde{\rho}^{3n-2}_{klr} - \mathbb{E}^{\omega^{3n-3}} \tilde{\rho}^{3n-2}_{klr} \right| \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \right) \\
\leq \;& 2K(K+1) \frac{\frac{1}{2\hat{M}}}{\frac{1}{\hat{M}^{\frac{2}{3}}}} = \frac{K(K+1)}{\hat{M}^{\frac{1}{3}}}.
\end{aligned}$$

Let $W^{3n-2} = \{\omega^{3n-2} \in \Omega^{3n-2} : \|\tilde{\rho}^{3n-2}(\omega^{3n-2}) - T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\}$. It is clear that

$$\begin{aligned}
Q^{3n-2}(W^{3n-2}) &= \int_{\Omega^{3n-3}} Q^{\omega^{3n-3}}_{3n-2} \left(\|\tilde{\rho}^{3n-2} - T_1\left(\tilde{\rho}^{3n-3}\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \right) dQ^{3n-3} \\
&\leq \frac{K(K+1)}{\hat{M}^{\frac{1}{3}}}. \qquad \text{(E.70)}
\end{aligned}$$

For the random matching step in period n, Lemma E.15 indicates that for any $\omega^{3n-1} \in \Omega^{3n-1}$,

$$||\tilde{\rho}^{3n-1}(\omega^{3n-1}) - T_2\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)||_\infty \leq \frac{2K}{\hat{M}}.$$

It is then clear that the set

$$W^{3n-1} = \{\omega^{3n-1} \in \Omega^{3n-1} : \|\tilde{\rho}^{3n-1}(\omega^{3n-1}) - T_2\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\}$$

is empty. Hence, we have

$$Q^{3n-1}(W^{3n-1}) = \int_{\Omega^{3n-2}} Q^{\omega^{3n-1}}_{3n-2} \left(\|\tilde{\rho}^{3n-1} - T_1\left(\tilde{\rho}^{3n-2}\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \right) dQ^{3n-2} = 0. \qquad \text{(E.71)}$$

For the type changing with break-up step in period n, fix any $\omega^{3n-1} \in \Omega^{3n-1}$. Let

$$C^{\omega^{3n-1}} = \{(i,j) \in I \times I : i < j, \,\hat{\pi}^{3n-1}(i, \omega^{3n-1}) = j\}.$$

For any $i, j \in I$ such that $i \neq j$ and $\hat{\pi}^{3n-1}(i, \omega^{3n-1}) \neq j$, and any $(k, l, r) \in \tilde{S}$, it is clear that $\mathbf{1}_{klr}(\tilde{\beta}_i^{3n})$ and $\mathbf{1}_{klr}(\tilde{\beta}_j^{3n})$ are independent on $(\Omega_{3n}, \mathcal{E}_{3n}, Q_{3n}^{\omega^{3n-1}})$. Therefore, we have

$$\begin{aligned}
\mathrm{Var}^{\omega^{3n-1}} \tilde{\rho}_{klr}^{3n} &= \mathrm{Var}^{\omega^{3n-1}} \frac{1}{\hat{M}} \sum_{i \in I} \mathbf{1}_{klr}(\tilde{\beta}_i^{3n}) \\
&= \frac{1}{\hat{M}^2} \sum_{i \in I} \mathrm{Var}^{\omega^{3n-1}} \mathbf{1}_{klr}(\tilde{\beta}_i^{3n}) + \frac{2}{\hat{M}^2} \sum_{(i,j) \in C^{\omega^{3n-1}}} \mathrm{Cov}\left(\mathbf{1}_{klr}(\tilde{\beta}_i^{3n}), \mathbf{1}_{klr}(\tilde{\beta}_j^{3n})\right) \\
&\leq \frac{1}{\hat{M}^2} \sum_{i \in I} \frac{1}{4} + \frac{2}{\hat{M}^2} \frac{\hat{M}}{2} \frac{1}{4} \\
&= \frac{1}{2\hat{M}}.
\end{aligned}$$

It follows from Lemmas D.2 and E.16 that

$$\begin{aligned}
& Q_{3n}^{\omega^{3n-1}} \left(\|\tilde{\rho}^{3n} - T_3\left(\tilde{\rho}^{3n-1}\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \right) \\
= \;& Q_{3n}^{\omega^{3n-1}} \left(\|\tilde{\rho}^{3n} - \mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n}\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \right) \\
\leq \;& \sum_{(k,l,r) \in \tilde{S}} Q_{3n}^{\omega^{3n-1}} \left(\left|\tilde{\rho}_{klr}^{3n} - \mathbb{E}^{\omega^{3n-1}} \tilde{\rho}_{klr}^{3n}\right| \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \right) \\
\leq \;& 2K(K+1) \frac{\frac{1}{2\hat{M}}}{\frac{1}{\hat{M}^{\frac{2}{3}}}} = \frac{K(K+1)}{\hat{M}^{\frac{1}{3}}}.
\end{aligned}$$

Let $W^{3n} = \{\omega^{3n} \in \Omega^{3n} : \|\tilde{\rho}^{3n}(\omega^{3n}) - T_3\left(\tilde{\rho}^{3n-1}(\omega^{3n-1})\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\}$. It is clear that

$$\begin{aligned}
Q^{3n}(W^{3n}) &= \int_{\Omega^{3n-1}} Q_{3n}^{\omega^{3n-1}} \left(\|\tilde{\rho}^{3n} - T_3\left(\tilde{\rho}^{3n-1}\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \right) dQ^{3n-1} \\
&\leq \frac{K(K+1)}{\hat{M}^{\frac{1}{3}}}.
\end{aligned} \tag{E.72}$$

For any $m \in \{0, 1, \ldots, 3M^2\}$, let

$$\overline{W}^m = \{\omega^m \in \Omega^m : \omega^{m'} \in W^{m'} \text{ for some } m' \text{ between 0 and } m\}.$$

By Equations (E.1), (E.70), (E.71) and (E.72), we have

$$Q^m(\overline{W}^m) \leq \sum_{m'=0}^{m} Q^{m'}(W^{m'}) \leq \frac{1}{\hat{M}^{\frac{1}{10}}} + \frac{3M^2 K(K+1)}{\hat{M}^{\frac{1}{3}}} \leq \epsilon_0.$$

Fix any $m \in \{0, 1, \ldots, 3M^2\}$ and $\omega^m \notin \overline{W}^m$. We have

$$\begin{aligned}
& \|\tilde{\rho}^m(\omega^m) - U_1^m(\mathbb{E}\tilde{\rho}^0)\|_\infty \\
\leq\ & \|\tilde{\rho}^m(\omega^m) - U_m^m(\tilde{\rho}^{m-1}(\omega^{m-1}))\|_\infty + \|U_m^m(\tilde{\rho}^{m-1}(\omega^{m-1})) - U_1^m(\mathbb{E}\tilde{\rho}^0)\|_\infty \\
\leq\ & \sum_{j=1}^{m} \|U_{j+1}^m(\tilde{\rho}^j(\omega^j)) - U_j^m(\tilde{\rho}^{j-1}(\omega^{j-1}))\|_\infty \\
& + \|U_1^m(\tilde{\rho}^0(\omega^0)) - U_1^m(\mathbb{E}\tilde{\rho}^0)\|_\infty \\
=\ & \sum_{j=1}^{m} \|U_{j+1}^m(\tilde{\rho}^j(\omega^j)) - U_{j+1}^m\left(U_j^j(\tilde{\rho}^{j-1}(\omega^{j-1}))\right)\|_\infty \\
& + \|U_1^m(\tilde{\rho}^0(\omega^0)) - U_1^m(\mathbb{E}\tilde{\rho}^0)\|_\infty.
\end{aligned}$$

By the definition of $\hat{M}$, we know that $\frac{1}{\hat{M}^{\frac{1}{10}}} \leq \xi_{3M^2+1}$. The fact that $\omega^m \notin \overline{W}^m$ leads to $\omega^j \notin W^j$ for any $j \in \{0, 1, \ldots, m\}$, which implies that

$$\|\tilde{\rho}^j(\omega^j) - U_j^j(\tilde{\rho}^{j-1}(\omega^{j-1}))\|_\infty < \frac{1}{\hat{M}^{\frac{1}{3}}} < \frac{1}{\hat{M}^{\frac{1}{10}}} \leq \xi_{3M^2+1},$$

$$\|\tilde{\rho}^0(\omega_0) - \mathbb{E}\tilde{\rho}^0\|_\infty < \frac{1}{\hat{M}^{\frac{1}{10}}} \leq \xi_{3M^2+1}.$$

By Lemma E.3, we have

$$\begin{aligned}
& \|\tilde{\rho}^m(\omega^m) - U_1^m(\mathbb{E}\tilde{\rho}^0)\|_\infty \\
\leq\ & \sum_{j=0}^{m-1} \xi_{3M^2+1-j} + \xi_{3M^2+1-m} \\
\leq\ & \sum_{j=0}^{m-1} \frac{1}{3M^2+1}\xi_0 + \frac{1}{3M^2+1}\xi_0 \leq \xi_0,
\end{aligned}$$

which implies that $\omega^m \notin V^m = \{\omega^m \in \Omega^m : \|\tilde{\rho}^m(\omega^m) - U_1^m(\mathbb{E}\tilde{\rho}^0)\|_\infty > \xi_0\}$. Since ω^m is an arbitrarily fixed element in $\Omega^m \setminus \overline{W}^m$, the fact that $\omega^m \notin V^m$ implies that $V^m \subseteq \overline{W}^m$. Therefore, we have $Q^m(V^m) \leq \epsilon_0$. ■

The following lemma shows that, when ω^{3n-2} is not in V^{3n-2}, $\hat{M}^{-\frac{1}{15}}$ is a lower bound for the population of single type-k agents after step $3n-2$.

Lemma E.18. *For any $n \in \{1, 2, \ldots, M^2\}$, $\omega^{3n-2} \notin V^{3n-2}$ and $k \in S$, we have $\tilde{\rho}_{kJ1}^{3n-2}(\omega^{3n-2}) \geq \hat{M}^{-\frac{1}{15}}$.*

Proof. : Fix any $n \in \mathbb{T}_0 = \{1, 2, \ldots, M^2\}$. When $n = 1$, our convention is that U_1^0 is the identity mapping, and hence we have

$$\sum_{k \in S} \left[U_1^{3n-3}(\mathbb{E}\tilde{\rho}^0)\right]_{kJ1} = \sum_{k \in S} \mathbb{E}\tilde{\rho}_{kJ1}^0 = \sum_{k \in S} \mathbb{E}\hat{\rho}_{kJ}^0.$$

A property stated above Lemma E.2 indicates that $\mathbb{E}\hat{\rho}^0_{kJ} \geq \frac{1}{M^2}$. Therefore, it is clear that $\sum_{k\in S}\left[U_1^{3n-3}(\mathbb{E}\tilde{\rho}^0)\right]_{kJ1} \geq \frac{1}{M^2}$.

When $n \geq 2$, the definition of T_3 implies the following identities:

$$
\begin{aligned}
&\sum_{k\in S}\left[U_1^{3n-3}(\mathbb{E}\tilde{\rho}^0)\right]_{kJ1} = \sum_{k\in S}\left[T_3\left(U_1^{3n-4}(\mathbb{E}\tilde{\rho}^0)\right)\right]_{kJ1} \\
= \quad & \sum_{k,k',l'\in S}(1-\hat{\xi}_{k'l'})\,\hat{\varsigma}_{k'l'}(k)[U_1^{3n-4}(\mathbb{E}\tilde{\rho}^0)]_{k'l'1} + \sum_{k,k',l'\in S}\hat{\vartheta}_{k'l'}\,\hat{\varsigma}_{k'l'}(k)[U_1^{3n-4}(\mathbb{E}\tilde{\rho}^0)]_{k'l'0} \\
& \quad + \sum_{k\in S}[U_1^{3n-4}(\mathbb{E}\tilde{\rho}^0)]_{kJ1} \\
= \quad & \sum_{k',l'\in S}(1-\hat{\xi}_{k'l'})[U_1^{3n-4}(\mathbb{E}\tilde{\rho}^0)]_{k'l'1} + \sum_{k',l'\in S}\hat{\vartheta}_{k'l'}[U_1^{3n-4}(\mathbb{E}\tilde{\rho}^0)]_{k'l'0} + \sum_{k\in S}[U_1^{3n-4}(\mathbb{E}\tilde{\rho}^0)]_{kJ1}.
\end{aligned}
$$

By the definition of $\hat{\vartheta}$ and $\hat{\xi}$, we know that $\hat{\vartheta}_{kl} \geq \frac{1}{M^2}$ and $\hat{\xi}_{kl} \leq 1 - \frac{1}{M^2}$ for any $k, l \in S$. Then, we can obtain that

$$
\begin{aligned}
& \sum_{k\in S}\left[U_1^{3n-3}(\mathbb{E}\tilde{\rho}^0)\right]_{kJ1} \\
\geq \quad & \frac{1}{M^2}\left(\sum_{k',l'\in S}[U_1^{3n-4}(\mathbb{E}\tilde{\rho}^0)]_{k'l'1} + \sum_{k',l'\in S}[U_1^{3n-4}(\mathbb{E}\tilde{\rho}^0)]_{k'l'0} + \sum_{k\in S}[U_1^{3n-4}(\mathbb{E}\tilde{\rho}^0)]_{kJ1}\right) \\
= \quad & \frac{1}{M^2}.
\end{aligned}
$$

Therefore, $\sum_{k\in S}\left[U_1^{3n-3}(\mathbb{E}\tilde{\rho}^0)\right]_{kJ1} \geq \frac{1}{M^2}$ for any $n \in \mathbb{T}_0$.

Note that $\hat{\eta}_{kl} \geq \frac{1}{M^2}$ for any $k, l \in S$ by its definition. The definition of T_1 implies that for any $k \in S$,

$$
\begin{aligned}
& \left[U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)\right]_{kJ1} = \left[T_1\left(U_1^{3n-3}(\mathbb{E}\tilde{\rho}^0)\right)\right]_{kJ1} \\
= \quad & \sum_{l\in S}\left[U_1^{3n-3}(\mathbb{E}\tilde{\rho}^0)\right]_{lJ1}\hat{\eta}_{lk} \geq \frac{1}{M^2}\sum_{l\in S}\left[U_1^{3n-3}(\mathbb{E}\tilde{\rho}^0)\right]_{lJ1} \geq \frac{1}{M^4}.
\end{aligned}
$$

Fix any $\omega^{3n-2} \notin V^{3n-2}$. We have $\|\tilde{\rho}^{3n-2}(\omega^{3n-2}) - U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)\|_\infty \leq \xi_0$, which implies that

$$
\tilde{\rho}^{3n-2}_{kJ1}(\omega^{3n-2}) \geq [U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)]_{kJ1} - \xi_0 \geq \frac{1}{M^4} - \xi_{-1}.
$$

Note that $\xi_{-1} = \frac{1}{M^{M^M}}$ and $\hat{M} \geq M^{M^M}$. It is clear that $\xi_{-1} \leq \frac{1}{2M^4}$ and $\hat{M}^{-\frac{1}{15}} \leq \frac{1}{2M^4}$. Therefore, we have

$$
\tilde{\rho}^{3n-2}_{kJ1}(\omega^{3n-2}) \geq \frac{1}{M^4} - \frac{1}{2M^4} = \frac{1}{2M^4} \geq \frac{1}{\hat{M}^{\frac{1}{15}}},
$$

which is the required inequality in the lemma. ■

The following lemma provides an approximation of the matching probabilities at step $3n-1$ using parameter $\hat{q}$.

Lemma E.19. *For any $i, j \in I$ with $i \neq j$, $\omega^{3n-2} \notin V^{3n-2}$ and $k_1, l_1, k_2, l_2 \in S$, if $\tilde{\beta}_i^{3n-2}\left(\omega^{3n-2}\right) = (k_1, J, 1)$ and $\tilde{\beta}_j^{3n-2}\left(\omega^{3n-2}\right) = (k_2, J, 1)$, then*

$$\begin{aligned}
&\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1) - \hat{q}_{k_1 l_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}}, \\
&\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = J) - \hat{q}_{k_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}}, \\
&\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1, \hat{g}_j^{3n-1} = l_2) - \hat{q}_{k_1 l_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2 l_2}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}}, \\
&\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = J, \hat{g}_j^{3n-1} = l_2) - \hat{q}_{k_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2 l_2}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}}, \\
&\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = J, \hat{g}_j^{3n-1} = J) - \hat{q}_{k_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}}.
\end{aligned}$$

Proof. Fix any $i, j \in I$ with $i \neq j$, $\omega^{3n-2} \notin V^{3n-2}$ and $k_1, l_1, k_2, l_2 \in S$. Assume that $\tilde{\beta}_i^{3n-2}\left(\omega^{3n-2}\right) = (k_1, J, 1)$ and $\tilde{\beta}_j^{3n-2}\left(\omega^{3n-2}\right) = (k_2, J, 1)$.

By Lemma E.18, we have $\tilde{\rho}_{k_1 J 1}^{3n-2}\left(\omega^{3n-2}\right) \geq \frac{1}{\hat{M}^{\frac{1}{15}}} > \frac{1}{\hat{M}^{\frac{1}{3}}}$, and $\tilde{\rho}_{k_2 J 1}^{3n-2}\left(\omega^{3n-2}\right) \geq \frac{1}{\hat{M}^{\frac{1}{15}}} > \frac{1}{\hat{M}^{\frac{1}{3}}}$. It follows from Lemma E.1 that

$$\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1) - \hat{q}_{k_1 l_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \leq \frac{2}{\hat{M}^{\frac{2}{3}}} \leq \frac{1}{\hat{M}^{\frac{1}{9}}}, \text{ and}$$

$$\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1) Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_j^{3n-1} = l_2) - \hat{q}_{k_1 l_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2 l_2}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \leq \frac{5}{\hat{M}^{\frac{2}{3}}} \leq \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

Next, we consider the case that agent i is not matched. We can obtain

$$\begin{aligned}
& \left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = J) - \hat{q}_{k_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\
= & \left|\sum_{l_1 \in S} Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1) - \sum_{l_1 \in S} \hat{q}_{k_1 l_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\
\leq & \frac{2K}{\hat{M}^{\frac{2}{3}}} \leq \frac{1}{\hat{M}^{\frac{1}{9}}}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
& \left|Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1} = J, \hat{g}_j^{3n-1} = l_2\right) - \hat{q}_{k_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2 l_2}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\
= & \left|\sum_{l' \in S} Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1} = l', \hat{g}_j^{3n-1} = l_2\right) - \sum_{l' \in S} \hat{q}_{k_1 l'}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2 l_2}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\
\leq & \frac{5K}{\hat{M}^{\frac{2}{3}}} \leq \frac{1}{\hat{M}^{\frac{1}{9}}}.
\end{aligned}$$

It remains to consider the case that agents i and j are not matched. We have

$$\begin{aligned}
& \left|Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=J,\hat{g}_j^{3n-1}=J\right)-\hat{q}_{k_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\
= & \left|\sum_{l'\in S}Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=J,\hat{g}_j^{3n-1}=l'\right)-\sum_{l'\in S}\hat{q}_{k_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2l'}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\
\leq & \frac{5K^2}{\hat{M}^{\frac{2}{3}}}\leq\frac{1}{\hat{M}^{\frac{1}{9}}}.
\end{aligned}$$

The proof is thus completed. ∎

The following lemma strengthens Lemma E.5 by providing a sharper bound, which will be used in the proof of Lemma E.21 below.

Lemma E.20. *For any $i\in I$, $k,l\in S$, $n\in\mathbb{T}_0$, and $F^{3n-2}\in\mathcal{F}^{3n-2}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-2}=(k,J,1)\right)\cap F^{3n-2}\right)>0$, we have*

$$\begin{aligned}
& \left|P_0\left(\tilde{\beta}_i^{3n-1}=(k,l,1)\,\middle|\,\left(\tilde{\beta}_i^{3n-2}=(k,J,1)\right)\cap F^{3n-2}\right)-\hat{q}_{kl}\left(U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)\right)\right| \\
\leq & \frac{2\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2}=(k,J,1)\right)\cap F^{3n-2}\right)}+\frac{1}{\hat{M}^{\frac{1}{9}}}+\xi_{-1}.
\end{aligned}$$

Proof. Fix any $i\in I$, $k,l\in S$, $n\in\mathbb{T}_0$, and $F^{3n-2}\in\mathcal{F}^{3n-2}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-2}=(k,J,1)\right)\cap F^{3n-2}\right)>0$. Let $a=(k,J,1)$, $b=(k,l,1)$, and

$$A=\left\{\omega^{3n-2}\in\Omega^{3n-2}:\tilde{\beta}_i^{3n-2}(\omega^{3n-2})=(k,J,1)\right\}\cap F^{3n-2}.$$

We know that $Q^{3n-2}(A)=P_0(A)>0$. It is clear that

$$\begin{aligned}
P_0\left(\tilde{\beta}_i^{3n-1}=b\,\middle|\,\left(\tilde{\beta}_i^{3n-2}=a\right)\cap F^{3n-2}\right) &= \frac{1}{Q^{3n-2}(A)}\int_A Q_{3n-1}^{\omega^{3n-2}}\left(\tilde{\beta}_i^{3n-1}=b\right)dQ^{3n-2} \\
&= \frac{1}{Q^{3n-2}(A)}\int_A Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=l\right)dQ^{3n-2}.
\end{aligned}$$

By Lemmas E.17 and E.19, we have

$$\begin{aligned}
& \left|P_0\left(\tilde{\beta}_i^{3n-1}=b\,\middle|\,\left(\tilde{\beta}_i^{3n-2}=a\right)\cap F^{3n-2}\right)-\frac{1}{Q^{3n-2}(A)}\int_A\hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)dQ^{3n-2}\right| \\
\leq & \frac{1}{Q^{3n-2}(A)}\int_{A\cap V^{3n-2}}\left|Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=l\right)-\hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right|dQ^{3n-2} \\
& +\frac{1}{Q^{3n-2}(A)}\int_{A\setminus V^{3n-2}}\left|Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=l\right)-\hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right|dQ^{3n-2} \\
\leq & \frac{Q^{3n-2}(V^{3n-2})}{Q^{3n-2}(A)}+\frac{Q^{3n-2}(A\setminus V^{3n-2})}{Q^{3n-2}(A)}\frac{1}{\hat{M}^{\frac{1}{9}}} \\
\leq & \frac{\epsilon_0}{Q^{3n-2}(A)}+\frac{1}{\hat{M}^{\frac{1}{9}}}.
\end{aligned} \tag{E.73}$$

By the definition of V^{3n-2} in Lemma E.17, we know that $\|\tilde{\rho}^{3n-2}(\omega^{3n-2}) - U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)\|_\infty \le \xi_0$ for any $\omega^{3n-2} \notin V^{3n-2}$. By Lemma E.3, we have $\left|\hat{q}_{kl}\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right) - \hat{q}_{kl}\left(U_1^{3n-2}(\tilde{\rho}^0)\right| \le \xi_{-1}$ for any $\omega^{3n-2} \notin V^{3n-2}$. It follows from Lemma E.17 that

$$\begin{aligned}
&\left|\hat{q}_{kl}\left(U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)\right) - \frac{1}{Q^{3n-2}(A)}\int_A \hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right) dQ^{3n-2}\right| \\
\le\; & \frac{1}{Q^{3n-2}(A)}\int_{A\cap V^{3n-2}} \left|q_{kl}\left(U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)\right) - \hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| dQ^{3n-2} \\
&+\frac{1}{Q^{3n-2}(A)}\int_{A\setminus V^{3n-2}} \left|\hat{q}_{kl}\left(U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)\right) - \hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| dQ^{3n-2} \\
\le\; & \frac{Q^{3n-2}(V^{3n-2})}{Q^{3n-2}(A)} + \frac{Q^{3n-2}(A\setminus V^{3n-2})}{Q^{3n-2}(A)}\xi_{-1} \\
\le\; & \frac{\epsilon_0}{Q^{3n-2}(A)} + \xi_{-1}.
\end{aligned} \tag{E.74}$$

By Equations (E.73) and (E.74), we have

$$\begin{aligned}
&\left|P_0\left(\tilde{\beta}_i^{3n-1} = (k,l,1)\,\middle|\, \left(\tilde{\beta}_i^{3n-2} = (k,J,1)\right) \cap F^{3n-2}\right) - \hat{q}_{kl}\left(U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)\right)\right| \\
=\; & \left|P_0\left(\tilde{\beta}_i^{3n-1} = b\,\middle|\, \left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) - \frac{1}{Q^{3n-2}(A)}\int_A \hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right) dQ^{3n-2}\right| \\
+\; & \left|\hat{q}_{kl}\left(U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)\right) - \frac{1}{Q^{3n-2}(A)}\int_A \hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right) dQ^{3n-2}\right| \\
\le\; & \frac{2\epsilon_0}{Q^{3n-2}(A)} + \frac{1}{\hat{M}^{\frac{1}{9}}} + \xi_{-1} \\
\le\; & \frac{2\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2} = (k,J,1)\right) \cap F^{3n-2}\right)} + \frac{1}{\hat{M}^{\frac{1}{9}}} + \xi_{-1},
\end{aligned}$$

which completes the proof. ∎

The following lemma improves the upper bound in Part (ii) of Lemma E.6.

Lemma E.21. *For any $i \in I$, $n \in \mathbb{T}_0$, $a, b \in \tilde{S}$, and $F^{3n-2} \in \mathcal{F}^{3n-2}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) > 0$, we have*

$$\begin{aligned}
&\left|P_0\left(\tilde{\beta}_i^{3n-1} = b\,\middle|\, \left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) - P_0\left(\tilde{\beta}_i^{3n-1} = b\,\middle|\, \tilde{\beta}_i^{3n-2} = a\right)\right| \\
\le\; & \frac{4K\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right)} + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}.
\end{aligned}$$

Proof. Fix any $i \in I$, $n \in \mathbb{T}_0$, $a \in \tilde{S}$, and $F^{3n-2} \in \mathcal{F}^{3n-2}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) > 0$. When $\tilde{\beta}_i^{3n-2} = (k,l,0)$ for some $k, l \in S$, agent i is already matched at the mutation step of $3n-2$. Thus, her expanded type at step $3n-1$ does not change with probability one. In other words, we have for any $b \in \tilde{S}$,

$$P_0\left(\tilde{\beta}_i^{3n-1} = b\,\middle|\, \left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) = P_0\left(\tilde{\beta}_i^{3n-1} = b\,\middle|\, \tilde{\beta}_i^{3n-2} = a\right), \tag{E.75}$$

which implies the inequality in the lemma. Since $P_0\left(\left(\tilde{\beta}_i^{3n-2}=a\right)\cap F^{3n-2}\right)>0$, it is not possible for a to be $(k,J,0)$ or $(k,l,1)$ for any $k,l\in S$. Hence, we only need to consider $a=(k,J,1)$. In this case, if b is neither $(k,J,1)$ nor $(k,l,1)$, then we must have $P_0\left(\tilde{\beta}_i^{3n-1}=b\right)=0$, which implies the identity in Equation (E.75). Therefore, the inequality in the lemma holds again. Thus, it remains to consider $b=(k,J,1)$ or $(k,l,1)$.

Let $b=(k,l,1)$. It follows from Lemma E.20 that

$$\begin{aligned}
&\left|P_0\left(\tilde{\beta}_i^{3n-1}=(k,l,1)\,\middle|\,\left(\tilde{\beta}_i^{3n-2}=(k,J,1)\right)\cap F^{3n-2}\right)-\hat{q}_{kl}\left(U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)\right)\right|\\
\leq\;&\frac{2\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2}=(k,J,1)\right)\cap F^{3n-2}\right)}+\frac{1}{\hat{M}^{\frac{1}{9}}}+\xi_{-1},\text{ and}
\end{aligned}$$

$$\begin{aligned}
&\left|P_0\left(\tilde{\beta}_i^{3n-1}=(k,l,1)\,\middle|\,\tilde{\beta}_i^{3n-2}=(k,J,1)\right)-\hat{q}_{kl}\left(U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)\right)\right|\\
\leq\;&\frac{2\epsilon_0}{P_0\left(\tilde{\beta}_i^{3n-2}=(k,J,1)\right)}+\frac{1}{\hat{M}^{\frac{1}{9}}}+\xi_{-1}.
\end{aligned}$$

By combining the above two inequalities, we obtain that

$$\begin{aligned}
&\left|P_0\left(\tilde{\beta}_i^{3n-1}=b\,\middle|\,\left(\tilde{\beta}_i^{3n-2}=a\right)\cap F^{3n-2}\right)-P_0\left(\tilde{\beta}_i^{3n-1}=b\,\middle|\,\tilde{\beta}_i^{3n-2}=a\right)\right|\\
\leq\;&\frac{4\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2}=a\right)\cap F^{3n-2}\right)}+\frac{2}{\hat{M}^{\frac{1}{9}}}+2\xi_{-1},
\end{aligned}$$

which implies the inequality in the lemma.

Assume $b=(k,J,1)$. Then, we have

$$\begin{aligned}
&\left|P_0\left(\tilde{\beta}_i^{3n-1}=(k,J,1)\,\middle|\,\left(\tilde{\beta}_i^{3n-2}=a\right)\cap F^{3n-2}\right)-P_0\left(\tilde{\beta}_i^{3n-1}=(k,J,1)\,\middle|\,\tilde{\beta}_i^{3n-2}=a\right)\right|\\
=\;&\left|\sum_{l=1}^{K}P_0\left(\tilde{\beta}_i^{3n-1}=(k,l,1)\,\middle|\,\left(\tilde{\beta}_i^{3n-2}=a\right)\cap F^{3n-2}\right)-\sum_{l=1}^{K}P_0\left(\tilde{\beta}_i^{3n-1}=(k,l,1)\,\middle|\,\tilde{\beta}_i^{3n-2}=a\right)\right|\\
\leq\;&\frac{4K\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2}=a\right)\cap F^{3n-2}\right)}+\frac{2K}{\hat{M}^{\frac{1}{9}}}+2K\xi_{-1}.
\end{aligned}$$

Hence, the proof is completed. ■

The following lemma is Part (i) of Lemma E.6.

Lemma E.22. *For any $i\in I$, $a,b\in\tilde{S}$, $n\in\mathbb{T}_0$, and $F^{3n-3}\in\mathcal{F}^{3n-3}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-3}=a\right)\cap F^{3n-3}\right)>0$, we have*

$$P_0\left(\tilde{\beta}_i^{3n-2}=b\,\middle|\,\left(\tilde{\beta}_i^{3n-3}=a\right)\cap F^{3n-3}\right)=P_0\left(\tilde{\beta}_i^{3n-2}=b\,\middle|\,\tilde{\beta}_i^{3n-3}=a\right).$$

Proof. Fix any $i \in I$, $a, b \in \tilde{S}$, $n \in \mathbb{T}_0$, and $F^{3n-3} \in \mathcal{F}^{3n-3}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-3} = a\right) \cap F^{3n-3}\right) > 0$. Let

$$D_1 = \{\omega^{3n-3} \in \Omega^{3n-3} : \tilde{\beta}_i^{3n-3}(\omega^{3n-3}) = a\} \cap F^{3n-3}.$$

We know that $P_0(D_1) = Q^{3n-3}(D_1) > 0$. By the construction of the mutation step at period n, it is easy to see that

$$\begin{aligned}
& P_0\left(\tilde{\beta}_i^{3n-2} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-3} = a\right) \cap F^{3n-3}\right) \\
= \;& \frac{1}{Q^{3n-3}(D_1)} \int_{D_1} Q_{3n-2}^{\omega^{3n-3}}(\tilde{\beta}_i^{3n-2} = b) dQ^{3n-3} \\
= \;& \frac{1}{Q^{3n-3}(D_1)} \int_{D_1} B_{ab} dQ^{3n-3} \\
= \;& B_{ab},
\end{aligned}$$

where

$$B_{ab} = \begin{cases} \hat{\eta}_{k_1 l_1} \hat{\eta}_{k_2 l_2} & \text{if } a = (k_1, k_2, 0), b = (l_1, l_2, 0) \\ \hat{\eta}_{k_1 l_1} & \text{if } a = (k_1, J, 1), b = (l_1, J, 1) \\ 0 & \text{otherwise.} \end{cases}$$

By taking F^{3n-3} to be Ω^{3n-3} , we have

$$P_0\left(\tilde{\beta}_i^{3n-2} = b \,\middle|\, \tilde{\beta}_i^{3n-3} = a\right) = B_{ab}.$$

Therefore, the identity in the lemma follows. ∎

The following lemma is Part (iii) of Lemma E.6.

Lemma E.23. *For any $i \in I$, $a, b \in \tilde{S}$, $n \in \mathbb{T}_0$, and $F^{3n-1} \in \mathcal{F}^{3n-1}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-1} = a\right) \cap F^{3n-1}\right) > 0$, we have*

$$P_0\left(\tilde{\beta}_i^{3n} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-1} = a\right) \cap F^{3n-1}\right) = P_0\left(\tilde{\beta}_i^{3n} = b \,\middle|\, \tilde{\beta}_i^{3n-1} = a\right).$$

Proof. Fix any $i \in I$, $a, b \in \tilde{S}$, $n \in \mathbb{T}_0$, and $F^{3n-1} \in \mathcal{F}^{3n-1}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-1} = a\right) \cap F^{3n-1}\right) > 0$. Let

$$D_1 = \{\omega^{3n-1} \in \Omega^{3n-1} : \tilde{\beta}_i^{3n-1}(\omega^{3n-1}) = a\} \cap F^{3n-1}.$$

By the construction of the type changing with break-up step at period n, it is clear that

$$\begin{aligned}
& P_0\left(\tilde{\beta}_i^{3n} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-1} = a\right) \cap F^{3n-1}\right) \\
= \;& \frac{1}{Q^{3n-1}(D_1)} \int_{D_1} Q_{3n}^{\omega^{3n-1}}(\tilde{\beta}_i^{3n} = b) dQ^{3n-1} \\
= \;& \frac{1}{Q^{3n-1}(D_1)} \int_{D_1} B_{ab} dQ^{3n-1} \\
= \;& B_{ab},
\end{aligned}$$

where

$$B_{ab} = \begin{cases} 1-\hat{\vartheta}_{k_1k_2} & \text{if } a=(k_1,k_2,0), b=(k_1,k_2,0) \\ \hat{\vartheta}_{k_1k_2}\varsigma_{k_1k_2}(l_1) & \text{if } a=(k_1,k_2,0), b=(l_1,J,1) \\ \hat{\xi}_{k_1k_2}\hat{\sigma}_{k_1k_2}(l_1,l_2) & \text{if } a=(k_1,k_2,1), b=(l_1,l_2,0) \\ (1-\hat{\xi}_{k_1k_2})\hat{\varsigma}_{k_1k_2}(l_1) & \text{if } a=(k_1,k_2,1), b=(l_1,J,1) \\ 1 & \text{if } a=(k_1,J,1), b=(k_1,J,1) \\ 0 & \text{otherwise.} \end{cases}$$

By taking F^{3n-1} to be Ω^{3n-1} , we have

$$P_0\left(\tilde{\beta}_i^{3n}=b \,\middle|\, \tilde{\beta}_i^{3n-1}=a\right)=B_{ab}.$$

Hence, we obtain the identity in the lemma. ■

The following lemma provides an upper bounded for the probability with which two agents are matched at the m-th step.

Lemma E.24. *For any $i,j\in I$ with $i\neq j$ and $m\in\{0,1,\ldots,3M^2\}$, we have*

$$P_0\left(\hat{\pi}_i^m=j\right)\leq P_0\left(\hat{\pi}_i^0=j\right)+m\epsilon_0+\frac{2m}{\hat{M}^{\frac{14}{15}}}.$$

Proof. Fix any $i,j\in I$ with $i\neq j$. It is clear that the inequality holds when $m=0$. Suppose the inequality holds when $m=m'$. It is easy to see that

$$\begin{aligned} P_0\left(\hat{\pi}_i^{m'+1}=j\right) &= P_0\left(\hat{\pi}_i^{m'+1}=j,\hat{\pi}_i^{m'}=j\right)+P_0\left(\hat{\pi}_i^{m'+1}=j,\hat{\pi}_i^{m'}\neq j\right) \\ &\leq P_0\left(\hat{\pi}_i^{m'}=j\right)+P_0\left(\hat{\pi}_i^{m'+1}=j,\hat{\pi}_i^{m'}\neq j\right). \end{aligned} \quad \text{(E.76)}$$

If $m'=3n-3$ or $3n-1$ for some $n\in\mathbb{T}_0$, it is clear that $P_0\left(\hat{\pi}_i^{m'+1}=j,\hat{\pi}_i^{m'}\neq j\right)=0$. Then, Equation (E.76) and the induction hypothesis imply that

$$P_0\left(\hat{\pi}_i^{m'+1}=j\right)\leq P_0\left(\hat{\pi}_i^{m'}=j\right)\leq P_0\left(\hat{\pi}_i^0=j\right)+(m'+1)\epsilon_0+\frac{2m'+2}{\hat{M}^{\frac{14}{15}}}.$$

If $m'=3n-2$ for some $n\in\mathbb{T}_0$, we have

$$P_0\left(\hat{\pi}_i^{m'+1}=j,\hat{\pi}_i^{m'}\neq j\right)=P_0\left(\hat{\pi}_i^{m'+1}=j,\hat{\pi}_i^{m'}=i\right).$$

Let $A^{3n-2}=\{\omega^{3n-2}\in\Omega^{3n-2}:\hat{\pi}_i^{3n-2}(\omega^{3n-2})=i\}$. Then, we obtain that

$$\begin{aligned} & P_0\left(\hat{\pi}_i^{m+1}=j,\hat{\pi}_i^m\neq j\right) \\ = & \int_{A^{3n-2}} Q_{3n-1}^{\omega^{3n-2}}(\hat{\pi}_i^{3n-1}=j)dQ^{3n-2} \\ = & \int_{A^{3n-2}\cap V^{3n-2}} Q_{3n-1}^{\omega^{3n-2}}(\hat{\pi}_i^{3n-1}=j)dQ^{3n-2}+\int_{A^{3n-2}\setminus V^{3n-2}} Q_{3n-1}^{\omega^{3n-2}}(\hat{\pi}_i^{3n-1}=j)dQ^{3n-2} \\ \leq & Q^{3n-2}\left(A^{3n-2}\cap V^{3n-2}\right)+\int_{A^{3n-2}\setminus V^{3n-2}} Q_{3n-1}^{\omega^{3n-2}}(\hat{\pi}_i^{3n-1}=j)dQ^{3n-2}. \end{aligned}$$

For any $\omega^{3n-2} \in A^{3n-2}$, if $\hat{\pi}_j^{3n-2}(\omega^{3n-2}) \neq j$, it is clear that $Q_{3n-1}^{\omega^{3n-2}}(\hat{\pi}_i^{3n-1} = j) = 0$; if $\hat{\pi}_j^{3n-2}(\omega^{3n-2}) = j$, Lemma E.1 (ii) implies that $Q_{3n-1}^{\omega^{3n-2}}(\hat{\pi}_i^{3n-1} = j) \leq \frac{2}{\hat{M}\hat{\rho}_{\alpha_i^{3n-2}(\omega^{3n-2})J}}$. It follows from Lemma E.18 that for any $\omega^{3n-2} \notin V^{3n-2}$ and $k \in S$, we have $\tilde{\rho}_{kJ1}^{3n-2}(\omega^{3n-2}) \geq \hat{M}^{-\frac{1}{15}}$. Therefore, Lemma E.17 leads to

$$\begin{aligned}
& P_0\left(\hat{\pi}_i^{m'+1} = j, \hat{\pi}_i^{m'} \neq j\right) \\
\leq\ & Q^{3n-2}\left(A^{3n-2} \cap V^{3n-2}\right) + Q^{3n-2}\left(A^{3n-2} \backslash V^{3n-2}\right) \frac{2}{\hat{M}\hat{M}^{-\frac{1}{15}}} \\
\leq\ & \epsilon_0 + \frac{2}{\hat{M}^{\frac{14}{15}}}.
\end{aligned}$$

The above inequality and Equation (E.76) together with the induction hypothesis imply that

$$P_0\left(\hat{\pi}_i^{m'+1} = j\right) \leq P_0\left(\hat{\pi}_i^{m'} = j\right) + \epsilon_0 + \frac{2}{\hat{M}^{\frac{14}{15}}} \leq P_0\left(\hat{\pi}_i^0 = j\right) + (m'+1)\epsilon_0 + \frac{2m'+2}{\hat{M}^{\frac{14}{15}}}.$$

By induction, we have

$$P_0\left(\hat{\pi}_i^m = j\right) \leq P_0\left(\hat{\pi}_i^0 = j\right) + m\epsilon_0 + \frac{2m}{\hat{M}^{\frac{14}{15}}}$$

for any $m \in \{0, 1, \ldots, 3M^2\}$. ∎

E.5.4 Proof of Lemma E.3

First, we work with T_2. Define a mapping $F : \mathbb{N} \times \tilde{\Delta} \to \mathbb{R}^{2K(K+1)}$ as follow: for any $(N, \tilde{p}) \in \mathbb{N} \times \tilde{\Delta}$,

$$[F(N, \tilde{p})]_{kl0} = \begin{cases} \tilde{p}_{kl0} & \text{if } l \neq J \\ 0 & \text{if } l = J, \end{cases}$$

$$[F(N, \tilde{p})]_{kl1} = \begin{cases} \frac{\tilde{p}_{kJ1}\theta_{kl}(\tilde{p})}{N} & \text{if } l \neq J \\ \tilde{p}_{kJ1}\left(1 - \sum_{l \in S} \frac{\theta_{kl}(\tilde{p})}{N}\right) & \text{if } l = J. \end{cases}$$

It is easy to see that $T_2(\tilde{\rho}) = {}^*F(M, \tilde{\rho})$.

For any fixed $N \in \mathbb{N}$, there exists a strictly increasing continuous bijection v_N on $\mathbb{R}_+$ with $v_N(0) = 0$ such that $||F(N, \tilde{p}) - F(N, \tilde{p}')||_\infty \leq v_N(||\tilde{p} - \tilde{p}'||_\infty)$ for any $\tilde{p}, \tilde{p}' \in \tilde{\Delta}$ (which is called a modulus of continuity of the function $F(N, \cdot)$).[12] By the Transfer Principle, for any

[12]Given a continuous function f from a compact metric space (X, d_X) to a metric space (Y, d_Y), f admits a (global) modulus of continuity ω in the sense that ω is a function from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $\lim_{t\to 0} \omega(t) = \omega(0) = 0$, and for any $x, x' \in X$, $d_Y(f(x), f(x')) \leq \omega(d_X(x, x'))$. Since the range of f is compact, we can assume with loss of generality that ω is a bounded function on $\mathbb{R}_+$. Following the wikipedia entry "Modulus of continuity" (https://en.wikipedia.org/wiki/Modulus_of_continuity), let $\omega'(t) := \frac{1}{t}\int_t^{2t}\left[\sup_{0\leq s'\leq s} \omega(s')\right] ds$ for $t > 0$ and $\omega'(0) = 0$. Then, it is easy to verify that ω' is increasing and continuous on $\mathbb{R}_+$. Let $\hat{\omega}(t) := \omega'(t) + t$ for any $t \in \mathbb{R}_+$, which is a modulus of continuity for f that is a strictly increasing continuous bijection on $\mathbb{R}_+$.

$N \in {}^*\mathbb{N}$, there exists an internal, strictly increasing bijection v_N on ${}^*\mathbb{R}_+$ such that $||{}^*F(N,\tilde{\rho}) - {}^*F(N,\tilde{\rho}')||_\infty \le v_N(||\tilde{\rho}-\tilde{\rho}'||_\infty)$ for any $\tilde{\rho},\tilde{\rho}' \in {}^*\tilde{\Delta}$. Let $N = M$, it is clear that $||T_2(\tilde{\rho}) - T_2(\tilde{\rho}')||_\infty \le v_M(||\tilde{\rho}-\tilde{\rho}'||_\infty)$.

For T_1, T_3, $\hat{q}$, we can derive their modulus of continuity in the same way. By taking the maximum, we can get an internal, strictly increasing bijection v on ${}^*\mathbb{R}_+$ which is a common modulus of continuity for all these mappings.

Let $\xi_{-1} = \frac{1}{M^{M^M}}$ and w be the inverse function v^{-1} on ${}^*\mathbb{R}_+$. Let $\xi_0 = \min(w(\xi_{-1}), \xi_{-1})$, $\xi_m = \min\left(w(\xi_{m-1}), \frac{\xi_0}{3M^2+1}\right)$ for any $m \in \{1,\ldots,3M^2\}$. Hence, it is clear that $(3M^2+1)\xi_m \le \xi_0 \le \xi_{-1}$ for any $m \in \{1,2,\ldots,3M^2+1\}$.

Fix any $m \in \{-1,0,\ldots,3M^2\}$, and $\tilde{\rho},\tilde{\rho}' \in {}^*\tilde{\Delta}$ with $\|\tilde{\rho}-\tilde{\rho}'\|_\infty \le \xi_{m+1}$. Then, we know that $\|\tilde{\rho}-\tilde{\rho}'\|_\infty \le w(\xi_m)$. The fact that v is a strictly increasing bijection on ${}^*\mathbb{R}_+$ implies that $v\left(\|\tilde{\rho}-\tilde{\rho}'\|_\infty\right) \le \xi_m$. Since v is a common modulus of continuity for T_1, T_2, T_3 and $\hat{q}$, we obtain that for any $r \in \{1,2,3\}$,

$$\|T_r(\tilde{\rho}) - T_r(\tilde{\rho}')\|_\infty \le \xi_m, \text{ and } \|\hat{q}(\tilde{\rho}) - \hat{q}(\tilde{\rho}')\|_\infty \le \xi_m,$$

which completes the proof.

E.5.5 Proof of Lemma E.4

By Lemma E.17, for any $m \in \{0,1,\ldots,3\hat{M}^2\}$, $P_0\left(\|\tilde{\rho}^m(\omega^m) - U_1^m(\mathbb{E}\tilde{\rho}^0)\|_\infty > \xi_0\right) \le \epsilon_0$. Note that ξ_0 and ϵ_0 are infinitesimals. It is clear that $\mathbb{E}\tilde{\rho}^m \simeq U_1^m(\mathbb{E}\tilde{\rho}^0)$ for any $m \in \{0,1,\ldots,3\hat{M}^2\}$.

E.5.6 Proof of Lemma E.5

By Lemma E.20, for any $i \in I$, $k,l \in S$, $n \in \mathbb{T}_0$ and $F^{3n-2} \in \mathcal{F}^{3n-2}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-2} = (k,J,1)\right) \cap F^{3n-2}\right) > 0$, we have

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{3n-1} = (k,l,1) \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = (k,J,1)\right) \cap F^{3n-2}\right) - \hat{q}_{k_2k_1}\left(U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)\right) \right| \\ \le \;& \frac{2\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2} = (k,J,1)\right) \cap F^{3n-2}\right)} + \frac{1}{\hat{M}^{\frac{1}{9}}} + \xi_{-1}. \end{aligned}$$

Note that $2\epsilon_0 = \frac{4}{\hat{M}^{\frac{1}{10}}} < \frac{1}{M^3}$ and $\frac{1}{\hat{M}^{\frac{1}{9}}} + \xi_{-1} = \frac{1}{\hat{M}^{\frac{1}{9}}} + \frac{1}{M^{M^M}} < \frac{1}{M^2}$. It is then clear that

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{3n-1} = (k,l,1) \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = (k,J,1)\right) \cap F^{3n-2}\right) - \hat{q}_{k_2k_1}\left(U_1^{3n-2}(\mathbb{E}\tilde{\rho}^0)\right) \right| \\ \le \;& \frac{1}{M^3 P_0\left(\left(\tilde{\beta}_i^{3n-2} = (k,J,1)\right) \cap F^{3n-2}\right)} + \frac{1}{M^2}, \end{aligned}$$

which is the required inequality in Lemma E.5.

E.5.7 Proof of Lemma E.6

Parts (i) and (iii) of Lemma E.6 have been shown in Lemmas E.22 and E.23 respectively.

Fix any $i \in I$, $a, b \in \tilde{S}$, $n \in \{1, 2, \ldots, M^2\}$, and $F^{3n-2} \in \mathcal{F}^{3n-2}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) > 0$. Lemma E.21 indicates that

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) - P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \tilde{\beta}_i^{3n-2} = a\right) \right| \\ \leq \; & \frac{4K\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right)} + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}. \end{aligned}$$

Since $4K\epsilon_0 = \frac{8K}{\hat{M}^{\frac{1}{10}}} < \frac{1}{M^3}$ and $\frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1} = \frac{2K}{\hat{M}^{\frac{1}{9}}} + \frac{2K}{M^{M^M}} < \frac{1}{M^2}$, we can obtain that

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) - P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \tilde{\beta}_i^{3n-2} = a\right) \right| \\ \leq \; & \frac{1}{M^3 P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right)} + \frac{1}{M^2}, \end{aligned}$$

which is Part (ii) of Lemma E.6.

E.5.8 Proof of Lemma E.7

For an internal random variable f and an internal event G on Ω, we shall use (from now onwards) the simplified notation $(f = a, G)$ to represent the event $(f = a) \cap G$ that event G happens while f takes value a.

We need to prove that there exists a sequence of nonnegative infinitesimals $\{c_m\}_{1 \leq m \leq 3M^2}$ such that for any $i \in I$, any $m, m_1 \in \{0, 1, \ldots, 3M^2\}$ with $m > m_1$, any expanded types $a, a_1 \in \tilde{S}$, and any $F_i^{m_1-1} \in \mathcal{F}_i^{m_1-1}$, we have

$$\begin{aligned} & \Big| P_0\left(\tilde{\beta}_i^{m} = a, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) \\ & \quad - P_0\left(\tilde{\beta}_i^{m} = a, \tilde{\beta}_i^{m_1} = a_1\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) \Big| \leq c_m. \end{aligned}$$

Fix any $i \in I$ and $m \in \{1, 2, \ldots, 3M^2\}$. When $m = 1$, it is clear that c_1 can be taken to be 0. Suppose that we have already defined c_m, we need to define c_{m+1} using c_m.

Fix any $m_1, \boldsymbol{m_2}$ with $m + 1 > m_1 > \boldsymbol{m_2}$, and expanded types $a, a_1, \boldsymbol{a_2}$. We first consider the case when $m > m_1$. It is clear that

$$\begin{aligned} & P_0\left(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) \\ = \; & \sum_{b \in \tilde{S}} P_0\left(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right). \end{aligned}$$

Let $A = \left\{ b \in \tilde{S} : P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) > 0 \right\}$. We have[13]

$$\begin{aligned} & P_0\left(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) \\ = & \sum_{b \in A} P_0\left(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) \\ = & \sum_{b \in A} P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) \\ & \quad P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right). \end{aligned}$$

Let $B = \sum_{b \in A} P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1\right) P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right)$. We can obtain that

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = a_2\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) - B \right| \\ = & \sum_{b \in A} \left| P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) - P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1\right) \right| \\ & \quad \times P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right). \end{aligned}$$

By Lemmas E.21, E.22 and E.23, we know that for any $m \in \{1, 2, \ldots, 3M^2\}$, and $b \in A$,

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) - P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b\right) \right| \\ \leq & \; K\left(\frac{4\epsilon_0}{P_0(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1})} + 2\xi_{-1} + \frac{2}{\hat{M}^{\frac{1}{9}}} \right), \text{ and} \end{aligned}$$

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1\right) - P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b\right) \right| \\ \leq & \; K\left(\frac{4\epsilon_0}{P_0(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1)} + 2\xi_{-1} + \frac{2}{\hat{M}^{\frac{1}{9}}} \right). \end{aligned}$$

[13]When A is empty, we follow the convention that summation over an empty set is zero.

It follows from the above inequalities that

$$\begin{aligned}
& \left|P_0\left(\tilde{\beta}_i^{m+1}=a, \tilde{\beta}_i^{m_1}=a_1, \tilde{\beta}_i^{\boldsymbol{m_2}}=a_2\right) P_0\left(\tilde{\beta}_i^{m_1}=a_1\right)-B\right| \\
= & \sum_{b \in A}\left|P_0\left(\tilde{\beta}_i^{m+1}=a \,\middle|\, \tilde{\beta}_i^m=b, \tilde{\beta}_i^{m_1}=a_1, F_i^{m_1-1}\right)-P_0\left(\tilde{\beta}_i^{m+1}=a \,\middle|\, \tilde{\beta}_i^m=b\right)\right. \\
& \quad\left.+P_0\left(\tilde{\beta}_i^{m+1}=a \,\middle|\, \tilde{\beta}_i^m=b\right)-P_0\left(\tilde{\beta}_i^{m+1}=a \,\middle|\, \tilde{\beta}_i^m=b, \tilde{\beta}_i^{m_1}=a_1\right)\right| \\
& \quad \times P_0\left(\tilde{\beta}_i^m=b, \tilde{\beta}_i^{m_1}=a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1}=a_1\right) \\
\leq & \sum_{b \in A} K\left(\frac{4 \epsilon_0}{P_0(\tilde{\beta}_i^m=b, \tilde{\beta}_i^{m_1}=a_1, F_i^{m_1-1})}+2 \xi_{-1}+\frac{2}{\hat{M}^{\frac{1}{9}}}+\frac{4 \epsilon_0}{P_0(\tilde{\beta}_i^m=b, \tilde{\beta}_i^{m_1}=a_1)}+2 \xi_{-1}+\frac{2}{\hat{M}^{\frac{1}{9}}}\right) \\
& \quad \times P_0\left(\tilde{\beta}_i^m=b, \tilde{\beta}_i^{m_1}=a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1}=a_1\right) \\
\leq & \sum_{b \in A} K\left(4 \epsilon_0+2 \xi_{-1}+\frac{2}{\hat{M}^{\frac{1}{9}}}+4 \epsilon_0+2 \xi_{-1}+\frac{2}{\hat{M}^{\frac{1}{9}}}\right) \\
\leq & 2 K^2(K+1)\left(8 \epsilon_0+4 \xi_{-1}+\frac{4}{\hat{M}^{\frac{1}{9}}}\right) .
\end{aligned} \tag{E.77}$$

The induction hypothesis implies that for any $b \in \tilde{S}$,

$$\left|P_0\left(\tilde{\beta}_i^m=b, \tilde{\beta}_i^{m_1}=a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1}=a_1\right)-P_0\left(\tilde{\beta}_i^m=b, \tilde{\beta}_i^{m_1}=a_1\right) P_0\left(\tilde{\beta}_i^{m_1}=a_1, F_i^{m_1-1}\right)\right| \leq c_m.$$

Then, we can obtain that

$$\begin{aligned}
& \left|B-P_0\left(\tilde{\beta}_i^{m+1}=a, \tilde{\beta}_i^{m_1}=a_1\right) P_0\left(\tilde{\beta}_i^{m_1}=a_1, F_i^{m_1-1}\right)\right| \\
= & \left|B-\sum_{b \in A} P_0\left(\tilde{\beta}_i^{m+1}=a \,\middle|\, \tilde{\beta}_i^m=b, \tilde{\beta}_i^{m_1}=a_1\right) P_0\left(\tilde{\beta}_i^m=b, \tilde{\beta}_i^{m_1}=a_1\right) P_0\left(\tilde{\beta}_i^{m_1}=a_1, F_i^{m_1-1}\right)\right| \\
\leq & \sum_{b \in A} P_0\left(\tilde{\beta}_i^{m+1}=a \,\middle|\, \tilde{\beta}_i^m=b, \tilde{\beta}_i^{m_1}=a_1\right)\left|P_0\left(\tilde{\beta}_i^m=b, \tilde{\beta}_i^{m_1}=a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1}=a_1\right)\right. \\
& \quad\left.-P_0\left(\tilde{\beta}_i^m=b, \tilde{\beta}_i^{m_1}=a_1\right) P_0\left(\tilde{\beta}_i^{m_1}=a_1, F_i^{m_1-1}\right)\right| \\
\leq & 2 K(K+1) c_m \leq 2 K^2(K+1) c_m .
\end{aligned} \tag{E.78}$$

By Equations (E.77) and (E.78), we have

$$\begin{aligned}
& \left|P_0\left(\tilde{\beta}_i^{m+1}=a, \tilde{\beta}_i^{m_1}=a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1}=a_1\right)\right. \\
& \left.-P_0\left(\tilde{\beta}_i^{m+1}=a, \tilde{\beta}_i^{m_1}=a_1\right) P_0\left(\tilde{\beta}_i^{m_1}=a_1, F_i^{m_1-1}\right)\right| \\
\leq & 2 K^2(K+1)\left(4 \xi_{-1}+8 \epsilon_0+\frac{4}{\hat{M}^{\frac{1}{9}}}+c_m\right) .
\end{aligned} \tag{E.79}$$

If $m=m_1$ and $P_0\left(\tilde{\beta}_i^m=a_1, F_i^{m_1-1}\right)=0$, then it is clear that

$$\begin{aligned}
& P_0\left(\tilde{\beta}_i^{m+1}=a, \tilde{\beta}_i^{m_1}=a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1}=a_1\right) \\
= & P_0\left(\tilde{\beta}_i^{m+1}=a, \tilde{\beta}_i^{m_1}=a_1\right) P_0\left(\tilde{\beta}_i^{m_1}=a_1, F_i^{m_1-1}\right) .
\end{aligned} \tag{E.80}$$

If $m = m_1$ and $P_0\left(\tilde{\beta}_i^m = a_1, F_i^{m_1-1}\right) > 0$, then it follows from Lemma E.21, E.22 and E.23 that

$$\begin{aligned}
& \left|P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) - P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^{m_1} = a_1\right)\right| \\
\le\ & \frac{4K\epsilon_0}{P_0\left(\tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right)} + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}.
\end{aligned}$$

Therefore, we can obtain that

$$\begin{aligned}
& \left|P_0\left(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right)\right. \\
& \quad \left. - P_0\left(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m_1} = a_1\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right)\right| \\
=\ & \left|P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) - P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^{m_1} = a_1\right)\right| \\
& \quad P_0\left(\tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) \\
\le\ & \left(\frac{4K\epsilon_0}{P_0\left(\tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right)} + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) \\
\le\ & 4K\epsilon_0 + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}. \qquad \text{(E.81)}
\end{aligned}$$

By Equations (E.79), (E.80) and (E.81), we can define c_{m+1} to be

$$2K^2(K+1)\left(4\xi_{-1} + 8\epsilon_0 + \frac{4}{\hat{M}^{\frac{1}{9}}} + c_m\right)$$

so that c_{m+1} has the desired inductive property.

Next, we use induction again to prove that for any $m \in \{1, 2, \ldots, 3M^2\}$,

$$c_m \le 2^{2m} K^{4m} (K+1)^{2m} \left(4\xi_{-1} + 8\epsilon_0 + \frac{4}{\hat{M}^{\frac{1}{9}}}\right). \qquad \text{(E.82)}$$

Note that $c_1 = 0$. It is clear that this inequality holds for $m = 1$. Suppose that this inequality holds for $m = m' \ge 1$. Then we have

$$\begin{aligned}
c_{m'+1} &= 2K^2(K+1)\left(4\xi_{-1} + 8\epsilon_0 + \frac{4}{\hat{M}^{\frac{1}{9}}} + c_{m'}\right) \\
&\le 2K^2(K+1)\left(4\xi_{-1} + 8\epsilon_0 + \frac{4}{\hat{M}^{\frac{1}{9}}}\right) + 2^{2m'+1}K^{4m'+2}(K+1)^{2m'+1}\left(4\xi_{-1} + 8\epsilon_0 + \frac{4}{\hat{M}^{\frac{1}{9}}}\right) \\
&\le 2^{2m'+2}K^{4m'+4}(K+1)^{2m'+2}\left(4\xi_{-1} + 8\epsilon_0 + \frac{4}{\hat{M}^{\frac{1}{9}}}\right).
\end{aligned}$$

Hence, Equation (E.82) holds. Since $\xi_{-1} = \frac{1}{M^{M^M}}$, $\epsilon_0 = \frac{2}{\hat{M}^{\frac{1}{10}}}$ and $\hat{M} \ge M^{M^M}$, it is easy to check that c_m is infinitesimal for any $m \in \{1, 2, \ldots, 3M^2\}$. Hence, Lemma E.7 holds.

E.5.9 Proof of Lemma E.8

Fix any $i, j \in I$ with $P_0\left(\hat{\pi}_i^0 = j\right) \leq \frac{1}{\hat{M}^{\frac{1}{5}}}$. It follows from Lemma E.24 that for any $m \in \{0, 1, \ldots, 3M^2\}$,

$$\begin{aligned} P_0\left(\hat{\pi}_i^m = j\right) &\leq P_0\left(\hat{\pi}_i^0 = j\right) + m\epsilon_0 + \frac{2m}{\hat{M}^{\frac{14}{15}}} \\ &\leq \frac{1}{\hat{M}^{\frac{1}{5}}} + \frac{6M^2}{\hat{M}^{\frac{1}{10}}} + \frac{6M^2}{\hat{M}^{\frac{14}{15}}} \\ &\leq \frac{1}{\hat{M}^{\frac{1}{11}}}. \end{aligned}$$

For any $m \in \{0, 1, \ldots, 3M^2\}$, let $F_{ij}^m = \{\omega^m \in \Omega^m : \hat{\pi}_i^m(\omega) = j\}$; then we have $P_0\left(F_{ij}^m\right) \leq \frac{1}{\hat{M}^{\frac{1}{11}}}$.

We need to prove that there exists a sequence of nonnegative infinitesimals $\{d_m\}_{0 \leq m \leq 3M^2}$ such that for any $m \in \{0, 1, \ldots, 3M^2\}$, $a_1, a_2 \in \tilde{S}$, and $F_i^{m-1} \in \mathcal{F}_i^{m-1}$ and $F_j^{m-1} \in \mathcal{F}_j^{m-1}$, we have

$$\begin{aligned} &\Big| P_0\left(\tilde{\beta}_i^m = a_1, \tilde{\beta}_j^m = a_2, F_i^{m-1}, F_j^{m-1}\right) \\ &\quad - P_0\left(\tilde{\beta}_i^m = a_1, F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^m = a_2, F_j^{m-1}\right)\Big| \leq d_m. \end{aligned} \tag{E.83}$$

Fix any $m \in \{0, 1, \ldots, 3M^2\}$. When $m = 0$, by Lemma E.2, we can take d_0 to be $\frac{1}{\hat{M}^{\frac{1}{10}}}$. Suppose that we have already defined d_m, we need to define d_{m+1} using d_m.

Fix any $a_1, a_2, b_1, b_2 \in \tilde{S}$, $F_i^{m-1} \in \mathcal{F}_i^{m-1}$ and $F_j^{m-1} \in \mathcal{F}_j^{m-1}$. We first estimate the following difference

$$\begin{aligned} &\Big| P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\ &\quad - P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1}\right)\Big|. \end{aligned}$$

For notational simplicity, we let

$$\begin{aligned} A &= \left\{\omega^m \in \Omega^m : \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2\right\} \cap F_i^{m-1} \cap F_j^{m-1}, \\ A' &= \{\omega^m \in \Omega^m : \tilde{\beta}_i^m = b_1\} \cap F_i^{m-1}, \\ A'' &= \{\omega^m \in \Omega^m : \tilde{\beta}_j^m = b_2\} \cap F_j^{m-1}. \end{aligned}$$

We can obtain that

$$\begin{aligned} &P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\ = &\int_A Q_{m+1}^{\omega^m}\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2\right) dQ^m. \end{aligned}$$

We first consider the case when $m = 3n - 2$ for some $n \in \mathbb{T}_0$, that is, the $m+1$ step is the matching step in the n-th period. We start by assuming $b_1 = (k, l, 0)$. When $\tilde{\beta}_i^m = (k, l, 0)$ for some $k, l \in S$, agent i is already matched at the mutation step in the n-th period. By the construction of the hyperfinite dynamic matching model, paired agents do not change their expanded types in the matching step. When $a_1 \neq (k, l, 0)$, it is clear that

$$\begin{aligned} & P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\ = \; & P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = (k,l,0), F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1}\right) = 0. \end{aligned} \tag{E.84}$$

When $a_1 = (k, l, 0)$, and $P_0\left(\tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) = 0$, the inductive hypothesis implies that

$$P_0\left(\tilde{\beta}_i^m = (k,l,0), F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^m = b_2, F_j^{m-1}\right) \leq d_m.$$

We can then obtain

$$\begin{aligned} & \Big| P_0\left(\tilde{\beta}_i^{m+1} = (k,l,0), \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\ & \quad - P_0\left(\tilde{\beta}_i^{m+1} = (k,l,0), \tilde{\beta}_i^m = (k,l,0), F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1}\right) \Big| \\ = \; & P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = (k,l,0), F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1}\right) \\ \leq \; & P_0\left(\tilde{\beta}_i^m = (k,l,0), F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^m = b_2, F_j^{m-1}\right) \leq d_m. \end{aligned} \tag{E.85}$$

When $a_1 = (k, l, 0)$, $P_0\left(\tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) > 0$, we have

$$\begin{aligned} & P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\ = \; & P_0\left(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\ = \; & P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\ & \quad P_0\left(\tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right). \end{aligned}$$

It follows from Lemma E.21 that

$$\begin{aligned} & \Big| P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) - P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_j^m = b_2\right) \Big| \\ \leq \; & \frac{4K\epsilon_0}{P_0\left(\tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right)} + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}, \text{ and} \end{aligned}$$

$$\begin{aligned} & \Big| P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_j^m = b_2, F_j^{m-1}\right) - P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_j^m = b_2\right) \Big| \\ \leq \; & \frac{4K\epsilon_0}{P_0\left(\tilde{\beta}_j^m = b_2, F_j^{m-1}\right)} + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}. \end{aligned}$$

Then, the above inequalities imply that

$$
\begin{aligned}
& \Big| P_0\left(\tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_i^m=(k,l,0), \tilde{\beta}_j^m=b_2, F_i^{m-1}, F_j^{m-1}\right) \\
& \qquad - P_0\left(\tilde{\beta}_j^{m+1}=a_2 \,|\, \tilde{\beta}_j^m=b_2\right) P_0\left(\tilde{\beta}_i^m=(k,l,0), \tilde{\beta}_j^m=b_2, F_i^{m-1}, F_j^{m-1}\right) \Big| \\
= & \Big| P_0\left(\tilde{\beta}_j^{m+1}=a_2 \,|\, \tilde{\beta}_i^m=(k,l,0), \tilde{\beta}_j^m=b_2, F_i^{m-1}, F_j^{m-1}\right) - P_0\left(\tilde{\beta}_j^{m+1}=a_2 \,|\, \tilde{\beta}_j^m=b_2\right) \Big| \\
& \qquad P_0\left(\tilde{\beta}_i^m=(k,l,0), \tilde{\beta}_j^m=b_2, F_i^{m-1}, F_j^{m-1}\right) \\
\leq & \; 4K\epsilon_0 + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}, \text{ and}
\end{aligned}
$$

$$
\begin{aligned}
& \Big| P_0\left(\tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_j^m=b_2, F_j^{m-1}\right) - P_0\left(\tilde{\beta}_j^{m+1}=a_2 \,|\, \tilde{\beta}_j^m=b_2\right) P_0\left(\tilde{\beta}_j^m=b_2, F_j^{m-1}\right) \Big| \\
= & \Big| P_0\left(\tilde{\beta}_j^{m+1}=a_2 \,|\, \tilde{\beta}_j^m=b_2, F_j^{m-1}\right) - P_0\left(\tilde{\beta}_j^{m+1}=a_2 \,|\, \tilde{\beta}_j^m=b_2\right) \Big| P_0\left(\tilde{\beta}_j^m=b_2, F_j^{m-1}\right) \\
\leq & \; 4K\epsilon_0 + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}.
\end{aligned}
$$

By the above two inequalities and the inductive hypothesis, we can obtain that

$$
\begin{aligned}
& \Big| P_0\left(\tilde{\beta}_i^{m+1}=(k,l,0), \tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_i^m=(k,l,0), \tilde{\beta}_j^m=b_2, F_i^{m-1}, F_j^{m-1}\right) \\
& \qquad - P_0\left(\tilde{\beta}_i^{m+1}=a_1, \tilde{\beta}_i^m=(k,l,0), F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_j^m=b_2, F_j^{m-1}\right) \Big| \\
= & \Big| P_0\left(\tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_i^m=(k,l,0), \tilde{\beta}_j^m=b_2, F_i^{m-1}, F_j^{m-1}\right) \\
& \qquad - P_0\left(\tilde{\beta}_j^{m+1}=a_2 \,|\, \tilde{\beta}_j^m=b_2\right) P_0\left(\tilde{\beta}_i^m=(k,l,0), \tilde{\beta}_j^m=b_2, F_i^{m-1}, F_j^{m-1}\right) \Big| \\
& \quad + \Big| P_0\left(\tilde{\beta}_j^{m+1}=a_2 \,|\, \tilde{\beta}_j^m=b_2\right) P_0\left(\tilde{\beta}_i^m=(k,l,0), \tilde{\beta}_j^m=b_2, F_i^{m-1}, F_j^{m-1}\right) \\
& \qquad - P_0\left(\tilde{\beta}_j^{m+1}=a_2 \,|\, \tilde{\beta}_j^m=b_2\right) P_0\left(\tilde{\beta}_i^m=(k,l,0), F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^m=b_2, F_j^{m-1}\right) \Big| \\
& \quad + \Big| P_0\left(\tilde{\beta}_j^{m+1}=a_2 \,|\, \tilde{\beta}_j^m=b_2\right) P_0\left(\tilde{\beta}_i^m=(k,l,0), F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^m=b_2, F_j^{m-1}\right) \\
& \qquad - P_0\left(\tilde{\beta}_i^m=(k,l,0), F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_j^m=b_2, F_j^{m-1}\right) \Big| \\
\leq & \; 4K\epsilon_0 + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1} + d_m + 4K\epsilon_0 + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1} \\
= & \; 8K\epsilon_0 + \frac{4K}{\hat{M}^{\frac{1}{9}}} + 4K\xi_{-1} + d_m.
\end{aligned}
\tag{E.86}
$$

Next, we assume that $b_1 = (k, J, 0)$ or $(k, l, 1)$ for some $k, l \in S$. It is clear that $P_0\left(\tilde{\beta}_i^m = b_1\right) = 0$. Then we have

$$
\begin{aligned}
& P_0\left(\tilde{\beta}_i^{m+1}=a_1, \tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_i^m=b_1, \tilde{\beta}_j^m=b_2, F_i^{m-1}, F_j^{m-1}\right) \\
= & \; P_0\left(\tilde{\beta}_i^{m+1}=a_1, \tilde{\beta}_i^m=b_1, F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_j^m=b_2, F_j^{m-1}\right) \\
= & \; 0.
\end{aligned}
\tag{E.87}
$$

One can exchange the positions of i and j to obtain exactly the same estimations as in Equations (E.84), (E.85), (E.86) and (E.87), when i is replaced by j, and the conditions on b_1 are restated on b_2. Thus, for the matching step, it remains to consider $b_1 = (k_1, J, 1)$ and $b_2 = (k_2, J, 1)$ for some $k_1, k_2 \in S$. In this case, if a_1 is neither $(k_1, J, 1)$ nor $(k_1, l, 1)$ $(l \in S)$, then we must have $P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, F_i^{m-1}\right) = 0$, which implies the identities in Equation (E.87). By the same reason, Equation (E.87) also holds when a_2 is neither $(k_2, J, 1)$ nor $(k_2, l, 1)$ $(l \in S)$. Hence, we only need to consider $a_1 = (k_1, l_1, 1)$ and $a_2 = (k_2, l_2, 1)$ for some $l_1, l_2 \in S \cup \{J\}$.

In this paragraph, we work with $a_1 = (k_1, l_1, 1)$, $a_2 = (k_2, l_2, 1)$, $b_1 = (k_1, J, 1)$, and $b_2 = (k_2, J, 1)$ for some $k_1, k_2 \in S$, and $l_1, l_2 \in S \cup \{J\}$. The inequalities in Lemma E.19 give symmetric treatment the cases for $l \in S$ or $l = J$. For the simplicity of applying this lemma, we introduce the notation $\hat{q}_{kJ}$ to represent $\hat{q}_k$ in the rest of the proof for Lemma E.8. By Lemmas E.17 and E.19, we have

$$\begin{aligned}
&\Big| P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\
&\qquad - \int_A \hat{q}_{k_1 l_1}(\tilde{\rho}^m)\hat{q}_{k_2 l_2}(\tilde{\rho}^m) dQ^m \Big| \\
= \;& \left| \int_A \left(Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2) - \hat{q}_{k_1 l_1}(\tilde{\rho}^m)\hat{q}_{k_2 l_2}(\tilde{\rho}^m) \right) dQ^m \right| \\
\leq \;& \int_{A \setminus V^m} \left| Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2) - \hat{q}_{k_1 l_1}(\tilde{\rho}^m)\hat{q}_{k_2 l_2}(\tilde{\rho}^m) \right| dQ^m + Q^m(V^m) \\
\leq \;& \frac{1}{\hat{M}^{\frac{1}{9}}} + \epsilon_0.
\end{aligned} \tag{E.88}$$

Next, we estimate the difference

$$\begin{aligned}
&\left| \int_A \hat{q}_{k_1 l_1}(\tilde{\rho}^m)\hat{q}_{k_2 l_2}(\tilde{\rho}^m) dQ^m - P_0(A)\hat{q}_{k_1 l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\hat{q}_{k_2 l_2}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right) \right| \\
\leq \;& \int_A \left| \hat{q}_{k_1 l_1}(\tilde{\rho}^m)\hat{q}_{k_2 l_2}(\tilde{\rho}^m) - \hat{q}_{k_1 l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\hat{q}_{k_2 l_2}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right) \right| dQ^m \\
\leq \;& \int_{A \setminus V^m} \left| \hat{q}_{k_1 l_1}(\tilde{\rho}^m)\hat{q}_{k_2 l_2}(\tilde{\rho}^m) - \hat{q}_{k_1 l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\hat{q}_{k_2 l_2}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right) \right| dQ^m + Q^m(V^m) \\
= \;& \int_{A \setminus V^m} \left| \hat{q}_{k_1 l_1}(\tilde{\rho}^m)\hat{q}_{k_2 l_2}(\tilde{\rho}^m) - \hat{q}_{k_1 l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\hat{q}_{k_2 l_2}(\tilde{\rho}^m) \right| dQ^m \\
&+ \int_{A \setminus V^m} \left| \hat{q}_{k_1 l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\hat{q}_{k_2 l_2}(\tilde{\rho}^m) - \hat{q}_{k_1 l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\hat{q}_{k_2 l_2}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right) \right| dQ^m + Q^m(V^m) \\
\leq \;& \int_{A \setminus V^m} \left| \hat{q}_{k_1 l_1}(\tilde{\rho}^m) - \hat{q}_{k_1 l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right) \right| dQ^m \\
&+ \int_{A \setminus V^m} \left| \hat{q}_{k_2 l_2}(\tilde{\rho}^m) - \hat{q}_{k_2 l_2}\left(U_1^m(\mathbb{E}\tilde{\rho}^0\right) \right| dQ^m + Q^m(V^m).
\end{aligned}$$

By Lemma E.17, for any $\omega^m \notin V^m$, we have $||\tilde{\rho}^m(\omega^m) - U_1^m(\mathbb{E}\tilde{\rho}^0)||_\infty \leq \xi_0$. Lemma E.3 implies that for any $\omega^m \notin V^m$,

$$\left|\hat{q}_{k_1l_1}(\tilde{\rho}^m(\omega^m)) - \hat{q}_{k_1l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\right| \leq \xi_{-1} \text{ and } \left|\hat{q}_{k_2l_2}(\tilde{\rho}^m(\omega^m)) - \hat{q}_{k_2l_2}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\right| \leq \xi_{-1}.$$

It follows from the above inequalities and Lemma E.17 that

$$\begin{aligned} & \left|\int_A \hat{q}_{k_1l_1}(\tilde{\rho}^m)\hat{q}_{k_2l_2}(\tilde{\rho}^m)dQ^m - P_0(A)\hat{q}_{k_1l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\hat{q}_{k_2l_2}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\right| \\ \leq \;& 2\xi_{-1} + \epsilon_0 \end{aligned} \tag{E.89}$$

By Equations (E.88) and (E.89), we have

$$\begin{aligned} & \Big|P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\ & \quad -P_0(A)\hat{q}_{k_1l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\hat{q}_{k_2l_2}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\Big| \leq 2\epsilon_0 + \frac{1}{\hat{M}^{\frac{1}{9}}} + 2\xi_{-1}. \end{aligned} \tag{E.90}$$

It follows from Lemmas E.17 and E.19 that

$$\begin{aligned} & \left|P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, F_i^{m-1}\right) - \int_{A'} \hat{q}_{k_1l_1}(\tilde{\rho}^m)dQ^m\right| \\ = \;& \left|\int_{A'}\left(Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1) - \hat{q}_{k_1l_1}(\tilde{\rho}^m)\right)dQ^m\right| \\ \leq \;& \int_{A'\setminus V^m}\left|Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1) - \hat{q}_{k_1l_1}(\tilde{\rho}^m)\right|dQ^m + Q^m(V^m) \\ \leq \;& \frac{1}{\hat{M}^{\frac{1}{9}}} + \epsilon_0. \end{aligned} \tag{E.91}$$

Next, we estimate the difference

$$\begin{aligned} & \left|\int_{A'} \hat{q}_{k_1l_1}(\tilde{\rho}^m)dQ^m - P_0(A')\hat{q}_{k_1l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\right| \\ \leq \;& \int_{A'}\left|\hat{q}_{k_1l_1}(\tilde{\rho}^m) - \hat{q}_{k_1l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\right|dQ^m \\ \leq \;& \int_{A'\setminus V^m}\left|\hat{q}_{k_1l_1}(\tilde{\rho}^m) - \hat{q}_{k_1l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\right|dQ^m + Q^m(V^m). \end{aligned}$$

It follows from Lemma E.17 that for any $\omega^m \notin V^m$, $||\tilde{\rho}^m(\omega^m) - U_1^m(\mathbb{E}\tilde{\rho}^0)||_\infty \leq \xi_0$. Lemma E.3 implies that for any $\omega^m \notin V^m$, $\left|\hat{q}_{k_1l_1}(\tilde{\rho}^m)(\omega^m) - \hat{q}_{k_1l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\right| \leq \xi_{-1}$. It is then obvious that

$$\begin{aligned} & \left|\int_{A'} \hat{q}_{k_1l_1}(\tilde{\rho}^m) - P_0(A')\hat{q}_{k_1l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\right| \\ \leq \;& \xi_{-1} + \epsilon_0 \end{aligned} \tag{E.92}$$

By combining Equations (E.91) and (E.92), we have

$$\begin{aligned} & \left|P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, F_i^{m-1}\right) - P_0(A')\hat{q}_{k_1l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\right| \\ \leq \;& 2\epsilon_0 + \frac{1}{\hat{M}^{\frac{1}{9}}} + \xi_{-1}. \end{aligned} \tag{E.93}$$

Equation (E.93) states an inequality for a general agent i, which can be restated for agent j as follows:

$$\begin{aligned}
&\left|P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^m=b_2,F_j^{m-1}\right)-P_0(A'')\hat{q}_{k_2l_2}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\right| \\
\leq\ & 2\epsilon_0+\frac{1}{\hat{M}^{\frac{1}{9}}}+\xi_{-1},
\end{aligned} \tag{E.94}$$

Based on Equations (E.93) and (E.94), we can obtain that

$$\begin{aligned}
&\left|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^m=b_1,F_i^{m-1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^m=b_2,F_j^{m-1}\right)\right. \\
&\quad\left.-P_0(A')P_0(A'')\hat{q}_{k_1l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\hat{q}_{k_2l_2}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\right| \\
\leq\ & \left|P_0(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^m=b_1,F_i^{m-1})P_0(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^m=b_2,F_j^{m-1})\right. \\
&\quad\left.-P_0(A')\hat{q}_{k_1l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)P_0(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^m=b_2,F_j^{m-1})\right| \\
&\quad+\left|P_0(A')\hat{q}_{k_1l_1}(U_1^m(\mathbb{E}\tilde{\rho}^0))P_0(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^m=b_2,F_j^{m-1})\right. \\
&\quad\quad\left.-P_0(A')P_0(A'')\hat{q}_{k_1l_1}(U_1^m(\mathbb{E}\tilde{\rho}^0))\hat{q}_{k_2l_2}(U_1^m(\mathbb{E}\tilde{\rho}^0))\ \right| \\
\leq\ & \left|P_0(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^m=b_1,F_i^{m-1})-P_0(A')\hat{q}_{k_1l_1}\left(U_1^m(\mathbb{E}\tilde{\rho}^0)\right)\right| \\
&\quad+\left|P_0(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^m=b_2,F_j^{m-1})-P_0(A'')\hat{q}_{k_2l_2}(U_1^m(\mathbb{E}\tilde{\rho}^0))\ \right| \\
\leq\ & 4\epsilon_0+\frac{2}{\hat{M}^{\frac{1}{9}}}+2\xi_{-1},
\end{aligned} \tag{E.95}$$

The induction hypothesis indicates that $|P_0(A)-P_0(A')P_0(A'')|\leq d_m$. By Equations (E.90) and (E.95), we have

$$\begin{aligned}
&\left|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^m=b_1,\tilde{\beta}_j^m=b_2,F_i^{m-1},F_j^{m-1}\right)\right. \\
&\quad\left.-P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^m=b_1,F_i^{m-1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^m=b_2,F_j^{m-1}\right)\right| \\
\leq\ & \left|P_0(A)\hat{q}_{k_1l_1}(U_1^m(\mathbb{E}\tilde{\rho}^0))\hat{q}_{k_2l_2}(U_1^m(\mathbb{E}\tilde{\rho}^0))-P_0(A')P_0(A'')\hat{q}_{k_1l_1}(U_1^m(\mathbb{E}\tilde{\rho}^0))\hat{q}_{k_2l_2}(U_1^m(\mathbb{E}\tilde{\rho}^0))\right| \\
&\quad+6\epsilon_0+\frac{3}{\hat{M}^{\frac{1}{9}}}+4\xi_{-1} \\
\leq\ & \left|P_0(A)-P_0(A')P_0(A'')\right|+6\epsilon_0+\frac{3}{\hat{M}^{\frac{1}{9}}}+4\xi_{-1} \\
\leq\ & 6\epsilon_0+\frac{3}{\hat{M}^{\frac{1}{9}}}+4\xi_{-1}+d_m.
\end{aligned} \tag{E.96}$$

By Equations (E.85), (E.86) and (E.96), we know that for $m=3n-2$, and for any $a_1,a_2,b_1,b_2\in\tilde{S}$, $F_i^{m-1}\in\mathcal{F}_i^{m-1}$ and $F_j^{m-1}\in\mathcal{F}_j^{m-1}$,

$$\begin{aligned}
&\left|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^m=b_1,\tilde{\beta}_j^m=b_2,F_i^{m-1},F_j^{m-1}\right)\right. \\
&\quad\left.-P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^m=(k,l,0),F_i^{m-1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^m=b_2,F_j^{m-1}\right)\right| \\
\leq\ & 8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{9}}}+4K\xi_{-1}+d_m.
\end{aligned} \tag{E.97}$$

Next, we consider the case when $m = 3n - 3$ for some $n \in \mathbb{T}_0$, that is, the $m+1$ step is the mutation step in the n-th period. Let

$$B_{ab} = \begin{cases} \hat{\eta}_{k_1 l_1} \hat{\eta}_{k_2 l_2} & \text{if } a = (k_1, k_2, 0), b = (l_1, l_2, 0) \\ \hat{\eta}_{k_1 l_1} & \text{if } a = (k_1, J, 1), b = (l_1, J, 1) \\ 0 & \text{otherwise.} \end{cases}$$

By the construction of the mutation step in the hyperfinite dynamic matching model, we know that if $\hat{\pi}_i^m(\omega^m) \neq j$, $\tilde{\beta}_i^m(\omega^m) = b_1$, $\tilde{\beta}_j^m(\omega^m) = b_2$,

$$Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2) = Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1) Q_{m+1}^{\omega^m}(\tilde{\beta}_j^{m+1} = a_2) = B_{b_1 a_1} B_{b_2 a_2}.$$

Recall from the beginning of this proof that $F_{ij}^m = \{\omega^m \in \Omega^m : \hat{\pi}_i^m(\omega) = j\}$ and $P_0\left(F_{ij}^m\right) \leq \frac{1}{\hat{M}^{\frac{1}{11}}}$. It then follows from Lemma E.17 that

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) - P_0(A) B_{b_1 a_1} B_{b_2 a_2} \right| \\ \leq & \int_A \left| Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2) - B_{b_1 a_1} B_{b_2 a_2} \right| dQ^m \\ = & \int_{A \setminus (F_{ij}^m \cup V^m)} \left| B_{b_1 a_1} B_{b_2 a_2} - B_{b_1 a_1} B_{b_2 a_2} \right| dQ^m \\ & \quad + \int_{A \cap (F_{ij}^m \cup V^m)} \left| Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2) - B_{b_1 a_1} B_{b_2 a_2} \right| dQ^m \\ \leq & P_0(F_{ij}^m \cup V^m) \leq P_0(F_{ij}^m) + P_0(V^m) \leq \frac{1}{\hat{M}^{\frac{1}{11}}} + \epsilon_0. \end{aligned} \tag{E.98}$$

By the construction of the mutation step in the hyperfinite dynamic matching model again, we have

$$\begin{aligned} P_0(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, F_i^{m-1}) &= P_0(A') B_{b_1 a_1}, \text{ and} \\ P_0(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1}) &= P_0(A'') B_{b_2 a_2}. \end{aligned}$$

It follows from the above identities, the induction hypothesis $\left| P_0(A) - P_0(D_{b_1}^i) P_0(D_{b_2}^j) \right| \leq d_m$, and Equation (E.98) that

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \right. \\ & \qquad \left. - P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1}\right) \right| \\ \leq & \left| P_0(A) B_{b_1 a_1} B_{b_2 a_2} - P_0(A') B_{b_1 a_1} P_0(A'') B_{b_2 a_2} \right| + \frac{1}{\hat{M}^{\frac{1}{11}}} + \epsilon_0 \\ = & \left| P_0(A) - P_0(A') P_0(A'') \right| B_{b_1 a_1} B_{b_2 a_2} + \frac{1}{\hat{M}^{\frac{1}{11}}} + \epsilon_0 \\ \leq & \epsilon_0 + \frac{1}{\hat{M}^{\frac{1}{11}}} + d_m. \end{aligned} \tag{E.99}$$

It remains to consider the case when $m = 3n - 1$ for some $n \in \mathbb{T}_0$, that is, the $m + 1$ step is the type changing with break-up step in the n-th period. Though the proof of this part is similar to the case of the mutation step, we present a full proof for the sake of completeness. Let

$$C_{ab} = \begin{cases} 1 - \hat{\vartheta}_{k_1 k_2} & \text{if } a = (k_1, k_2, 0), b = (k_1, k_2, 0) \\ \hat{\vartheta}_{k_1 k_2} \varsigma_{k_1 k_2}(l_1) & \text{if } a = (k_1, k_2, 0), b = (l_1, J, 1) \\ \hat{\xi}_{k_1 k_2} \hat{\sigma}_{k_1 k_2}(l_1, l_2) & \text{if } a = (k_1, k_2, 1), b = (l_1, l_2, 0) \\ (1 - \hat{\xi}_{k_1 k_2}) \hat{\varsigma}_{k_1 k_2}(l_1) & \text{if } a = (k_1, k_2, 1), b = (l_1, J, 1) \\ 1 & \text{if } a = (k_1, J, 1), b = (k_1, J, 1) \\ 0 & \text{otherwise.} \end{cases}$$

By the construction of the type changing with break-up step in the hyperfinite dynamic matching model, we know that if $\hat{\pi}_i^m(\omega^m) \neq j$, $\tilde{\beta}_i^m(\omega^m) = b_1$, $\tilde{\beta}_j^m(\omega^m) = b_2$,

$$Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2) = Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1) Q_{m+1}^{\omega^m}(\tilde{\beta}_j^{m+1} = a_2) = C_{b_1 a_1} C_{b_2 a_2}.$$

Lemma E.17 implies that

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) - P_0(A) C_{b_1 a_1} C_{b_2 a_2} \right| \\ \leq \; & \int_A \left| Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2) - C_{b_1 a_1} C_{b_2 a_2} \right| dQ^m \\ = \; & \int_{A \setminus (F_{ij}^m \cup V^m)} \left| C_{b_1 a_1} C_{b_2 a_2} - C_{b_1 a_1} C_{b_2 a_2} \right| dQ^m \\ & \quad + \int_{A \cap (F_{ij}^m \cup V^m)} \left| Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2) - C_{b_1 a_1} C_{b_2 a_2} \right| dQ^m \\ \leq \; & P_0(F_{ij}^m \cup V^m) \leq P_0(F_{ij}^m) + P_0(V^m) \leq \frac{1}{\hat{M}^{\frac{1}{11}}} + \epsilon_0. \end{aligned} \tag{E.100}$$

By the construction of the type changing with break-up step in the hyperfinite dynamic matching model again, we have

$$\begin{aligned} P_0(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, F_i^{m-1}) &= P_0(A') C_{b_1 a_1}, \text{ and} \\ P_0(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1}) &= P_0(A'') C_{b_2 a_2}. \end{aligned}$$

By the above identities, the induction hypothesis $\left| P_0(A) - P_0(D_{b_1}^i) P_0(D_{b_2}^j) \right| \leq d_m$, and Equation (E.100), we obtain that

$$\begin{aligned}
&\Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_j^{m}=b_2,F_i^{m-1},F_j^{m-1}\right)\\
&\qquad -P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=b_1,F_i^{m-1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,F_j^{m-1}\right)\Big|\\
\le\ & \left|P_0(A)C_{b_1a_1}C_{b_2a_2}-P_0(A')C_{b_1a_1}\,P_0(A'')C_{b_2a_2}\right|+\frac{1}{\hat{M}^{\frac{1}{11}}}+\epsilon_0\\
=\ & \left|P_0(A)-P_0(A')P_0(A'')\right|C_{b_1a_1}C_{b_2a_2}+\frac{1}{\hat{M}^{\frac{1}{11}}}+\epsilon_0\\
\le\ & \epsilon_0+\frac{1}{\hat{M}^{\frac{1}{11}}}+d_m. \qquad\qquad (E.101)
\end{aligned}$$

By combining Equations (E.97), (E.99), (E.101), we obtain that

$$\begin{aligned}
&\Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_j^{m}=b_2,F_i^{m-1},F_j^{m-1}\right)\\
&-P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=b_1,F_i^{m-1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,F_j^{m-1}\right)\Big|\\
\le\ & 8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{11}}}+4K\xi_{-1}+d_m. \qquad\qquad (E.102)
\end{aligned}$$

Fix any $F_i^m\in\mathcal{F}_i^m$. There exist $F_{ib}^{m-1}\in\mathcal{F}_i^{m-1}$ for $b\in\tilde{S}$ such that $F_i^m=\bigcup_{b\in\tilde{S}}\left(\left(\tilde{\beta}_i^m=b\right)\cap F_{ib}^{m-1}\right)$. Similarly, for any fixed $F_j^m\in\mathcal{F}_j^m$, there exist $F_{jb}^{m-1}\in\mathcal{F}_j^{m-1}$ for $b\in\tilde{S}$ such that $F_j^m=\bigcup_{b\in\tilde{S}}\left(\left(\tilde{\beta}_j^m=b\right)\cap F_{jb}^{m-1}\right)$. Therefore, by Equation (E.102), we can obtain that

$$\begin{aligned}
&\Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,F_i^{m},F_j^{m}\right)\\
&\qquad -P_0\left(\tilde{\beta}_i^{m+1}=a_1,F_i^{m}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,F_j^{m}\right)\Big|\\
\le\ & \sum_{b_1,b_2\in\tilde{S}}\Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_j^{m}=b_2,F_{ib_1}^{m-1},F_{jb_2}^{m-1}\right)\\
&\qquad -P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=b_1,F_{ib_1}^{m-1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,F_{jb_2}^{m-1}\right)\Big|\\
\le\ & 4K^2(K+1)^2\left(8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{11}}}+4K\xi_{-1}+d_m\right). \qquad\qquad (E.103)
\end{aligned}$$

Thus, we can define d_{m+1} to be $4K^2(K+1)^2\left(8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{11}}}+4K\xi_{-1}+d_m\right)$.

Next, we prove that for any $m\in\{0,2,\ldots,3M^2\}$,

$$d_m\le 2^{4m}K^{4m}(K+1)^{4m}\left(8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{11}}}+4K\xi_{-1}\right). \qquad (E.104)$$

Since $d_0=\frac{1}{\hat{M}^{\frac{1}{10}}}$ and $\epsilon_0=\frac{2}{\hat{M}^{\frac{1}{10}}}$, it is clear that Equation (E.104) holds for $m=0$. Suppose that Equation (E.104) holds for $m = m'$. Then

$$d_{m'+1} = 4K^2(K+1)^2\left(8K\epsilon_0 + \frac{4K}{\hat{M}^{\frac{1}{11}}} + 4K\xi_{-1} + d_{m'}\right)$$
$$\leq \; 4K^2(K+1)^2\left(8K\epsilon_0 + \frac{4K}{\hat{M}^{\frac{1}{11}}} + 4K\xi_{-1}\right) + 2^{4m'+2}K^{4m'+2}(K+1)^{4m'+2}\left(8K\epsilon_0 + \frac{4K}{\hat{M}^{\frac{1}{11}}} + 4K\xi_{-1}\right)$$
$$\leq \; 2^{4m'+4}K^{4m'+4}(K+1)^{4m'+4}\left(8K\epsilon_0 + \frac{4K}{\hat{M}^{\frac{1}{11}}} + 4K\xi_{-1}\right).$$

Therefore, Equation (E.104) holds for any $m \in \{0, 2, \ldots, 3M^2\}$ by mathematical induction. Since $\xi_{-1} = \frac{1}{M^{M^M}}$, $\epsilon_0 = \frac{2}{\hat{M}^{\frac{1}{10}}}$ and $\hat{M} \geq M^{M^M}$, d_m is always infinitesimal for any $m \in \{0, 1, \ldots, 3M^2\}$.

Fix any $F_i^m \in \mathcal{F}_i^m$ and $F_j^m \in \mathcal{F}_j^m$. Equation (E.103) implies that

$$\begin{aligned}
&\left|P_0\left(F_i^m \cap F_j^m\right) - P_0\left(F_i^m\right)P_0\left(F_j^m\right)\right| \\
= \; &\left|\sum_{a_1,a_2\in\tilde{S}} P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, F_i^m, F_j^m\right)\right. \\
&\left. - \sum_{a_1,a_2\in\tilde{S}} P_0\left(\tilde{\beta}_i^{m+1} = a_1, F_i^m\right)P_0\left(\tilde{\beta}_j^{m+1} = a_2, F_j^m\right)\right| \\
\leq \; & 4K^2(K+1)^2 d_{m+1},
\end{aligned}$$

which is an infinitesimal. Hence, Lemma E.8 is proved.

E.5.10 Proof of Lemma E.9

Fix any $i \in I$, $m, \Delta m \in \{0, \ldots, 3M^2\}$ and $F^m \in \mathcal{F}^m$ such that $m + \Delta m \leq 3M^2$, $\frac{\Delta m}{M}$ is finite, and $P_0(F^m) > 0$.

We first consider the case when $m+\Delta m = 3n-2$ for some $n \in \mathbb{T}_0$. Fix any $\omega^{3n-3} \in \Omega^{3n-3}$. Denote $\hat{\alpha}_i^{3n-3}(\omega^{3n-3})$ and $\hat{g}_i^{3n-3}(\omega^{3n-3})$ by $k \in S$ and $l \in S \cup \{J\}$ respectively. If $l \neq J$, by the construction of the mutation step in the hyperfinite dynamical matching model, we have

$$Q_{3n-2}^{\omega^{3n-3}}\left(\hat{X}_i^{3n-2} = \hat{X}_i^{3n-3}\right) = \hat{\eta}_{kk}\hat{\eta}_{ll} \geq \left(1 - \frac{K\bar{a}}{M}\right)^2.$$

If $l = J$, we have

$$Q_{3n-2}^{\omega^{3n-3}}\left(\hat{X}_i^{3n-2} = \hat{X}_i^{3n-3}\right) = \hat{\eta}_{kk} \geq 1 - \frac{K\bar{a}}{M} \geq \left(1 - \frac{K\bar{a}}{M}\right)^2.$$

Let $A^{3n-3} = \left(\hat{X}_i^{3n-3} = \hat{X}_i^m\right) \cap F^m$. If $P_0(A^{3n-3}) > 0$, then

$$\begin{aligned}
& P_0(\hat{X}_i^{3n-2} = \hat{X}_i^{3n-3} | \hat{X}_i^{3n-3} = \hat{X}_i^m, F^m) \\
= & \frac{\int_{A^{3n-3}} Q_{3n-2}^{\omega^{3n-3}} \left(\hat{X}_i^{3n-2} = \hat{X}_i^{3n-3}\right) dQ^{3n-3}}{P_0(A^{3n-3})} \\
\geq & \frac{\int_{A^{3n-3}} \left(1 - \frac{K\bar{a}}{M}\right)^2 dQ^{3n-3}}{P_0(A^{3n-3})} \\
= & \left(1 - \frac{K\bar{a}}{M}\right)^2.
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
& P_0(\hat{X}_i^{3n-2} = \hat{X}_i^m | F^m) \\
= & P_0(\hat{X}_i^{3n-3} = \hat{X}_i^m | F^m) P_0(\hat{X}_i^{3n-2} = \hat{X}_i^{3n-3} | \hat{X}_i^{3n-3} = \hat{X}_i^m, F^m) \\
\geq & P_0(\hat{X}_i^{3n-3} = \hat{X}_i^m | F^m) \left(1 - \frac{K\bar{a}}{M}\right)^2. \qquad \text{(E.105)}
\end{aligned}$$

If $P_0(\hat{X}_i^{3n-3} = \hat{X}_i^m, F^m) = 0$, then the above inequality is trivially satisfied.

Next, we consider the case when $m + \Delta m = 3n - 1$ for some $n \in \mathbb{T}_0$. Fix any $\omega^{3n-2} \in \Omega^{3n-2}$. Denote $\hat{\alpha}_i^{3n-2}(\omega^{3n-2})$ and $\hat{g}_i^{3n-2}(\omega^{3n-2})$ by $k \in S$ and $l \in S \cup \{J\}$ respectively. The construction of the matching step in the hyperfinite dynamical matching model and Lemma E.1 allows us to claim that

$$Q_{3n-1}^{\omega^{3n-2}} \left(\hat{X}_i^{3n-1} = \hat{X}_i^{3n-2}\right) = 1, \quad \text{if } l \neq J,$$

$$Q_{3n-1}^{\omega^{3n-2}} \left(\hat{X}_i^{3n-1} > \hat{X}_i^{3n-2}\right) \leq \sum_{l' \in S} \hat{q}_{kl'} \leq \frac{K\bar{a}}{M}, \quad \text{if } l = J.$$

It is then clear that

$$Q_{3n-1}^{\omega^{3n-2}} \left(\hat{X}_i^{3n-1} = \hat{X}_i^{3n-2}\right) \geq 1 - \sum_{l' \in S} \hat{q}_{kl'} \geq 1 - \frac{K\bar{a}}{M},$$

Let $A^{3n-2} = \left(\hat{X}_i^{3n-2} = \hat{X}_i^m\right) \cap F^m$. If $P_0(A^{3n-2}) > 0$, then

$$\begin{aligned}
& P_0(\hat{X}_i^{3n-1} = \hat{X}_i^{3n-2} | \hat{X}_i^{3n-2} = \hat{X}_i^m, F^m) \\
= & \frac{\int_{A^{3n-2}} Q_{3n-1}^{\omega^{3n-2}} \left(\hat{X}_i^{3n-1} = \hat{X}_i^{3n-2}\right) dQ^{3n-2}}{P_0(A^{3n-2})} \\
\geq & \frac{\int_{A^{3n-2}} \left(1 - \frac{K\bar{a}}{M}\right) dQ^{3n-2}}{P_0(A^{3n-2})} \\
= & \left(1 - \frac{K\bar{a}}{M}\right).
\end{aligned}$$

Hence, we can derive the following estimation

$$\begin{aligned}
& P_0(\hat{X}_i^{3n-1} = \hat{X}_i^m | F^m) \\
= \;& P_0(\hat{X}_i^{3n-2} = \hat{X}_i^m | F^m) P_0(\hat{X}_i^{3n-1} = \hat{X}_i^{3n-2} | \hat{X}_i^{3n-2} = \hat{X}_i^m, F^m) \\
\geq \;& P_0(\hat{X}_i^{3n-2} = \hat{X}_i^m | F^m) \left(1 - \frac{K\bar{a}}{M}\right).
\end{aligned} \tag{E.106}$$

If $P_0(\hat{X}_i^{3n-2} = \hat{X}_i^m, F^m) = 0$, then the above inequality is trivially satisfied.

It remains to consider the case when $m + \Delta m = 3n$ for some $n \in \mathbb{T}_0$. Fix any $\omega^{3n-1} \in \Omega^{3n-1}$. Denote $\hat{\alpha}_i^{3n-1}(\omega^{3n-1})$ and $\hat{g}_i^{3n-1}(\omega^{3n-1})$ and $\hat{h}_i^{3n-1}(\omega^{3n-1})$ by $k \in S$, $l \in S \cup \{J\}$ and $r \in \{0, 1\}$ respectively. The construction of the type changing and break-up step in the hyperfinite dynamical matching model says that

$$Q_{3n}^{\omega^{3n-1}} \left(\hat{X}_i^{3n} = \hat{X}_i^{3n-1}\right) = 1, \quad \text{if } l = J \text{ or } r = 1,$$

$$Q_{3n}^{\omega^{3n-1}} \left(\hat{X}_i^{3n} = \hat{X}_i^{3n-1}\right) = \left(1 - \vartheta_{kl}\right) \geq 1 - \frac{\bar{a}}{M}, \quad \text{if } l \neq J \text{ and } r = 0.$$

Let $A^{3n-1} = \left(\hat{X}_i^{3n-1} = \hat{X}_i^m\right) \cap F^m$. If $P_0(A^{3n-1}) > 0$, then

$$\begin{aligned}
& P_0(\hat{X}_i^{3n} = \hat{X}_i^{3n-1} | \hat{X}_i^{3n-1} = \hat{X}_i^m, F^m) \\
= \;& \frac{\int_{A^{3n-1}} Q_{3n}^{\omega^{3n-1}} \left(\hat{X}_i^{3n} = \hat{X}_i^{3n-1}\right) dQ^{3n-1}}{P_0(A^{3n-1})} \\
\geq \;& \frac{\int_{A^{3n-1}} \left(1 - \frac{\bar{a}}{M}\right) dQ^{3n-1}}{P_0(A^{3n-1})} \\
= \;& \left(1 - \frac{\bar{a}}{M}\right).
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned}
& P_0(\hat{X}_i^{3n} = \hat{X}_i^m | F^m) \\
= \;& P_0(\hat{X}_i^{3n-1} = \hat{X}_i^m | F^m) P_0(\hat{X}_i^{3n} = \hat{X}_i^{3n-1} | \hat{X}_i^{3n-1} = \hat{X}_i^m, F^m) \\
\geq \;& P_0(\hat{X}_i^{3n-1} = \hat{X}_i^m | F^m) \left(1 - \frac{\bar{a}}{M}\right).
\end{aligned} \tag{E.107}$$

If $P_0(\hat{X}_i^{3n-1} = \hat{X}_i^m, F^m) = 0$, then the above inequality is trivially satisfied.

By Equations (E.105), (E.106) and (E.107), we can derive

$$\begin{aligned}
& P_0(\hat{X}_i^{m+\Delta m} = \hat{X}_i^m | F^m) \\
\geq\ & P_0(\hat{X}_i^{m+\Delta m-1} = \hat{X}_i^m | F^m) \left(1 - \frac{K\bar{a}}{M}\right)^2 \\
\geq\ & P_0(\hat{X}_i^m = \hat{X}_i^m | F^m) \prod_{m'=m+1}^{m+\Delta m} \left(1 - \frac{K\bar{a}}{M}\right)^2 \\
\geq\ & \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m} \\
\simeq\ & e^{-\frac{2K\bar{a}\Delta m}{M}},
\end{aligned}$$

which is the required inequality in Lemma E.9.

E.5.11 Proof of Lemma E.10

Fix any $i \in I$, $m, \Delta m \in \{0, \ldots, 3M^2\}$ such that $m + \Delta m \leq 3M^2$, $\frac{\Delta m}{M}$ is finite and $P_0(F^m) > 0$. It is clear that

$$\begin{aligned}
& P_0\left(\hat{X}_i^{m+\Delta m} - \hat{X}_i^m \geq 2 \,\middle|\, F^m\right) \\
=\ & \sum_{r=m+1}^{m+\Delta m-1} P_0\left(\hat{X}^{m+\Delta m} - \hat{X}_i^r \geq 1, \hat{X}_i^r = \hat{X}_i^{r-1} + 1, \hat{X}^{r-1} = \hat{X}^m \,\middle|\, F^m\right). \quad \text{(E.108)}
\end{aligned}$$

Fix any $r \in \{m+1, m+2, \ldots, m+\Delta m-1\}$. Assume that $P_0\left(\hat{X}_i^r = \hat{X}_i^{r-1} + 1, \hat{X}^{r-1} = \hat{X}^m, F^m\right) > 0$. By Lemma E.9, we can obtain that

$$\begin{aligned}
& P_0\left(\hat{X}^{m+\Delta m} = \hat{X}_i^r \,\middle|\, \hat{X}_i^r = \hat{X}_i^{r-1} + 1, \hat{X}^{r-1} = \hat{X}^m, F^m\right) \\
\gtrsim\ & e^{-\frac{2K\bar{a}(m+\Delta m-r)}{M}} \geq e^{-\frac{2K\bar{a}\Delta m}{M}},
\end{aligned}$$

which implies that

$$\begin{aligned}
& P_0\left(\hat{X}^{m+\Delta m} - \hat{X}_i^r \geq 1 \,\middle|\, \hat{X}_i^r = \hat{X}_i^{r-1} + 1, \hat{X}^{r-1} = \hat{X}^m, F^m\right) \\
\lesssim\ & 1 - e^{-\frac{2K\bar{a}\Delta m}{M}}.
\end{aligned}$$

It follows from the above inequality that

$$\begin{aligned}
& P_0\left(\hat{X}^{m+\Delta m} - \hat{X}_i^r \geq 1, \hat{X}_i^r = \hat{X}_i^{r-1} + 1, \hat{X}^{r-1} = \hat{X}^m \,\middle|\, F^m\right) \\
=\ & P_0\left(\hat{X}^{m+\Delta m} - \hat{X}_i^r \geq 1 \,\middle|\, \hat{X}_i^r = \hat{X}_i^{r-1} + 1, \hat{X}^{r-1} = \hat{X}^m, F^m\right) \\
& \quad P_0\left(\hat{X}_i^r = \hat{X}_i^{r-1} + 1, \hat{X}^{r-1} = \hat{X}^m \,\middle|\, F^m\right) \\
\lesssim\ & \left(1 - e^{-\frac{2K\bar{a}\Delta m}{M}}\right) P_0\left(\hat{X}_i^r = \hat{X}_i^{r-1} + 1, \hat{X}^{r-1} = \hat{X}^m \,\middle|\, F^m\right). \quad \text{(E.109)}
\end{aligned}$$

When $P_0\left(\hat{X}_i^r = \hat{X}_i^{r-1}+1, \hat{X}^{r-1} = \hat{X}^m, F^m\right) = 0$, the above inequality is trivially satisfied. Hence, Equations (E.108) and (E.109) together with Lemma E.9 imply that

$$\begin{aligned}
& P_0\left(\hat{X}_i^{m+\Delta m} - \hat{X}_i^m \geq 2 \,\middle|\, F^m\right) \\
\lesssim\ & \left(1-e^{-\frac{2K\bar{a}\Delta m}{M}}\right) \sum_{r=m+1}^{m+\Delta m} P_0\left(\hat{X}_i^r = \hat{X}_i^{r-1}+1, \hat{X}^{r-1} = \hat{X}^m \,\middle|\, F^m\right) \\
=\ & \left(1-e^{-\frac{2K\bar{a}\Delta m}{M}}\right) P_0\left(\hat{X}_i^{m+\Delta m} \geq \hat{X}^m + 1 \,\middle|\, F^m\right) \\
=\ & \left(1-e^{-\frac{2K\bar{a}\Delta m}{M}}\right)\left(1 - P_0\left(\hat{X}_i^{m+\Delta m} = \hat{X}^m \,\middle|\, F^m\right)\right) \\
\lesssim\ & \left(1-e^{-\frac{2K\bar{a}\Delta m}{M}}\right)^2,
\end{aligned}$$

which is the required inequality in Lemma E.10.

E.5.12 Proof of Lemma E.11

Fix any $i \in I$. For any $N \in \mathbb{N}$, let $A_i^N = \{\omega \in \Omega : \hat{X}_i^{NM}(\omega) \text{ is finite}\}$. It is clear that $A_i = \cap_{N=1}^{\infty} A_i^N$.

Fix any $N \in \mathbb{N}$. For any $n \in \mathbb{N}$ and $j \in \{0, 1, \ldots, n\}$, let m_j be the integer part of $\frac{jNM}{n}$. Then $m_0 = 0$, $m_n = NM$ and $m_j - m_{j-1} < \frac{2NM}{n}$ for any $j \in \{1, \ldots, n\}$.

Fix any $n \in \mathbb{N}$. For any $\omega \notin A_i^N$, $\hat{X}_i^{NM}(\omega)$ is infinite, which implies that there exists $j \in \{1, \ldots, n\}$ such that $\hat{X}_i^{m_j}(\omega) - \hat{X}_i^{m_{j-1}}(\omega) \geq 2$. Therefore, we know that

$$\Omega \setminus A_i^N \subseteq \bigcup_{j=1}^{n} \{\omega \in \Omega : \hat{X}_i^{m_j}(\omega) - \hat{X}_i^{m_{j-1}}(\omega) \geq 2\},$$

which implies that

$$P_0\left(\Omega \setminus A_i^N\right) \leq \sum_{j=1}^{n} P_0\left(\hat{X}_i^{m_j} - \hat{X}_i^{m_{j-1}} \geq 2\right).$$

It follows from Lemma E.10 that

$$P_0\left(\hat{X}_i^{m_j} - \hat{X}_i^{m_{j-1}} \geq 2\right) \lesssim \left(1 - e^{-\frac{2K(m_j - m_{j-1})\bar{a}}{M}}\right)^2 \leq \left(1 - e^{-\frac{4KN\bar{a}}{n}}\right)^2.$$

By combining the above two inequalities, we obtain that

$$P_0\left(\Omega \setminus A_i^N\right) \leq n\left(1 - e^{-\frac{4KN\bar{a}}{n}}\right)^2.$$

Note that $n\left(1 - e^{-\frac{4KN\bar{a}}{n}}\right)^2 \to 0$ as $n \to \infty$. Then $P\left(\Omega \setminus A_i^N\right) = 0$, which implies that $P\left(A_i^N\right) = 1$. Therefore, we have $P\left(A_i\right) = P\left(\cap_{N=1}^{\infty} A_i^N\right) = 1$. By the Fubini property, $\lambda \boxtimes P(A) = \int_I P(A_i) d\lambda = 1$. Hence, the lemma is proven.

E.5.13 Proof of Lemma E.12

Fix any $(k,l,r) \in \tilde{S}$. By the definition of $\tilde{\rho}$, we obtain that

$$\begin{aligned}
& \left|\mathbb{E}\left(\tilde{\rho}_{klr}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}_{klr}^{m}\right)\right| \\
= & \left|\mathbb{E}\left(\frac{1}{\hat{M}}\sum_{i\in I}\mathbf{1}_{klr}\left(\tilde{\beta}_i^{m+\Delta m}\right)\right) - \mathbb{E}\left(\frac{1}{\hat{M}}\sum_{i\in I}\mathbf{1}_{klr}\left(\tilde{\beta}_i^{m}\right)\right)\right| \\
\leq & \frac{1}{\hat{M}}\sum_{i\in I}\mathbb{E}\left|\mathbf{1}_{klr}\left(\tilde{\beta}_i^{m+\Delta m}\right) - \mathbf{1}_{klr}\left(\tilde{\beta}_i^{m}\right)\right| \\
= & \frac{1}{\hat{M}}\sum_{i\in I}P_0\left(\left|\mathbf{1}_{klr}\left(\tilde{\beta}_i^{m+\Delta m}\right) - \mathbf{1}_{klr}\left(\tilde{\beta}_i^{m}\right)\right| = 1\right). \qquad \text{(E.110)}
\end{aligned}$$

For any $\omega \in \Omega$, if $\left|\mathbf{1}_{klr}\left(\tilde{\beta}_i^{m+\Delta m}(\omega)\right) - \mathbf{1}_{klr}\left(\tilde{\beta}_i^{m}(\omega)\right)\right| = 1$, then $\hat{X}_i^{m+\Delta m}(\omega) > \hat{X}_i^{m}(\omega)$. Thus, we can obtain from Equation (E.110) that

$$\left|\mathbb{E}\left(\tilde{\rho}_{klr}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}_{klr}^{m}\right)\right| \leq \frac{1}{\hat{M}}\sum_{i\in I}P_0\left(\hat{X}_i^{m+\Delta m} > \hat{X}_i^{m}\right).$$

By Lemma E.9, we have

$$P_0\left(\hat{X}_i^{m+\Delta m} > \hat{X}_i^{m}\right) \lesssim 1 - e^{-\frac{2K\bar{a}\Delta m}{M}}.$$

Therefore, we can obtain that

$$\left|\mathbb{E}\left(\tilde{\rho}_{klr}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}_{klr}^{m}\right)\right| \lesssim 1 - e^{-\frac{2K\bar{a}\Delta m}{M}},$$

which implies that

$$\|\mathbb{E}\left(\tilde{\rho}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}^{m}\right)\|_\infty \lesssim 1 - e^{-\frac{2K\bar{a}\Delta m}{M}}.$$

Hence, Lemma E.12 is proven.

E.5.14 Proof of Lemma E.13

Fix any $m \in \{0,1,\ldots,3M^2\}$. It follows from Lemmas E.4 and E.17 that for any $\omega^m \in \Omega^m \backslash V^m$, $\tilde{\rho}^m(\omega^m) \simeq U_1^m(\mathbb{E}\tilde{\rho}^0) \simeq \mathbb{E}\tilde{\rho}^m$. Since f is continuous on the compact set $\tilde{\Delta}$, ${}^*f\left(\tilde{\rho}^m(\omega^m)\right) \simeq {}^*f\left(\mathbb{E}\tilde{\rho}^m\right)$ for any $\omega^m \in \Omega^m \backslash V^m$.

Since f is continuous on the compact set $\tilde{\Delta}$, it is bounded. Then, by Lemma E.17, we have

$$\begin{aligned}
& |\mathbb{E}^*f(\tilde{\rho}^m) - {}^*f(\mathbb{E}\tilde{\rho}^m)| \\
= & \left|\int_{\Omega^m}\left({}^*f(\tilde{\rho}^m) - {}^*f(\mathbb{E}\tilde{\rho}^m)\right)dQ^m\right| \\
= & \left|\int_{\Omega^m\backslash V^m}\left({}^*f(\tilde{\rho}^m) - {}^*f(\mathbb{E}\tilde{\rho}^m)\right)dQ^m\right| + \left|\int_{V^m}\left({}^*f(\tilde{\rho}^m) - {}^*f(\mathbb{E}\tilde{\rho}^m)\right)dQ^m\right| \\
\simeq & \left|\int_{\Omega^m\backslash V^m}\left({}^*f(\tilde{\rho}^m) - {}^*f(\mathbb{E}\tilde{\rho}^m)\right)dQ^m\right| \simeq 0.
\end{aligned}$$

Fix any $\Delta m \in \{0,\ldots,3M^2\}$ such that $m+\Delta m \leq 3M^2$, and $\frac{\Delta m}{M}$ is infinitesimal. The above equation implies that

$$\mathbb{E}^* f\left(\tilde{\rho}^{m+\Delta m}\right) - \mathbb{E}^* f\left(\tilde{\rho}^{m}\right) \simeq {}^*f\left(\mathbb{E}\tilde{\rho}^{m+\Delta m}\right) - {}^*f\left(\mathbb{E}\tilde{\rho}^{m}\right).$$

By Lemma E.12, $||\mathbb{E}\tilde{\rho}^{m+\Delta m} - \mathbb{E}\tilde{\rho}^{m}||_\infty$ is infinitesimal. Since f is continuous on the compact set $\tilde{\Delta}$, we know that ${}^*f\left(\mathbb{E}\tilde{\rho}^{m+\Delta m}\right) - {}^*f\left(\mathbb{E}\tilde{\rho}^{m}\right)$ is infinitesimal, which implies $\mathbb{E}^* f\left(\tilde{\rho}^{m+\Delta m}\right) - \mathbb{E}^* f\left(\tilde{\rho}^{m}\right)$ is also infinitesimal. Hence, the lemma is proven.

F Proofs of Results in Section 2

The continuous-time random matching model with immediate break-up described in Section 2 can be treated as a special case of the model of random matching with enduring partnership in Appendix A by taking the enduring probabilities ξ_{kl} to be 0 for any $k, l \in S$. It is natural to define other parameters for the random matching model with enduring partnership. For any $k, l \in S$, extend θ_{kl} from its domain Δ to $\hat{\Delta}$ by letting $\theta_{kl}(\hat{p}) = \theta_{kl}(p)$, $\sigma_{kl} = \delta_{(k,l)}$, $\vartheta_{kl} = 1$, and η_{kl} and ς_{kl} remain the same.

In Section 2, at any given time t, any agent i has no partner with probability one. It means that the process g is the constant J with probability one. Hence, we can obtain the properties and results on the type process α in Section 2 directly from the corresponding properties and results on the extended type process (α, g) in Appendix A. In other words, Properties 1, 2, 3 and 4 of the random matching model $\mathbb{D}$ in Section 2, are special cases of the corresponding properties for the random matching model $\hat{\mathbb{D}}$ in Appendix A, while Parts (1), (2), (4), (5) of Theorem 2.1 are direct implications of Theorems A.1 and A.2. It remains to verify Property 5 of the dynamical matching model $\mathbb{D}$ and to prove Theorem 2.1 (3).

To check Property 5 of the dynamical matching model $\mathbb{D}$, we shall need to study the properties of agents' last partners. Suppose that the hyperfinite dynamic matching model described in Section E.2 has been constructed. Fix any agent $i \in I$ and any standard natural number $n \in \mathbb{N}$. For any $\omega \in \Omega$, let $\hat{d}_i^n(\omega)$ be the n-th matching period of agent i. That is, $\hat{d}_i^n(\omega)$-th period is the period when agent i's n-th partner arrives. If the total number of matching periods is less than n, we let $\hat{d}_i^n(\omega)$ be J); otherwise $1 \leq \hat{d}_i^n(\omega) \leq M^2$. The real time for the n-th matching of agent i is defined by

$$d_i^n(\omega) = \begin{cases} \mathrm{st}\left(\frac{\hat{d}_i^n(\omega)}{M}\right) & \text{if } \hat{d}_i^n(\omega) \neq J \text{ and } \frac{\hat{d}_i^n(\omega)}{M} \text{ is limited} \\ \infty & \text{if } \hat{d}_i^n(\omega) = J \text{ or } \frac{\hat{d}_i^n(\omega)}{M} \text{ is unlimited} \end{cases}$$

Note that $[tM]$ denotes the integer part of tM. For any $\omega \in \Omega$ and $t \in \mathbb{R}_+$, agent i's last matching period before time t is define by

$$\hat{\tau}_i^t(\omega) = \max\{n' \in \mathbb{T}_0 : \hat{\pi}_i^{3n'-1}(\omega^{3n'-1}) \neq i \text{ and } n' \leq [tM]\}$$

when the set $\{n' \in \mathbb{T}_0 : \hat{\pi}_i^{3n'-1}(\omega^{3n'-1}) \neq i \text{ and } n' \leq [tM]\}$ is nonempty; otherwise, $\hat{\tau}_i^t(\omega)$ is defined to be J.

Next, we define the process φ for agents' last partners. Fix any $i \in I$. For any $\omega \in \Omega$ and $t \in \mathbb{R}_+$, let

$$\varphi_i'(\omega, t) = \begin{cases} \hat{\pi}_i^{3\hat{\tau}_i^t(\omega)-1}(\omega) & \text{if } \hat{\tau}_i^t(\omega) \neq J \\ i & \text{if } \hat{\tau}_i^t(\omega) = J. \end{cases}$$

Then, $\varphi_i'(\omega, t)$ is agent i's last partner up to the $[tM]$-th period. Since $\hat{\tau}_i^t(\omega) = J$ means that agent i has not been matched up to the $[tM]$-th period, agent i's last partner is simply defined to be herself in this case. Note that $\varphi_i'(t)$ may not be RCLL. Recall that the set

$$A_i = \{\omega' \in \Omega : \hat{X}_i^m(\omega') \text{ is finite for any positive hyperinteger } m \text{ such that } \frac{m}{M} \text{ is finite}\}$$

has probability one by Lemma E.11. For any $\omega \notin A_i$ and $t \in \mathbb{R}_+$, define $\varphi_i(\omega, t)$ to be i; it is obvious that $\varphi_i(\omega, t)$ is RCLL in t. Fix any $\omega \in A_i$ and $t \in \mathbb{R}_+$. By the definition of A_i, we know that agent i matches finitely many times up to the $[(t+1)M]$-th period. For any t' in the real time interval $[0, t+1]$, since $\varphi_i'(\omega, t')$ is agent i's last partner up to the $[t'M]$-th period, we know that there exists $j, j' \in I$ and $\epsilon \in \mathbb{R}_{++}$ such that $\varphi_i'(\omega, t')$ is j on $(t, t+\epsilon)$ and j' on $(t-\epsilon, t)$. Define $\varphi_i(\omega, t)$ to be j. For any $t' \in (t, t+\epsilon)$, we know that $\varphi_i'(\omega, t'')$ is j for $t'' \in (t', t+\epsilon')$. According to the definition of φ, we obtain that $\varphi_i(\omega, t')$ is still j for $t' \in (t, t+\epsilon)$. Therefore, $\varphi_i(\omega, t')$ is right continuous at real time t. Similarly, for any $t' \in (t-\epsilon, t)$, $\varphi_i'(\omega, t'')$ is j' for $t'' \in (t', t)$. The definition of φ implies that $\varphi_i(\omega, t')$ is j' for $t' \in (t-\epsilon, t)$. Therefore, the left limit of $\varphi_i(\omega, t')$ exists at time t. For simplicity, let $\varphi_i(\omega, \infty) = i$ for any $\omega \in \Omega$.

For any $i \in I$, let $B(i) = \{\omega \in \Omega : \varphi(\varphi(i, \omega, d_i^n(\omega)), \omega, d_i^n(\omega)) = i\}$. It is clear that $\{\omega \in \Omega : d_i^n(\omega) = \infty\} \subseteq B(i)$. We are going to show that $P(B(i)) = 1$. For any $N \in \mathbb{T}_0$, let

$$B_N(i) = \{\omega \in \Omega : \quad \hat{d}_i^n(\omega) \neq J,\ \hat{\pi}^{3n'-1}(i, \omega) = i,\ \hat{\pi}^{3n'-1}\left(\hat{\pi}^{3\hat{d}_i^n(\omega)-1}(i, \omega), \omega\right) = \hat{\pi}^{3\hat{d}_i^n-1}(i, \omega)$$
$$\text{for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n(\omega) < n' \leq \hat{d}_i^n(\omega) + N\}.$$

Then $B_N(i)$ is the event that agent i and her n-th partner do not match in period $\hat{d}_i^n + 1, \hat{d}_i^n + 2, \ldots, \hat{d}_i^n + N$. The following lemma shows a relationship between $B_N(i)$ and $B(i)$.

Lemma F.1. *For any $N \in \mathbb{T}_0$ such that $\frac{N}{M}$ is limited and* $\mathrm{st}\left(\frac{N}{M}\right) > 0$*, $B_N(i) \subseteq B(i)$.*

Proof. Fix any $N \in \mathbb{T}_0$ such that $\frac{N}{M}$ is limited and $\mathrm{st}\left(\frac{N}{M}\right) > 0$, and any $\omega \in B_N(i)$. If $\omega \notin A_i$, by the definition of φ, $\varphi(i, \omega, d_i^n(\omega)) = i$. It is clear that

$$\varphi(\varphi(i, \omega, d_i^n(\omega)), \omega, d_i^n(\omega)) = i.$$

If $\frac{\hat{d}_i^n(\omega)}{M}$ is unlimited, we have $d_i^n(\omega) = \infty$. By the definition of φ, we have

$$\varphi(\varphi(i, \omega, d_i^n(\omega)), \omega, d_i^n(\omega)) = i.$$

Next, we consider the case when $\omega \in A_i$ and $\frac{\hat{d}_i^n(\omega)}{M}$ is limited. Since $\omega \in B_N(i)$, we have $\hat{d}_i^n(\omega) \neq J, \quad \hat{\pi}^{3n'-1}(i,\omega) = i$,

$$\hat{\pi}^{3n'-1}\left(\hat{\pi}^{3\hat{d}_i^n(\omega)-1}(i,\omega), \omega\right) = \hat{\pi}^{3\hat{d}_i^n(\omega)-1}(i,\omega)$$

for any $n' \in \mathbb{T}_0$ such that $\hat{d}_i^n < n' \leq \hat{d}_i^n + N$. Therefore, for any $t' \in \left(d_i^n(\omega), d_i^n(\omega) + \mathrm{st}\left(\frac{N}{M}\right)\right)$, $\varphi'(i,\omega,t') = \hat{\pi}^{3\hat{d}_i^n(\omega)-1}(i,\omega)$ and $\varphi'(\hat{\pi}^{3\hat{d}_i^n(\omega)-1}(i,\omega), \omega, t') = i$. By the definition of φ, we have $\varphi(i,\omega,d_i^n(\omega)) = \hat{\pi}^{3\hat{d}_i^n(\omega)-1}(i,\omega)$ and $\varphi(\hat{\pi}^{3\hat{d}_i^n(\omega)-1}(i), \omega, d_i^n) = i$, which implies

$$\varphi\left(\varphi\left(i,\omega,d_i^n(\omega)\right), \omega, d_i^n(\omega)\right) = i.$$

Hence, we have $\omega \in B(i)$. By the arbitrary choice of ω in $B_N(i)$, we know that $B_N(i)$ is a subset of $B(i)$. ■

The following lemma verifies Property 5 of the dynamical matching model $\mathbb{D}$ in Section 2 , which says that for any agent i, her partner's partner at her n-th matching time d_i^n is agent i with probability one.

Lemma F.2. *For any $i \in I$ and $n \in \mathbb{N}$, we have $\varphi\left(\varphi\left(i, d_i^n\right), d_i^n\right) = i$ P-almost surely.*

Proof. Fix any $i \in I$, $n \in \mathbb{N}$, and any $N \in \mathbb{T}_0$ such that $\frac{N}{M}$ is limited and $\mathrm{st}\left(\frac{N}{M}\right) > 0$. It follows from the definition of $B_N(i)$ that

$$\begin{aligned}
P_0\left(B_N(i)\right) &= \sum_{j\in I}\sum_{r\in\mathbb{T}_0} P_0\Big(\hat{d}_i^n = r, \hat{\pi}^{3r-1}(i) = j, \hat{\pi}^{3n'-1}(i) = i, \hat{\pi}^{3n'-1}(j) = j \\
&\qquad\qquad \text{for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n < n' \leq \hat{d}_i^n + N\Big) \\
&= \sum_{j\in I}\sum_{r\in\mathbb{T}_0} P_0\left(\hat{d}_i^n = r, \hat{\pi}^{3r-1}(i) = j\right) \\
&\qquad P_0\Big(\hat{\pi}^{3n'-1}(i) = i, \hat{\pi}^{3n'-1}(j) = j \\
&\qquad\qquad \text{for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n < n' \leq \hat{d}_i^n + N \,\Big|\, \hat{d}_i^n = r, \hat{\pi}^{3r-1}(i) = j\Big).
\end{aligned}$$

It follows from Lemma E.9 that

$$P_0\left(\hat{\pi}^{3n'-1}(i) = i \text{ for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n < n' \leq \hat{d}_i^n + N \,\Big|\, \hat{d}_i^n = r, \hat{\pi}^{3r-1}(i) = j\right) \gtrsim e^{-\frac{6K\bar{a}N}{M}},$$

$$P_0\left(\hat{\pi}^{3n'-1}(j) = j \text{ for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n < n' \leq \hat{d}_i^n + N \,\Big|\, \hat{d}_i^n = r, \hat{\pi}^{3r-1}(i) = j\right) \gtrsim e^{-\frac{6K\bar{a}N}{M}}.$$

Then, we can obtain that

$$\begin{aligned}
& P_0\left(\hat{\pi}^{3n'-1}(i)=i, \hat{\pi}^{3n'-1}(j,\omega)=j\right. \\
& \qquad \left.\text{for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n < n' \le \hat{d}_i^n + N \,\middle|\, \hat{d}_i^n = r, \hat{\pi}^{3r-1}(i)=j\right) \\
\ge\; & P_0\left(\hat{\pi}^{3n'-1}(i)=i \text{ for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n < n' \le \hat{d}_i^n + N \,\middle|\, \hat{d}_i^n = r, \hat{\pi}^{3r-1}(i)=j\right) \\
& + P_0\left(\hat{\pi}^{3n'-1}(j)=j \text{ for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n < n' \le \hat{d}_i^n + N \,\middle|\, \hat{d}_i^n = r, \hat{\pi}^{3r-1}(i)=j\right) - 1 \\
\gtrsim\; & 2e^{-\frac{6K\bar{a}N}{M}} - 1.
\end{aligned}$$

Therefore, we can derive that

$$\begin{aligned}
P_0\left(B_N(i)\right) &\gtrsim \sum_{j\in I}\sum_{r\in\mathbb{T}_0} P_0\left(\hat{d}_i^n = r, \hat{\pi}^{3r-1}(i)=j\right)\left(2e^{-\frac{6K\bar{a}N}{M}}-1\right) \\
&= P_0\left(\hat{d}_i^n \ne J\right)\left(2e^{-\frac{6K\bar{a}N}{M}}-1\right).
\end{aligned}$$

It is clear that

$$\begin{aligned}
P_0\left(B_N(i) \cup \left(\hat{d}_i^n = J\right)\right) &= P_0\left(B_N(i)\right) + P_0\left(\hat{d}_i^n = J\right) \\
&\gtrsim P_0\left(\hat{d}_i^n \ne J\right)\left(2e^{-\frac{6K\bar{a}N}{M}}-1\right) + P_0\left(\hat{d}_i^n = J\right). \qquad \text{(F.1)}
\end{aligned}$$

Since $\left(\hat{d}_i^n = J\right) \subseteq \{\omega \in \Omega : d_i^n(\omega) = \infty\}$ and $\{\omega \in \Omega : d_i^n(\omega) = \infty\} \subseteq B(i)$, we know that $\left(\hat{d}_i^n = J\right) \subseteq B(i)$. Hence, Lemma F.1 implies that $B_N(i) \cup \left(\hat{d}_i^n = J\right) \subseteq B(i)$. Therefore, by Equation (F.1), we obtain that

$$P_0\left(B(i)\right) \gtrsim P_0\left(\hat{d}_i^n \ne J\right)\left(2e^{-\frac{6K\bar{a}N}{M}}-1\right) + P_0\left(\hat{d}_i^n = J\right). \qquad \text{(F.2)}$$

If $\operatorname{st}\left(\frac{N}{M}\right) \to 0$, then $\left(2e^{-\frac{6K\bar{a}N}{M}}-1\right) \to 1$, which implies that the right hand side of Equation (F.2) tends to $P_0\left(\hat{d}_i^n \ne J\right) + P_0\left(\hat{d}_i^n = J\right) = 1$. Therefore, we can claim that $P\left(B(i)\right) = 1$, which implies that $\varphi\left(\varphi\left(i, d_i^n\right), d_i^n\right) = i$ P-almost surely. ■

For any $k, l \in S$, and $1 \le m \le 3M^2$, the number of matches by agent i up to the m-step, when of type k, to an agent of type l is defined to be

$$\hat{N}_{ikl}^m(\omega) = |\{n \in \mathbb{T}_0 : \hat{\alpha}_i^{3n-1}(\omega) = k, \hat{g}_i^{3n-2}(\omega) \ne \hat{g}_i^{3n-1}(\omega) = l, 3n-1 \le m\}|.$$

The following defines the counting process for the number of matches by agent i, when of type k, to an agent of type l:

$$N_{ikl}(\omega,t) = \begin{cases} \hat{N}_{ikl}^{3[tM]}(\omega) & \text{if } \omega \in A_i \\ 0 & \text{if } \omega \notin A_i. \end{cases}$$

Recall that $\Theta_{kl}(t) = \int_I N_{ikl}(\omega,t)\, d\lambda(i)$ denotes the cumulative total quantity of matches of agents of any given type k with agents of another given type l, by time t.

Finally, we are ready to prove Part (3) of Theorem 2.1.

Proof of Theorem 2.1 (3): Fix any $t \in \mathbb{R}_+$, $k, l \in S$, and non-negative standard integers n and n'. For any $i, j \in I$ with $i \neq j$ and $P_0\left(\hat{\pi}_i^0 = j\right) \leq \frac{1}{\hat{M}^{\frac{1}{5}}}$, it is clear that the events $\left(\hat{N}_{ikl}^{3[tM]} = n\right)$ and $\left(\hat{N}_{jkl}^{3[tM]} = n'\right)$ are in $\mathcal{F}_i^{3[tM]}$ and $\mathcal{F}_j^{3[tM]}$ respectively. It follows from Lemma E.8 that

$$P\left(\hat{N}_{ikl}^{3[tM]} = n, \hat{N}_{jkl}^{3[tM]} = n'\right) = P\left(\hat{N}_{ikl}^{3[tM]} = n\right) P\left(\hat{N}_{jkl}^{3[tM]} = n'\right).$$

Since A_i has probability one, it is obvious that the events $\left(\hat{N}_{ikl}(t) = n\right)$ and $\left(\hat{N}_{jkl}(t) = n'\right)$ are independent. By the arbitrary choices of n and n', we know that the random variables $N_{ikl}(t)$ and $N_{jkl}(t)$ are independent.

For any $i, j \in I$ and $1 \leq m \leq 3M^2$, let $F_{ij}^m = \{\omega \in \Omega : \hat{\pi}_i^m(\omega) = j\}$. It is clear that for any $i \in I$, $F_{ij}^m \cap F_{ij'}^m = \emptyset$ if $j \neq j'$ in I. For any $i \in I$, let $F_i^m = \{j \in I : P_0(F_{ij}^m) \geq \frac{1}{\hat{M}^{\frac{1}{5}}}\}$. Then, $\lambda_0(F_i^m) \leq \frac{1}{\hat{M}^{\frac{4}{5}}}$. For any $i \in I$, and any j not in the λ-null set F_i^m, the random variables $N_{ikl}(t)$ and $N_{jkl}(t)$ are independent. By the exact law of large numbers (Corollary 2.10 in Sun (2006)), we have $\Theta_{kl}(\omega, t) = \mathbb{E}\Theta_{kl}(t)$ for P-almost all $\omega \in \Omega$.

Fix any $\Delta t \in \mathbb{R}_+$. Let n and $n+\Delta n$ be the integer part of tM and $(t+\Delta t)M$ respectively. For any $k, l \in S$, it follows from the Fubini property and the definition of Θ_{kl} that

$$\begin{aligned}
& \frac{1}{\Delta t}\left(\mathbb{E}\Theta_{kl}(t+\Delta t) - \mathbb{E}\Theta_{kl}(t)\right) \\
= & \frac{1}{\Delta t}\int_I \left(\mathbb{E}\hat{N}_{ikl}(t+\Delta t) - \mathbb{E}\hat{N}_{ikl}(t)\right) d\lambda \\
\simeq & \frac{1}{\Delta t}\int_I \left(\mathbb{E}\hat{N}_{ikl}^{3(n+\Delta n)} - \mathbb{E}\hat{N}_{ikl}^{3n}\right) d\lambda_0 \\
= & \frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_I \mathbb{E}\left(\hat{N}_{ikl}^{3n'} - \hat{N}_{ikl}^{3(n'-1)}\right) d\lambda_0 \\
= & \frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_I \mathbb{E}\left(\hat{N}_{ikl}^{3n'-1} - \hat{N}_{ikl}^{3n'-2}\right) d\lambda_0 \\
= & \frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_I P_0\left(\hat{N}_{ikl}^{3n'-1} - \hat{N}_{ikl}^{3n'-2} = 1\right) d\lambda_0 \\
= & \frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_I\int_{\Omega^{3n'-2}} Q_{3n'-1}^{\omega^{3n'-2}}\left(\hat{N}_{ikl}^{3n'-1} - \hat{N}_{ikl}^{3n'-2} = 1\right) dQ^{3n'-2} d\lambda_0 \\
= & \frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_{\Omega^{3n'-2}}\int_I Q_{3n'-1}^{\omega^{3n'-2}}\left(\hat{N}_{ikl}^{3n'-1} - \hat{N}_{ikl}^{3n'-2} = 1\right) d\lambda_0 dQ^{3n'-2}. \qquad \text{(F.3)}
\end{aligned}$$

For any $i \in I$ and $\omega^{3n'-2} \in \Omega^{3n'-2} \setminus V^{3n'-2}$, if $\hat{\beta}_i^{3n'-2}(\omega^{3n'-2}) = (k, J)$ (i.e., $\tilde{\beta}_i^{3n'-2}(\omega^{3n'-2}) =$

$(k, J, 1))$, then Lemma E.19 implies that

$$Q^{\omega^{3n'-2}}_{3n'-1}\left(\hat{N}^{3n'-1}_{ikl} - \hat{N}^{3n'-2}_{ikl} = 1\right) = Q^{\omega^{3n'-2}}_{3n'-1}\left(\hat{g}^{3n'-1}_{i} = l\right) \simeq \hat{q}_{kl}\left(\tilde{\rho}^{3n'-2}\left(\omega^{3n'-2}\right)\right);$$

if $\hat{\beta}^{3n'-2}_i(\omega^{3n'-2}) \neq (k, J)$, then the construction of the hyperfinite dynamic matching model implies that $Q^{\omega^{3n'-2}}_{3n'-1}\left(\hat{N}^{3n'-1}_{ikl} - \hat{N}^{3n'-2}_{ikl} = 1\right) = 0$. By Equation (F.3) and Lemma E.17, we can obtain that

$$\begin{aligned}
& \frac{1}{\Delta t}\left(\mathbb{E}\Theta_{kl}(t+\Delta t) - \mathbb{E}\Theta_{kl}(t)\right) \\
\simeq\ & \frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_{\Omega^{3n'-2}\setminus V^{3n'-2}}\int_I Q^{\omega^{3n'-2}}_{3n'-1}\left(\hat{N}^{3n'-1}_{ikl} - \hat{N}^{3n'-2}_{ikl} = 1\right) d\lambda_0 dQ^{3n'-2} \\
\simeq\ & \frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_{\Omega^{3n'-2}\setminus V^{3n'-2}} \hat{\rho}^{3n'-2}_{kJ}\left(\omega^{3n'-2}\right)\hat{q}_{kl}\left(\tilde{\rho}^{3n'-2}\left(\omega^{3n'-2}\right)\right) dQ^{3n'-2} \\
\simeq\ & \frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_{\Omega^{3n'-2}} \hat{\rho}^{3n'-2}_{kJ}\left(\omega^{3n'-2}\right)\hat{q}_{kl}\left(\tilde{\rho}^{3n'-2}\left(\omega^{3n'-2}\right)\right) dQ^{3n'-2} \\
=\ & \frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\mathbb{E}\left[\hat{\rho}^{3n'-2}_{kJ}\left(\omega^{3n'-2}\right)\hat{q}_{kl}\left(\tilde{\rho}^{3n'-2}\left(\omega^{3n'-2}\right)\right)\right] \\
\simeq\ & \frac{1}{\Delta n}\sum_{n'=n+1}^{n+\Delta n}\mathbb{E}\left[\hat{\rho}^{3n'-2}_{kJ}\left(\omega^{3n'-2}\right){}^*\theta_{kl}\left(\hat{\rho}^{3n'-2}\left(\omega^{3n'-2}\right)\right)\right].
\end{aligned} \tag{F.4}$$

Fix any $\Delta n' \in \mathbb{T}_0$ such that $\frac{\Delta n'}{M}$ is infinitesimal. For any $\tilde{p} \in \tilde{\Delta}$, let $\hat{p}$ be the marginal probability distribution of $\tilde{p}$ on $\hat{\Delta}$. Then, it is clear that the function $\hat{p}_{kJ}\theta_{kl}(\hat{p})$ is continuous on $\tilde{\Delta}$. For any n' between $n+1$ and $n+\Delta n'$, Lemma E.13 implies that

$$\mathbb{E}\left[\hat{\rho}^{3n'-2}_{kJ}\left(\omega^{3n'-2}\right){}^*\theta_{kl}\left(\hat{\rho}^{3n'-2}\left(\omega^{3n'-2}\right)\right)\right] \simeq \mathbb{E}\left[\hat{\rho}^{3n}_{kJ}\left(\omega^{3n}\right){}^*\theta_{kl}\left(\hat{\rho}^{3n}\left(\omega^{3n}\right)\right)\right].$$

Thus, we can obtain that

$$\begin{aligned}
& \frac{1}{\Delta n'}\sum_{n'=n+1}^{n+\Delta n'}\mathbb{E}\left[\hat{\rho}^{3n'-2}_{kJ}\left(\omega^{3n'-2}\right){}^*\theta_{kl}\left(\hat{\rho}^{3n'-2}\left(\omega^{3n'-2}\right)\right)\right] \\
\simeq\ & \frac{1}{\Delta n'}\sum_{n'=n+1}^{n+\Delta n'}\mathbb{E}\left[\hat{\rho}^{3n}_{kJ}\left(\omega^{3n}\right){}^*\theta_{kl}\left(\hat{\rho}^{3n}\left(\omega^{3n}\right)\right)\right] \\
=\ & \mathbb{E}\left[\hat{\rho}^{3n}_{kJ}\left(\omega^{3n}\right){}^*\theta_{kl}\left(\hat{\rho}^{3n}\left(\omega^{3n}\right)\right)\right].
\end{aligned} \tag{F.5}$$

Lemma E.17 implies that for P-almost all $\omega^{3n} \in \Omega^{3n}$,

$$\tilde{\rho}^{3n}\left(\omega^{3n}\right) \simeq U^{3n}_1\left(\mathbb{E}\tilde{\rho}^0\right) \simeq \mathbb{E}\tilde{\rho}^{3n},$$

which implies that $\hat{\rho}^{3n}\left(\omega^{3n}\right) \simeq \mathbb{E}\hat{\rho}^{3n}$ for P-almost all $\omega^{3n} \in \Omega^{3n}$. Since $\hat{p}_{kJ}\theta_{kl}(\hat{p})$ is also continuous on $\hat{\Delta}$, Equation (F.5) implies that

$$\frac{1}{\Delta n'}\sum_{n'=n+1}^{n+\Delta n'} \mathbb{E}\left[\hat{\rho}_{kJ}^{3n'-2}\left(\omega^{3n'-2}\right){}^{*}\theta_{kl}\left(\hat{\rho}^{3n'-2}\left(\omega^{3n'-2}\right)\right)\right] \simeq \left(\mathbb{E}\hat{\rho}_{kJ}^{3n}\right){}^{*}\theta_{kl}\left(\mathbb{E}\hat{\rho}^{3n}\right).$$

It follows from Equation (E.5) that $\check{p}(t) = \mathbb{E}\hat{p}(t) \simeq \mathbb{E}\hat{\rho}^{3n}$. Hence, we have

$$\begin{aligned} &\frac{1}{\Delta n'}\sum_{n'=n+1}^{n+\Delta n'} \mathbb{E}\left[\hat{\rho}_{kJ}^{3n'-2}\left(\omega^{3n'-2}\right){}^{*}\theta_{kl}\left(\hat{\rho}^{3n'-2}\left(\omega^{3n'-2}\right)\right)\right] \\ \simeq\ & \check{p}_{kJ}(t)\,\theta_{kl}\left(\check{p}(t)\right). \end{aligned}$$

Note that $\check{p}_{kJ}(t) = \bar{p}_k(t)$ and $\theta_{kl}\left(\check{p}(t)\right) = \theta_{kl}\left(\bar{p}(t)\right)$. By Lemma D.1 and Equation (F.4), we obtain that

$$\lim_{\Delta t \to 0}\frac{1}{\Delta t}\left(\mathbb{E}\Theta_{kl}(t+\Delta t) - \mathbb{E}\Theta_{kl}(t)\right) = \bar{p}_k(t)\theta_{kl}\left(\bar{p}(t)\right),$$

which implies that $\mathbb{E}\Theta_{kl}(t)$ is differentiable and $\frac{d\mathbb{E}\Theta_{kl}(t)}{dt} = \bar{p}_k(t)\theta_{kl}\left(\bar{p}(t)\right).$ ■

References

R. M. Anderson, A nonstandard representation for Brownian motion and Ito integration, *Israel Journal of Mathematics* **25** (1976), 15-46.

D. Duffie and Y. N. Sun, Existence of independent random matching, *Annals of Applied Probability* **17** (2007), 386–419.

D. Duffie, L. Qiao and Y. N. Sun, Dynamic directed random matching, *Journal of Economic Theory* **174** (2018), 124–183.

I. Karatzas and S. Shreve, *Brownian motion and stochastic calculus*, Second Edition, New York, Springer, 1991.

H. J. Keisler, Hyperfinite model theory, in *Logic Colloquium 76* (R. O. Gandy and J. M. E. Hyland eds.) North-Holland, Amsterdam, 1977.

P. A. Loeb, Conversion from nonstandard to standard measure spaces and applications in probability theory, *Transactions of the American Mathematical Society* **211** (1975), 113–122.

P. A. Loeb, *Real Analysis*, Birkhäuser, Switzerland, 2016.

P. A. Loeb and M. Wolff, eds. *Nonstandard Analysis for the Working Mathematician*, Second Edition, Springer, Berlin, 2015.

Y. N. Sun, A theory of hyperfinite processes: the complete removal of individual uncertainty via exact LLN, *Journal of Mathematical Economics* **29** (1998), 419–503.

Y. N. Sun, The exact law of large numbers via Fubini extension and characterization of insurable risks, *Journal of Economic Theory* **126** (2006), 31–69.