Continuous-Time Random Matching

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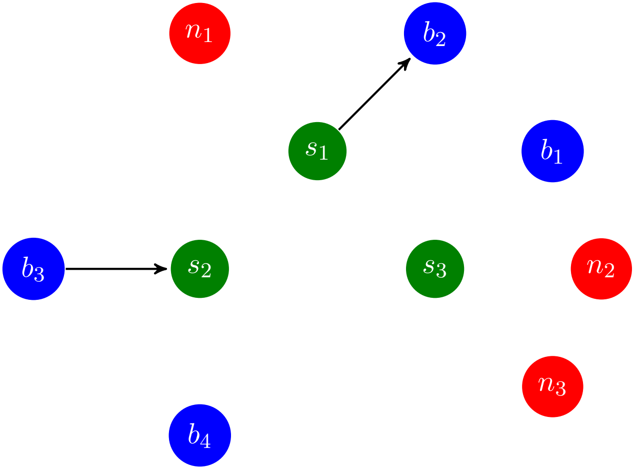
Random matching markets



Reliance on continuous-time independent matching

- Many researchers have appealed to a "law of large numbers" for continuous-time independent random matching among an atomless measure space of agents.
- Based on this, the fraction p_{tk} at time t of agents of any type k is presumed to evolve deterministically, almost surely, with naturally conjectured dynamics.
- The optimal strategy of each agent, given the path of p_t, is then much easier to solve than in a finite-agent model with random population dynamics [Boylan (1994)].
- Assuming this works, the equilibrium evolution of p_t can be analyzed.
- But there has been no result justifying the proposed application of the law of large numbers and conjectured dynamics. [Gilboa and Matsui (1992) have an example based on finitely-additive measures.]

Research areas relying on continuous-time random matching

- **Monetary theory.** Hellwig (1976), Diamond-Yellin (1990), Diamond (1993), Trejos-Wright (1995), Shi (1997), Zhou (1997), Postel-Vinay-Robin (2002), Moscarini (2005).
- **Labor markets.** Pissarides (1985), Hosios (1990), Mortensen-Pissarides (1994), Acemoglu-Shimer (1999), Shimer (2005), Flinn (2006), Kiyotaki-Lagos (2007).
- **Over-the-counter financial markets.** Duffie-Gârleanu-Pedersen (2005), Weill (2008), Vayanos-Wang (2007), Vayanos-Weill (2008), Weill (2008), Lagos-Rocheteau (2009), Hugonnier-Lester-Weill (2014), Lester, Rocheteau, Weill (2015), Üslü (2016).
- **Biology (genetics, molecular dynamics, epidemiology).** Hardy-Weinberg (1908), Crow-Kimura (1970), Eigen (1971), Shashahani (1978), Schuster-Sigmund (1983), Bomze (1983).
- **Game theory.** Mortensen (1982), Foster-Young (1990), Binmore-Samuelson (1999), Battalio-Samuelson-Van Huycjk (2001), Burdzy-Frankel-Pauzner (2001), Benaïm-Weibull (2003), Currarini-Jackson-Pin (2009), Hofbauer-Sandholm (2007).
- **Social learning.** Börgers (1997), Hopkins (1999), Duffie-Manso (2007), Duffie-Malamud-Manso (2009).

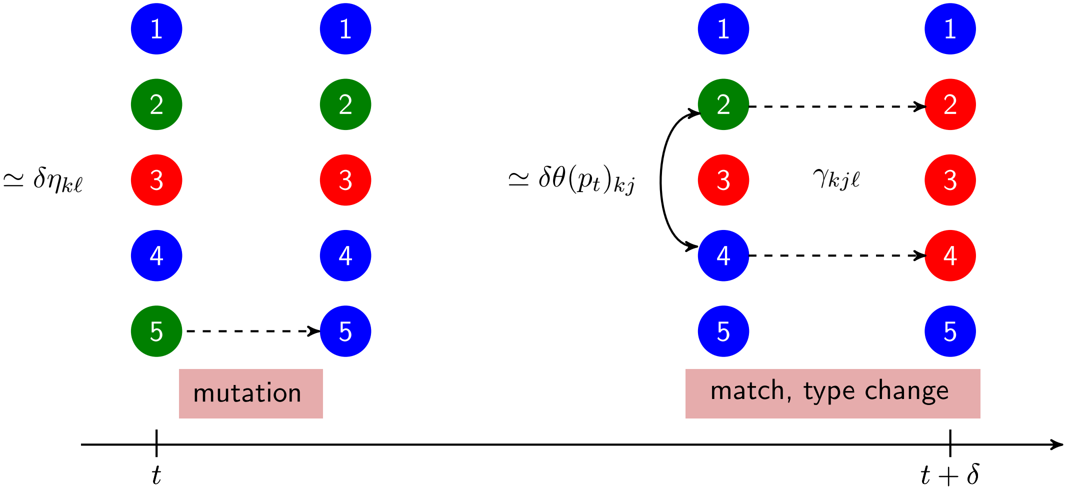
Parameters of the most basic model

- Type space $S = \{1, \ldots, K\}$.
- Initial cross-sectional distribution $p^0 \in \Delta(S)$ of agent types.
- For each pair (k, ℓ) of types:
  - Mutation intensity $\eta_{k\ell}$.
  - Matching intensity $\theta_{k\ell} : \Delta(S) \to \mathbb{R}_+$, continuous, satisfying the balance identity

$$p_k \theta_{k\ell}(p) = p_\ell \theta_{\ell k}(p).$$

  - Match-induced type probability distribution $\gamma_{k\ell} \in \Delta(S)$.

Mutation, matching, and match-induced type changes



Key solution processes

For a probability space $(\Omega, \mathcal{F}, P)$, atomless agent space $(I, \mathcal{I}, \lambda)$, and measurability on $I \times \Omega \times \mathbb{R}_+$ to be specified:

- Agent type $\alpha(i, \omega, t)$, for $\alpha : I \times \Omega \times \mathbb{R}_+ \to S$.
- Latest counterparty $\pi(i, \omega, t)$, for $\pi : I \times \Omega \times \mathbb{R}_+ \to I$.
- Cross-sectional type distribution $p : \Omega \times \mathbb{R}_+ \to \Delta(S)$. That is,

$$p(\omega, t)_k = \lambda(\{i \in I : \alpha(i, \omega, t) = k\})$$

  is the fraction of agents of type k.

Evolution of the cross-sectional distribution p_t of agent types

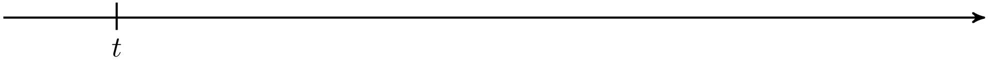
Existence of a model with independence conditions under which

$$\dot{p}_t = p_t R(p_t) \quad \text{almost surely,}$$

where $R(p_t)$ is also the agent-level Markov-chain infinitesimal generator:

$$R(p_t)_{k\ell} = \eta_{k\ell} + \sum_{j=1}^{K} \theta_{kj}(p_t)\gamma_{kj\ell}$$

$$R(p_t)_{kk} = -\sum_{\ell \neq k}^{K} R_{k\ell}(p_t).$$



A Fubini extension

A probability space $(I \times \Omega, \mathcal{W}, Q)$ extending the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \times P)$ is a **Fubini extension** if, for any real-valued integrable function f on $I \times \Omega$,

$$\int_{I \times \Omega} f \, dQ = \int_I \left(\int_\Omega f(i, \omega) \, dP(\omega) \right) \, d\lambda(i) = \int_\Omega \left(\int_I f(i, \omega) \, d\lambda(i) \right) dP(\omega).$$

Such a Fubini extension is denoted $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

Sun's exact law of large numbers

Suppose a measurable $f : (I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P) \to \mathbb{R}$ is essentially pairwise independent. Sun (2006) provides an existence result.

That is, for almost every agent i, the agent-level random variables $f(i) = f(i, \cdot)$ and $f(j)$ are independent for almost every agent j.

The cross-sectional distribution G of f at $x \in \mathbb{R}$ in state ω is $G(x, \omega) = \lambda(\{i : f(i, \omega) \leq x\})$.

Proposition (Sun, 2006)

For P-almost every ω,

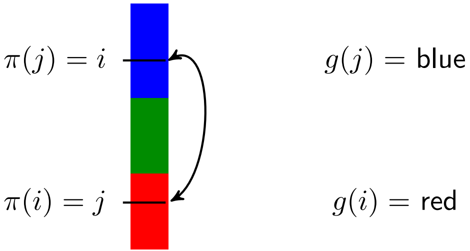
$$G(x, \omega) = \int_I P(f(i) \leq x) \, d\lambda(i).$$

In particular, if the probability distribution F of $f(i)$ does not depend on i, then the cross-sectional distribution G is equal to F almost surely.

Random matching

- A *random matching* $\pi : I \times \Omega \to I$ assigns a unique agent $\pi(i)$ to agent i, with $\pi(\pi(i)) = i$. If $\pi(i) = i$, agent i is not matched.

- Let $g(i) = \alpha(\pi(i))$ be the type of the agent to whom i is matched. (If i is not matched, let $g(i) = J$.)



Independent random matching with given probabilities

- Given: A measurable $\alpha : I \to S$ with distribution $p \in \Delta(S)$ and matching probabilities $(q_{k\ell})$ satisfying $p_k q_{k\ell} = p_\ell q_{\ell k}$.
- A random matching π is said to be independent with parameters (p, q) if the counterparty type g is $\mathcal{I} \boxtimes \mathcal{F}$-measurable and essentially pairwise independent with

$$P(g(i) = \ell) = q_{\alpha(i),\ell} \quad \lambda - a.e.$$

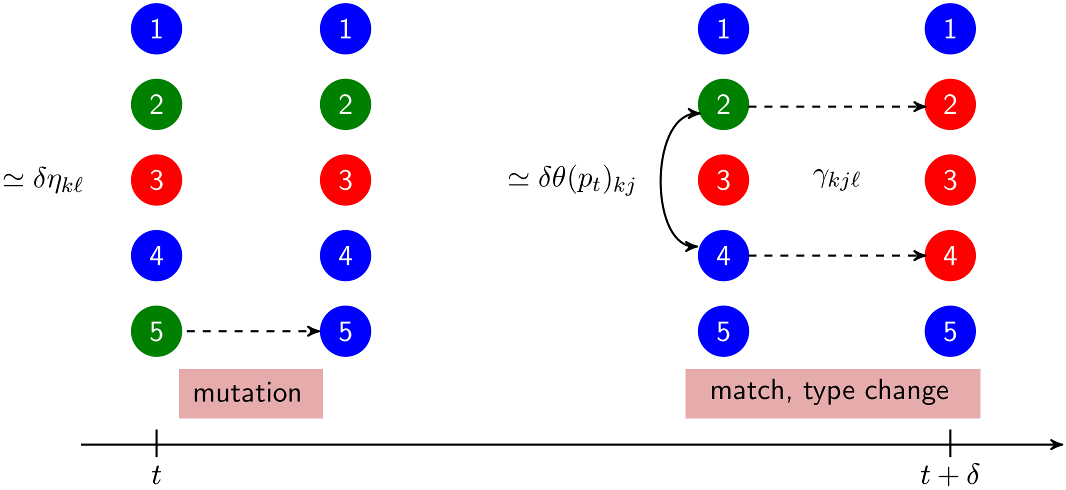
- In this case, the exact law of large numbers implies, for any k and ℓ, that

$$\lambda(\{i : \alpha(i) = k, g(i) = \ell\}) = p_k q_{k\ell} \quad a.s.$$

Proposition (Duffie, Qiao, and Sun, 2015)

For any given (p, q), there exists an independent matching π.

Loeb transfer of hyperfinite model



Continuous-time random matching

Theorem

For any parameters $(p^0, \eta, \theta, \gamma)$, there exists a continuous-time system (α, π) of agent type and last-counterparty processes such that

- *The agent type process α and last-counterparty type process $g = \alpha \circ \pi$ are measurable with respect to $(\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}(\mathbb{R}_+)$ and essentially pairwise independent.*
- *The cross-sectional type distribution process $\{p_t : t \geq 0\}$ satisfies $\dot{p}_t = p_t R(p_t)$ almost surely.*
- *The agent-level type processes $\{\alpha(i) : i \in I\}$ are Markov chains with infinitesimal generator $\{R(p_t) : t \geq 0\}$.*

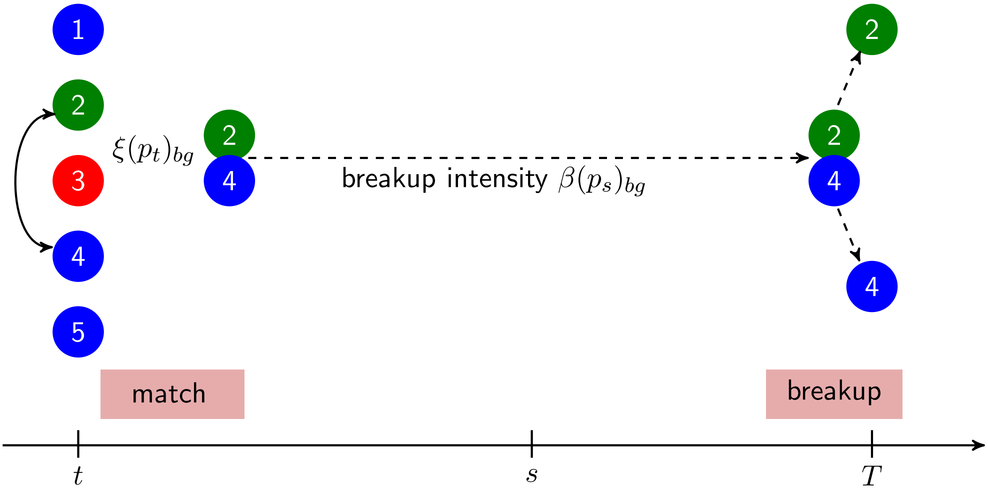
Stationary case

Proposition

For any (η, θ, γ)*, there is an initial type distribution* p^0 *such that the continuous-time system* (α, π) *associated with parameters* $(p^0, \eta, \theta, \gamma)$ *has constant cross-sectional type distribution* $p_t = p^0$*.*

If the initial agent types $\{\alpha_0(i) : i \in I\}$ *are essentially pairwise independent with probability distribution* p^0*, then the probability distribution of the agent type* $\alpha_t(i)$ *is also constant and equal to* p^0*, for* λ*-a.e. agent.*

With enduring match probability $\xi(p_t)$



Further generality

- When agents of types k and ℓ form an enduring match at time t, their new types are drawn with a given joint probability distribution $\sigma(p_t)_{k\ell} \in \Delta(S \times S)$.
- While enduringly matched, the mutation parameters of an agent may depend on both the agent's own type and the counterparty's type.
- Time-dependent parameters $(\eta_t, \theta_t, \gamma_t, \xi_t, \beta_t, \sigma_t)$, subject to continuity.
- The agent type space can be infinite, for example $S = \mathbb{Z}_+$ or $S = [0,1]^m$.