Continuous Time Random Matching*

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Abstract

We show the existence of independent random matching of a large population in a continuous-time dynamical system, where the matching intensities could be general non-negative jointly continuous functions on the space of type distributions and the time line. In particular, we construct a continuum of independent continuous-time Markov processes that is derived from random mutation, random matching, random type changing and random break up with time-dependent and distribution dependent parameters. It follows from the exact law of large numbers that the deterministic evolution of the agents' realized type distribution for such a continuous-time dynamical system can be determined by a system of differential equations. The results provide the first mathematical foundation for a large literature on continuous-time search-based models of labor markets, money, and over-the-counter markets for financial products.

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1 Introduction

Continuous-time independent random matching in a continuum population is a convenient and powerful modeling approach that is applied extensively in the economics literature.[1] Throughout, the literature exploits the idea that independence should lead, by the law of large numbers, to an almost-surely constant cross-sectional distribution of population types (or, in more general models, deterministic time-varying cross-sectional type distributions). Mathematical foundations for this result, however, have not been available. The existence of a model with independent matching has simply been assumed, along with the desired law of large numbers. The main aim of this paper is to provide the first treatment of the existence of continuous-time independent random matching in a continuum population.[2] In particular, we construct a joint agent-probability space on which there is a continuum of independent continuous-time Markov chains describing the types of each agent, respecting properties derived structurally from random mutation, pair-wise random matching between agents, random break-up of pairs, and random type changes induced by matching, and with general time-dependent and distribution dependent parameters. We state and prove an exact law of large numbers for this continuous-time dynamic system, by which there is an almost-sure constant (or, more generally, deterministically evolving) cross-sectional distribution of types.

For a finite space S of agent types, let $p(t)$ denote the cross-sectional distribution of types at time t. That is, $p_k(t)$ denotes the fraction of agents that are currently of type $k \in S$. A key primitive of the model is the intensity $\theta_{kl}(p(t), t)$ with which a specific agent of type k is matched to some agent of type l. More precisely, letting $\alpha(i, t)$ denote the stochastic type of agent i at time t, the cumulative number of matches of agent i with partners of type-l is a counting process with intensity $\theta_{\alpha(i,t),\, l}(p(t), t)$. By allowing these intensities to depend on the underlying cross-sectional type distribution p_t, the model accommodates the "matching-function" approach that has been popular in the labor literature. For technical reasons, we assume that the matching intensity $\theta_{kl}(p(t), t)$ depends continuously on $p(t)$ and t. The specified intensity function θ must of course satisfy the identity that the aggregate rate $p_k(t)\theta_{kl}(p(t), t)$ of matches of agents of type k to agents of type l is always equal to the aggregate rate $p_l(t)\theta_{lk}(p(t), t)$ of agents of type l matched to agents of type k.

[1]See, for example, [9], [10], [23], [41], [43], [44], and [52] in monetary theory; [1], [2], [20], [26], [28], [33], [35], [34], [38], [40], [45], and [46] in labor economics; [12], [13], [27], [29], [30], [49], [50], [51] in over-the-counter financial markets; [3], [8], and [24] in game theory; and [7], [25], [15], and [14] in social learning theory. The same sort of "ansatz" is applied without mathematical foundations in the natural sciences, including genetics and biological molecular dynamics, as explained by [6], [19], and [42].

[2] In [22], Gilboa and Matsui presented a particular matching model of two countable populations with a countable number of encounters in the time interval $[0, 1)$, where both the agent space $\mathbb{N}$ and the sample space are endowed with purely finitely additive measures.

In many practical applications, and in many labor-market models, once two agents are matched they may form a long-term relationship rather than immediately breaking up. For instance, when a worker and a firm meet, they form a job match with some probability. At this point, the worker may stop searching for new jobs until he or she becomes unemployed again. (See, for example, [37].) To this end, for any pair of agents of types k and l, we introduce a probability ξ_{kl} that an enduring partnership is formed at the time of the match. If formed, this partnership ends at a time whose arrival intensity ϑ_{kl} may change over time with changes in the cross-sectional distribution of agent types. The special case without enduring partnerships, more popular in monetary economics and financial models, is obtained by taking $\xi_{kl} = 0$.

Random-matching models often allow for the random mutation of agent types. We allow for independent random mutation, along the lines of [17], [18], and [16]. In some models, a matched pair of agents may have their respective types changed, possibly randomly, by the match or by the break-up. We allow this, and permit the post-break-up type probability distributions to depend on the current type distribution p_t.

All of the relevant parameters of the continuous-time dynamical system may be time-dependent. For the special time-homogeneous case, we obtain a stationary deterministic joint cross-sectional distribution of unmatched agent types and pairs of currently matched types.

Previous work ([17], [18] and [16]) provides related results for discrete-time Markov independent dynamical systems with random matching. This continuous-time setting involves an extra underlying layer of analysis based on methods of nonstandard analysis for hyperfinite dynamical systems.[3] This allows mutation, pairwise random matching, and random match-induced type changes to occur at successive infinitesimal time periods. The final results, however, are provided in the form of standard continuous-time processes (e.g., [39]) that are defined on the usual real time line.

The remainder of the paper is organized as follows. In Section 2, we provide some mathematical preliminaries. Section 3 defines an independent continuous-time dynamical system with random mutation, random partial matching, random break-up and random type changing. The main results on the existence and exact law of large numbers for a continuous-time dynamical system with enduring partnerships are presented in the first subsection of Section 4. When enduring partnerships are not possible, the notation and structure of the continuous-time dynamical system are much simpler. In order to allow easier access to this case, we present it separately in Subsection 4.2. In Section 5, we present three examples of applications to some main models in labor markets, over-the-counter financial markets, and monetary theory. The proofs of Theorems 1 and 2 are presented respectively in Sections 6 and 7.

[3]For the basics of nonstandard analysis, see the first three chapters of [32].

2 Mathematical Preliminaries

Let $(I, \mathcal{I}, \lambda)$ be an atomless probability space representing the space of agents. Let $(\Omega, \mathcal{F}, P)$ be a sample probability space. We fix a filtration $\{\mathcal{F}_t : t \geq 0\}$ of sub-σ-algebras of $\mathcal{F}$ satisfying the usual conditions ([39]). We may view $\mathcal{F}_t$ as the information available at time t.

We model a continuum of independent stochastic processes based on the index space $(I, \mathcal{I}, \lambda)$ and sample space $(\Omega, \mathcal{F}, P)$. As noted in Proposition 2.1 of [47], joint measurability with respect to the usual product probability space is in general incompatible with independence. A Fubini extension, defined as follows, is proposed in [47] to handle such a problem.

Definition 1 *A probability space $(I \times \Omega, \mathcal{W}, Q)$ extending the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ is said to be a* Fubini extension *of $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ if for any real-valued Q-integrable function g on $(I \times \Omega, \mathcal{W})$, the functions $g_i = g(i, \cdot)$ and $g_\omega = g(\cdot, \omega)$ are integrable respectively on $(\Omega, \mathcal{F}, P)$ for λ-almost all $i \in I$ and on $(I, \mathcal{I}, \lambda)$ for P-almost all $\omega \in \Omega$; and if, moreover, $\int_\Omega g_i \, dP$ and $\int_I g_\omega \, d\lambda$ are integrable, respectively, on $(I, \mathcal{I}, \lambda)$ and on $(\Omega, \mathcal{F}, P)$, with $\int_{I \times \Omega} g \, dQ = \int_I \left(\int_\Omega g_i \, dP \right) d\lambda = \int_\Omega \left(\int_I g_\omega \, d\lambda \right) dP$. To reflect the fact that the probability space $(I \times \Omega, \mathcal{W}, Q)$ has $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ as its marginal spaces, as required by the Fubini property, this space is denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.*

Definition 2 *Let $S = \{1, 2, \ldots, K\}$ be a finite set of agent types and J be a special type representing no matching.*

(i) A full matching ϕ is a one-to-one mapping from I onto I such that, for each $i \in I$, $\phi(i) \neq i$ and $\phi(\phi(i)) = i$.

(ii) A (partial) matching ψ is a mapping from I to I such that for some subset B of I, the restriction of ψ to B is a full matching on B, and $\psi(i) = i$ on $I \setminus B$. This means that agent i is matched with agent $\psi(i)$ for $i \in B$, whereas any agent i not in B is unmatched, that is $\psi(i) = i$.

(iii) A random matching π is a mapping from $I \times \Omega$ to I such that π_ω is a matching for each $\omega \in \Omega$.

3 The Model

Let $\hat{S} = S \times (S \cup \{J\})$ be the set of extended types. An agent with an extended type of the form (k, l) has type $k \in S$ and is currently matched to some agent of type l in S. If an agent's extended type is of the form (k, J), then the agent is "unmatched." The space $\hat{\Delta}$ of extended type distributions is the set of probability distributions $\hat{p}$ on $\hat{S}$ satisfying $\hat{p}(k, l) = \hat{p}(l, k)$ for all

k and l in S. A time is an element of $\mathbb{R}_+$, the set of non-negative real numbers, with its Borel σ-algebra $\mathcal{B}$.

The main objects of our model are $\alpha : I \times \Omega \times \mathbb{R}_+ \to S$, $\pi : I \times \Omega \times \mathbb{R}_+ \to I$, and $g : I \times \Omega \times \mathbb{R}_+ \to S \cup \{J\}$ specifying, for any agent i, state ω, and time t, the agent's type $\alpha(i, \omega, t)$, the agent's partner $\pi(i, \omega, t)$, and the partner's type $g(i, \omega, t)$. As usual, we let $\alpha(i)$ (or α_i) and $g(i)$ (or g_i) denote the type processes for agent i and her partners, and we let $\alpha(i, t)$ and $g(i, t)$ denote the random types of agent i and of the partner of agent i at time t, respectively. Our objective is to model the type processes α and g, as well as random matching between agents in a manner consistent with given parameters for independent random mutation, independent directed random matching among agents, independent random type changes at each matching and break-up, and independent random break-up for matched pairs.

The parameters of the model are the initial extended type distribution $\hat{p}^0 \in \hat{\Delta}$ and, for any k and l in S:

(i) A continuous mutation intensity function $\eta_{kl} : \hat{\Delta} \times \mathbb{R}_+ \to \mathbb{R}_+$, specifying the intensity $\eta_{kl}(\hat{p}, t)$, given the extended type distribution $\hat{p}$ at time t, with which any type-k agent mutates to type l.

(ii) A continuous matching intensity function $\theta_{kl} : \hat{\Delta} \times \mathbb{R}_+ \to \mathbb{R}_+$, specifying the intensity $\theta_{kl}(\hat{p}, t)$ at time t with which any type-k agent is matched with a type-l agent, if the cross-sectional agent extended type distribution at time t is $\hat{p} \in \hat{\Delta}$. This function satisfies the mass-balancing requirement $\hat{p}_{kJ} \cdot \theta_{kl}(\hat{p}, t) = \hat{p}_{lJ} \cdot \theta_{lk}(\hat{p}, t)$ that the total aggregate rate of matches of type-k agents to type l agents is of course equal to the aggregate rate of matches of type-l agents to type-k agents.

(iii) A continuous function $\xi_{kl} : \hat{\Delta} \times \mathbb{R}_+ \to [0, 1]$ specifying the probability $\xi_{kl}(\hat{p}, t)$ that a match between a type-k agent and a type-l agent causes a long-term relationship between the two agents after a match given the extended type distribution $\hat{p}$ at time t.

(iv) A continuous function $\sigma_{kl} : \hat{\Delta} \times \mathbb{R}_+ \to \mathcal{M}(S \times S)$ specifying the probability distribution $\sigma_{kl}(t)$ of the new types of a type-k agent and a type-l agent who have been matched, conditional on the event that the match causes an enduring relationship between them, given the extended type distribution $\hat{p}$ at time t.

(v) A continuous function $\varsigma_{kl} : \hat{\Delta} \times \mathbb{R}_+ \to \mathcal{M}(S)$ specifying the probability distribution $\varsigma_{kl}(\hat{p}, t)$ of the new type of a type-k agent who is matched with a type-l agent at time t, conditional on the event that there is no enduring relationship, and the match is dissolved immediately, given the extended type distribution $\hat{p}$ at time t.

(vi) A continuous function $\vartheta_{kl} : \hat{\Delta} \times \mathbb{R}_+ \to \mathbb{R}_+$ specifying the break-up rate $\vartheta_{kl}(\hat{p}, t)$ of the long term relationship between a type-k agent and a type-l agent, given the extended type distribution $\hat{p}$ at time t. In this case, if the type-k agent and type-l agent eventually break up at time s, they emerge with new types drawn from the probability distributions $\varsigma_{kl}(\hat{p}_s, s)$ and $\varsigma_{lk}(\hat{p}_s, s)$ respectively.

For given parameters $(\hat{p}^0, \eta, \theta, \xi, \sigma, \varsigma, \vartheta)$, a continuous-time dynamical system $\mathbb{D}$ with enduring partnership, if it exists, is a triple (α, π, g) defined by the properties:

1. $\alpha(i, \omega, t)$ and $g(i, \omega, t)$ are $(\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}$-measurable. The stochastic processes α_i and g_i have sample paths that are right-continuous with left limits (RCLL), a standard regularity property of stochastic processes, found for example, in [39]. For any $t \in \mathbb{R}_+$, $\pi(\cdot, \cdot, t)$ is a random matching on $I \times \Omega$. For any $i \in I$ and $t \in \mathbb{R}_+$,

$$g(i, \omega, t) = \begin{cases} \alpha(\pi(i, \omega, t)) & \text{if } \pi(i, \omega, t) \neq i \\ J & \text{if } \pi(i, \omega, t) = i \end{cases}$$

for P-almost all $\omega \in \Omega$, where $\alpha(J, \omega, t)$ is defined to be J.

2. The cross-sectional extended type distribution $\hat{p}(t)$ at time t is defined by

$$\hat{p}_{kl}(t) = \lambda(\{i : \alpha(i, t) = k, g(i, t) = l\}).$$

Let $\check{p}(t) = \mathbb{E}(\hat{p}(t))$. For any agent $i \in I$, the extended type process $(\alpha(i), g(i))$ of agent i is a continuous-time Markov chain in $S \times (S \cup \{J\})$ whose generator (transition-rate matrix) Q is defined at time t by:

$$Q^t_{(k_1 l_1)(k_2 l_2)} = \eta_{k_1 k_2}(\check{p}(t), t)\delta_{l_1}(l_2) + \eta_{l_1 l_2}(\check{p}(t), t)\delta_{k_1}(k_2), \quad (1)$$

$$Q^t_{(k_1 l_1)(k_2 J)} = \vartheta_{k_1 l_1}(\check{p}(t), t)[\varsigma_{(k_1, l_1)}(\check{p}(t), t)](k_2), \quad (2)$$

$$Q^t_{(k_1 J)(k_2 l_2)} = \sum_{l_1=1}^{K} \theta_{k_1 l_1}(\check{p}(t), t)\xi_{k_1 l_1}(t)[\sigma_{k_1 l_1}(\check{p}(t), t)](k_2, l_2), \quad (3)$$

$$Q^t_{(k_1 J)(k_2 J)} = \eta_{k_1 k_2}(\check{p}(t), t) + \sum_{l_1=1}^{K} \theta_{k_1 l_1}(\check{p}(t), t)(1 - \xi_{k_1 l_1}(\check{p}(t), t))[\varsigma_{k_1 l_1}(\check{p}(t), t)](k_2), \quad (4)$$

$$Q^t_{(kl)(kl)} = - \sum_{(k', l') \neq (k, l)} Q^t_{(kl)(k'l')}, \quad (5)$$

where $\delta_{k_1}(k_2) = 0$ for $k_1 \neq k_2$, whereas $\delta_{k_1}(k_1) = 1$.

3. The stochastic processes $\{(\alpha_i, g_i), i \in I\}$ are essentially pairwise independent in the sense that for λ-almost all $i \in I$, (α_i, g_i) and (α_j, g_j) are independent for λ-almost all $j \in I$.

4 The Main Results

4.1 Continuous time random matching with enduring partnerships

The exact law of large numbers (Theorem 2.16 of [47]) will be used to show that the cross-sectional type distribution $\hat{p}(t)$ is deterministic almost surely, and equal to the solution $\check{p}(t)$ of the ordinary differential equation for the expected cross-sectional type distribution, $\mathbb{E}(\hat{p}(t))$, given by

$$\frac{d\check{p}(t)}{dt} = \check{p}(t)Q^t, \qquad \check{p}(0) = \hat{p}^0. \tag{6}$$

We are now ready to state the properties of an independent continuous-time dynamical system with random mutation, directed random matching and random type changing and random break-up.

Theorem 1 *If $\mathbb{D}$ is a continuous-time dynamical system with parameters $(\hat{p}^0, \eta, \theta, \xi, \sigma, \varsigma, \vartheta)$, then:*

(1) For P-almost all $\omega \in \Omega$, the cross-sectional type process $(\alpha_\omega, g_\omega)$ is a continuous-time Markov chain with generator Q.

(2) For P-almost all $\omega \in \Omega$, the realized cross-sectional extended type distribution $\hat{p}(t)$ is equal to the expected cross-sectional type distribution $\mathbb{E}(\hat{p}(t))$.

(3) Suppose that the parameters $(\eta, \theta, \xi, \sigma, \varsigma, \vartheta)$ are time independent. Then there exists a probability distribution $\hat{p}^*$ on $\hat{S}$ such that the dynamical system $\mathbb{D}$ with parameters $(\hat{p}^*, \eta, \theta, \xi, \sigma, \varsigma, \vartheta)$ has $\hat{p}^*$ as a stationary type distribution. In particular, the realized cross-sectional type distribution $\hat{p}(t)$ at any time t is almost surely $\hat{p}^*$ and the generator $Q^t(\hat{p}(t))$ is equal to $Q^0(\hat{p}^*)$.

The following result shows the general existence of an continuous-time dynamical system with random mutation, directed random matching, random type changing and random break-up.

Theorem 2 *For any given parameters $(\hat{p}^0, \eta, \theta, \xi, \sigma, \varsigma, \vartheta)$, there exists a Fubini extension on which is defined an dynamical system $\mathbb{D}$ with these parameters.*

4.2 Continuous time random matching with immediate break-up

The assumption that agents break up immediately after meeting is widely used in the economics literature, including that for finance and monetary economics. In this section, we consider a special case of the general model presented in the previous section in which agents do not form long-term partnerships. This special case can be derived by letting the enduring probabilities to be 0 and all the other parameters depend on type distribution rather than extended type distribution. In order to allow easier access to this case with much simpler notation and structure, we state the model and results of this special case for the convenience of the reader and for applications.

As in the previous section, let $S = \{1, 2, \ldots, K\}$ be a finite set of agent types, Δ the set of probability measures on S, and $\mathbb{R}_+$ the set of non-negative real numbers with its Borel σ-algebra $\mathcal{B}$. Since agents do not have enduring partnerships, the parameters and results are simplified dramatically.

The parameters of the model are the initial type distribution $p^0 \in \Delta$ and, for any k and l in S:

(i) A continuous mutation intensity function $\eta_{kl} : \mathbb{R}_+ \to \mathbb{R}_+$ specifying the intensity $\eta_{kl}(t)$ at time t with which any type k agent mutates to type l.

(ii) A continuous matching intensity function $\theta_{kl} : \Delta \times \mathbb{R}_+ \to \mathbb{R}_+$ specifying the intensity $\theta_{kl}(p, t)$ at time t with which any type-k agent is matched with a type-l agent, if the cross-sectional agent type distribution at time t is $p \in \Delta$. This function satisfies the mass-balancing requirement $p_k \cdot \theta_{kl}(p, t) = p_l \cdot \theta_{lk}(p, t)$ that the total aggregate rate of matches of type-k agents to type l agents is of course equal to the aggregate rate of matches of type-l agents to type-k agents.

(iii) A continuous function $\varsigma_{kl} : \mathbb{R}_+ \to \Delta$ specifying the probability distribution $\varsigma_{kl}(t)$ of the new type of a type-k agent who has been matched at time t to a type-l agent. We denote $\varsigma_{klr}(t) = [\varsigma_{kl}(t)](\{r\})$.

For given parameters $(p^0, \eta, \theta, \varsigma)$, a continuous-time dynamical system $\mathbb{D}$ with independent random mutation, independent directed random matching and independent random type changing is required to have the following properties:

1. The cross-sectional type distribution $p(t)$ at time t is defined by $p_k(t) = \lambda(\{i : \alpha(i, t) = k\})$ with initial condition $p(0) = p^0$. Let $\bar{p}(t) = \mathbb{E}\left(p(t)\right)$. For λ-almost every agent i, the type process $\alpha(i)$ of agent i is a continuous-time Markov chain in S whose transition intensity from any state k to any state $r \neq k$ is given almost surely by

$$R^t_{kr} = \eta_{kr}(t) + \sum_{l=1}^{K} \theta_{kl}(\bar{p}(t), t)\varsigma_{klr}(t). \tag{7}$$

2. The type $\alpha(i, \omega, t)$ is $(\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}$-measurable.

3. The agents' stochastic type processes $\{\alpha_i : i \in I\}$ are essentially pairwise independent in the sense that for λ almost all $i \in I$, α_i and α_j are independent for λ almost all $j \in I$.

The exact law of large numbers will be used to show that the cross-sectional type distribution $p(t)$ is deterministic almost surely, and given by the solution $\bar{p}^t$ of the ordinary differential equation for the expected cross-sectional type distribution given by

$$\frac{d\bar{p}^t}{dt} = \bar{p}^t R^t, \qquad \bar{p}^0 = p^0,$$

where R_{kr} is specified by (7) and $R_{kk} = -\sum_{l \neq k} R_{kl}$. That is, the probability distribution of each agent's type evolves according to the same dynamics as those of the cross-sectional type distribution. These distributions differ only with respect to their initial conditions.

The following results include the exact law of large numbers and the existence of a stationary deterministic type distribution, which follow from Theorem 1 directly.

Corollary 1 *If $\mathbb{D}$ is an independent dynamical system with parameters $(p^0, \eta, \theta, \varsigma)$, then*

(1) For P-almost all $\omega \in \Omega$, the cross-sectional type process α_ω is a Markov chain with generator R^t.

(2) For P-almost all $\omega \in \Omega$, the realized cross-sectional type distribution $p(t)$ is equal to the expected cross-sectional type distribution $\bar{p}^t$.

(3) Suppose that the parameters $(\eta, \theta, \varsigma)$ are time independent. Then there exists a probability distribution p^* on S such that any independent dynamical system $\mathbb{D}$ with parameters $(p^*, \eta, \theta, \varsigma)$ has p^* as a stationary type distribution. In particular, the realized cross-sectional type distribution $p(t)$ at any time t is almost surely p^* and the generator R^t is constant and equal to R^0.

The following result, which states the existence of a continuous-time dynamical system with immediate break-up, is a direct corollary of Theorem 2.

Corollary 2 *For any given parameters $(p^0, \eta, \theta, \varsigma)$, there exists a Fubini extension on which is defined an independent dynamical system $\mathbb{D}$ with these parameters.*

5 Applications

The section offers some illustrative applications, drawn respectively from labor economics, financial economics, and monetary theory.

5.1 The DMP model in labor economics

Our first example is from [37]. The agents are workers and firms. Each firm has a single job position. Our results for continuous-time random matching with enduring partnerships provide a foundation for the equilibrium unemployment rate, modeled as Equation (1) of [37].

The type space of the agents is $S = \{W, F, D\}$. Here, W, D, and F represent respectively workers, dormant firms, and firms that are active in the labor market. Dormant firms are neither matched with a worker nor immediately open to a match. The proportion[4] of agents that are workers is $w > 0$.

In Section 1 of [37], the fraction $v(t)$ of unmatched active firms at time t is exogenously given. Following the notation in Section 3, we can let $v(t) = \mathbb{E}(\hat{p}^t_{FJ})$.

There are frictions in the labor market, which make it impossible for all the unemployed workers to find jobs instantaneously. At time t, for any unemployed worker, the next vacant firm arrives with intensity $\bar{\theta}_1(\hat{p}^t, t)$; for any vacant firm, the next unemployed worker arrives with intensity $\bar{\theta}_2(\hat{p}^t, t)$, where $\hat{p}^t$ is the extended type distribution at time t.[5] When a firm and a worker meet,[6] they form a (long term) job match with probability $\bar{\xi}$. Furthermore, each matched job-worker pair faces a randomly timed separation at an exogenous intensity $\bar{\vartheta}$.

Viewed in terms of our model, the corresponding parameters are given as follows. Vacant and dormant firms could mutate to each other while workers and active firms do not mutate. Mutation intensities[7] are defined, for any k and $l \in S$, by

$$\eta_{kl}(\hat{p}, t) = \begin{cases} \frac{\max(\dot{v} - (w - \hat{p}^t_{WJ})\bar{\vartheta} + \bar{\xi}\theta_1(\hat{p}, t)\hat{p}^t_{WJ}, 0)}{\hat{p}_{DJ}} & \text{if } (k, l) = (D, F) \\ \frac{\max(-\dot{v} + (w - \hat{p}^t_{WJ})\bar{\vartheta} - \bar{\xi}\theta_1(\hat{p}, t)\hat{p}^t_{WJ}, 0)}{\hat{p}_{FJ}} & \text{if } (k, l) = (F, D) \\ 0 & \text{otherwise,} \end{cases}$$

where $\dot{v}$ is the derivative of v with respect to t.

Matching occurs only between unemployed workers and firms with vacant jobs. For matching intensities, we define

$$\theta_{kl}(\hat{p}, t) = \begin{cases} \bar{\theta}_1(\hat{p}, t) & \text{if } (k, l) = (W, F) \\ \bar{\theta}_2(\hat{p}, t) & \text{if } (k, l) = (F, W) \\ 0 & \text{otherwise.} \end{cases}$$

[4] In [37], the measure of workers is 1. In order to stay with our convention that an agent space has total mass 1, we rescale without loss of generality so that the measure of workers is w.

[5] Obviously, one needs to have the mass balance identity $p^t_{WJ} \cdot \bar{\theta}_1(\hat{p}^t, t) = p^t_{FJ} \cdot \bar{\theta}_2(\hat{p}^t, t)$.

[6] Details regarding the job-creation mechanism are provided in Section 1 of [37].

[7] The mutation intensities proposed here guarantee that the population of firms with unfilled job openings is always $v(t)$, which is exogenous, as given in [37].

For enduring-relationship probabilities, we define for any $k, l \in S$,

$$\xi^t_{kl} = \begin{cases} \bar{\xi} & \text{if } (k,l) = (W,F) \text{ or } (F,W) \\ 0 & \text{otherwise.} \end{cases}$$

The match-induced type-change probabilities are

$$\sigma^t_{kl}(k', l') = \delta_k(k')\delta_l(l')$$

and

$$\varsigma^t_{kl}(k') = \delta_k(k').$$

The mean separation rates are

$$\vartheta^t_{kl} = \begin{cases} \bar{\vartheta} & \text{if } (k,l) = (W,F) \text{ or } (F,W) \\ 0 & \text{otherwise.} \end{cases}$$

By Equation (6)

$$\begin{aligned} \frac{d\mathbb{E}(\hat{p}^t_{WJ})}{dt} &= \sum_{(k,l)\in\hat{S}} Q^t_{(kl)(WJ)}\mathbb{E}(\hat{p}^t_{kl}) \\ &= \left(w - \mathbb{E}(\hat{p}^t_{WJ}\right)\bar{\vartheta} - \bar{\xi}\theta_1\left(\mathbb{E}(\hat{p}^t), t\right)\mathbb{E}\left(\hat{p}^t_{WJ}\right). \end{aligned}$$

Letting $u(t)$ be the fraction of unemployed workers at time t, as in [37], we have

$$u(t) = \frac{1}{w}\,\mathbb{E}(\hat{p}^t_{WJ}).$$

We can therefore derive that

$$\frac{du(t)}{dt} = (1 - u(t))\,\bar{\vartheta} - \bar{\xi}\,\theta_1 u(t),$$

which is Equation (1) in [37].

5.2 Over-The-Counter financial markets

Our second example is from [12]. There are two classes of agents, investors and marketmakers. Each agent consumes a single nonstorable consumption good that is used as a numeraire. The masses[8] of investors and marketmakers are each 1/2.

Investors can hold 0 or 1 unit of the asset. A fraction s of investors are initially endowed with 1 unit of the asset. An investor is characterized as an asset owner or non-owner, and also by an intrinsic preference for ownership that is high (h) or low (l). A high-type investor has no

[8]In [12], the measures of investors and marketmakers are both 1. In order to work with a probability agent space, we rescale these masses to 1/2.

such holding cost. A low type switches from low to high with intensity λ_u, and switches back with intensity λ_d.

The type space is thus $S = \{ho, hn, lo, ln, m\}$, where the letters "$h$" and "$l$" designate the investor's intrinsic preference, "o" and "n" indicate whether the investor owns the asset or not, and "m" indicates a marketmaker. Marketmakers never change their type. When a high-preference non-owner meets an low-preference owner, they endogenously choose to trade the asset, generating a change of types for each. Other investor-to-investor matches generate no trade, thus no type changes. Trades generated by contact with a marketmaker will be characterized shortly.

Investors meet by independent random search, as follows. At the successive event times of a Poisson process with some intensity parameter λ, an investor contacts another investor chosen at random, uniformly from the entire investor population. Thus, letting

$$\mu_k(t) = \frac{p_k(t)}{p_{ho}(t) + p_{hn}(t) + p_{lo}(t) + p_{ln}(t)} = 2p_k(t)$$

denote the relative fraction of investors (among all the investors) of type k at time t, the intensity for any investor of contacting an investor of type k is $\lambda\mu_k(t)$. In [12], contact is directional, in the sense that the event of a type k investor contacting a type l investor is distinguished from the event of a type l investor contacting a type k investor. Thus the total meeting intensity for specific type-k investor with some type-l investor is

$$\theta_{kl}(p_t, t) = 2\lambda\mu_l(t) = 4\lambda p_l(t).$$

This directional-contact formulation implies that the derived matching intensity function θ automatically satisfies the mass-balance condition. Directional contact also allows in principle for a difference in the terms of trade in the asset bargaining outcome, depending on which of a pair contacts the other, but that difference plays no role here.

Each investor also contacts some randomly drawn marketmaker at the event times of a Poisson process with a fixed intensity of ρ.

Viewed in terms of our model, the corresponding parameters are given as follows. For mutation intensities, we have for any k and $r \in S$ such that $k \neq r$,

$$\eta_{kr}(p, t) = \begin{cases} \lambda_u & \text{if } (k, r) = (lo, ho) \text{ or } (ln, hn) \\ \lambda_d & \text{if } (k, r) = (ho, lo) \text{ or } (hn, ln) \\ 0 & \text{otherwise.} \end{cases}$$

For matching intensities, we have for any k and $r \in S$,

$$\theta_{kr}(p,t) = \begin{cases} 4\lambda p_r & \text{if } k, r \in \{ho, lo, hn, ln\} \\ \rho & \text{if } k \in \{ho, lo, hn, ln\} \text{ and } r = m \\ \rho p_r & \text{if } k = m \text{ and } r \in \{ho, lo, hn, ln\} \\ 0 & \text{if } k = r = m. \end{cases}$$

Because agents in this model do not form long-term partnerships after matching, we have enduring-relationship parameters

$$\xi_{kr} \equiv 0,\ \sigma^t_{kr}(k', r') = \delta_k(k')\delta_l(r') \text{ and } \vartheta_{kr} \equiv 0 \text{ for any } k, k', r, r' \in S.$$

When a type-hn investor meets a type-lo investor, the type-hn investor, having purchased the asset, becomes a type-ho investor. Likewise the type-lo investor becomes a type-ln investor.

When a type-hn investor meets a marketmaker, the community of marketmakers may be experiencing an excess of buyer contacts relative to seller contacts. Marketmakers are able to instantly lay off their trades in the inter-dealer market, but do not absorb excess order flow into their own accounts. In that case, each marketmaker will ration trades by randomizing whether it will accept the position of at contacting buyer. Specifically, at marketmaker contact, a type-hn investor becomes a type-ho investor with probability

$$\frac{\min(p_{hn}, p_{lo})}{p_{hn}}.$$

Similarly, when a type-lo investor meets a marketmaker, the type-lo investor becomes a type-ln investor with probability

$$\frac{\min(p_{hn}, p_{lo})}{p_{lo}}.$$

Thus,

$$[\varsigma_{kr}(\hat{p}, t)](k') = \begin{cases} \delta_{ho}(k') & \text{if } (k, r) = (hn, lo) \\ \delta_{ln}(k') & \text{if } (k, r) = (lo, hn) \\ \frac{\min(p_{hn}, p_{lo})}{p_k}\delta_{ho}(k') + \left(1 - \frac{\min(p_{hn}, p_{lo})}{p_k}\right)\delta_{hn}(k') & \text{if } (k, r) = (hn, m) \\ \frac{\min(p_{hn}, p_{lo})}{p_k}\delta_{ln}(k') + \left(1 - \frac{\min(p_{hn}, p_{lo})}{p_k}\right)\delta_{lo}(k') & \text{if } (k, r) = (lo, m) \\ \delta_k(k') & \text{otherwise.} \end{cases}$$

By Corollary 1,

$$\begin{aligned} Q^t_{(ho)(lo)} &= \lambda_d \\ Q^t_{(hn)(lo)} &= 0 \\ Q^t_{(lo)(lo)} &= -\lambda_u - 4\lambda\bar{p}^t_{hn} - \frac{\rho \min(\bar{p}_{hn}, \bar{p}_{lo})}{\bar{p}_{lo}} \\ Q^t_{(ln)(lo)} &= 0. \end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d\bar{p}^t_{lo}}{dt} &= Q^t_{(ho)(lo)}\bar{p}^t_{ho} + Q^t_{(hn)(lo)}\bar{p}^t_{hn} + Q^t_{(lo)(lo)}\bar{p}^t_{lo} + Q^t_{(ln)(lo)}\bar{p}^t_{ln} \\ &= \lambda_d \bar{p}^t_{ho} - (\lambda_u + 4\lambda\bar{p}^t_{hn} + \frac{\rho\min(\bar{p}_{hn}, \bar{p}_{lo})}{\bar{p}_{lo}})\bar{p}^t_{lo} \\ &= \lambda_d \bar{p}^t_{ho} - \lambda_u \bar{p}^t_{lo} - 4\lambda\bar{p}^t_{hn}\bar{p}^t_{lo} - \rho\min(\bar{p}_{hn}, \bar{p}_{lo}).\end{aligned}$$

Because $\mu_k(t) = 2\bar{p}^t_k$, we have

$$\frac{d\mu_{lo}(t)}{dt} = \lambda_d\mu_{ho}(t) - \lambda_u\mu_{lo}(t) - 2\lambda\mu_{hn}(t)\mu_{lo}(t) - \rho\min(\mu_{hn}(t), \mu_{lo}(t)),$$

which is Equation (3) of [12]. The remaining population-distribution evolution equations of [12] follow similarly.

5.3 The Kiyotaki-Wright money model in continuous time

Our third example is the continuous-time version of the Kiyotaki-Wright Model from [52]. The economy is populated by a continuum of infinitely-lived agents of unit total mass. Agents are from two regions, Home and Foreign. Let $n \in (0, 1)$ be the size of Home population.

Within each of the two regions, there are equal proportions of agents with K respective traits, for some $K \geq 3$. The trait space is $\{1, \ldots, K, 1^*, \ldots, K^*\}$, where i denotes a home trait and i^* denotes a foreign trait.

There are K kinds of indivisible commodities in each region. The commodity space is also $\{1, \ldots, K, 1^*, \ldots, K^*\}$. An agent with trait k derives utility only from consumption of commodity k or k^*. After he consumes commodity k, he is able to produce one and only one unit of commodity $k+1 \pmod K$ costlessly, and can also store up to one unit of his production good costlessly. He can neither produce nor store other types of goods.

An agent of type k has random preferences between goods of types k and k'. One can think of goods k and k' as a pair of goods with different features over which a consumer's taste switches from time to time. Let l describe the preference state of an agent with type k in which he prefers his local consumption good k over k^*,[9] and let n be the preference state in which he prefers the non-local consumption good k^*. The preference state process of each agent a two-state Markov chain with constant transition intensity b_{ln} from l to n and intensity b_{nl} from n to l.

In addition to the commodities described above, there are two distinguishable fiat monies, objects with zero intrinsic worth, which we call the Home currency 0 and the Foreign currency 0^*. Each currency is indivisible and can be stored costlessly in amounts of up to one unit

[9]When k is a foreign trait i^*, k^* is simply i.

by any agent, provided that the agent is not carrying his own production good or the foreign currency. This implies that, at any date, the inventory of each agent consists of one unit of the Home currency, one unit of the Foreign currency, or one unit of his production good, but does not include more than one of these three objects in total at any one time.

Agents meet pairwise randomly. Any agent's potential trading partners arrive at the event times of a Poisson process with parameter ν.

The type space S is the set of ordered tuples of the form (a, b, c), where

$$a \in \{1, \ldots, K, 1^*, \ldots, K^*\},\ b \in \{0, 1, \ldots, K, 0^*, 1^*, \ldots, K^*\},\ c \in \{l, n\}.$$

For example, a agent of type $(1, 2, l)$ is a trait-1 agent who holds one unit of the type-2 good and who prefers local goods.

An agent chooses a trading strategy to maximize his expected discounted utility, taking as given the strategies of other agents and the distribution of inventories. In [52], the author focused on pure strategies that depend only on an agent's trait, preference state, and the objects that he and his counterparty have as inventories. Thus, the trading strategy of a trait-a agent with preference state c can be described simply as

$$\tau_{bb'}^{ac} = \begin{cases} 1 & \text{if he agrees to trade object } b \text{ for object } b' \\ 0 & \text{otherwise,} \end{cases}$$

where b and b' are in $\{0, 1, \ldots, K, 0^*, 1^*, \ldots, K^*\}$.

We can apply our model of continuous time random matching with immediate break-up to give a mathematical foundation for the matching model in [52] by choosing suitable parameters $(\eta, \theta, \vartheta)$ governing random mutation, random matching, and match-induced type changing. The mutation intensities are

$$\eta_{(a_1,b_1,c_1)(a_2,b_2,c_2)} = \begin{cases} \delta_{a_1}(a_2)\delta_{b_1}(b_2)b_{ln} & \text{if } c_1 = l, c_2 = n \\ \delta_{a_1}(a_2)\delta_{b_1}(b_2)b_{nl} & \text{if } c_1 = n, c_2 = l \\ 0 & \text{otherwise.} \end{cases}$$

The matching intensities are simply proportional, and given by

$$\theta_{(a_1,b_1,c_1)(a_2,b_2,c_2)}(p) = \nu p_{(a_2,b_2,c_2)}.$$

for a cross-sectional agent type distribution $p \in \Delta$.

Because the consumption traits of agents do not change, a matched agent cannot change to a type with a different trait. Suppose that agent i is of type (a_1, b_1, c_1) and is matched with agent j who has type (a_2, b_2, c_2). The probability that agent i changes type to (a_3, b_3, c_3) is $\nu_{(a_1,b_1,c_1)(a_2,b_2,c_2)}(a_3, b_3, c_3)$. Because the consumption traits and preference of agents are not changed by meetings, $\nu_{(a_1,b_1,c_1)(a_2,b_2,c_2)}(a_3, b_3, c_3) = 0$ if $(a_1, c_1) \neq (a_3, c_3)$.

If an agent of type (a_1, b_1, c_1) obtains one unit of good a_1 or of a_1^*, then she will consume the good and produce one unit of good $a_1 + 1$. Thus, there is no agent with type (a_1, a_1, c_1) or (a_1, a_1^*, c_1) in the market.

If $b_3 \neq b_1$, trade occurs between these two types of agents, so

$$\varsigma_{(a_1,b_1,c_1)(a_2,b_2,c_2)}(a_1, b_3, c_1) = \begin{cases} \tau^{a_1c_1}_{b_1b_2}\tau^{a_2c_2}_{b_2b_1}\delta_{b_2}(b_3) & \text{if } b_2 \neq a_1 \text{ or } a_1^* \\ \\ \tau^{a_1c_1}_{b_1b_2}\tau^{a_2c_2}_{b_2b_1}\delta_{a_1+1}(b_3) & \text{if } b_2 = a_1 \text{ or } a_1^* \text{ and } b_1 \neq a_1 + 1 \\ \\ 0 & \text{if } b_2 = a_1 \text{ or } a_1^* \text{ and } b_1 = a_1 + 1, \end{cases}$$

which implies that

$$\varsigma_{(a_1,b_1,c_1)(a_2,b_2,c_2)}(a_1, b_1, c_1) = 1 - \sum_{b_3 \neq b_1} \varsigma_{(a_1,b_1,c_1)(a_2,b_2,c_2)}(a_1, b_3, c_1).$$

6 Proof of Theorem 1

Let $\beta_i = (\alpha_i, g_i)$ be the extended type process for agent i. Note that for any $t > t_1 > \ldots, > t_n$ and $\Delta t > 0$, if $(k, l) \neq (k', l')$,

$$\begin{aligned}
& \lambda \boxtimes P\left(\beta^{t+\Delta t}(i,\omega) = (k', l'), \beta^t(i,\omega) = (k,l), \beta^{t_1}(i,\omega) = (k_1,l_1), \ldots, \beta^{t_n}(i,\omega) = (k_n,l_n)\right) \\
= & \int_I P\left(\beta_i^{t+\Delta t}(\omega) = (k', l'), \beta_i^t(\omega) = (k,l), \beta_i^{t_1}(\omega) = (k_1,l_1), \ldots, \beta_i^{t_n}(\omega) = (k_n,l_n)\right) d\lambda \\
= & \int_I P\left(\beta_i^t(\omega) = (k,l), \beta_i^{t_1}(\omega) = (k_1,l_1), \ldots, \beta_i^{t_n}(\omega) = (k_n,l_n)\right) \\
& \quad P\left(\beta_i^{t+\Delta t}(\omega) = (k', l') \,|\, \beta_i^t(\omega) = (k,l), \beta_i^{t_1}(\omega) = (k_1,l_1), \ldots, \beta_i^{t_n}(\omega) = (k_n,l_n)\right) d\lambda \\
= & \int_I P\left(\beta_i^t(\omega) = (k,l), \beta_i^{t_1}(\omega) = (k_1,l_1), \ldots, \beta_i^{t_n}(\omega) = (k_n,l_n)\right) \\
& \quad P\left(\beta_i^{t+\Delta t}(\omega) = (k', l') \,|\, \beta_i^t(\omega) = (k,l)\right) d\lambda \\
= & \int_I P\left(\beta^t(i,\omega) = (k,l), \beta_i^{t_1}(\omega) = (k_1,l_1), \ldots, \beta_i^{t_1}(\omega) = (k_n,l_n)\right) \\
& \quad \left((Q^t_{(kl)(k'l')}\Delta t + R_i(\Delta t)\right) d\lambda \\
= & \; Q^t_{(kl)(k'l')}\Delta t \lambda \boxtimes P\left(\beta^t(i,\omega) = (k,l), \beta_i^{t_1}(\omega) = (k_1,l_1), \ldots, \beta_i^{t_n}(\omega) = (k_n,l_n)\right) \\
& + \int_I P\left(\beta^t(i,\omega) = (k,l), \beta_i^{t_1}(\omega) = (k_1,l_1), \ldots, \beta_i^{t_n}(\omega) = (k_n,l_n)\right) R_i(\Delta t) d\lambda,
\end{aligned}$$

where $\lim_{\Delta t \to 0} \frac{R_i(\Delta t)}{\Delta t} = 0$. Note that the generator and initial distribution determine the finite dimensional distributions, then R_i has at most $K(K+1)$ different forms since there are only $K(K+1)$ initial extended types. Then

$$\lim_{\Delta t \to 0} \frac{P\left(\beta^t(i,\omega) = (k,l), \beta_i^{t_1}(\omega) = (k_1,l_1), \ldots, \beta_i^{t_n}(\omega) = (k_n,l_n)\right) R_i(\Delta t) d\lambda}{\Delta t} = 0.$$

Therefore,

$$\begin{aligned}
& \lambda \boxtimes P\left(\beta^{t+\Delta t}(i,\omega)=(k',l') \,|\, \beta^{t}(i,\omega)=(k,l), \beta^{t_1}(i,\omega)=(k_1,l_1),\ldots,\beta^{t_n}(i,\omega)=(k_n,l_n)\right) \\
= \;& Q^t_{(kl)(k'l')}\Delta t + o(\Delta t).
\end{aligned}$$

Note that the right hand side does not depend on $t_1,\ldots,t_n$. Then β viewed as a stochastic process with sample space $(I\times\Omega, \mathcal{I}\boxtimes\mathcal{F}, \lambda\boxtimes P)$ is also a Markov process with transition rate matrix Q.

By the exact law of large numbers in Theorem 2.16 of [47], we know that for P-almost all $\omega\in\Omega$, $(\beta^{t_1}_\omega,\ldots,\beta^{t_n}_\omega)$ and $(\beta^{t_1},\ldots,\beta^{t_n})$ (viewed as random vectors) have the same distribution. Note that finite dimensional distributions determines whether a process is a Markov chain and also the transition rate matrix. Then for P-almost all $\omega\in\Omega$, β_ω is also a Markov chain with transition rate matrix Q_t at time t.

Note that for any $\omega\in\Omega$, the realized extended type distribution $\hat{p}^t(\omega)$ is also the distribution β^t_ω. By (1), $\hat{p}^t(\omega)$ is equal to the distribution of β^t P-almost surely. Then $\hat{p}^t(\omega)$ is equal to $\mathbb{E}\left(\hat{p}^t\right)$ P-almost surely

For part (3), for any $\hat{p},\hat{q}\in\hat{\Delta}$, define $\hat{q}Q(\hat{p})\in\hat{\Delta}$ such that

$$[\hat{q}Q(\hat{p})]_{kl} = \sum_{(k',l')\in\hat{S}} \hat{q}_{k'l'}Q(\hat{p})_{(k'l')(kl)}.$$

It is sufficient to show that there exists a $\hat{p}^*\in\hat{\Delta}$ such that $\hat{p}^*Q(\hat{p}^*)=0$. Since $Q_{(kl)(k'l')}$ is continuous on $\hat{\Delta}$, and $\hat{\Delta}$ is compact, we can find a positive real number c such that $\left|cQ_{(kl)(k'l')}(\hat{p})\right|\le 1$ for any $\hat{p}\in\hat{\Delta}$ and $(k,l),(k',l')\in\hat{S}$.

It is easy to see that $\hat{p}Q(\hat{p})=0$ is equivalent to the statement that $f(\hat{p})\triangleq\hat{p}+c\hat{p}Q(\hat{p})$ has a fixed point. Note that

$$f(\hat{p})_{kl} = \hat{p}_{kl} + \sum_{(k',l')\in\hat{S}} c\hat{p}_{k'l'}Q(\hat{p})_{(k',l')(k,l)} = \left(1+cQ(\hat{p})_{(k,l)(k,l)}\right)\hat{p}_{kl} + \sum_{(k',l')\neq(k,l)} c\hat{p}_{k'l'}Q(\hat{p})_{(k',l')(k,l)},$$

$\left(1+cQ(\hat{p})_{(k,l)(k,l)}\right)\ge 0$ and $Q(\hat{p})_{(k',l')(k,l)}\ge 0$ if $(k',l')\neq(k,l)$. Then $[f(\hat{p})]_{kl}\ge 0$ for any $(k,l)\in\hat{S}$. One can check that

$$\begin{aligned}
\sum_{(k,l)\in\hat{S}} f(\hat{p})_{(k,l)} &= \sum_{(k,l)\in\hat{S}}\sum_{(k',l')\in\hat{S}} \hat{p}_{k'l'} + c\hat{p}_{k'l'}Q(\hat{p})_{(k',l')(k,l)} \\
&= \sum_{(k',l')\in\hat{S}}\sum_{(k,l)\in\hat{S}} \hat{p}_{k'l'} + c\hat{p}_{k'l'}Q(\hat{p})_{(k',l')(k,l)} \\
&= \sum_{(k',l')\in\hat{S}} \hat{p}_{(k',l')} = 1.
\end{aligned}$$

Hence, f is a continuous function from $\hat{\Delta}$ to $\hat{\Delta}$. By Kakutani's Fixed Point Theorem, there exists a $\hat{p}^*\in\hat{\Delta}$ such that $\hat{p}^*+c\hat{p}^*Q(\hat{p}^*)=\hat{p}^*$. Therefore, $\hat{p}^*$ is a stationary distribution.

7 Proof of Theorem 2

This section is organized as follows. Lemma 1 in Subsection 7.1 presents a static model for internal random matching as well as some estimations on the relevant matching probabilities. Such a static matching model will be used in the construction of a hyperfinite dynamic matching model in Subsection 7.2. Lemmas 2 – 5 in Subsection 7.3 state some properties of the hyperfinite dynamic matching model. Based on Lemmas 2 – 5, Theorem 2 is proved in Subsection 7.4. For the convenience of the reader, we leave the technical proofs of Lemmas 1 – 5 in Subsection 7.5. Elementary nonstandard analysis is used extensively in this section; the reader is referred to [32] for the details.

7.1 Static internal matching model

Let $I = \{1, \ldots, \hat{M}\}$ be a hyperfinite set with $\hat{M}$ an unlimited hyperfinite integer in ${}^*\mathbb{N}_\infty$, $\mathcal{I}_0$ the internal power set on I, λ_0 the internal counting probability measure on $\mathcal{I}_0$.

Lemma 1 *Let $(I, \mathcal{I}_0, \lambda_0)$ be the hyperfinite internal counting probability space. Then, there exists a hyperfinite internal set Ω with its internal power set $\mathcal{F}_0$ such that for any initial internal type function α^0 from I to S and initial internal partial matching π^0 from I to $I \cup \{J\}$ with $g^0 = \alpha^0 \circ \pi^0$, and for any internal matching probability function q from $S \times S$ to ${}^*\mathbb{R}_+$ with $\sum_{r \in S} q_{kr} \leq 1$ and $\hat{\rho}_{kJ} q_{kl} = \hat{\rho}_{lJ} q_{lk}$ for any $k, l \in S$, where $\hat{\rho} = \lambda_0 \left(\alpha^0, g^0\right)^{-1}$ is the internal extended type distribution, there exists an internal random matching π from $I \times \Omega$ to I and an internal probability measure P_0 on $(\Omega, \mathcal{F}_0)$ with the following properties.*

(i) Let $H = \{i : \pi^0(i) \neq J\}$. Then $P_0\left(\{\omega \in \Omega : \pi_\omega(i) = \pi^0(i) \text{ for any } i \in H\}\right) = 1$.

(ii) Let g be the internal mapping from $I \times \Omega$ to $S \cup \{J\}$, define by

$$g(i, \omega) = \begin{cases} \alpha^0(\pi(i, \omega)) & \text{if } \pi(i, \omega) \neq i \\ J & \text{if } \pi(i, \omega) = i \end{cases}$$

for any $(i, \omega) \in I \times \Omega$. Suppose $i \neq j$, $(\alpha_i^0, \pi_i^0) = (k_1, i)$ and $(\alpha_j^0, \pi_j^0) = (k_2, j)$, where $k_1, k_2 \in S$. For any $l_1, l_2 \in S$, if $\hat{\rho}_{k_1 J} > \frac{1}{\hat{M}^{\frac{1}{3}}}$,

$$q_{k_1 l_1} - \frac{1}{\hat{M}^{\frac{2}{3}}} \leq P_0(g_i = l_1) \leq q_{k_1 l_1};$$

if, in addition, $\hat{\rho}_{k_2 J} \geq \frac{1}{\hat{M}^{\frac{1}{3}}}$, we will also have

$$q_{k_1 l_1} q_{k_2 l_2} - \frac{5}{\hat{M}^{\frac{2}{3}}} \leq P_0(g_i = l_1, g_j = l_2) \leq q_{k_1 l_1} q_{k_2 l_2} + \frac{1}{\hat{M}^{\frac{2}{3}}}.$$

To reflect their dependence on (α^0, π^0, q), π and P_0 will also be denoted by $\pi_{(\alpha^0, \pi^0, q)}$ and $P_{(\alpha^0, \pi^0, q)}$.

7.2 Hyperfinite dynamic matching model

What we need to do is to construct a hyperfinite sequence of internal transition probabilities and a hyperfinite sequence of internal type functions. Since we need to consider random mutation, random matching and random type changing at each infinitesimal time period, three internal measurable spaces with internal transition probability will be constructed at each time period.

Let M and $\hat{M}$ be fixed unlimited hyperfinite natural numbers in ${}^*\mathbb{N}_\infty$ and $\hat{M}$ is sufficiently larger than M. Let $I_0 = \{1, 2, \ldots, \hat{M}\}$, $\mathcal{I}_0$ the internal power set on I, and λ_0 the internal counting probability measure on $\mathcal{I}_0$.

We define the parameters for the hyperfinite dynamical system as follows. Let

$$\hat{\eta}^n_{kl}(\hat{\rho}) = \begin{cases} \frac{1}{M}({}^*\eta_{kl})(\hat{\rho}, \frac{n}{M}) + \frac{1}{M^2} & \text{if } k \neq l \\ 1 - \sum_{l \neq k} \hat{\eta}^n_{kl}(\hat{\rho}) & \text{if } k = l, \end{cases}$$

$$\hat{q}^n_{kl}(\hat{\rho}) = \frac{1}{M}({}^*\theta_{kl})(\hat{\rho}, \frac{n}{M}) \text{ and } \hat{q}^n_k(\hat{\rho}) = 1 - \sum_{l \in S} \hat{q}^n_{kl}(\hat{\rho}),$$

$$\hat{\xi}^n_{kl}(\hat{\rho}) = \min\{({}^*\xi_{kl})(\hat{\rho}, \frac{n}{M}), 1 - \frac{1}{M^2}\},$$

$$[\hat{\sigma}^n_{kl}(\hat{\rho})](k', l') = [({}^*\sigma_{kl})(\hat{\rho}, \frac{n}{M})](k', l'),$$

$$\hat{\vartheta}^n_{kl}(\hat{\rho}) = \frac{1}{M}({}^*\vartheta_{kl})(\hat{\rho}, \frac{n}{M}) + \frac{1}{M^2},$$

$$[\hat{\varsigma}^n_{kl}(\hat{\rho})](k') = [({}^*\varsigma_{kl})(\hat{\rho}, \frac{n}{M})](k')$$

for any $k, k', l, l' \in S$, $n \in {}^*\mathbb{N}$ and $\hat{\rho} \in {}^*\hat{\Delta}$. Denote

$$\bar{\eta}^{n_0} = M \max\{\hat{\eta}^n_{kl}(\hat{\rho}) : n \leq n_0, k, l \in S, k \neq l, \hat{\rho} \in {}^*\hat{\Delta}\},$$

$$\bar{q}^{n_0} = M \max\{\hat{q}^n_{kl}(\hat{\rho}) : n \leq n_0, k, l \in S, \hat{\rho} \in {}^*\hat{\Delta}\},$$

$$\bar{\vartheta}^{n_0} = M \max\{\hat{\vartheta}^n_{kl}(\hat{\rho}) : n \leq n_0, k, l \in S, \hat{\rho} \in {}^*\hat{\Delta}\}.$$

It is clear that $\bar{\eta}^{n_0}$, $\bar{q}^{n_0}$ and $\bar{\vartheta}^{n_0}$ are finite if $\frac{n_0}{M}$ is finite. Note that $\frac{\bar{\eta}^{nM}}{M}, \frac{\bar{q}^{nM}}{M}, \frac{\bar{\vartheta}^{nM}}{M} \leq \frac{1}{K}$ for any $n \in \mathbb{N}$. Then there there exists a hyperfinite natural number $C \in {}^*\mathbb{N}_\infty$ such that $C \leq M$ and $\frac{\bar{\eta}^{CM}}{M}, \frac{\bar{q}^{CM}}{M}, \frac{\bar{\vartheta}^{CM}}{M} \leq \frac{1}{K}$. Let T_0 be the hyperfinite discrete time line $\{n/M\}_{n=0}^{CM}$.

Let $\{A_{kl}\}_{(k,l) \in \tilde{S}}$ be an internal partition of I such that $\frac{|A_{kl}|}{\hat{M}} \simeq \hat{p}^0_{kl}$ for any $k \in S$ and $l \in S \cup \{J\}$, and $|A_{kl}| = |A_{lk}|$ and $|A_{kk}|$ are even for any $k, l \in S$. Let α^0 be an internal function from $(I, \mathcal{I}_0, \lambda_0)$ to S such that $\alpha^0(i) = k$ if $i \in \bigcup_{l \in S \cup \{J\}} A_{kl}$. Let π^0 be an internal partial matching from I to I such that $\pi^0(i) = i$ on $\bigcup_{k \in S} A_{kJ}$, and the restriction $\pi^0|_{A_{kl}}$ is an internal bijection from A_{kl} to A_{lk} for any $k, l \in S$. Let $g^0(i) = \alpha^0(\pi^0(i))$. It is clear that $\lambda_0(\{i : \alpha^0(i) = k, g^0(i) = l\}) \simeq \hat{p}^0_{kl}$ for any $k \in S$ and $l \in S \cup \{J\}$.

Suppose that the construction for the dynamical system $\mathbb{D}$ has been done up to time period $n-1$. Thus, $\{(\Omega_m, \mathcal{E}_m, Q_m)\}_{m=1}^{3n-3}$ and $\{\hat{\alpha}^l, \hat{\pi}^l\}_{l=0}^{3n-3}$ have been constructed, where each Ω_m is a hyperfinite internal set with its internal power set $\mathcal{E}_m$, Q_m an internal transition probability from Ω^{m-1} to $(\Omega_m, \mathcal{F}_m)$, and α^l an internal type function from $I \times \Omega^{l-1}$ to the type space S, and π^l an internal random matching from $I \times \Omega^{l-1}$ to I. Here, $\Omega^m = \prod_{j=1}^m \Omega_j$, and $\{\omega_j\}_{j=1}^m$ will also be denoted by ω^m when there is no confusion. Denote the internal product transition probability $Q_1 \otimes Q_2 \otimes \cdots \otimes Q_m$ by Q^m, and $\otimes_{j=1}^m \mathcal{E}_j$ by $\mathcal{E}^m$ (which is simply the internal power set on Ω^m). Then, Q^m is the internal product of the internal transition probability Q_m with the internal probability measure Q^{m-1}.

We shall now consider the constructions for time n. We first work with the random mutation step. Let $\Omega_{3n-2} = S^I$ (the space of all internal functions from I to S) with its internal power set $\mathcal{E}_{3n-2}$. For each $i \in I$, $\omega^{3n-3} \in \Omega^{3n-3}$, if $\hat{\alpha}^{3n-3}(i, \omega^{3n-3}) = k$, define a probability measure $\gamma_i^{\omega^{3n-3}}$ on S by letting $\gamma_i^{\omega^{3n-3}}(l) = \hat{\eta}_{kl}^n(\hat{\rho}_{\omega^{3n-3}}^{3n-3})$ for each $l \in S$, where $\hat{\rho}_{\omega^{3n-3}}^{3n-3} = \lambda_0 \left(\hat{\alpha}_{\omega^{3n-3}}^{3n-3}, \hat{g}_{\omega^{3n-3}}^{3n-3}\right)^{-1}$. Define an internal probability measure $Q_{3n-2}^{\omega^{3n-3}}$ on $(S^I, \mathcal{E}_{3n-2})$ to be the internal product measure $\prod_{i \in I} \gamma_i^{\omega^{3n-3}}$. Let $\hat{\alpha}^{3n-2} : \left(I \times \prod_{m=1}^{3n-2} \Omega_m\right) \to S$ be such that $\hat{\alpha}^{3n-2}\left(i, \omega^{3n-2}\right) = \omega_{3n-2}(i)$. Let $\hat{\pi}^{3n-2} : \left(I \times \prod_{m=1}^{3n-2} \Omega_m\right) \to I$ be such that $\hat{\pi}^{3n-2}\left(i, \omega^{3n-2}\right) = \hat{\pi}^{3n-3}\left(i, \omega^{3n-3}\right)$. Let $\hat{g}^{3n-2} : \left(I \times \prod_{m=1}^{3n-2} \Omega_m\right) \to S \cup \{J\}$ be such that

$$\hat{g}^{3n-2}\left(i, \omega^{3n-2}\right) = \begin{cases} \hat{\alpha}^{3n-2}(\pi^{3n-2}(i, \omega^{3n-2}), \omega^{3n-2}) & \text{if } \pi^{3n-2}(i, \omega^{3n-2}) \neq i \\ J & \text{if } \pi^{3n-2}(i, \omega^{3n-2}) = i. \end{cases}$$

Let $\hat{\rho}_{\omega^{3n-2}}^{3n-2} = \lambda_0 \left(\hat{\alpha}_{\omega^{3n-2}}^{3n-2}, \hat{g}_{\omega^{3n-2}}^{3n-2}\right)^{-1}$ be the internal cross-sectional extended type distribution after random mutation.

Next, we consider the step of directed random matching. Let $(\Omega_{3n-1}, \mathcal{E}_{3n-1}) = (\bar{\Omega}, \bar{\mathcal{E}})$, where $(\bar{\Omega}, \bar{\mathcal{E}})$ is the measurable space constructed in the proof of Lemma 1. For any given $\omega^{3n-2} \in \Omega^{3n-2}$, the type function is $\hat{\alpha}_{\omega^{3n-2}}^{3n-2}(\cdot)$ while the partial matching function is $\hat{\pi}_{\omega^{3n-3}}^{3n-3}(\cdot)$. We can construct an internal probability measure $Q_{3n-1}^{\omega^{3n-2}} = P_{\hat{\alpha}_{\omega^{3n-2}}^{3n-2}, \hat{\pi}_{\omega^{3n-3}}^{3n-3}, \hat{q}^n(\hat{\rho}_{\omega^{3n-2}}^{3n-2})}$ and a directed random matching $\pi_{\hat{\alpha}_{\omega^{3n-2}}^{3n-2}, \hat{\pi}_{\omega^{3n-3}}^{3n-3}, \hat{q}^n(\hat{\rho}_{\omega^{3n-2}}^{3n-2})}$ by Lemma 1. Let $\hat{\alpha}^{3n-1} : \left(I \times \prod_{m=1}^{3n-1} \Omega_m\right) \to S$, $\hat{\pi}^{3n-1} : \left(I \times \prod_{m=1}^{3n-1} \Omega_m\right) \to I$ and $\hat{g}^{3n-1} : \left(I \times \prod_{m=1}^{3n-1} \Omega_m\right) \to S \cup \{J\}$ be such that

$$\hat{\alpha}^{3n-1}\left(i, \omega^{3n-1}\right) = \hat{\alpha}^{3n-2}\left(i, \omega^{3n-2}\right)$$

$$\hat{\pi}^{3n-1}\left(i, \omega^{3n-1}\right) = \pi_{\hat{\alpha}_{\omega^{3n-2}}^{3n-2}, \hat{\pi}_{\omega^{3n-3}}^{3n-3}, \hat{q}^n(\hat{\rho}_{\omega^{3n-2}}^{3n-2})},$$

$$\hat{g}^{3n-1}\left(i, \omega^{3n-1}\right) = \begin{cases} \hat{\alpha}^{3n-2}(\pi^{3n-1}(i, \omega^{3n-1}), \omega^{3n-2}) & \text{if } \pi^{3n-1}(i, \omega^{3n-1}) \neq i \\ J & \text{if } \pi^{3n-1}(i, \omega^{3n-1}) = i. \end{cases}$$

Let $\hat{\rho}_{\omega^{3n-1}}^{3n-1} = \lambda_0 \left(\hat{\alpha}_{\omega^{3n-1}}^{3n-1}, \hat{g}_{\omega^{3n-1}}^{3n-1}\right)^{-1}$ be the internal cross-sectional extended type distribution after random matching.

Now, we consider the final step of random type changing with break-up for matched agents. Let $\Omega_{3n} = (S \times \{0,1\})^I$ with its internal power set $\mathcal{F}_{3n}$, where 0 represents "unmatched" and 1 represents "paired"; each point $\omega_{3n} = (\omega^1_{3n}, \omega^2_{3n}) \in \Omega_{3n}$ is an internal function from I to $S \times \{0,1\}$. Define a new type function $\hat{\alpha}^{3n} : (I \times \Omega^{3n}) \to S$ by letting $\hat{\alpha}^{3n}(i, \omega^{3n}) = \omega^1_{3n}(i)$. Fix $\omega^{3n-1} \in \Omega^{3n-1}$. For each $i \in I$, (1) if $\hat{\pi}^{3n-1}(i, \omega^{3n-1}) = i$ (i is not paired after the matching step at time n), let $\tau_i^{\omega^{3n-1}}$ be the probability measure on the type space $S \times \{0,1\}$ that gives probability one to the type $\left(\hat{\alpha}^{3n-2}(i, \omega^{3n-2}), 0\right)$ and zero for the rest; (2) if $\hat{\pi}^{3n-1}(i, \omega^{3n-1}) \neq i$ and $\hat{\pi}^{3n-3}(i, \omega^{3n-3}) = i$ (i is newly paired after the matching step at time n), $\hat{\alpha}^{3n-2}(i, \omega^{3n-2}) = k$, $\hat{\pi}^{3n-1}(i, \omega^{3n-1}) = j$ and $\hat{\alpha}^{3n-2}(j, \omega^{3n-2}) = l$, define a probability measure $\tau_{ij}^{\omega^{3n-1}}$ on $(S \times \{0,1\}) \times (S \times \{0,1\})$ such that

$$\tau_{ij}^{\omega^{3n-1}}\left((k',1),(l',1)\right) = \xi^n_{kl}(\hat{\rho}^{3n-1}_{\omega^{3n-1}})[\sigma^n_{kl}(\hat{\rho}^{3n-1}_{\omega^{3n-1}})](k',l')$$

and

$$\tau_{ij}^{\omega^{3n-1}}\left((k',0),(l',0)\right) = \left(1 - \xi^n_{kl}(\hat{\rho}^{3n-1}_{\omega^{3n-1}})\right)[\varsigma^n_{kl}(\hat{\rho}^{3n-1}_{\omega^{3n-1}})](k',l')$$

for $k', l' \in S$, and zero for the rest; (3) if $\hat{\pi}^{3n-1}(i, \omega^{3n-1}) \neq i$ and $\hat{\pi}^{3n-3}(i, \omega^{3n-3}) \neq i$ (i is already paired at time $n-1$), $\hat{\alpha}^{3n-2}(i, \omega^{3n-2}) = k$, $\hat{\pi}^{3n-1}(i, \omega^{3n-1}) = j$ and $\hat{\alpha}^{3n-2}(j, \omega^{3n-2}) = l$, define a probability measure $\tau_{ij}^{\omega^{3n-1}}$ on $(S \times \{0,1\}) \times (S \times \{0,1\})$ such that

$$\tau_{ij}^{\omega^{3n-1}}\left((k',1),(l',1)\right) = \left(1 - \hat{\vartheta}^n_{kl}(\hat{\rho}^{3n-1}_{\omega^{3n-1}})\right)\delta_k(k')\delta_l(l')$$

and

$$\tau_{ij}^{\omega^{3n-1}}\left((k',0),(l',0)\right) = \hat{\vartheta}^n_{kl}(\hat{\rho}^{3n-1}_{\omega^{3n-1}})[\hat{\varsigma}^n_{kl}(\hat{\rho}^{3n-1}_{\omega^{3n-1}})](k')[\hat{\varsigma}^n_{lk}(\hat{\rho}^{3n-1}_{\omega^{3n-1}})](l')$$

for $k', l' \in S$, and zero for the rest. Let $A^n_{\omega^{3n-1}} = \{(i,j) \in I \times I : i < j, \bar{\pi}^n(i, \omega^{3n-1}) = j\}$ and $B^n_{\omega^{3n-1}} = \{i \in I : \bar{\pi}^n(i, \omega^{3n-1}) = i\}$. Define an internal probability measure $Q_{3n}^{\omega^{3n-1}}$ on $(S \times \{0,1\})^I$ to be the internal product measure

$$\prod_{i \in B^n_{\omega^{3n-1}}} \tau_i^{\omega^{3n-1}} \otimes \prod_{(i,j) \in A^n_{\omega^{3n-1}}} \tau_{ij}^{\omega^{3n-1}}.$$

Let

$$\hat{\pi}^{3n}(i, \omega^{3n}) = \begin{cases} i & \text{if } \hat{\pi}^{3n-1}(i, \omega^{3n-1}) = i \text{ or } \omega^2_{3n}(i) = 0 \text{ or } \omega^2_{3n}(\bar{\pi}^n(i, \omega^{3n-1})) = 0 \\ \hat{\pi}^{3n-1}(i, \omega^{3n-1}) & \text{otherwise.} \end{cases}$$

and $\hat{g}^{3n}(i, \omega^{3n}) = \hat{\alpha}^{3n}(\hat{\pi}^{3n}(i, \omega^{3n}), \omega^{3n})$.

Keep repeating the construction. we can construct a hyperfinite sequence $\{(\Omega_m, \mathcal{E}_m, Q_m)\}_{m=1}^{3CM}$ of internal transition probability and a hyperfinite sequence $\{\hat{\alpha}^l\}_{l=0}^{CM}$ of internal type functions.

Let $(I \times \Omega^{3CM}, \mathcal{I}_0 \otimes \mathcal{E}^{3CM}, \lambda_0 \otimes Q^{3CM})$ be the internal product probability space of $(I, \mathcal{I}_0, \lambda_0)$ and $(\Omega^{3CM}, \mathcal{E}^{3CM}, Q^{3CM})$. Let $(I \times \Omega^{3CM}, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be the Loeb space of the internal product. For simplicity, let Ω^{3CM} be denoted by Ω. Let $\mathcal{F}^m = \{F \in \mathcal{E}^{3CM} : F = F^m \times \Pi_{m'=m+1}^{3CM} \Omega_{m'}$ and $F^m \in \mathcal{E}^m\}$. For any random variable f on $(\Omega^{m+1}, \mathcal{E}^{m+1}, Q^{m+1})$ and $\omega^m \in \Omega^m$, let $\mathbb{E}^{\omega^m} f = \int_{\Omega_{m+1}} f(\omega^{m+1}) dQ_{m+1}^{\omega^m}$ and $\mathrm{Var}^{\omega^m} f = \int_{\Omega_{m+1}} \left(f(\omega^{m+1}) - \mathbb{E}^{\omega^m} f \right)^2 dQ_{m+1}^{\omega^m}$.

For any $m \in \{1, \ldots, 3CM\}$ and $A \subseteq \Omega^m$, let $\bar{A} = \{\omega \in \Omega : \omega^m \in \Omega^m\}$. In the following, we will often work with functions or sets that are measurable in $(\Omega^m, \mathcal{F}^m, Q^m)$ or its Loeb space for some $m \leq 3CM$, which may be viewed as functions or sets based on $(\Omega^{3CM}, \mathcal{F}^{3CM}, Q^{3CM})$ or its Loeb space by allowing for dummy components for the tail part. We can thus continue to use P to denote the Loeb measure generated by Q^m for convenience. Since all the type functions, random matchings and the partners' type functions are internal in the relevant hyperfinite settings, they are all $\mathcal{I} \boxtimes \mathcal{F}$-measurable when viewed as functions on $I \times \Omega$.

7.3 Properties of the hyperfinite dynamic matching model

As above, we decompose each period into three steps. Let $e(m) = [\frac{m+2}{3}]$ and $f(m) = m - 3e(m) + 3$, then the m-th step in the hyperfinite dynamical system is also the $f(m)$-th step in the $e(m)$-th period.

Let $\tilde{\beta}_i^m = (\hat{\alpha}_i^m, \hat{g}_i^m, \hat{h}_i^m)$, where

$$\hat{h}_i^m = \begin{cases} 0 & \text{if } g_i^m \neq J \text{ and } g_i^{m-1} \neq J \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that $\hat{h}_i^m = 0$ if and only if i has been matched with another agent for at least two steps. Let $\tilde{\rho}_{kl0}^m$ be the fraction of agent that is of type k, matched with type l agent at the m-th step and paired in the last period; let $\tilde{\rho}_{kl1}^m$ be the fraction of agent that is of type k, matched with type l agent at the m-th step and single in the last period. Note that $\hat{\rho}_{kl}^m$ is the proportion of type k agent matched with type l agent at the m-th step, which implies $\hat{\rho}_{kl}^m = \tilde{\rho}_{kl0}^m + \tilde{\rho}_{kl1}^m$.

Let $\tilde{\Delta}$ be the space of all the probability measure $\tilde{p}$ on $\tilde{S} = S \times (S \cup \{J\}) \times \{0, 1\}$ such that $\tilde{p}_{klr} = \tilde{p}_{lkr}$ for any $k, l \in S$ and $r \in \{0, 1\}$. For each $k, l \in S$ and $n \leq CM$, we use the same notation $\hat{\eta}_{kl}^n$ to denote the mutation rate from ${}^*\tilde{\Delta} \to {}^*\mathbb{R}$ by letting $\hat{\eta}_i^n(\tilde{\rho}) = \hat{\eta}^n(\hat{\rho})$. We also extend the domain of $\hat{q}_{kl}^n$, $\hat{q}_k^n$, $\hat{\xi}_{kl}^n$, $\hat{\sigma}_{kl}^n$, $\hat{\varsigma}_{kl}^n$, $\hat{\vartheta}_{kl}^n$ in the same way.

For any finite dimensional hyper integer vector $\boldsymbol{m} = (m^1, \ldots, m^r)$ such that $3CM \geq m^1 > \ldots m^r \geq 0$, define $\tilde{\beta}_i^{\boldsymbol{m}} = (\tilde{\beta}^{m^1}, \ldots, \tilde{\beta}^{m^r})$. We say $m' > \boldsymbol{m}$ if $m' > m^1$. We define $\tilde{\beta}^m = \tilde{\beta}^0$ if $m < 0$. Then we have the following lemma.

Lemma 2 *$\tilde{\beta}_i$ satisfies the Markov property for any $i \in I$, that is*

$$P_0(\tilde{\beta}_i^m = a, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) P_0(\tilde{\beta}_i^{m_1} = a_1) - P_0(\tilde{\beta}_i^m = a, \tilde{\beta}_i^{m_1} = a_1) P_0(\tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2})$$

is infinitesimal for any $i \in I$ if $3CM \geq m > m_1 > \boldsymbol{m_2}$.

The following lemma shows that $\tilde{\beta}$ is, in some sense, essentially pairwisely independent.

Lemma 3 *For any $i \in I$, the following statement holds for P-almost all agent $j \in I$: for any m and $\boldsymbol{m_0}$ such that $\boldsymbol{m_0} < m$,*

$$P_0(\tilde{\beta}_i^m = a_1, \tilde{\beta}_j^m = a_2, \tilde{\beta}_i^{\boldsymbol{m_0}} = \boldsymbol{c_1}, \tilde{\beta}_j^{\boldsymbol{m_0}} = \boldsymbol{c_2}) - P_0(\tilde{\beta}_i^m = a_1, \tilde{\beta}_i^{\boldsymbol{m_0}} = \boldsymbol{c_1})P_0(\tilde{\beta}_j^m = a_2, \tilde{\beta}_i^{\boldsymbol{m_0}} = \boldsymbol{c_2})$$

is infinitesimal.

Let

$$\hat{H}_i^m = |\{n : \hat{\alpha}_i^{3n-2} \neq \hat{\alpha}_i^{3n-3} \text{ or } \hat{g}_i^{3n-2} \neq \hat{g}_i^{3n-3}, 3n-2 \leq m\}|,$$
$$\hat{N}_i^m = |\{n : \hat{g}_i^{3n-1} \neq \hat{g}^{3n-2}, 3n-1 \leq m\}|$$

and

$$\hat{R}_i^m = |\{n : \hat{g}_i^{3n} = J \text{ and } \hat{g}^{3n-1} \neq J, 3n \leq m\}|.$$

Then $\hat{H}_i^m$, $\hat{N}_i^m$, $\hat{R}_i^m$ are agent i's numbers of mutations, matchings and break-ups up to m-th step respectively. Let $\hat{X}_i^m = \hat{H}_i^m + \hat{N}_i^m + \hat{R}_i^m$. The following lemma provides a lower bound of the probability that there is no jump for the counting process $\hat{X}_i$ between two different steps.

Lemma 4

$$\begin{aligned} & P_0(\hat{X}_i^{m+\Delta m} = \hat{X}_i^m | F^m) \\ \geq\ & \left(1 - \frac{K\bar{\eta}^{e(m+\Delta m)}}{M}\right)^{2\Delta m} \left(1 - \frac{K\bar{q}^{e(m+\Delta m)}}{M}\right)^{\Delta m} \left(1 - \frac{K\bar{\vartheta}^{e(m+\Delta m)}}{M}\right)^{\Delta m} \\ \simeq\ & e^{-\frac{K\Delta m\left(2\bar{\eta}^{e(m+\Delta m)} + \bar{q}^{e(m+\Delta m)} + \bar{\vartheta}^{e(m+\Delta m)}\right)}{M}} \end{aligned}$$

holds for any $m, \Delta m \in \{0, \ldots, 3CM\}$ and $F^m \in \mathcal{F}^m$ such that $m + \Delta m \leq 3CM$, $\frac{\Delta m}{M}$ is finite and $P_0(F^m) > 0$.

By Lemma 4, it is easy to prove the following result.

Lemma 5

$$\|\mathbb{E}\left(\tilde{\rho}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}^m\right)\|_\infty \lesssim 1 - e^{-\frac{K\Delta m\left(2\bar{\eta}^{e(m+\Delta m)} + \bar{q}^{e(m+\Delta m)} + \bar{\vartheta}^{e(m+\Delta m)}\right)}{M}}$$

holds for any $m, \Delta m \in \{0, \ldots, 3CM\}$ and $F^m \in \mathcal{F}^m$ such that $m + \Delta m \leq 3CM$, $\frac{\Delta m}{M}$ is finite and $P_0(F^m) > 0$.

7.4 Existence of continuous time random matching

Let

$$\alpha_i(t) = \begin{cases} \lim_{t_0 \to t^+} \min\{\hat{\alpha}_i^{3n} : \frac{n}{M} \in monad(t_0)\} & \text{if the limit exists} \\ min\{\hat{\alpha}_i^{3n} : \frac{n}{M} \in monad(t)\} & \text{otherwise,} \end{cases}$$

$$\pi_i(t) = \hat{\pi}^{3n}, \text{ where } n \text{ is the integer part of } tM,$$

$$g_i(t) = \begin{cases} \lim_{t_0 \to t^+} \min\{\hat{g}_i^{3n} : \frac{n}{M} \in monad(t_0)\} & \text{if the limit exists} \\ min\{\tilde{g}_i^{3n} : \frac{n}{M} \in monad(t)\} & \text{otherwise,} \end{cases}$$

Let $E_t = \{n : \frac{n}{M} \in monad(t)\}$. Fix $i \in I$ and $\omega \in \Omega$, if $\hat{\alpha}_i^{3n}(\omega) \equiv C$ on E_t, by Spillover Principle, there exists n_1, n_2 such that $n_1 < t < n_2$, $n_1, n_2 \notin E_t$ and $\hat{\alpha}_i^{3n}(\omega) \equiv C$ on $\{n_1, n_1+1, \ldots, n_2\}$. Hence for any $t' \in \left(\text{st}(\frac{n_1}{M}), \text{st}(\frac{n_2}{M})\right)$, $min\{\hat{\alpha}_i^{3n} : \frac{n}{M} \in monad(t)\} = C$. Therefore,

$$\alpha_i(\omega, t) = \lim_{t_0 \to t^+} \min\{\hat{\alpha}_i^{3n} : \frac{n}{M} \in monad(t_0)\} = C.$$

Fix any $n_0 \in E_t$. If $\hat{\alpha}_i^{3n_0}(\omega) \neq \alpha_i(\omega, t)$, by the argument above, $\hat{\alpha}_i^{3n}(\omega)$ can not be constant on E_t. Hence there is a mutation or matching or break up in E_t. Therefore, by Lemma 4, for any n_1, n_2 such that $n_1 < t < n_2$, $n_1, n_2 \notin E_t$

$$\begin{aligned} P\left(\hat{\alpha}_i^{3n_0}(\omega) \neq \alpha_i(\omega, t)\right) &\leq P\left(\hat{X}_i^{3n_1}(\omega) \neq \hat{X}_i^{3n_2}(\omega)\right) \\ &\leq 1 - \text{st}\left(e^{-\frac{3K(n_2-n_1)\left(2\bar{\eta}^{e(n_2)} + \bar{q}^{e(n_2)} + \bar{\vartheta}^{e(n_2)}\right)}{M}}\right). \end{aligned}$$

Let $\frac{n_2-n_1}{M} \to 0$, $\text{st}\left(e^{-\frac{3K(n_2-n_1)\left(2\bar{\eta}^{e(n_2)} + \bar{q}^{e(n_2)} + \bar{\vartheta}^{e(n_2)}\right)}{M}}\right) \to 1$. Then $P\left(\hat{\alpha}_i^{3n_0}(\omega) \neq \alpha_i(\omega, t)\right) = 0$, which implies $P\left(\hat{\alpha}_i^{3n_0}(\omega) = \alpha_i(\omega, t)\right) = 1$. Similarly, we can prove $P\left(\hat{g}_i^{3n_0}(\omega) = g_i(\omega, t)\right) = 1$ for any n_0 such that $\frac{n_0}{M} \in monad(t)$. By Lemmas 2 and 3, $(\hat{\alpha}^{3n}, \hat{g}^{3n})$ is Markovian and independent. Therefore, (α, g) is also Markovian and independent.

Suppose $\frac{n}{M} \in monad(t)$ and $\frac{n+\Delta n}{M} \in monad(t + \Delta t)$. Then

$$\begin{aligned} & P_0\left(\hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} \geq 2 \,|\, (\hat{\alpha}_i^{3n}, \hat{g}_i^{3n}) = (k, l)\right) \\ \leq & \sum_{r=3n+1}^{3n+3\Delta n} \Big[P_0\left(\hat{X}_i^r = \hat{X}_i^{r-1} + 1, \hat{X}^{r-1} = \hat{X}^{3n} \,|\, (\hat{\alpha}_i^{3n}, \hat{g}_i^{3n}) = (k, l)\right) \\ & P_0\left(\hat{X}^{3n+3\Delta n} - \hat{X}_i^r \geq 1 \,|\, \hat{X}_i^r = \hat{X}_i^{r-1} + 1, \hat{X}^{r-1} = \hat{X}^{3n}, (\hat{\alpha}_i^{3n}, \hat{g}_i^{3n}) = (k, l)\right) \Big] \end{aligned}$$

Note that

$$\sum_{r=3n+1}^{3n+3\Delta n} P_0\left(\hat{X}_i^r = \hat{X}_i^{r-1}+1, \hat{X}^{r-1} = \hat{X}^{3n} \,|\, (\hat{\alpha}_i^{3n}, \hat{g}_i^{3n}) = (k,l)\right) = P_0\left(\hat{X}_i^{3n+3\Delta n} > \hat{X}^{3n} \,|\, (\hat{\alpha}_i^{3n}, \hat{g}_i^{3n}) = (k,l)\right)$$

and

$$\begin{aligned} & P_0\left(\hat{X}^{3n+3\Delta n} - \hat{X}_i^r \geq 1 \,|\, \hat{X}_i^r = \hat{X}_i^{r-1}+1, \hat{X}^{r-1} = \hat{X}^{3n}, (\hat{\alpha}_i^{3n}, \hat{g}_i^{3n}) = (k,l)\right) \\ \lesssim\ & 1 - e^{-\frac{3K\Delta n\left(2\bar{\eta}^{n+\Delta n}+\bar{q}^{n+\Delta n}+\bar{\vartheta}^{n+\Delta n}\right)}{M}} \\ \simeq\ & 1 - e^{-3K\Delta t\left(2\bar{\eta}^{n+\Delta n}+\bar{q}^{n+\Delta n}+\bar{\vartheta}^{n+\Delta n}\right)} \end{aligned}$$

Then

$$\begin{aligned} & P_0\left(\hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} \geq 2 \,|\, (\hat{\alpha}_i^{3n}, \hat{g}_i^{3n}) = (k,l)\right) \\ \lesssim\ & \left(1 - e^{-3K\Delta t\left(2\bar{\eta}^{n+\Delta n}+\bar{q}^{n+\Delta n}+\bar{\vartheta}^{n+\Delta n}\right)}\right) \sum_{r=3n+1}^{3n+3\Delta n} P_0\left(\hat{X}_i^r = \hat{X}_i^{r-1}+1, \hat{X}^{r-1} = \hat{X}^{3n} \,|\, (\hat{\alpha}_i^{3n}, \hat{g}_i^{3n}) = (k,l)\right) \\ =\ & \left(1 - e^{-3K\Delta t\left(2\bar{\eta}^{n+\Delta n}+\bar{q}^{n+\Delta n}+\bar{\vartheta}^{n+\Delta n}\right)}\right) P_0\left(\hat{X}_i^{3n+3\Delta n} > \hat{X}^{3n} \,|\, (\hat{\alpha}_i^{3n}, \hat{g}_i^{3n}) = (k,l)\right) \\ \lesssim\ & \left(1 - e^{-3K\Delta t\left(2\bar{\eta}^{n+\Delta n}+\bar{q}^{n+\Delta n}+\bar{\vartheta}^{n+\Delta n}\right)}\right)^2 \\ =\ & O(\Delta t^2) \end{aligned} \tag{8}$$

It remains to check Equations (1) to (4). For Equation (1), fix any $k, l, k' \in S$ such that $k \neq k'$. By Equation (8),

$$\begin{aligned} & P\left(\beta_i(t+\Delta t) = (k', l) \,|\, \beta_i(t) = (k,l)\right) \\ \simeq\ & P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', l), \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,|\, \hat{\beta}_i^{3n} = (k,l)\right) + O(\Delta t^2). \end{aligned}$$

For any agent $i \in I$ with extended type (k, l), mutation is the only way for her to become an agent with extended type (k', l) given $\hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1$. Therefore,

$$\begin{aligned}
& P\left(\beta_i(t+\Delta t) = (k', l) \,|\, \beta_i(t) = (k, l)\right) \\
\simeq\ & P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', l), \hat{H}_i^{3n+3\Delta n} - \hat{H}_i^{3n} = \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,|\, \hat{\beta}_i^{3n} = (k, l)\right) + O(\Delta t^2) \\
\simeq\ & \sum_{r=n}^{n+\Delta n-1} P_0\Big(\hat{\beta}_i^{3r+1} = (k', l), \hat{H}_i^{3n+3\Delta n} = \hat{H}_i^{3r} + 1 = \hat{H}_i^{3n} + 1, \\
& \hat{X}_i^{3n+3\Delta n} = \hat{X}_i^{3r} + 1 = \hat{X}_i^{3n} + 1 \,|\, \hat{\beta}_i^{3n} = (k, l)\Big) + O(\Delta t^2) \\
\simeq\ & \sum_{r=n}^{n+\Delta n-1} \Big[P_0\Big(\hat{\beta}_i^{3r+1} = (k', l), \hat{H}_i^{3r} = \hat{H}_i^{3n}, \hat{X}_i^{3r} = \hat{X}_i^{3n} \,|\, \hat{\beta}_i^{3n} = (k, l)\Big) \\
& P_0\Big(\hat{X}_i^{3n+3\Delta n} = \hat{X}_i^{3r+1} \,|\, \hat{\beta}_i^{3r+1} = (k', l), \hat{H}_i^{3r} = \hat{H}_i^{3n}, \hat{X}_i^{3r} = \hat{X}_i^{3n}, \hat{\beta}_i^{3n} = (k, l)\Big)\Big] + O(\Delta t^2) \\
\simeq\ & \sum_{r=n}^{n+\Delta n-1} \Big[P_0\Big(\hat{H}_i^{3r} = \hat{H}_i^{3n}, \hat{X}_i^{3r} = \hat{X}_i^{3n} \,|\, \hat{\beta}_i^{3n} = (k, l)\Big) \\
& P_0\left(\hat{\beta}_i^{3r+1} = (k', l) \,|\, \hat{H}_i^{3r} = \hat{H}_i^{3n}, \hat{X}_i^{3r} = \hat{X}_i^{3n}, \hat{\beta}_i^{3n} = (k, l)\right) \\
& P_0\Big(\hat{X}_i^{3n+3\Delta n} = \hat{X}_i^{3r+1} \,|\, \hat{\beta}_i^{3r+1} = (k', l), \hat{H}_i^{3r} = \hat{H}_i^{3n}, \hat{X}_i^{3r} = \hat{X}_i^{3n}, \hat{\beta}_i^{3n} = (k, l)\Big)\Big] + O(\Delta t^2)
\end{aligned}$$

By Equation (14),

$$\begin{aligned}
& \left|P_0\left(\hat{\beta}_i^{3r+1} = (k', l) \,|\, \hat{H}_i^{3r} = \hat{H}_i^{3n}, \hat{X}_i^{3r} = \hat{X}_i^{3n}, \hat{\beta}_i^{3n} = (k, l)\right) - \hat{\eta}_{kk'}^{r+1}\left(U_1^{3r+1}(\tilde{\rho}^0)\right)\hat{\eta}_{ll}^{r+1}\left(U_1^{3r+1}(\tilde{\rho}^0)\right)\right| \\
\leq\ & \frac{\epsilon_0}{P_0\left(\hat{H}_i^{3r} = \hat{H}_i^{3n}, \hat{X}_i^{3r} = \hat{X}_i^{3n}, \hat{\beta}_i^{3n} = (k, l)\right)} + \xi_{-1}.
\end{aligned}$$

Then

$$\begin{aligned}
& \left|P_0\left(\hat{\beta}_i^{3r+1} = (k', l) \,|\, \hat{H}_i^{3r} = \hat{H}_i^{3n}, \hat{X}_i^{3r} = \hat{X}_i^{3n}, \hat{\beta}_i^{3n} = (k, l)\right) - \hat{\eta}_{kk'}^{r+1}\left(U_1^{3r+1}(\tilde{\rho}^0)\right)\right| \\
\leq\ & \left|\hat{\eta}_{kk'}^{r+1}\left(U_1^{3r+1}(\tilde{\rho}^0)\right)\hat{\eta}_{ll}^{r+1}\left(U_1^{3r+1}(\tilde{\rho}^0)\right) - \hat{\eta}_{kk'}^{r+1}\left(U_1^{3r+1}(\tilde{\rho}^0)\right)\right| \\
& + \frac{\epsilon_0}{P_0\left(\hat{H}_i^{3r} = \hat{H}_i^{3n}, \hat{X}_i^{3r} = \hat{X}_i^{3n}, \hat{\beta}_i^{3n} = (k, l)\right)} + \xi_{-1} \\
\leq\ & K\left(\frac{\overline{\eta}^{r+1}}{M}\right)^2 + \frac{\epsilon_0}{P_0\left(\hat{H}_i^{3r} = \hat{H}_i^{3n}, \hat{X}_i^{3r} = \hat{X}_i^{3n}, \hat{\beta}_i^{3n} = (k, l)\right)} + \xi_{-1}.
\end{aligned}$$

Therefore

$$
\begin{aligned}
& \left| P\left(\beta_i(t+\Delta t)=(k',l) \,|\, \beta_i(t)=(k,l)\right) - \sum_{r=n}^{n+\Delta n-1} \hat{\eta}_{kk'}^{r+1}\left(U_1^{3r+1}(\tilde{\rho}^0)\right) \right| \\
\lesssim \; & \sum_{r=n}^{n+\Delta n-1} \Big| P_0\Big(\hat{H}_i^{3r}=\hat{H}_i^{3n}, \hat{X}_i^{3r}=\hat{X}_i^{3n} \,|\, \hat{\beta}_i^{3n}=(k,l)\Big) \\
& \Big(P_0\left(\hat{\beta}_i^{3r+1}=(k',l) \,|\, \hat{H}_i^{3r}=\hat{H}_i^{3n}, \hat{X}_i^{3r}=\hat{X}_i^{3n}, \hat{\beta}_i^{3n}=(k,l)\right) - \hat{\eta}_i^{r+1}\left(U_1^{3r+1}(\tilde{\rho}^0)\right)\Big) \\
& P_0\Big(\hat{X}_i^{3n+3\Delta n}=\hat{X}_i^{3r+1} \,|\, \hat{\beta}_i^{3r+1}=(k',l), \hat{H}_i^{3r}=\hat{H}_i^{3n}, \hat{X}_i^{3r}=\hat{X}_i^{3n}, \hat{\beta}_i^{3n}=(k,l)\Big) \Big| \\
& + \sum_{r=n}^{n+\Delta n-1} \hat{\eta}_{kk'}^{r+1}\left(U_1^{3r+1}(\tilde{\rho}^0)\right) \Big| P_0\Big(\hat{H}_i^{3r}=\hat{H}_i^{3n}, \hat{X}_i^{3r}=\hat{X}_i^{3n} \,|\, \hat{\beta}_i^{3n}=(k,l)\Big) \\
& P_0\left(\hat{X}_i^{3n+3\Delta n}=\hat{X}_i^{3r+1} \,|\, \hat{\beta}_i^{3r+1}=(k',l), \hat{H}_i^{3r}=\hat{H}_i^{3n}, \hat{X}_i^{3r}=\hat{X}_i^{3n}, \hat{\beta}_i^{3n}=(k,l)\right) - 1 \Big| + O(\Delta t^2) \\
\lesssim \; & \sum_{r=n}^{n+\Delta n-1} \hat{\eta}_{kk'}^{r+1}\left(U_1^{3r+1}(\tilde{\rho}^0)\right) \Big| P_0\Big(\hat{H}_i^{3r}=\hat{H}_i^{3n}, \hat{X}_i^{3r}=\hat{X}_i^{3n} \,|\, \hat{\beta}_i^{3n}=(k,l)\Big) \\
& P_0\left(\hat{X}_i^{3n+3\Delta n}=\hat{X}_i^{3r+1} \,|\, \hat{\beta}_i^{3r+1}=(k',l), \hat{H}_i^{3r}=\hat{H}_i^{3n}, \hat{X}_i^{3r}=\hat{X}_i^{3n}, \hat{\beta}_i^{3n}=(k,l)\right) - 1 \Big| \\
& + \sum_{r=n}^{n+\Delta n-1} \left(K\left(\frac{\bar{\eta}^{r+1}}{M}\right)^2 + \frac{\epsilon_0+\xi_{-1}}{P_0\left(\hat{\beta}_i^{3n}=(k,l)\right)} \right) \\
\lesssim \; & \sum_{r=n}^{n+\Delta n-1} \frac{\bar{\eta}^{n+\Delta n}}{M} \left| e^{-\frac{3K\Delta n\left(2\bar{\eta}^{n+\Delta n}+\bar{q}^{n+\Delta n}+\bar{\vartheta}^{n+\Delta n}\right)}{M}} e^{-\frac{3K\Delta n\left(2\bar{\eta}^{n+\Delta n}+\bar{q}^{n+\Delta n}+\bar{\vartheta}^{n+\Delta n}\right)}{M}} - 1 \right| \\
& + \frac{K\left(\bar{\eta}^{n+\Delta n}\right)^2}{M} + \frac{M\epsilon_0 + M\xi_{-1}}{P_0\left(\hat{\beta}_i^{3n}=(k,l)\right)} + O(\Delta t^2) \\
\lesssim \; & \bar{\eta}^{n+\Delta n}\Delta t \left| e^{-6K\Delta t\left(2\bar{\eta}^{n+\Delta n}+\bar{q}^{n+\Delta n}+\bar{\vartheta}^{n+\Delta n}\right)} - 1 \right| + O(\Delta t^2) \\
= \; & O(\Delta t^2).
\end{aligned}
$$

Note that $P\left(\beta_i^t = \hat{\beta}_i^{3n}\right) = 1$ for any $i \in I$. Then $\mathbb{E}\left(\hat{p}^t\right) \simeq \mathbb{E}\left(\hat{\rho}^{3n}\right)$. Therefore,

$$
\begin{aligned}
& \frac{1}{\Delta t}\left| \sum_{r=n}^{n+\Delta n-1} \hat{\eta}_{kk'}^{r+1}\left(U_1^{3r+1}(\tilde{\rho}^0)\right) - \eta_{kk'}\left(t, \mathbb{E}\left(\hat{p}^t\right)\right)\Delta t \right| \\
\simeq \; & \frac{1}{\Delta t}\left| \sum_{r=n}^{n+\Delta n-1} \frac{1}{M}{}^*\eta_{kk'}\left(\frac{r+1}{M}, U_1^{3r+1}(\tilde{\rho}^0)\right) - \frac{{}^*\eta_{kk'}\left(\frac{n}{M}, \mathbb{E}\left(\hat{\rho}^{3n}\right)\right)}{M}\Delta n \right| \\
\lesssim \; & \frac{\Delta n}{M\Delta t}\frac{1}{\Delta n}\sum_{r=n}^{n+\Delta n-1} \left| {}^*\eta_{kk'}\left(\frac{r+1}{M}, \mathbb{E}\left(\hat{\rho}^{3r+1}\right)\right) - {}^*\eta_{kk'}\left(\frac{n}{M}, \mathbb{E}\left(\hat{\rho}^{3n}\right)\right) \right|.
\end{aligned}
$$

By Lemma 5,

$$
\|\mathbb{E}\left(\hat{\rho}^{3r+1}\right) - \mathbb{E}\left(\hat{\rho}^{3n}\right)\|_\infty \leq 1 - e^{-\frac{3K\Delta n\left(2\bar{\eta}^{e(n+\Delta n)}+\bar{q}^{e(n+\Delta n)}+\bar{\vartheta}^{e(n+\Delta n)}\right)}{M}} \doteq \epsilon(\Delta n)
$$

for any r such that $n \leq r \leq n + \Delta n - 1$. Let

$$W(\Delta n) = \{(t', \hat{\rho}') : |t' - \frac{n}{M}| < \frac{\Delta n}{M}, ||\hat{\rho}' - \mathbb{E}\left(\hat{\rho}^{3n}\right)||_\infty < \epsilon(\Delta n)\}$$

Then

$$\begin{aligned} & \frac{1}{\Delta t}\left|\sum_{r=n}^{n+\Delta n-1} \hat{\eta}_{kk'}^{r+1}\left(U_1^{3r+1}(\tilde{\rho}^0)\right) - \eta_{kk'}\left(t, \mathbb{E}\left(\hat{p}^t\right)\right)\Delta t\right| \\ \lesssim\ & \frac{\Delta n}{M\Delta t}\frac{1}{\Delta n}\sum_{r=n}^{n+\Delta n-1} \sup_{(t',\hat{\rho}')\in W(\Delta n)}\left|{}^*\eta_{kk'}\left(t', \hat{\rho}'\right) - {}^*\eta_{kk'}\left(\frac{n}{M}, \mathbb{E}\left(\hat{\rho}^{3n}\right)\right)\right| \\ \lesssim\ & \frac{\Delta n}{M\Delta t}\sup_{(t',\hat{\rho}')\in W(\Delta n)}\left|{}^*\eta_{kk'}\left(t', \hat{\rho}'\right) - {}^*\eta_{kk'}\left(\frac{n}{M}, \mathbb{E}\left(\hat{\rho}^{3n}\right)\right)\right| \end{aligned}$$

By the continuity of ${}^*\eta$, $\sup_{(t',\hat{\rho}')\in W(\Delta n)}\left|{}^*\eta_{kk'}\left(t', \hat{\rho}'\right) - {}^*\eta_{kk'}\left(\frac{n}{M}, \mathbb{E}\left(\hat{\rho}^{3n}\right)\right)\right| \to 0$. as $\Delta t \to 0$, which implies

$$\left|\sum_{r=n}^{n+\Delta n-1} \hat{\eta}_{kk'}^{r+1}\left(U_1^{3r+1}(\tilde{\rho}^0)\right) - \eta_{kk'}\left(t, \mathbb{E}\left(\hat{p}^t\right)\right)\Delta t\right| = o(\Delta t).$$

Therefore,

$$P\left(\beta_i(t+\Delta t) = (k', l) \,|\, \beta_i(t) = (k, l)\right) = \eta_{kk'}\left(t, \mathbb{E}\left(\hat{p}^t\right)\right)\Delta t + o(\Delta t).$$

We can prove

$$P\left(\beta_i(t+\Delta t) = (k, l') \,|\, \beta_i(t) = (k, l')\right) = \eta_{ll'}\left(t, \mathbb{E}\left(\hat{p}^t\right)\right)\Delta t + o(\Delta t).$$

and

$$P\left(\beta_i(t+\Delta t) = (k', l') \,|\, \beta_i(t) = (k, l)\right) = o(\Delta t).$$

in the same way, which implies (1). Similarly, we can prove (2), (3), (4).

Therefore $\mathbb{D}$ is a dynamical system with parameters $(\hat{p}^0, \eta, \theta, \xi, \sigma, \varsigma, \vartheta)$ that is Markovian and independent.

7.5 Proofs of Lemmas 1 – 5

The proof of Lemma 1 is given in Subsection 7.5.1. In order to prove Lemmas 2 – 5, some additional lemmas are presented in Subsection 7.5.2. Lemmas 2 – 5 are then proved in Subsections 7.5.3 – 7.5.6 respectively.

7.5.1 Proof of Lemma 1

For each $k \in S$, let $\eta_k = 1 - \sum_{r \in S} q_{kr}$ (the no-matching probability for a type-k agent), and $I_k = \{i \in I : \alpha^0(i) = k, \pi^0(i) = i\}$ (the set of type-k agents who are initially unmatched). Let

$$\Omega_0 = \{(A_{kl})_{k,l \in S} : \quad A_{kl} \subseteq I_k,\ A_{kl} \text{ is internal, } |A_{kl}| \text{ is the largest even integer less equal than } |I_k| q_{kl},$$
$$A_{kl} \text{ and } A_{kl'} \text{ are disjoint for different } l \text{ and } l'\}.$$

Let μ_0 be the internal counting probability measure on $(\Omega_0, \mathcal{A}_0)$, where $\mathcal{A}_0$ is the internal power set of Ω_0.

For any fixed $\omega_0 = (A_{kl})_{k,l \in S} \in \Omega_0$, we consider internal partial matchings on I that match agents from A_{kl} to A_{lk}. We only need to consider those sets A_{kl} which are nonempty. For each $k \in S$, let $\Omega_{kk}^{\omega_0}$ be the internal set of all the internal full matchings on A_{kk}. Let $\mu_{kk}^{\omega_0}$ be the internal counting probability measure on $\Omega_{kk}^{\omega_0}$. For $k, l \in S$ with $k < l$, let $\Omega_{kl}^{\omega_0}$ be the internal set of all the internal bijections from A_{kl} to A_{lk}. Let $\mu_{kl}^{\omega_0}$ the internal counting probability on A_{kl}. Let Ω_1 be the internal set of all the internal partial matchings from I to I. Define $\Omega_1^{\omega_0}$ to be the set of $\phi \in \Omega_1$, with

(i) the restriction $\phi|_H = \pi^0|_H$, where H is the set $\{i : \pi^0(i) \neq i\}$ of initially matched agents;

(ii) $\{i \in I_k : \phi(i) = i\} = I_k \setminus \left(\cup_{l=1}^K A_{kl}\right)$ for each $k \in S$;

(iii) the restriction $\phi|_{A_{kk}} \in \Omega_{kk}^{\omega_0}$ for $k \in S$;

(iv) for $k, l \in S$ with $k < l$, $\phi|_{A_{kl}} \in \Omega_{kl}^{\omega_0}$.

(i) means that initially matched agents remain matched with the same partners. The rest is clear.

Define an internal probability measure $\mu_1^{\omega_0}$ on Ω_1 such that such that

(i) for $\phi \in \Omega_1^{\omega_0}$,
$$\mu_1^{\omega_0}(\phi) = \prod_{1 \le k \le l \le K, A_{kl} \neq \emptyset} \mu_{kl}^{\omega_0}(\phi|_{A_{kl}});$$

(ii) $\phi \notin \Omega_1^{\omega_0}$, $\mu_1^{\omega_0}(\phi) = 0$.

The purpose of introducing the space $\Omega_1^{\omega_0}$ and the internal probability measure $\mu_1^{\omega_0}$ is to match the agents in A_{kl} to the agents A_{lk} randomly. The probability measure $\mu_1^{\omega_0}$ is trivially extended to the common sample space Ω_1.

Define an internal probability measure P_0 on $\Omega = \Omega_0 \times \Omega_1$ with the internal power set $\mathcal{F}_0$ by letting
$$P_0\left((\omega_0, \omega_1)\right) = \mu_0(\omega_0) \times \mu_1^{\omega_0}(\omega_1).$$

For $(i,\omega) \in I \times \Omega$, let $\pi(i,(\omega_0,\omega_1)) = \omega_1(i)$, and $g(i,\omega) = \begin{cases} \alpha^0(\pi(i,\omega)) & \text{if } \pi(i,\omega) \neq i \\ J & \text{if } \pi(i,\omega) = i. \end{cases}$ Denote the corresponding Loeb probability spaces of the internal probability spaces $(\Omega, \mathcal{F}_0, P_0)$ and $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes P_0)$ respectively by $(\Omega, \mathcal{F}, P)$ and $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Since π is an internal function from $I \times \Omega$ to I, it is $\mathcal{I} \boxtimes \mathcal{F}$-measurable.

Denote the internal set $\{(\omega_0,\omega_1) \in \Omega : \omega_0 \in \Omega_0,\ \omega_1 \in \Omega_1^{\omega_0,}\}$ by $\hat{\Omega}$. By the construction of P_0, it is clear that $P_0\left(\hat{\Omega}\right) = 1$. By its construction, it is clear that π is an internal random matching and satisfies part (i) of the lemma.

It remains to prove part (ii)of the lemma. Suppose $\alpha^0(i) = k_1$, $\pi^0(i) = i$ and $\hat{\rho}_{k_1J} > \hat{M}^{\frac{-1}{3}}$. Let M_k and m_{kl} be the internal cardinality of I_k and A_{kl} respectively. Let $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ denote the binomial coeffiecient. Then we have

$$P_0(g(i) = l_1) = \mu_0(\{(A_{kl})_{k,l\in S} : i \in A_{k_1l_1}\}) = \frac{\binom{M_{k_1}-1}{m_{k_1l_1}-1}}{\binom{M_{k_1}}{m_{k_1l_1}}} = \frac{m_{k_1l_1}}{M_{k_1}}.$$

It is clear that $P_0(g(i) = l_1) \leq \frac{M_{k_1}q_{k_1l_1}}{M_{k_1}} = q_{k_1l_1}$. Note that

$$\begin{aligned} P_0(g(i) = l_1) &\geq \frac{M_{k_1}q_{k_1l_1} - 2}{M_{k_1}} = q_{k_1l_1} - \frac{2}{M_{k_1}} \\ &= q_{k_1l_1} - \frac{2}{M_{k_1}} = q_{k_1l_1} - \frac{2}{\hat{M}\hat{\rho}_{k_1J}} \geq q_{k_1l_1} - \frac{2}{\hat{M}^{\frac{2}{3}}}. \end{aligned}$$

Then

$$q_{k_1l_1} - \frac{2}{\hat{M}^{\frac{2}{3}}} \leq P_0(g(i) = l_1) \leq q_{k_1l_1} \tag{9}$$

In addition, suppose $\alpha^0(j) = k_2$, $\pi^0(j) = j$, $j \neq i$ and $\hat{\rho}_{k_2J} > \hat{M}^{\frac{-1}{3}}$

If $k_1 \neq k_2$, then

$$\begin{aligned} P_0(g(i) = l_1, g(j) = l_2) &= \mu_0(\{(A_{kl})_{k,l\in S} : i \in A_{k_1l_1}, j \in A_{k_2l_2}\}) \\ &= \frac{\binom{M_{k_1}-1}{m_{k_1l_1}-1}}{\binom{M_{k_1}}{m_{k_1l_1}}} \frac{\binom{M_{k_2}-1}{m_{k_2l_2}-1}}{\binom{M_{k_2}}{m_{k_2l_2}}} = P_0(g(i) = l_1)P_0(g(j) = l_2) \end{aligned}$$

By Equation (9),

$$q_{k_1l_1}q_{k_2l_2} \geq P_0(g(i) = l_1, g(j) = l_2) \geq (q_{k_1l_1} - \frac{2}{\hat{M}^{\frac{2}{3}}})(q_{k_2l_2} - \frac{2}{\hat{M}^{\frac{2}{3}}}) \geq q_{k_1l_1}q_{k_2l_2} - \frac{4}{\hat{M}^{\frac{2}{3}}} \tag{10}$$

If $k_1 = k_2$ but $l_1 \neq l_2$, then

$$P_0(g(i) = l_1, g(j) = l_2) = \mu_0(\{(A_{kl})_{k,l\in S} : i \in A_{k_1l_1}, j \in A_{k_1l_2}\}) = \frac{\binom{M_{k_1}-2}{m_{k_1l_1}-1, m_{k_1l_2}-1}}{\binom{M_{k_1}}{m_{k_1l_1}, m_{k_1l_2}}},$$

where $\binom{a}{b,c} = \frac{a!}{b!c!(a-b-c)!}$ is the multinomial coefficient. It is clear that

$$\begin{aligned}
P_0(g(i) = l_1, g(j) = l_2) &= \frac{m_{k_1 l_1} m_{k_1 l_2}}{M_{k_1}(M_{k_1} - 1)} \leq \frac{m_{k_1 l_1}(m_{k_1 l_2} + 1)}{M_{k_1}^2} \\
&\leq q_{k_1 l_1} q_{k_2 l_2} + q_{k_1 l_1} \frac{1}{M_{k_1}} \leq q_{k_1 l_1} q_{k_2 l_2} + \frac{1}{M_{k_1}} \\
&\leq q_{k_1 l_1} q_{k_2 l_2} + \frac{1}{\hat{M} \hat{\rho}_{k_1 J}} \leq q_{k_1 l_1} q_{k_2 l_2} + \frac{1}{\hat{M}^{\frac{2}{3}}}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\frac{m_{k_1 l_1} m_{k_1 l_2}}{M_{k_1}(M_{k_1} - 1)} \\
\geq\ & \frac{(M_{k_1} q_{k_1 l_1} - 2)}{M_{k_1}} \frac{(M_{k_1} q_{k_1 l_2} - 2)}{M_{k_1}} \\
\geq\ & q_{k_1 l_1} q_{k_2 l_2} - \frac{2}{M_{k_1}} q_{k_1 l_1} - \frac{2}{M_{k_1}} q_{k_1 l_2} \\
\geq\ & q_{k_1 l_1} q_{k_2 l_2} - \frac{4}{M_{k_1}} \\
\geq\ & q_{k_1 l_1} q_{k_2 l_2} - \frac{4}{\hat{M} \hat{\rho}_{k_1 J}} \\
\geq\ & q_{k_1 l_1} q_{k_2 l_2} - \frac{4}{\hat{M}^{\frac{2}{3}}}.
\end{aligned}$$

Therefore,

$$q_{k_1 l_1} q_{k_2 l_2} + \frac{1}{\hat{M}^{\frac{2}{3}}} \geq P_0(g(i) = l_1, g(j) = l_2) \geq q_{k_1 l_1} q_{k_2 l_2} - \frac{4}{\hat{M}^{\frac{2}{3}}} \tag{11}$$

If $k_1 = k_2$ and $l_1 = l_2$, then

$$P_0(g(i) = l_1, g(j) = l_1) = \mu_0(\{(A_{kl})_{k,l \in S} : i, j \in A_{k_1 l_1}\}) = \frac{\binom{M_{k_1} - 2}{m_{k_1 l_1} - 2}}{\binom{M_{k_1}}{m_{k_1 l_1}}}.$$

It is clear that

$$\begin{aligned}
P_0(g(i) = l_1, g(j) = l_1) &= \frac{(m_{k_1 l_1})(m_{k_1 l_1} - 1)}{M_{k_1}(M_{k_1} - 1)} \\
&\leq \frac{m_{k_1 l_1}^2}{M_{k_1}^2} \\
&\leq q_{k_1 l_1}^2
\end{aligned}$$

On the other hand,

$$\begin{aligned}
& \frac{(m_{k_1l_1})(m_{k_1l_1}-1)}{M_{k_1}(M_{k_1}-1)} \\
\geq \quad & \frac{(M_{k_1}q_{k_1l_1}-2)}{M_{k_1}}\frac{(M_{k_1}q_{k_1l_1}-3)}{M_{k_1}} \\
\geq \quad & q_{k_1l_1}q_{k_2l_2} - \frac{5}{M_{k_1}}q_{k_1l_1} \\
\geq \quad & q_{k_1l_1}q_{k_2l_2} - \frac{5}{\hat{M}^{\frac{2}{3}}}.
\end{aligned}$$

Therefore,

$$q_{k_1l_1}^2 \geq P_0(g(i)=l_1, g(j)=l_2) \geq q_{k_1l_1}^2 - \frac{5}{\hat{M}^{\frac{2}{3}}} \tag{12}$$

Combining Equation (10), (11) and (13), for any $(k_1,l_1),(k_2,l_2) \in S^2$,

$$q_{k_1l_1}q_{k_2l_2} + \frac{1}{\hat{M}^{\frac{2}{3}}} \geq P_0(g(i)=l_1, g(j)=l_2) \geq q_{k_1l_1}q_{k_2l_2} - \frac{5}{\hat{M}^{\frac{2}{3}}} \tag{13}$$

7.5.2 Some additional lemmas

First we define three transformations T_1^n, T_2^n, T_3^n on $\tilde{\Delta}$ as follows:

$$[T_1^n(\tilde{\rho})]_{kl0} = \begin{cases} \sum_{k',l'\in S} \tilde{\rho}_{k'l'0}\hat{\eta}^n_{k'k}(\tilde{\rho})\hat{\eta}^n_{l'l}(\tilde{\rho}) & \text{if } l \neq J \\ 0 & \text{if } l = J, \end{cases}$$

$$[T_1^n(\tilde{\rho})]_{kl1} = \begin{cases} 0 & \text{if } l \neq J \\ \sum_{r\in S} \tilde{\rho}_{rJ1}\hat{\eta}^n_{rk}(\tilde{\rho}) & \text{if } l = J, \end{cases}$$

$$[T_2^n(\tilde{\rho})]_{kl0} = \begin{cases} \tilde{\rho}_{kl0} & \text{if } l \neq J \\ 0 & \text{if } l = J, \end{cases}$$

$$[T_2^n(\tilde{\rho})]_{kl1} = \begin{cases} \tilde{\rho}_{kJ1}\hat{q}^n_{kl}(\tilde{\rho}) & \text{if } l \neq J \\ \tilde{\rho}_{kJ1}\hat{q}^n_{k}(\tilde{\rho}) & \text{if } l = J, \end{cases}$$

$$[T_3^n(\tilde{\rho})]_{kl0} = \begin{cases} \tilde{\rho}_{kl0}\left(1-\hat{\vartheta}^n_{kl}(\tilde{\rho})\right) + \sum_{k',l'\in S} \tilde{\rho}_{k'l'1}\hat{\xi}^n_{k'l'}(\tilde{\rho})[\hat{\sigma}^n_{k'l'}(\tilde{\rho})](k,l) & \text{if } l \neq J \\ 0 & \text{if } l = J, \end{cases}$$

$$[T_3^n(\tilde{\rho})]_{kl1} = \begin{cases} 0 & \text{if } l \neq J \\ \sum_{k',l'\in S} \tilde{\rho}_{k'l'1}\left(1-\hat{\xi}^n_{k'l'}(\tilde{\rho})\right)[\hat{\varsigma}^n_{k'l'}(\tilde{\rho})](k) + \sum_{k',l'\in S} \tilde{\rho}_{k'l'0}\hat{\vartheta}^n_{k'l'}(\tilde{\rho})[\hat{\varsigma}^n_{k'l'}(\tilde{\rho})](k) + \tilde{\rho}^n_{kJ1} & \text{if } l = J. \end{cases}$$

We use $U^{m_2}_{m_1}$ to represent $T^{e(m_2)}_{f(m_2)} \circ T^{e(m_2-1)}_{f(m_2-1)} \circ \cdots \circ T^{e(m_1)}_{f(m_1)}$.

For any $a, b \in \tilde{S}$ and $n \in \{1, \ldots, CM\}$, define

$$B^{3n-2}_{ab}(\tilde{\rho}) = \begin{cases} \hat{\eta}^n_{k_1 l_1}(\tilde{\rho})\hat{\eta}^n_{k_2 l_2}(\tilde{\rho}) & \text{if } a = (k_1, k_2, 0), b = (l_1, l_2, 0) \\ \hat{\eta}^n_{k_1 l_1}(\tilde{\rho}) & \text{if } a = (k_1, J, 1), b = (l_1, J, 1) \\ 0 & \text{otherwise.} \end{cases}$$

$$B^{3n}_{ab}(\tilde{\rho}) = \begin{cases} 1 - \hat{\vartheta}^n_{k_1 k_2}(\tilde{\rho}) & \text{if } a = (k_1, k_2, 0), b = (k_1, k_2, 0) \\ \hat{\vartheta}^n_{k_1 k_2}(\tilde{\rho})[\varsigma_{k_1 k_2}(\tilde{\rho})](l_1) & \text{if } a = (k_1, k_2, 0), b = (l_1, J, 1) \\ \hat{\xi}^n_{k_1 k_2}(\tilde{\rho})[\hat{\sigma}^n_{k_1 k_2}(\tilde{\rho})](l_1, l_2) & \text{if } a = (k_1, k_2, 1), b = (l_1, l_2, 0) \\ (1 - \hat{\xi}^n_{k_1 k_2}(\tilde{\rho}))[\hat{\varsigma}^n_{k_1 k_2}(\tilde{\rho})](l_1) & \text{if } a = (k_1, k_2, 1), b = (l_1, J, 1) \\ 1 & \text{if } a = (k_1, J, 1), b = (k_1, J, 1) \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 6 *There exists a sequence* $\{\xi_m\}^{3CM}_{m=-1}$ *with* $\xi_{-1} = \frac{1}{M^{M^M}}$ *and* $3CM\xi_m \leq \xi_0$ *for any* $m \in \{1, \ldots, 3CM\}$ *such that for any* $m \in \{-1, 0, \ldots, 3CM\}$*,* $i \in 1, 2, 3$*,* $a_1, b_1, a_2, b_2 \in \tilde{S}$ *and* $n \in \{1, \ldots, CM\}$*,* $\|\tilde{\rho} - \tilde{\rho}'\|_\infty \leq \xi_{m+1}$ *implies*

$$\|T^n_i(\tilde{\rho}) - T^n_i(\tilde{\rho}')\|_\infty \leq \xi_m,$$

$$\|\hat{q}^n(\tilde{\rho}) - \hat{q}^n(\tilde{\rho}')\|_\infty \leq \xi_m,$$

$$|B^{3n-2}_{a_1 b_1}(\tilde{\rho}) - B^{3n-2}_{a_1 b_1}(\tilde{\rho}')| \leq \xi_m,$$

$$|B^{3n}_{a_1 b_1}(\tilde{\rho}) - B^{3n}_{a_1 b_1}(\tilde{\rho}')| \leq \xi_m,$$

$$|B^{3n-2}_{a_1 b_1}(\tilde{\rho})B^{3n-2}_{a_1 b_1}(\tilde{\rho}) - B^{3n-2}_{a_2 b_2}(\tilde{\rho}')B^{3n-2}_{a_2 b_2}(\tilde{\rho}')| \leq \xi_m,$$

$$|B^{3n}_{a_1 b_1}(\tilde{\rho})B^{3n}_{a_1 b_1}(\tilde{\rho}) - B^{3n}_{a_2 b_2}(\tilde{\rho}')B^{3n}_{a_2 b_2}(\tilde{\rho}')| \leq \xi_m.$$

Proof. First we work with T^n_2. Define $F : \mathbb{N} \times \mathbb{N} \times \tilde{\Delta} \to \tilde{\Delta}$ as follow,

$$[F(N, n, \tilde{p})]_{kl0} = \begin{cases} \tilde{p}_{kl0} & \text{if } l \neq J \\ 0 & \text{if } l = J, \end{cases}$$

$$[F(N, n, \tilde{p})]_{kl1} = \begin{cases} \frac{\tilde{p}_{kJ}\theta_{kl}(\tilde{p}, \frac{n}{N})\tilde{p}_{lJ}}{N} & \text{if } l \neq J \\ \tilde{p}_{kJ}(1 - \sum_{l \in S} \frac{\theta_{kl}(\tilde{p}, \frac{n}{N})\tilde{p}_{lJ}}{N}) & \text{if } l = J. \end{cases}$$

It is easy to see that $T^n_2(\tilde{\rho}) = {}^*F(M, n, \tilde{\rho})$.

For any $N, N' \in \mathbb{N}$, there exists a strictly increasing function $v_{NN'}$ of $\{F(N, n, \cdot)\}_{1 \leq n \leq N'}$ (which is called modulus of continuity) such that $||F(N, n, \tilde{p}) - F(N, n, \tilde{p}')||_\infty \leq v_{NN'}(||\tilde{p} - \tilde{p}'||_\infty)$ for any $\tilde{p}, \tilde{p}' \in \tilde{\Delta}$. By Transfer Principle, for any $N, N' \in {}^*\mathbb{N}$, there exists a strictly increasing function $v_{NN'}$ of $\{{}^*F(N, n, \cdot)\}_{1 \leq n \leq N'}$ such that $||{}^*F(N, n, \tilde{\rho}) - {}^*F(N, n, \tilde{\rho}')||_\infty \leq v_{NN'}(||\tilde{\rho} -$

$\tilde{\rho}'||_\infty)$ for any $\tilde{\rho}, \tilde{\rho}' \in {}^*\tilde{\Delta}$. Let $N = M$, $N' = CM$, it is clear that $||T_2^n(\tilde{\rho}) - T_2^n(\tilde{\rho'})||_\infty \leq v_{NN'}(||\tilde{\rho} - \tilde{\rho}'||_\infty)$. For T_1^n, T_3^n, $\hat{q}^n$, B_{ab}^{3n-2}, B_{ab}^{3n}, $B_{ab}^{3n-2}B_{ab}^{3n-2}$ and $B_{ab}^{3n}B_{ab}^{3n}$, we can derive the modulus of continuity in the same way. By taking maximal, we can get the common modulus of continuity v for all these processes.

Let $\xi_{-1} = \frac{1}{M^{M^M}}$ and $w = v^{-1}$. Let $\xi_0 = w(\xi_{-1})$, $\xi_m = \min\left(w(\xi_{m-1}), \frac{\xi_0}{3CM}\right)$ for any $m \in \{1, \ldots, 3CM\}$. It is clear that $\|\tilde{\rho} - \tilde{\rho}'\|_\infty \leq \xi_{m+1}$ implies

$$\|T_i^n(\tilde{\rho}) - T_i^n(\tilde{\rho}')\|_\infty \leq \xi_m,$$

$$\|\hat{q}^n(\tilde{\rho}) - \hat{q}^n(\tilde{\rho}')\|_\infty \leq \xi_m,$$

$$|B_{a_1b_1}^{3n-2}(\tilde{\rho}) - B_{a_1b_1}^{3n-2}(\tilde{\rho}')| \leq \xi_m,$$

$$|B_{a_1b_1}^{3n}(\tilde{\rho}) - B_{a_1b_1}^{3n}(\tilde{\rho})'| \leq \xi_m,$$

$$|B_{a_1b_1}^{3n-2}(\tilde{\rho})B_{a_1b_1}^{3n-2}(\tilde{\rho}) - B_{a_2b_2}^{3n-2}(\tilde{\rho}')B_{a_2b_2}^{3n-2}(\tilde{\rho}')| \leq \xi_m,$$

$$|B_{a_1b_1}^{3n}(\tilde{\rho})B_{a_1b_1}^{3n}(\tilde{\rho}) - B_{a_2b_2}^{3n}(\tilde{\rho}')B_{a_2b_2}^{3n}(\tilde{\rho}')| \leq \xi_m.$$

■

Let $\hat{M}$ be the smallest hyper integer greater equal to $\left(\frac{1}{\xi_{3CM}}\right)^3$, we have the following estimation.

Lemma 7 *Let* $V^m = \{\omega^m \in \Omega^m : \|\tilde{\rho}^m(\omega) - U_1^m(\tilde{\rho}^0)\|_\infty \geq \xi_0\}$ *and* $V = \cup_{m=1}^{3m}\bar{V}^m$, *then* $P_0(V) < \epsilon_0$, *where* $\epsilon_0 = \frac{3CMK(K+1)}{\hat{M}^{\frac{1}{3}}}$. *It is clear that*

$$Q^m\left(\|\tilde{\rho}^m - U_1^m(\tilde{\rho}^0)\|_\infty \geq \xi_0\right) < \epsilon_0,$$

for any m *between* 0 *and* $3CM$.

Proof. For the mutation step in period n, fix any $k, l \in S$,

$$\begin{aligned}
\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}_{kl0}^{3n-2} &= \int_{\Omega_{3n-2}} \tilde{\rho}_{kl0}^{3n-2}(\omega^{3n-2})dQ_{3n-2}^{\omega^{3n-3}} \\
&= \int_{\Omega_{3n-2}} \frac{1}{\hat{M}}\sum_{i\in I}\mathbf{1}_{kl0}(\tilde{\beta}_i^{3n-2})dQ_{3n-2}^{\omega^{3n-3}} \\
&= \sum_{k',l'\in S}\frac{1}{\hat{M}}\sum_{\tilde{\beta}_i^{3n-3}=(k',l',0)}\int_{\Omega_{3n-2}}\mathbf{1}_{kl0}(\tilde{\beta}_i^{3n-2})dQ_{3n-2}^{\omega^{3n-3}} \\
&= \sum_{k',l'\in S}\tilde{\rho}_{k'l'0}^{3n-3}(\omega^{3n-3})\hat{\eta}_{kk'}^n\hat{\eta}_{ll'}^n \\
&= [T_1^n\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)]_{kl0}.
\end{aligned}$$

By the same method, we can prove

$$\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}^{3n-2} = T_1^n\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right).$$

Given ω^{3n-3} and $i \neq j$, it is clear that $\mathbf{1}_{klr}(\tilde{\beta}_i^{3n-2})$ and $\mathbf{1}_{klr}(\tilde{\beta}_j^{3n-2})$ are independent on $(\Omega_{3n-2}, \mathcal{E}_{3n-2}, Q_{3n-2}^{\omega^{3n-3}})$. Therefore,

$$\begin{aligned} \mathrm{Var}^{\omega^{3n-3}} \tilde{\rho}_{klr}^{3n-2} &= \mathrm{Var}^{\omega^{3n-3}} \frac{1}{\hat{M}} \sum_{i \in I} \mathbf{1}_{klr}(\tilde{\beta}_i^{3n-2}) \\ &= \frac{1}{\hat{M}^2} \sum_{i \in I} \mathrm{Var}^{\omega^{3n-3}} \mathbf{1}_{klr}(\tilde{\beta}_i^{3n-2}) \\ &\leq \frac{1}{\hat{M}^2} \sum_{i \in I} \frac{1}{4} \\ &= \frac{1}{4\hat{M}} \end{aligned}$$

By Chebyshev's Inequality,

$$\begin{aligned} & Q_{3n-2}^{\omega^{3n-3}} \left(\|\tilde{\rho}^{3n-2} - T_1^n \left(\tilde{\rho}^{3n-3}\right) \|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \right) \\ \leq & \sum_{(k,l,r) \in \tilde{S}} Q_{3n-2}^{\omega^{3n-3}} \left(\left| \tilde{\rho}_{klr}^{3n-2} - [T_1^n(\tilde{\rho}^{3n-3})]_{klr} \right| \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \right) \\ \leq & \frac{K(K+1)}{2\hat{M}^{\frac{1}{3}}}. \end{aligned}$$

Let $W^{3n-2} = \{\omega^{3n-2} \in \Omega^{3n-2} : \|\tilde{\rho}^{3n-2}(\omega^{3n-2}) - T_1^n \left(\tilde{\rho}^{3n-3}\right) \|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\}$, then

$$P_0(W^{3n-2}) = \int_{\Omega^{3n-3}} Q_{3n-2}^{\omega^{3n-3}} \left(\|\tilde{\rho}^{3n-2} - T_1^n \left(\tilde{\rho}^{3n-3}\right) \|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \right) dQ^{3n-3} \leq \frac{K(K+1)}{2\hat{M}^{\frac{1}{3}}}$$

For the step of type changing with break-up step in period n, fix any $k, l \in S$,

$$\begin{aligned} \mathbb{E}^{\omega^{3n-1}} \tilde{\rho}_{kl0}^{3n} &= \int_{\Omega_{3n}} \tilde{\rho}_{kl0}^{3n}(\omega^{3n}) dQ_{3n}^{\omega^{3n-1}} \\ &= \int_{\Omega_{3n}} \frac{1}{\hat{M}} \sum_{i \in I} \mathbf{1}_{kl0}(\tilde{\beta}_i^{3n}) dQ_{3n}^{\omega^{3n-1}} \\ &= \frac{1}{\hat{M}} \sum_{\tilde{\beta}_i^{3n-1} = (k,l,0)} \int_{\Omega_{3n}} \mathbf{1}_{kl0}(\tilde{\beta}_i^{3n}) dQ_{3n}^{\omega^{3n-1}} \\ &\quad + \frac{1}{\hat{M}} \sum_{k',l' \in S} \sum_{\tilde{\beta}_i^{3n-1} = (k',l',1)} \int_{\Omega_{3n}} \mathbf{1}_{kl0}(\tilde{\beta}_i^{3n}) dQ_{3n}^{\omega^{3n-1}} \\ &= \tilde{\rho}_{kl0}^{3n-1}(\omega^{3n-1})(1 - \hat{\vartheta}_{kl}^n) + \sum_{k',l' \in S} \tilde{\rho}_{k'l'1}^{3n-1}(\omega^{3n-1}) \hat{\xi}_{k'l'}^n \hat{\sigma}_{k'l'}^n(k,l) \\ &= [T_3^n \left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)]_{kl0}. \end{aligned}$$

By the same method, we can prove

$$\mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n} = T_3^n \left(\tilde{\rho}^{3n-1}(\omega^{3n-1})\right).$$

Given ω^{3n-1} and (i,j) such that $\hat{\pi}^{3n-1}(i,\omega^{3n-1}) \neq j$, it is clear that $\mathbf{1}_{klr}(\tilde{\beta}_i^{3n})$ and $\mathbf{1}_{klr}(\tilde{\beta}_j^{3n})$ are independent on $(\Omega_{3n}, \mathcal{E}_{3n}, Q_{3n}^{\omega^{3n-1}})$. Let $A^n_{\omega^{3n-1}} = \{(i,j) \in I \times I : i < j,\, \bar{\pi}^n(i,\omega^{3n-1}) = j\}$, then

$$\begin{aligned}
\mathrm{Var}^{\omega^{3n-1}} \tilde{\rho}^{3n}_{klr} &= \mathrm{Var}^{\omega^{3n-1}} \frac{1}{\hat{M}} \sum_{i\in I} \mathbf{1}_{klr}(\tilde{\beta}_i^{3n}) \\
&= \frac{1}{\hat{M}^2} \sum_{i\in I} \mathrm{Var}^{\omega^{3n-3}} \mathbf{1}_{klr}(\tilde{\beta}_i^{3n-2}) + \frac{2}{\hat{M}^2} \sum_{(i,j)\in A^n_{\omega^{3n-1}}} \mathrm{Cov}\left(\mathbf{1}_{klr}(\tilde{\beta}_i^{3n-2}), \mathbf{1}_{klr}(\tilde{\beta}_j^{3n-2})\right) \\
&\le \frac{1}{\hat{M}^2} \sum_{i\in I} \frac{1}{4} + \frac{2}{\hat{M}^2} \frac{\hat{M}}{2} \frac{1}{4} \\
&= \frac{1}{2\hat{M}}
\end{aligned}$$

By Chebyshev's Inequality,

$$\begin{aligned}
& Q_{3n}^{\omega^{3n-1}} \left(\|\tilde{\rho}^{3n} - T_1^n\left(\tilde{\rho}^{3n-1}\right)\|_\infty \ge \frac{1}{\hat{M}^{\frac{1}{3}}} \right) \\
\le & \sum_{(k,l,r)\in \tilde{S}} Q_{3n}^{\omega^{3n-1}} \left(\left|\tilde{\rho}^{3n}_{klr} - [T_1^n(\tilde{\rho}^{3n-1})]_{klr}\right| \ge \frac{1}{\hat{M}^{\frac{1}{3}}} \right) \\
\le & \frac{K(K+1)}{\hat{M}^{\frac{1}{3}}}.
\end{aligned}$$

Let $W^{3n} = \{\omega^{3n} \in \Omega^{3n} : \|\tilde{\rho}^{3n}(\omega^{3n}) - T_1^n\left(\tilde{\rho}^{3n-1}\right)\|_\infty \ge \frac{1}{\hat{M}^{\frac{1}{3}}}\}$, then

$$P_0(W^{3n}) = \int_{\Omega^{3n-1}} Q_{3n}^{\omega^{3n-1}} \left(\|\tilde{\rho}^{3n} - T_3^n\left(\tilde{\rho}^{3n-1}\right)\|_\infty \ge \frac{1}{\hat{M}^{\frac{1}{3}}} \right) dQ^{3n-1} \le \frac{K(K+1)}{\hat{M}^{\frac{1}{3}}}$$

For the step of random matching in period n, It is clear that $\tilde{\rho}^{3n-1}_{kl0} = \tilde{\rho}^{3n-2}_{kl0}$ for any $l \in S$. By the construction of static random matching, for any $\omega \in \Omega$ and $k,l \in S$,

$$\left|\tilde{\rho}^{3n-1}_{kl1} - [T_2^n(\tilde{\rho}^{3n-1})]_{kl1}\right| \le \frac{1}{\hat{M}}.$$

Then

$$\left|\tilde{\rho}^{3n-1}_{kJ1} - [T_2^n(\tilde{\rho}^{3n-1})]_{kJ1}\right| \le \frac{K}{\hat{M}},$$

for any $k \in S$. Let $W^{3n-1} = \{\omega^{3n-1} \in \Omega^{3n-1} : \|\tilde{\rho}^{3n-1}(\omega^{3n-1}) - T_2^n(\tilde{\rho}^{3n-2})\|_\infty \ge \frac{1}{\hat{M}^{\frac{1}{3}}}\}$. Then $W^{3n-1} = \emptyset$.

Let $W = \{\omega \in \Omega : \|\tilde{\rho}^{m+1}(\omega) - T^{e(m+1)}_{f(m+1)}(\tilde{\rho}^m)\|_\infty \ge \frac{1}{\hat{M}^{\frac{1}{3}}}$ for some m between 0 and $3CM\}$, then $W = \cup_{m=1}^{3CM} \overline{W}^m$ which implies $P_0(W) \le \sum_{m=1}^{3CM} P_0(\overline{W}^m) \le \frac{3CMK(K+1)}{\hat{M}^{\frac{1}{3}}} = \epsilon_0$.

Note that $\hat{M} \geq \left(\frac{1}{\xi_{3CM}}\right)^3$, then $\frac{1}{\hat{M}^{\frac{1}{3}}} \leq \xi_{3CM}$. Therefore, if $\omega \notin W$, we have

$$\begin{aligned}
& \|\tilde{\rho}^m - U_1^m(\tilde{\rho}^0)\|_\infty \\
\leq\ & \|\tilde{\rho}^m - U_m^m(\tilde{\rho}^{m-1})\|_\infty + \|U_m^m(\tilde{\rho}^{m-1}) - U_1^m(\tilde{\rho}^0)\|_\infty \\
\leq\ & \|\tilde{\rho}^m - U_m^m(\tilde{\rho}^{m-1})\|_\infty + \|U_m^m(\tilde{\rho}^{m-1}) - U_{m-1}^m(\tilde{\rho}^{m-2})\|_\infty + \|U_{m-1}^m(\tilde{\rho}^{m-2}) - U_1^m(\tilde{\rho}^0)\|_\infty \\
\leq\ & \|\tilde{\rho}^m - U_m^m(\tilde{\rho}^{m-1})\|_\infty + \sum_{j=1}^{m-1} \|U_{j+1}^m(\tilde{\rho}^j) - U_j^m(\tilde{\rho}^{j-1})\|_\infty \\
\leq\ & \frac{m}{\hat{M}^{\frac{1}{3}}} \leq 3CM\xi_{3CM} \leq 3CM\xi_1 \leq \xi_0.
\end{aligned}$$

Then ω is also in V, which implies $V \subseteq W$. Therefore, $P_0(V) \leq \frac{3CMK(K+1)}{\hat{M}^{\frac{1}{3}}} = \epsilon_0$. ∎

Lemma 8 *For any $\omega \notin V$, $\tilde{\rho}_{kJ1}^{3n-2} \geq \frac{1}{\hat{M}^{\frac{1}{15}}}$ for any integer n between 0 and CM.*

Proof. : Note that $\hat{\vartheta}_{kl}^n \geq \frac{1}{M^2}$ and $\hat{\xi}_{kl}^n \leq 1 - \frac{1}{M^2}$, then

$$\begin{aligned}
& \sum_{k\in S}[U_1^{3n}(\tilde{\rho}^0)]_{kJ1} = \sum_{k\in S} T_3^n\left(U_1^{3n-1}(\tilde{\rho}^0)\right) \\
=\ & \sum_{k,k',l'\in S}(1-\hat{\xi}_{k'l'}^n)[U_1^{3n-1}(\tilde{\rho}^0)]_{k'l'1}\hat{\varsigma}_{k'l'}^n(k) + \sum_{k,k',l'\in S}\hat{\vartheta}_{k'l'}^n[U_1^{3n-1}(\tilde{\rho}^0)]_{k'l'0}\hat{\varsigma}_{k'l'}^n(k) + \sum_{k\in S}[U_1^{3n-1}(\tilde{\rho}^0)]_{kJ1} \\
=\ & \sum_{k',l'\in S}(1-\hat{\xi}_{k'l'}^n)[U_1^{3n-1}(\tilde{\rho}^0)]_{k'l'1} + \sum_{k',l'\in S}\hat{\vartheta}_{k'l'}^n[U_1^{3n-1}(\tilde{\rho}^0)]_{k'l'0} + \sum_{k\in S}[U_1^{3n-1}(\tilde{\rho}^0)]_{kJ1} \\
\geq\ & \frac{1}{M^2}\left(\sum_{k',l'\in S}[U_1^{3n-1}(\tilde{\rho}^0)]_{k'l'1} + \sum_{k',l'\in S}[U_1^{3n-1}(\tilde{\rho}^0)]_{k'l'0} + \sum_{k\in S}[U_1^{3n-1}(\tilde{\rho}^0)]_{kJ1}\right) \\
=\ & \frac{1}{M^2}.
\end{aligned}$$

Since $\hat{\eta}_{kl}^n \geq \frac{1}{M^2}$ for any $k, l \in S$ with $k \neq l$. Then

$$[U_1^{3n-2}(\tilde{\rho}^0)]_{kJ1} = \sum_{r\in S}[U_1^{3n-3}(\tilde{\rho}^0)]_{rJ1}\hat{\eta}_{rk}^n \geq \frac{1}{M^2}\sum_{r\in S}[U_1^{3n-3}(\tilde{\rho}^0)]_{rJ1} \geq \frac{K}{M^4}$$

For any fixed $\omega \notin V$, we have $\|\tilde{\rho}^{3n-2}(\omega) - U_1^{3n-2}(\tilde{\rho}^0)\|_\infty \leq \xi_0$, then

$$\tilde{\rho}_{kJ1}^{3n-2} \geq [U_1^{3n-2}(\tilde{\rho})]_{kJ1} - \xi_0 \geq \frac{K}{M^4} - \xi_{-1}.$$

Note that $\xi_{-1} = \frac{1}{M^{M^M}}$ and $\frac{1}{\hat{M}^{\frac{1}{15}}} \leq \left(\frac{\xi_{3CM}}{2K}\right)^{\frac{1}{5}} \leq \left(\frac{\xi_{-1}}{2K}\right)^{\frac{1}{5}} = \left(\frac{1}{2KM^{M^M}}\right)^{\frac{1}{5}}$. It is clear that $\xi_{-1} \leq \frac{K}{2M^4}$ and $\frac{1}{\hat{M}^{\frac{1}{15}}} \leq \frac{K}{2M^4}$, then $\tilde{\rho}_{kJ1}^{3n-2} \geq \frac{1}{\hat{M}^{\frac{1}{15}}}$. ∎

Lemma 9 *For any $\omega^{3n-2} \notin V^{3n-2}$, if $\tilde{\beta}_i^{3n-2} = (k_1, J, 1)$ and $\tilde{\beta}_j^{3n-2} = (k_2, J, 1)$, then*

$$\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1) - \hat{q}_{k_1k_1}^n(\tilde{\rho}^{3n-2})\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}}$$

$$\left|Q^{\omega^{3n-2}}_{3n-1}(\hat{g}^{3n-1}_i = J) - \hat{q}^n_{k_1}(\tilde{\rho}^{3n-2})\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}}$$

$$\left|Q^{\omega^{3n-2}}_{3n-1}(\hat{g}^{3n-1}_i = l_1, \hat{g}^{3n-1}_j = l_2) - \hat{q}^n_{k_1 l_1}(\tilde{\rho}^{3n-2})\hat{q}^n_{k_2 l_2}(\tilde{\rho}^{3n-2})\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

$$\left|Q^{\omega^{3n-2}}_{3n-1}(\hat{g}^{3n-1}_i = l_1, \hat{g}^{3n-1}_j = J) - \hat{q}^n_{k_1 l_1}(\tilde{\rho}^{3n-2})\hat{q}^n_{k_2 J}(\tilde{\rho}^{3n-2})\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

$$\left|Q^{\omega^{3n-2}}_{3n-1}(\hat{g}^{3n-1}_i = J, \hat{g}^{3n-1}_j = J) - \hat{q}^n_{k_1}(\tilde{\rho}^{3n-2})\hat{q}^n_{k_2}(\tilde{\rho}^{3n-2})\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

Proof. If $\tilde{\rho}^{3n-2}_{k_1 J1}\hat{q}^n_{k_1 l_1} > \frac{1}{\hat{M}^{\frac{1}{5}}}$, then $\tilde{\rho}^{3n-2}_{k_1 J1} > \frac{1}{\hat{M}^{\frac{1}{5}}} > \frac{1}{\hat{M}^{\frac{1}{3}}}$. By Lemma 1,

$$\left|Q^{\omega^{3n-2}}_{3n-1}(\hat{g}^{3n-1}_i = l_1) - \hat{q}^n_{k_1 l_1}\right| \leq \frac{1}{\hat{M}^{\frac{1}{3}}} \leq \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

Note that $\left|Q^{\omega^{3n-2}}_{3n-1}(\hat{g}^{3n-1}_i = J) - \hat{q}^n_{k_1}\right| \leq \left|\sum_{l_1\in S} Q^{\omega^{3n-2}}_{3n-1}(\hat{g}^{3n-1}_i = l_1) - \sum_{l_1\in S}\hat{q}^n_{k_1 l_1}\right|$, then

$$\left|Q^{\omega^{3n-2}}_{3n-1}(\hat{g}^{3n-1}_i = J) - \hat{q}^n_{k_1}\right| \leq \frac{K}{\hat{M}^{\frac{1}{3}}} \leq \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

if $\tilde{\rho}^{3n-2}_{k_2 J1}\hat{q}^n_{k_2 k_1}(\tilde{\rho}^{3n-2}) \leq \frac{1}{\hat{M}^{\frac{1}{5}}}$, it is easy to prove $\hat{q}^n_{k_2 k_1}(\tilde{\rho}^{3n-2}) \leq \frac{1}{\hat{M}^{\frac{2}{15}}}$ since $\tilde{\rho}^{3n-2}_{k_2} \geq \frac{1}{\hat{M}^{\frac{1}{15}}}$. Note that $Q^{\omega^{3n-2}}_{3n-1}(\hat{g}^{3n-1}_i = l_1) \leq \hat{q}^n_{k_1 l_1}$, then

$$\left|Q^{\omega^{3n-2}}_{3n-1}(\hat{g}^{3n-1}_i = k_2) - \hat{q}^n_{k_1 k_2}(\tilde{\rho}^{3n-2})\right| = \hat{q}^n_{k_1 k_2}(\tilde{\rho}^{3n-2}) - Q^{\omega^{3n-2}}_{3n-1}(\hat{g}^{3n-1}_i = k_2) \leq \hat{q}^n_{k_1 k_2}(\tilde{\rho}^{3n-2}) \leq \frac{1}{\hat{M}^{\frac{2}{15}}} \leq \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

Similarly, we can derive

$$\left|Q^{\omega^{3n-2}}_{3n-1}(\hat{g}^{3n-1}_i = J) - \hat{q}^n_{k_1}\right| \leq \frac{K}{\hat{M}^{\frac{2}{15}}} \leq \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

We can prove the rest parts in the same way. ■

For any $A, B \in \mathcal{E}^{3CM}$, define

$$P_0(A|B) = \begin{cases} \frac{P_0(A\cap B)}{P_0(B)} & \text{if } P_0(B) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

We have the following estimation.

Lemma 10 *For any $a, b \in \tilde{S}$ and $F^m \in \mathcal{F}^m$ such that $P_0(\tilde{\beta}^m_i = b, \mathbf{1}_{F^m} = 1) > 0$,*

$$\left|P_0(\tilde{\beta}^{m+1}_i = b|\tilde{\beta}^m_i = a, \mathbf{1}_{F^m} = 1) - P_0(\tilde{\beta}^{m+1}_i = b|\tilde{\beta}^m_i = a)\right| \leq \frac{2\xi_{-1} + 4\epsilon_0}{P_0(\{\tilde{\beta}^m_i = a\}\cap F^m)} + \frac{2}{\hat{M}^{\frac{1}{9}}},$$

Proof. Let $D_1 = \{\omega^m \in \Omega^m : \tilde{\beta}^m_i = a\} \cap F^m$, $D_2 = \{\omega^m \in \Omega^m : \tilde{\beta}^m_i = a\}$, then

$$P_0(\tilde{\beta}^{m+1}_i = b|\{\tilde{\beta}^m_i = a\}\cap F^m) = \frac{1}{Q^m(D_1)}\int_{D_1} Q^{\omega^m}_{m+1}(\tilde{\beta}^{m+1}_i = b)dQ^m$$

and

$$P_0(\tilde{\beta}_i^{m+1}=a|\tilde{\beta}_i^m=b)=\frac{1}{Q^m(D_2)}\int_{D_2}Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1}=a)dQ^m.$$

For the step of random matching in period n, we can assume $a=(k_1,k_2,1)$ and $b=(k_1,J,1)$ or $(k_1,J,1)$, otherwise $P_0(\tilde{\beta}_i^{3n-1}=b|\tilde{\beta}_i^{3n-2}=a,\mathbf{1}_{F^{3n-2}}=1)$ and $P_0(\tilde{\beta}_i^{3n-1}=a|\tilde{\beta}_i^{3n-2}=b)$ are both 0 or 1. If $b=(k_1,k_2,1)$, by Lemma 7 and Lemma 9,

$$\begin{aligned}
&\left|P_0(\tilde{\beta}_i^{3n-1}=b|\tilde{\beta}_i^{3n-2}=a,\mathbf{1}_{F^{3n-2}}=1)-\frac{1}{Q^m(D_1)}\int_{D_1}\hat{q}_{k_1k_2}^n(\tilde{\rho}^{3n-2})dQ^m\right|\\
\le\ &\left|\frac{1}{Q^m(D_1)}\int_{D_1\cap V}Q_{3n-1}^{\omega^{3n-2}}(\tilde{\beta}_i^{3n-1}=b)-\hat{q}_{k_1k_2}^n(\tilde{\rho}_{\omega^{3n-2}}^{3n-2})dQ^m\right|\\
&+\left|\frac{1}{Q^m(D_1)}\int_{D_1\setminus V}Q_{3n-1}^{\omega^{3n-2}}(\tilde{\beta}_i^{3n-1}=b)-\hat{q}_{k_1k_2}^n(\tilde{\rho}_{\omega^{3n-2}}^{3n-2})dQ^m\right|\\
\le\ &\left|\frac{P_0(V)}{Q^m(D_1)}\right|+\left|\frac{P_0(D_1\setminus V)}{Q^m(D_1)}\right|\frac{1}{\hat{M}^{\frac{1}{9}}}\\
\le\ &\frac{\epsilon_0}{Q^m(D_1)}+\frac{1}{\hat{M}^{\frac{1}{9}}}.
\end{aligned}$$

Note that $\|\tilde{\rho}^{3n-2}(\omega^{3n-2})-U_1^{3n-2}(\tilde{\rho}^0)\|_\infty\le\xi_0$ for all $\omega\notin V$, by Lemma 6,

$$\begin{aligned}
&\left|\hat{q}_{k_1k_2}^n\left(U_1^{3n-2}(\tilde{\rho}^0)\right)-\frac{1}{Q^m(D_1)}\int_{D_1}\hat{q}_{k_1k_2}^n(\tilde{\rho}_{\omega^{3n-2}}^{3n-2})dQ^m\right|\\
\le\ &\left|\frac{1}{Q^m(D_1)}\int_{D_1\cap V}\hat{q}_{k_1k_2}^n\left(U_1^{3n-2}(\tilde{\rho}^0)\right)-\hat{q}_{k_1k_2}^n(\tilde{\rho}_{\omega^{3n-2}}^{3n-2})dQ^m\right|\\
&+\left|\frac{1}{Q^m(D_1)}\int_{D_1\setminus V}\hat{q}_{k_1k_2}^n\left(U_1^{3n-2}(\tilde{\rho}^0)\right)-\hat{q}_{k_1k_2}^n(\tilde{\rho}_{\omega^{3n-2}}^{3n-2})dQ^m\right|\\
\le\ &\left|\frac{P_0(V)}{Q^m(D_1)}\right|+\left|\frac{P_0(D_1\setminus V)}{Q^m(D_1)}\right|\xi_{-1}\\
\le\ &\frac{\epsilon_0}{Q^m(D_1)}+\xi_{-1}.
\end{aligned}$$

Therefore,

$$\left|P_0(\tilde{\beta}_i^{3n-1}=b|\tilde{\beta}_i^{3n-2}=a,\mathbf{1}_{F^{3n-2}}=1)-\hat{q}_{k_2k_1}^n\left(U_1^{3n-2}(\tilde{\rho}^0)\right)\right|\le\frac{2\epsilon_0}{Q^m(D_1)}+\frac{1}{\hat{M}^{\frac{1}{9}}}+\xi_{-1}.$$

We can prove

$$\left|P_0(\tilde{\beta}_i^{3n-1}=b|\tilde{\beta}_i^{3n-2}=a)-\hat{q}_{k_1k_2}^n\left(U_1^{3n-2}(\tilde{\rho}^0)\right)\right|\le\frac{2\epsilon_0}{Q^m(D_2)}+\frac{1}{\hat{M}^{\frac{1}{9}}}+\xi_{-1}$$

in the same way. Then

$$\left|P_0\left(\tilde{\beta}_i^{3n-1}=b|\tilde{\beta}_i^{3n-2}=a,\mathbf{1}_{F^{3n-2}}=1\right)-P_0\left(\tilde{\beta}_i^{3n-1}=b|\tilde{\beta}_i^{3n-2}=a\right)\right|\le\frac{2\xi_{-1}+4\epsilon_0}{P_0\left(\tilde{\beta}_i^{3n-2}=a,\mathbf{1}_{F^{3n-2}}=1\right)}+\frac{2}{\hat{M}^{\frac{1}{9}}}.$$

If $b=(k_1,J,1)$, we can derive the above inequality in the same way.

For the mutation step in period n, it is easy to prove

$$\begin{aligned}
&P_0\left(\tilde{\beta}_i^{3n-2}=b|\tilde{\beta}_i^{3n-3}=a,\mathbf{1}_{F^{3n-3}}=1\right)\\
=&\frac{1}{Q^m(D_1)}\int_{D_1}Q_{3n-2}^{\omega^{3n-3}}(\tilde{\beta}_i^{3n-2}=a)dQ^m\\
=&\frac{1}{Q^m(D_1)}\int_{D_1}B_{ab}^{3n-2}(\tilde{\rho}_{\omega^{3n-3}}^{3n-3})dQ^m
\end{aligned}$$

where

$$B_{ab}^{3n-2}(\tilde{\rho})=\begin{cases}\hat{\eta}_{k_1l_1}^n(\tilde{\rho})\tilde{\eta}_{k_2l_2}^n(\tilde{\rho}) & \text{if } a=(k_1,k_2,0),b=(l_1,l_2,0)\\ \hat{\eta}_{k_1l_1}^n(\tilde{\rho}) & \text{if } a=(k_1,J,1),b=(l_1,J,1)\\ 0 & \text{otherwise.}\end{cases}$$

Therefore,

$$\begin{aligned}
&\left|P_0(\tilde{\beta}_i^{3n-2}=b|\tilde{\beta}_i^{3n-3}=a,\mathbf{1}_{F^{3n-3}}=1)-B_{ab}^{3n-2}\left(U_1^{3n-3}(\tilde{\rho}^0)\right)\right|\\
\leq&\frac{1}{Q^m(D_1)}\int_{D_1}\left|B_{ab}^{3n-2}(\tilde{\rho}_{\omega^{3n-3}}^{3n-3})-B_{ab}^{3n-2}\left(U_1^{3n-3}(\tilde{\rho}^0)\right)\right|dQ^m\\
=&\frac{1}{Q^m(D_1)}\int_{D_1\cap V}\left|B_{ab}^{3n-2}(\tilde{\rho}_{\omega^{3n-3}}^{3n-3})-B_{ab}^{3n-2}\left(U_1^{3n-3}(\tilde{\rho}^0)\right)\right|dQ^m\\
&+\frac{1}{Q^m(D_1)}\int_{D_1\setminus V}\left|B_{ab}^{3n-2}(\tilde{\rho}_{\omega^{3n-3}}^{3n-3})-B_{ab}^{3n-2}\left(U_1^{3n-3}(\tilde{\rho}^0)\right)\right|dQ^m
\end{aligned}$$

By Lemma 6 and 7, we know

$$\left|B_{ab}^{3n-2}(\tilde{\rho}_{\omega^{3n-3}}^{3n-3})-B_{ab}^{3n-2}\left(U_1^{3n-3}(\tilde{\rho}^0)\right)\right|\leq\xi_{-1}$$

for any $\omega\notin V$. Then

$$\left|P_0(\tilde{\beta}_i^{3n-2}=b|\tilde{\beta}_i^{3n-3}=a,\mathbf{1}_{F^{3n-3}}=1)-B_{ab}^{3n-2}\left(U_1^{3n-3}(\tilde{\rho}^0)\right)\right|\leq\frac{\epsilon_0}{Q^m(D_1)}+\xi_{-1} \tag{14}$$

We can prove

$$\left|P_0(\tilde{\beta}_i^{3n-2}=b|\tilde{\beta}_i^{3n-3}=a)-B_{ab}^{3n-2}\left(U_1^{3n-3}(\tilde{\rho}^0)\right)\right|\leq\frac{\epsilon_0}{Q^m(D_1)}+\xi_{-1}$$

in the same way. Therefore

$$P_0(\tilde{\beta}_i^{3n-2}=b|\tilde{\beta}_i^{3n-3}=a,\mathbf{1}_{F^{3n-3}}=1)-P_0(\tilde{\beta}_i^{3n-2}=b|\{\tilde{\beta}_i^{3n-3}=a\})\leq\frac{2\xi_{-1}+2\epsilon_0}{Q^m(D_1)}.$$

For the step of type changing with break-up step in period n, it is easy to prove

$$\begin{aligned}
&P_0\left(\tilde{\beta}_i^{3n}=b|\tilde{\beta}_i^{3n-1}=a,\mathbf{1}_{F^{3n-1}}=1\right)\\
=&\frac{1}{Q^m(D_1)}\int_{D_1}Q_{3n}^{\omega^{3n-1}}(\tilde{\beta}_i^{3n}=a)dQ^m\\
=&\frac{1}{Q^m(D_1)}\int_{D_1}B_{ab}^{3n}(\tilde{\rho}_{\omega^{3n-1}}^{3n-1})dQ^m
\end{aligned}$$

where

$$B_{ab}^{3n}(\tilde{\rho}) = \begin{cases} 1-\hat{\vartheta}^n_{k_1k_2}(\tilde{\rho}) & \text{if } a=(k_1,k_2,0), b=(k_1,k_2,0) \\ \hat{\vartheta}^n_{k_1k_2}(\tilde{\rho})[\varsigma_{k_1k_2}(\tilde{\rho})](l_1) & \text{if } a=(k_1,k_2,0), b=(l_1,J,1) \\ \hat{\xi}^n_{k_1k_2}(\tilde{\rho})[\hat{\sigma}^n_{k_1k_2}(\tilde{\rho})](l_1,l_2) & \text{if } a=(k_1,k_2,1), b=(l_1,l_2,0) \\ (1-\hat{\xi}^n_{k_1k_2}(\tilde{\rho}))[\hat{\varsigma}^n_{k_1k_2}(\tilde{\rho})](l_1) & \text{if } a=(k_1,k_2,1), b=(l_1,J,1) \\ 1 & \text{if } a=(k_1,J,1), b=(k_1,J,1) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} & \left|P_0(\tilde{\beta}_i^{3n}=b|\tilde{\beta}_i^{3n-1}=a,\mathbf{1}_{F^{3n-1}}=1)-B_{ab}^{3n}\left(U_1^{3n-1}(\tilde{\rho}^0)\right)\right| \\ \leq\ & \frac{1}{Q^m(D_1)}\int_{D_1}\left|B_{ab}^{3n}(\tilde{\rho}^{3n-1}_{\omega^{3n-1}})-B_{ab}^{3n}\left(U_1^{3n-1}(\tilde{\rho}^0)\right)\right|dQ^m \\ =\ & \frac{1}{Q^m(D_1)}\int_{D_1\cap V}\left|B_{ab}^{3n}(\tilde{\rho}^{3n-1}_{\omega^{3n-1}})-B_{ab}^{3n}\left(U_1^{3n-1}(\tilde{\rho}^0)\right)\right|dQ^m \\ & +\frac{1}{Q^m(D_1)}\int_{D_1\setminus V}\left|B_{ab}^{3n}(\tilde{\rho}^{3n-1}_{\omega^{3n-1}})-B_{ab}^{3n}\left(U_1^{3n-1}(\tilde{\rho}^0)\right)\right|dQ^m \end{aligned}$$

By Lemma 6 and 7, we know

$$\left|B_{ab}^{3n}(\tilde{\rho}^{3n-1}_{\omega^{3n-1}})-B_{ab}^{3n}\left(U_1^{3n-1}(\tilde{\rho}^0)\right)\right|\leq\xi_{-1}$$

for any $\omega\notin V$. Then

$$\left|P_0(\tilde{\beta}_i^{3n}=b|\tilde{\beta}_i^{3n-1}=a,\mathbf{1}_{F^{3n-1}}=1)-B_{ab}^{3n}\left(U_1^{3n-1}(\tilde{\rho}^0)\right)\right|\leq\frac{\epsilon_0}{Q^m(D_1)}+\xi_{-1} \tag{15}$$

We can prove

$$\left|P_0(\tilde{\beta}_i^{3n}=b|\tilde{\beta}_i^{3n-1}=a)-B_{ab}^{3n}\left(U_1^{3n-1}(\tilde{\rho}^0)\right)\right|\leq\frac{\epsilon_0}{Q^m(D_1)}+\xi_{-1}$$

in the same way. Therefore

$$P_0(\tilde{\beta}_i^{3n}=b|\tilde{\beta}_i^{3n-1}=a,\mathbf{1}_{F^{3n-1}}=1)-P_0(\tilde{\beta}_i^{3n}=b|\tilde{\beta}_i^{3n-1}=a)\leq\frac{2\xi_{-1}+2\epsilon_0}{Q^m(D_1)}.$$

By combining the results of step 1, 2 and 3 in each period, we derive

$$\left|P_0(\tilde{\beta}_i^{m+1}=b|\tilde{\beta}_i^{m}=a,\mathbf{1}_{F^m}=1)-P_0(\tilde{\beta}_i^{m+1}=b|\tilde{\beta}_i^{m}=a)\right|\leq\frac{2\xi_{-1}+4\epsilon_0}{P_0(\{\tilde{\beta}_i^m=a\}\cap F^m)}+\frac{2}{\hat{M}^{\frac{1}{9}}},$$

for any integer $m<3CM$ ■

7.5.3 Proof of Lemma 2

We only need to prove that we can find a sequence of positive number $\{c_m\}_{1\leq n\leq 3CM}$ such that c_m is infinitesimal for every m and

$$\left|P_0(\tilde{\beta}^m=a,\tilde{\beta}^{m_1}=a_1,\tilde{\beta}^{\boldsymbol{m_2}}=\boldsymbol{a_2})P_0(\tilde{\beta}^{m_1}=a_1)-P_0(\tilde{\beta}^m=a,\tilde{\beta}^{m_1}=a_1)P_0(\tilde{\beta}^{m_1}=a_1,\tilde{\beta}^{\boldsymbol{m_2}}=\boldsymbol{a_2})\right|\leq c_m$$

for any $m, m_1, \boldsymbol{m_2}$ such that $m > m_1 > \boldsymbol{m_2}$. Note that $c_1 = 0$. Suppose we already derive c_m, we need to find out the relationship between c_m and c_{m+1}. It is easy to see that

$$\begin{aligned}
& P_0(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) P_0(\tilde{\beta}_i^{m_1} = a_1) \\
= & \sum_{b \in \tilde{S}} P_0(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) P_0(\tilde{\beta}_i^{m_1} = a_1) \\
= & \sum_{b \in \tilde{S}} P_0(\tilde{\beta}_i^{m+1} = a | \tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) P_0(\tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) P_0(\tilde{\beta}_i^{m_1} = a_1).
\end{aligned}$$

Let

$$Q = \sum_{b \in \tilde{S}} P_0(\tilde{\beta}_i^{m+1} = a | \tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1) P_0(\tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) P_0(\tilde{\beta}_i^{m_1} = a_1)$$

and $A = \{b \in \tilde{S} : P_0(\tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) > 0\}$. It is easy to prove

$$\begin{aligned}
& \left| P_0(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = a_2) P_0(\tilde{\beta}_i^{m_1} = a_1) - Q \right| \\
= & \sum_{b \in \tilde{S}} \left| P_0(\tilde{\beta}_i^{m+1} = a | \tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) - P_0(\tilde{\beta}_i^{m+1} = a | \tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1) \right| \\
& \quad P_0(\tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) P_0(\tilde{\beta}_i^{m_1} = a_1) \\
= & \sum_{b \in A} \left| P_0(\tilde{\beta}_i^{m+1} = a | \tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) - P_0(\tilde{\beta}_i^{m+1} = a | \tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1) \right| \\
& \quad P_0(\tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) P_0(\tilde{\beta}_i^{m_1} = a_1)
\end{aligned}$$

By Lemma 10,

$$\begin{aligned}
& \left| P_0(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = a_2) P_0(\tilde{\beta}_i^{m_1} = a_1) - Q \right| \\
= & \sum_{b \in A} \Big| P_0(\tilde{\beta}_i^{m+1} = a | \tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) - P_0(\tilde{\beta}_i^{m+1} = a | \tilde{\beta}_i^{m} = b) \\
& \quad + P_0(\tilde{\beta}_i^{m+1} = a | \tilde{\beta}_i^{m} = b) - P_0(\tilde{\beta}_i^{m+1} = a | \tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1) \Big| \\
& \quad P_0(\tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) P_0(\tilde{\beta}_i^{m_1} = a_1) \\
\leq & \sum_{b \in A} \left(\frac{2\xi_{-1} + 4\epsilon_0}{P_0(\tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2})} + \frac{2}{\hat{M}^{\frac{1}{9}}} + \frac{2\xi_{-1} + 4\epsilon_0}{P_0(\tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1)} + \frac{2}{\hat{M}^{\frac{1}{9}}} \right) \\
& \quad P_0(\tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) P_0(\tilde{\beta}_i^{m_1} = a_1) \\
\leq & 2K(K+1)(4\xi_{-1} + 8\epsilon_0) + \frac{4}{\hat{M}^{\frac{1}{9}}}.
\end{aligned}$$

Note that

$$\left| P_0(\tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) P_0(\tilde{\beta}_i^{m_1} = a_1) - P_0(\tilde{\beta}_i^{m} = b, \tilde{\beta}_i^{m_1} = a_1) P_0(\tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = \boldsymbol{a_2}) \right| \leq c_m,$$

we can prove

$$\begin{aligned}
& |Q-\sum_{b\in\tilde{S}} P_0(\tilde{\beta}_i^{m+1}=a|\tilde{\beta}_i^{m}=b,\tilde{\beta}_i^{m_1}=a_1)P_0(\tilde{\beta}_i^{m}=b,\tilde{\beta}_i^{m_1}=a_1)P_0(\tilde{\beta}_i^{m_1}=a_1,\tilde{\beta}_i^{\boldsymbol{m_2}}=\boldsymbol{a_2})| \\
= & |Q-\sum_{b\in\tilde{S}} P_0(\tilde{\beta}_i^{m}=b,\tilde{\beta}_i^{m+1}=a,\tilde{\beta}_i^{m_1}=a_1)P_0(\tilde{\beta}_i^{m_1}=a_1,\tilde{\beta}_i^{\boldsymbol{m_2}}=\boldsymbol{a_2})| \\
\leq & 2K(K+1)c_m.
\end{aligned}$$

Then

$$\begin{aligned}
& |P_0(\tilde{\beta}_i^{n+1}=a,\tilde{\beta}_i^{m_1}=a_1,\tilde{\beta}_i^{\boldsymbol{m_2}}=\boldsymbol{a_2})P_0(\tilde{\beta}_i^{m_1}=a_1)-P_0(\tilde{\beta}_i^{n+1}=a,\tilde{\beta}_i^{m_1}=a_1)P_0(\tilde{\beta}_i^{m_1}=a_1,\tilde{\beta}_i^{\boldsymbol{m_2}}=\boldsymbol{a_2})| \\
\leq & 2K(K+1)(4\xi_{-1}+8\epsilon_0+c_m)+\frac{4}{\hat{M}^{\frac{1}{9}}}.
\end{aligned}$$

Let $c_{m+1}=2K(K+1)(4\xi_{-1}+8\epsilon_0+c_m)+\frac{4}{\hat{M}^{\frac{1}{9}}}$. By mathematical induction, it is easy to prove that

$$c_m \leq 2^{2m}K^{2m}(K+1)^{2m}\left(4\xi_{-1}+8\epsilon_0+\frac{4}{\hat{M}^{\frac{1}{9}}}\right).$$

Note that $\xi_{-1}=\frac{1}{M^{M^M}}$, $\epsilon_0=\frac{3CMK^2}{\hat{M}^{\frac{1}{3}}}$ and $\hat{M}\geq M^{M^M}$, c_m is infinitesimal for all $m\leq 3CM$.

7.5.4 Proof of Lemma 3

For any $i\in I$ and $m\leq 3CM$, let $F_{ij}^m=\{\omega\in\Omega:\hat{\pi}_i^m(\omega)=j\}$. It is clear that for any $i\in I$, $F_{ij}^m\cap F_{ij'}^m=\emptyset$ if $j\neq j'$. For any $i\in I$, let $F_i^m=\{j\in I:P_0(F_{ij}^m)\geq\frac{1}{\hat{M}^{\frac{1}{9}}}\}$. Then $\lambda_0(F_i^m)\leq\frac{1}{\hat{M}^{\frac{8}{9}}}$. Let $F_i=\{j\in I:\exists m\leq 3CM$ such that $P_0(F_{ij}^m)\geq\frac{1}{\hat{M}^{\frac{1}{9}}}\}$, then

$$\lambda_0(F_i)\leq\frac{3CM}{\hat{M}^{\frac{8}{9}}}\simeq 0.$$

We only need to find a sequence of positive number $\{d_m\}_{0\leq n\leq 3CM}$ such that d_m is infinitesimal for every m and

$$\begin{aligned}
\Big| & P_0\left(\tilde{\beta}_i^m=a_1,\tilde{\beta}_j^m=a_2,\tilde{\beta}_i^{m-1}=b_1,\tilde{\beta}_j^{m-1}=b_2,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1},\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2}\right) \\
& -P_0\left(\tilde{\beta}_i^m=a_1,\tilde{\beta}_i^{m-1}=b_1,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1}\right)P_0\left(\tilde{\beta}_j^m=a_2,\tilde{\beta}_j^{m-1}=b_2,\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2}\right)\Big|\leq d_m
\end{aligned}$$

for any $i,j\in I$ such that $j\notin F_i$ and $\boldsymbol{m_0}<m-1$. It is clear that $d_0=0$. Suppose we already derive d_m, we need to find out the relationship between d_m and d_{m+1}. It is clear that

$$\begin{aligned}
& P_0(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1},\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2}) \\
= & \int_{D_{b_1b_2}^{ij}} Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2)dQ^m,
\end{aligned}$$

where $D_{b_1b_2}^{ij}=\{\tilde{\beta}_i^m(\omega)=b_1,\tilde{\beta}_j^m(\omega)=b_2,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1},\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2}\}$.

For the step of random matching in period n, $m = 3n-2$. If $b_1 = (k_1, J, 1)$, $b_2 = (k_2, J, 1)$, we can assume $a_1 = (k_1, l_1, 1)$, $a_2 = (k_2, l_2, 1)$, where $l_1, l_2 \in S \cup \{J\}$, otherwise

$$\begin{aligned}
& P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, \tilde{\beta}_i^{\boldsymbol{m_0}} = \boldsymbol{c_1}, \tilde{\beta}_j^{\boldsymbol{m_0}} = \boldsymbol{c_2}\right) \\
= \; & P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, \tilde{\beta}_i^{\boldsymbol{m_0}} = \boldsymbol{c_1}\right) P_0\left(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, \tilde{\beta}_j^{\boldsymbol{m_0}} = \boldsymbol{c_2}\right) \\
= \; & 0.
\end{aligned}$$

By Lemma 9,

$$\begin{aligned}
& \Big| P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, \tilde{\beta}_i^{\boldsymbol{m_0}} = \boldsymbol{c_1}, \tilde{\beta}_j^{\boldsymbol{m_0}} = \boldsymbol{c_2}\right) \\
& \qquad - \int_{D^{ij}_{b_1 b_2}} \hat{q}^n_{k_1 l_1}(\tilde{\rho}^m) \hat{q}^n_{k_2 l_2}(\tilde{\rho}^m) dQ^m \Big| \\
= \; & \left| \int_{D^{ij}_{b_1 b_2}} \left(Q^{\omega^m}_{m+1}(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2) - \hat{q}^n_{k_1 l_1}(\tilde{\rho}^m) \hat{q}^n_{k_2 l_2}(\tilde{\rho}^m) \right) dQ^m \right| \\
\leq \; & \left| \int_{D^{ij}_{b_1 b_2} \setminus V} \left(Q^{\omega^m}_{m+1}(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2) - \hat{q}^n_{k_1 l_1}(\tilde{\rho}^m) \hat{q}^n_{k_2 l_2}(\tilde{\rho}^m) \right) dQ^m \right| + P_0(V) \\
\leq \; & \epsilon_0 + \frac{1}{\hat{M}^{\frac{1}{9}}}.
\end{aligned}$$

By Lemmas 6 and 7,

$$\begin{aligned}
& \left| \int_{D^{ij}_{b_1 b_2}} \hat{q}^n_{k_1 l_1}(\tilde{\rho}^m) \hat{q}^n_{k_2 l_2}(\tilde{\rho}^m) dQ^m - P_0(D^{ij}_{b_1 b_2}) \hat{q}^n_{k_1 l_1}(U^m_1(\tilde{\rho}^0)) \hat{q}^n_{k_2 l_2}(U^m_1(\tilde{\rho}^0)) \right| \\
\leq \; & \left| \int_{D^{ij}_{b_1 b_2}} \hat{q}^n_{k_1 l_1}(\tilde{\rho}^m) \hat{q}^n_{k_2 l_2}(\tilde{\rho}^m) - \hat{q}^n_{k_1 l_1}(U^m_1(\tilde{\rho}^0)) \hat{q}^n_{k_2 l_2}(U^m_1(\tilde{\rho}^0)) dQ^m \right| \\
\leq \; & \left| \int_{D^{ij}_{b_1 b_2} \setminus V} \hat{q}^n_{k_1 l_1}(\tilde{\rho}^m) \hat{q}^n_{k_2 l_2}(\tilde{\rho}^m) - \hat{q}^n_{k_1 l_1}(U^m_1(\tilde{\rho}^0)) \hat{q}^n_{k_2 l_2}(U^m_1(\tilde{\rho}^0)) dQ^m \right| + P_0(V) \\
= \; & \left| \int_{D^{ij}_{b_1 b_2} \setminus V} \hat{q}^n_{k_1 l_1}(\tilde{\rho}^m) \hat{q}^n_{k_2 l_2}(\tilde{\rho}^m) - \hat{q}^n_{k_1 l_1}(U^m_1(\tilde{\rho}^0)) \hat{q}^n_{k_2 l_2}(\tilde{\rho}^m) dQ^m \right| \\
& + \left| \int_{D^{ij}_{b_1 b_2} \setminus V} \hat{q}^n_{k_1 l_1}(U^m_1(\tilde{\rho}^0)) \hat{q}^n_{k_2 l_2}(\tilde{\rho}^m) - \hat{q}^n_{k_1 l_1}(U^m_1(\tilde{\rho}^0)) \hat{q}^n_{k_2 l_2}(U^m_1(\tilde{\rho}^0) dQ^m \right| + P_0(V) \\
\leq \; & \int_{D^{ij}_{b_1 b_2} \setminus V} \left| \hat{q}^n_{k_1 l_1}(\tilde{\rho}^m) - \hat{q}^n_{k_1 l_1}(U^m_1(\tilde{\rho}^0)) \right| dQ^m \\
& + \int_{D^{ij}_{b_1 b_2} \setminus V} \left| \hat{q}^n_{k_2 l_2}(\tilde{\rho}^m) - \hat{q}^n_{k_2 l_2}(U^m_1(\tilde{\rho}^0) \right| dQ^m + P_0(V) \\
\leq \; & 2\xi_{-1} + \epsilon_0
\end{aligned}$$

Therefore,

$$\Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1},\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2}\right)$$
$$-P_0(D^{ij}_{b_1b_2})\hat{q}^n_{k_1l_1}(U_1^m(\tilde{\rho}^0))\hat{q}^n_{k_2l_2}(U_1^m(\tilde{\rho}^0))\Big|\leq 2\epsilon_0+\frac{1}{\hat{M}^{\frac{1}{9}}}+2\xi_{-1}.$$

We can prove

$$\Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1}\right)-P_0(D^i_{b_1})\hat{q}^n_{k_1l_1}(U_1^m(\tilde{\rho}^0))\Big|\leq 2\epsilon_0+\frac{1}{\hat{M}^{\frac{1}{9}}}+2\xi_{-1},$$

in a similar way, where $D^i_{b_1}=\{\tilde{\beta}_i^m=b_1,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1}\}$. Then

$$\begin{aligned}
&\Big|P_0(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1})P_0(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2})\\
&-P_0(D^i_{b_1})P_0(D^j_{b_2})\hat{q}^n_{k_1l_1}(U_1^m(\tilde{\rho}^0))\hat{q}^n_{k_2l_2}(U_1^m(\tilde{\rho}^0))\Big|\\
\leq\;&\Big|P_0(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1})P_0(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2})\\
&-P_0(D^i_{b_1})\hat{q}^n_{k_1l_1}(U_1^m(\tilde{\rho}^0))P_0(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2})\Big|\\
&+\Big|P_0(D^i_{b_1})\hat{q}^n_{k_1l_1}(U_1^m(\tilde{\rho}^0))P_0(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2})\\
&-P_0(D^i_{b_1})P_0(D^j_{b_2})\hat{q}^n_{k_1l_1}(U_1^m(\tilde{\rho}^0))\hat{q}^n_{k_2l_2}(U_1^m(\tilde{\rho}^0))\Big|\\
\leq\;&4\epsilon_0+\frac{2}{\hat{M}^{\frac{1}{9}}}+4\xi_{-1},
\end{aligned}$$

Let $D^{ij}_{b_1b_2b_1'b_2'}=\{\tilde{\beta}_i^m(\omega)=b_1,\tilde{\beta}_j^m(\omega)=b_2,\tilde{\beta}_i^{m-1}(\omega)=b_1',\tilde{\beta}_j^{m-1}(\omega)=b_2',\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1},\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2}\}$, $D_{i,b_1b_1'}=\{\tilde{\beta}_i^m(\omega)=b_1,\tilde{\beta}_i^{m-1}(\omega)=b_1',\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1}\}$. Then

$$\begin{aligned}
&\Big|P_0(D^{ij}_{b_1b_2})-P_0(D^i_{b_1})P_0(D^j_{b_2})\Big|\\
=\;&\Big|\sum_{b_1',b_2'\in\tilde{S}}\Big(P_0(D^{ij}_{b_1b_2b_1'b_2'})-P_0(D_{i,b_1b_1'})P_0(D_{j,b_2b_2'})\Big)\Big|\\
\leq\;&4K^2(K+1)^2d_m.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1},\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2}\right)\\
&-P_0(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1})P_0(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2})\Big|\\
\leq\;&\Big|P_0(D^{ij}_{b_1b_2})\hat{q}^n_{k_1l_1}(U_1^m(\tilde{\rho}^0))\hat{q}^n_{k_2l_2}(U_1^m(\tilde{\rho}^0))-P_0(D^i_{b_1})P_0(D^j_{b_2})\hat{q}^n_{k_1l_1}(U_1^m(\tilde{\rho}^0))\hat{q}^n_{k_2l_2}(U_1^m(\tilde{\rho}^0))\Big|\\
&+6\epsilon_0+\frac{3}{\hat{M}^{\frac{1}{9}}}+6\xi_{-1}\\
\leq\;&\Big|P_0(D^{ij}_{b_1b_2})-P_0(D^i_{b_1})P_0(D^j_{b_2})\Big|+6\epsilon_0+\frac{3}{\hat{M}^{\frac{1}{9}}}+6\xi_{-1}\\
\leq\;&6\epsilon_0+\frac{3}{\hat{M}^{\frac{1}{9}}}+6\xi_{-1}+4K^2(K+1)^2d_m. \qquad (16)
\end{aligned}$$

If $b_1 = (k_1, J, 1)$, $b_2 = (k_2, l_2, 0)$, we can assume $a_1 = (k_1, l_1, 1)$, $a_2 = b_2 = (k_2, l_2, 0)$, where $l_1 \in S \cup \{J\}$, otherwise

$$\begin{aligned}
& P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, \tilde{\beta}_i^{\boldsymbol{m_0}} = \boldsymbol{c_1}, \tilde{\beta}_j^{\boldsymbol{m_0}} = \boldsymbol{c_2}\right) \\
= \; & P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, \tilde{\beta}_i^{\boldsymbol{m_0}} = \boldsymbol{c_1}\right) P_0\left(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, \tilde{\beta}_j^{\boldsymbol{m_0}} = \boldsymbol{c_2}\right) \\
= \; & 0.
\end{aligned}$$

By Lemma 9,

$$\begin{aligned}
& \Big| P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, \tilde{\beta}_i^{\boldsymbol{m_0}} = \boldsymbol{c_1}, \tilde{\beta}_j^{\boldsymbol{m_0}} = \boldsymbol{c_2}\right) \\
& \quad - \int_{D_{b_1 b_2}^{ij}} \hat{q}_{k_1 l_1}^n(\tilde{\rho}^m) dQ^m \Big| \\
= \; & \left| \int_{D_{b_1 b_2}^{ij}} \left(Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1) - \hat{q}_{k_1 l_1}^n(\tilde{\rho}^m)\right) dQ^m \right| \\
\leq \; & \left| \int_{D_{b_1 b_2}^{ij} \setminus V} \left(Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1) - \hat{q}_{k_1 l_1}^n(\tilde{\rho}^m)\right) dQ^m \right| + P_0(V) \\
\leq \; & \epsilon_0 + \frac{1}{\hat{M}^{\frac{1}{9}}}.
\end{aligned}$$

By Lemma 6 and 7,

$$\begin{aligned}
& \left| \int_{D_{b_1 b_2}^{ij}} \hat{q}_{k_1 l_1}^n(\tilde{\rho}^m) dQ^m - P_0(D_{b_1 b_2}^{ij}) \hat{q}_{k_1 l_1}^n(U_1^m(\tilde{\rho}^0)) \right| \\
\leq \; & \left| \int_{D_{b_1 b_2}^{ij}} \hat{q}_{k_1 l_1}^n(\tilde{\rho}^m) - \hat{q}_{k_1 l_1}^n(U_1^m(\tilde{\rho}^0)) dQ^m \right| \\
\leq \; & \left| \int_{D_{b_1 b_2}^{ij} \setminus V} \hat{q}_{k_1 l_1}^n(\tilde{\rho}^m) - \hat{q}_{k_1 l_1}^n(U_1^m(\tilde{\rho}^0)) dQ^m \right| + P_0(V) \\
\leq \; & \xi_{-1} + \epsilon_0
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \Big| P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, \tilde{\beta}_i^{\boldsymbol{m_0}} = \boldsymbol{c_1}, \tilde{\beta}_j^{\boldsymbol{m_0}} = \boldsymbol{c_2}\right) \\
& - P_0(D_{b_1 b_2}^{ij}) \hat{q}_{k_1 l_1}^n(U_1^m(\tilde{\rho}^0)) \Big| \leq 2\epsilon_0 + \frac{1}{\hat{M}^{\frac{1}{9}}} + \xi_{-1}.
\end{aligned}$$

We can prove

$$\left| P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, \tilde{\beta}_i^{\boldsymbol{m_0}} = \boldsymbol{c_1}\right) - P_0(D_{b_1}^i) \hat{q}_{k_1 l_1}^n(U_1^m(\tilde{\rho}^0)) \right| \leq 2\epsilon_0 + \frac{1}{\hat{M}^{\frac{1}{9}}} + \xi_{-1},$$

in a similar way, where $D^i_{b_1} = \{\tilde{\beta}^m_i = b_1, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}\}$. Then

$$\begin{aligned}
& \Big| P_0(\tilde{\beta}^{m+1}_i = a_1, \tilde{\beta}^m_i = b_1, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}) P_0(\tilde{\beta}^{m+1}_j = a_2, \tilde{\beta}^m_j = b_2, \tilde{\beta}^{\boldsymbol{m_0}}_j = \boldsymbol{c_2}) \\
& - P_0(D^i_{b_1}) P_0(D^j_{b_2}) \hat{q}^n_{k_1 l_1}(U^m_1(\tilde{\rho}^0)) \Big| \\
\leq & \Big| P_0(\tilde{\beta}^{m+1}_i = a_1, \tilde{\beta}^m_i = b_1, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}) P_0(D^j_{b_2}) - P_0(D^i_{b_1}) P_0(D^j_{b_2}) \hat{q}^n_{k_1 l_1}(U^m_1(\tilde{\rho}^0)) \Big| \\
\leq & 2\epsilon_0 + \frac{1}{\hat{M}^{\frac{1}{9}}} + \xi_{-1},
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \Big| P_0\left(\tilde{\beta}^{m+1}_i = a_1, \tilde{\beta}^{m+1}_j = a_2, \tilde{\beta}^m_i = b_1, \tilde{\beta}^m_j = b_2, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}, \tilde{\beta}^{\boldsymbol{m_0}}_j = \boldsymbol{c_2}\right) \\
& - P_0(\tilde{\beta}^{m+1}_i = a_1, \tilde{\beta}^m_i = b_1, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}) P_0(\tilde{\beta}^{m+1}_j = a_2, \tilde{\beta}^m_j = b_2, \tilde{\beta}^{\boldsymbol{m_0}}_j = \boldsymbol{c_2}) \Big| \\
\leq & \Big| P_0(D^{ij}_{b_1 b_2}) \hat{q}^n_{k_1 l_1}(U^m_1(\tilde{\rho}^0)) - P_0(D^i_{b_1}) P_0(D^j_{b_2}) \hat{q}^n_{k_1 l_1}(U^m_1(\tilde{\rho}^0)) \Big| \\
& + 4\epsilon_0 + \frac{2}{\hat{M}^{\frac{1}{9}}} + 2\xi_{-1} \\
\leq & \Big| P_0(D^{ij}_{b_1 b_2}) - P_0(D^i_{b_1}) P_0(D^j_{b_2}) \Big| + 4\epsilon_0 + \frac{2}{\hat{M}^{\frac{1}{9}}} + 2\xi_{-1} \\
\leq & 4\epsilon_0 + \frac{2}{\hat{M}^{\frac{1}{9}}} + 2\xi_{-1} + 4K^2(K+1)^2 d_m. \qquad (17)
\end{aligned}$$

If $b_1 = (k_1, l_1, 0)$, $b_2 = (k_2, l_2, 0)$, it is easy to see that

$$\begin{aligned}
& P_0\left(\tilde{\beta}^{m+1}_i = a_1, \tilde{\beta}^{m+1}_j = a_2, \tilde{\beta}^m_i = b_1, \tilde{\beta}^m_j = b_2, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}, \tilde{\beta}^{\boldsymbol{m_0}}_j = \boldsymbol{c_2}\right) \\
= & \delta_{b_1}(a_1) \delta_{b_2}(a_2) P_0\left(\tilde{\beta}^m_i = b_1, \tilde{\beta}^m_j = b_2, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}, \tilde{\beta}^{\boldsymbol{m_0}}_j = \boldsymbol{c_2}\right),
\end{aligned}$$

and

$$P_0\left(\tilde{\beta}^{m+1}_i = a_1, \tilde{\beta}^m_i = b_1, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}\right) = \delta_{b_1}(a_1) \delta_{b_2}(a_2) P_0\left(\tilde{\beta}^m_i = b_1, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}\right).$$

Then

$$\begin{aligned}
& \Big| P_0\left(\tilde{\beta}^{m+1}_i = a_1, \tilde{\beta}^{m+1}_j = a_2, \tilde{\beta}^m_i = b_1, \tilde{\beta}^m_j = b_2, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}, \tilde{\beta}^{\boldsymbol{m_0}}_j = \boldsymbol{c_2}\right) \\
& - P_0(\tilde{\beta}^{m+1}_i = a_1, \tilde{\beta}^m_i = b_1, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}) P_0(\tilde{\beta}^{m+1}_j = a_2, \tilde{\beta}^m_j = b_2, \tilde{\beta}^{\boldsymbol{m_0}}_j = \boldsymbol{c_2}) \Big| \\
\leq & \Big| P_0(D^{ij}_{b_1 b_2}) - P_0(D^i_{b_1}) P_0(D^j_{b_2}) \Big| \\
\leq & 4K^2(K+1)^2 d_m. \qquad (18)
\end{aligned}$$

For the step of random matching, by combining Equations (16), (17) and (18), we can derive

$$\begin{aligned}
& \Big| P_0\left(\tilde{\beta}^{m+1}_i = a_1, \tilde{\beta}^{m+1}_j = a_2, \tilde{\beta}^m_i = b_1, \tilde{\beta}^m_j = b_2, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}, \tilde{\beta}^{\boldsymbol{m_0}}_j = \boldsymbol{c_2}\right) \\
\leq & 6\epsilon_0 + \frac{3}{\hat{M}^{\frac{1}{9}}} + 6\xi_{-1} + 4K^2(K+1)^2 d_m. \qquad (19)
\end{aligned}$$

For the step of random mutation in period n, $m = 3n-3$. Let

$$B^{3n-2}_{ab}(\tilde{\rho}) = \begin{cases} \hat{\eta}^n_{k_1 l_1}(\tilde{\rho})\hat{\eta}^n_{k_2 l_2}(\tilde{\rho}) & \text{if } a = (k_1, k_2, 0), b = (l_1, l_2, 0) \\ \hat{\eta}^n_{k_1 l_1}(\tilde{\rho}) & \text{if } a = (k_1, J, 1), b = (l_1, J, 1) \\ 0 & \text{otherwise.} \end{cases}$$

By the construction, it is clear that

$$Q^{\omega^m}_{m+1}(\tilde{\beta}^{m+1}_i = a_1, \tilde{\beta}^{m+1}_j = a_2) = Q^{\omega^m}_{m+1}(\tilde{\beta}^{m+1}_i = a_1)Q^{\omega^m}_{m+1}(\tilde{\beta}^{m+1}_j = a_2)$$

if $\pi^m_i(\omega) \neq j$. Then

$$\begin{aligned}
& \Big| P_0\left(\tilde{\beta}^{m+1}_i = a_1, \tilde{\beta}^{m+1}_j = a_2, \tilde{\beta}^m_i = b_1, \tilde{\beta}^m_j = b_2, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}, \tilde{\beta}^{\boldsymbol{m_0}}_j = \boldsymbol{c_2}\right) \\
& -P_0(D^{ij}_{b_1 b_2})B^{3n-2}_{b_1 a_1}\left(U^m_1(\tilde{\rho})\right)B^{3n-2}_{b_2 a_2}\left(U^m_1(\tilde{\rho})\right)\Big| \\
\leq & \int_{D^{ij}_{b_1 b_2}} \left| Q^{\omega^m}_{m+1}(\tilde{\beta}^{m+1}_i = a_1, \tilde{\beta}^{m+1}_j = a_2) - B^{3n-2}_{b_1 a_1}\left(U^m_1(\tilde{\rho})\right)B^{3n-2}_{b_2 a_2}\left(U^m_1(\tilde{\rho})\right)\right| dQ^m \\
= & \int_{D^{ij}_{b_1 b_2}\setminus(F^m_{ij}\cup V)} \left| B^{3n-2}_{b_1 a_1}\left(\tilde{\rho}^m_{\omega^m}\right)B^{3n-2}_{b_2 a_2}\left(\tilde{\rho}^m_{\omega^m}\right) - B^{3n-2}_{b_1 a_1}\left(U^m_1(\tilde{\rho})\right)B^{3n-2}_{b_2 a_2}\left(U^m_1(\tilde{\rho})\right)\right| dQ^m \\
& + \int_{D^{ij}_{b_1 b_2}\cap(F^m_{ij}\cup V)} \left| B^{3n-2}_{b_1 a_1}\left(\tilde{\rho}^m_{\omega^m}\right)B^{3n-2}_{b_2 a_2}\left(\tilde{\rho}^m_{\omega^m}\right) - B^{3n-2}_{b_1 a_1}\left(U^m_1(\tilde{\rho})\right)B^{3n-2}_{b_2 a_2}\left(U^m_1(\tilde{\rho})\right)\right| dQ^m \\
\leq & \int_{D^{ij}_{b_1 b_2}\setminus(F^m_{ij}\cup V)} \left| B^{3n-2}_{b_1 a_1}\left(\tilde{\rho}^m_{\omega^m}\right)B^{3n-2}_{b_2 a_2}\left(\tilde{\rho}^m_{\omega^m}\right) - B^{3n-2}_{b_1 a_1}\left(U^m_1(\tilde{\rho})\right)B^{3n-2}_{b_2 a_2}\left(U^m_1(\tilde{\rho})\right)\right| dQ^m + P_0(F^m_{ij}\cup V).
\end{aligned}$$

By Lemma 6 and 7,

$$\left| B^{3n-2}_{b_1 a_1}\left(\tilde{\rho}^m_{\omega^m}\right)B^{3n-2}_{b_2 a_2}\left(\tilde{\rho}^m_{\omega^m}\right) - B^{3n-2}_{b_1 a_1}\left(U^m_1(\tilde{\rho})\right)B^{3n-2}_{b_2 a_2}\left(U^m_1(\tilde{\rho})\right)\right| \leq \xi_{-1}$$

for any $\omega \notin V$. Then

$$\begin{aligned}
& \Big| P_0\left(\tilde{\beta}^{m+1}_i = a_1, \tilde{\beta}^{m+1}_j = a_2, \tilde{\beta}^m_i = b_1, \tilde{\beta}^m_j = b_2, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}, \tilde{\beta}^{\boldsymbol{m_0}}_j = \boldsymbol{c_2}\right) \\
& -P_0(D^{ij}_{b_1 b_2})B^{3n-2}_{b_1 a_1}\left(U^m_1(\tilde{\rho})\right)B^{3n-2}_{b_2 a_2}\left(U^m_1(\tilde{\rho})\right)\Big| \\
\leq & P_0(D^{ij}_{b_1 b_2})\xi_{-1} + \frac{1}{\hat{M}^{\frac{1}{9}}} + \epsilon_0 \\
\leq & \xi_{-1} + \frac{1}{\hat{M}^{\frac{1}{9}}} + \epsilon_0
\end{aligned}$$

By Equation (14)

$$\left| P_0(\tilde{\beta}^{m+1}_i = a_1, \tilde{\beta}^m_i = b_1, \tilde{\beta}^{\boldsymbol{m_0}}_i = \boldsymbol{c_1}) - P_0(D^i_{b_1})B^{3n-2}_{b_1 a_1}\left(U^m_1(\tilde{\rho}^0)\right)\right| \leq \epsilon_0 + \xi_{-1}.$$

Therefore,

$$\begin{aligned}
& \Big| P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2}\right) \\
& -P_0(D_{b_1}^i)P_0(D_{b_2}^j)B_{b_1a_1}^{3n-2}\left(U_1^m(\tilde{\rho}^0)\right)B_{b_2a_2}^{3n-2}\left(U_1^m(\tilde{\rho}^0)\right)\Big| \\
\le\ & \Big| P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2}\right) \\
& -P_0(D_{b_1}^i)B_{b_1a_1}^{3n-2}\left(U_1^m(\tilde{\rho}^0)\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2}\right)\Big| \\
& +\Big|P_0(D_{b_1}^i)B_{b_1a_1}^{3n-2}\left(U_1^m(\tilde{\rho}^0)\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2}\right) \\
& -P_0(D_{b_1}^i)P_0(D_{b_2}^j)\left(U_1^m(\tilde{\rho}^0)\right)B_{b_1a_1}^{3n-2}\left(U_1^m(\tilde{\rho}^0)\right)B_{b_2a_2}^{3n-2}\left(U_1^m(\tilde{\rho}^0)\right)\Big| \\
\le\ & 2\epsilon_0+2\xi_{-1}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1},\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2}\right) \\
& -P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2}\right)\Big| \\
\le\ & \Big|P_0(D_{b_1b_2}^{ij})-P_0(D_{b_1}^i)P_0(D_{b_2}^j)\Big|B_{b_1a_1}^{3n-2}\left(U_1^m(\tilde{\rho}^0)\right)B_{b_2a_2}^{3n-2}\left(U_1^m(\tilde{\rho}^0)\right)+3\xi_{-1}+\frac{1}{\hat{M}^{\frac{1}{9}}}+3\epsilon_0 \\
\le\ & 3\xi_{-1}+\frac{1}{\hat{M}^{\frac{1}{9}}}+3\epsilon_0+4K^2(K+1)^2d_m. \qquad (20)
\end{aligned}$$

For the step of type changing with break up, we can derive Equation (20) in the same way. Therefore,

$$\begin{aligned}
& \Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_j^{m}=b_2,\tilde{\beta}_i^{\boldsymbol{m_0}}=\boldsymbol{c_1},\tilde{\beta}_j^{\boldsymbol{m_0}}=\boldsymbol{c_2}\right) \\
\le\ & 6\epsilon_0+\frac{3}{\hat{M}^{\frac{1}{9}}}+6\xi_{-1}+4K^2(K+1)^2d_m \qquad (21)
\end{aligned}$$

for any $m\le 3CM$. Let $d_{m+1}=6\epsilon_0+\frac{3}{\hat{M}^{\frac{1}{9}}}+6\xi_{-1}+4K^2(K+1)^2d_m$. By mathematical induction, it is easy to prove that Then

$$d_m\le 4^{2m}K^{4m}(K+1)^{4m}\left(6\epsilon_0+\frac{3}{\hat{M}^{\frac{1}{9}}}+6\xi_{-1}\right).$$

Note that $\xi_{-1}=\frac{1}{M^{M^M}}$, $\epsilon_0=\frac{3CMK^2}{\hat{M}^{\frac{1}{3}}}$ and $\hat{M}\ge M^{M^M}$, d_m is infinitesimal for all m.

7.5.5 Proof of Lemma 4

First, it is easy to see that

$$P_0(\hat{X}_i^{m+\Delta m}=\hat{X}_i^m|F^m)=P_0(\hat{X}_i^{m+\Delta m-1}=\hat{X}_i^m|F^m)P_0(\hat{X}_i^{m+\Delta m}=\hat{X}_i^m|\hat{X}_i^{m+\Delta m-1}=\hat{X}_i^m,F^m)$$

For the step of random mutation, we take $m+\Delta m = 3n-2$. If $P_0(\hat{X}_i^{m+\Delta m-1} = \hat{X}_i^m, F^m) > 0$, then

$$\begin{aligned}
& P_0(\hat{X}_i^{m+\Delta m} = \hat{X}_i^m | \hat{X}_i^{m+\Delta m-1} = \hat{X}_i^m, F^m) \\
= \;& \frac{\int_{\{\omega : \hat{X}_i^{m+\Delta m-1} = \hat{X}_i^m\} \cap F^m} Q_{m+\Delta m}^{\omega^{m+\Delta m-1}}(\hat{\alpha}_i^{m+\Delta m} = \hat{\alpha}_i^{m+\Delta m-1}, \hat{g}_i^{m+\Delta m} = \hat{g}_i^{m+\Delta m-1}) dQ^m}{P_0(\hat{X}_i^{m+\Delta m-1} = \hat{X}_i^m, F^m)} \\
\geq \;& \frac{\int_{\{\omega : \hat{X}_i^{m+\Delta m-1} = \hat{X}_i^m\} \cap F^m} \left(1 - \frac{K\bar{\eta}^{e(m+\Delta m)}}{M}\right)^2 dQ^m}{P_0(\hat{X}_i^{m+\Delta m-1} = \hat{X}_i^m, F^m)} \\
= \;& \left(1 - \frac{K\bar{\eta}^{e(m+\Delta m)}}{M}\right)^2 .
\end{aligned}$$

Therefore

$$P_0(\hat{X}_i^{m+\Delta m} = \hat{X}_i^m | F^m) \geq P_0(\hat{X}_i^{m+\Delta m-1} = \hat{X}_i^m | F^m) \left(1 - \frac{K\bar{\eta}^{e(m+\Delta m)}}{M}\right)^2 . \tag{22}$$

If $P_0(\hat{X}_i^{m+\Delta m-1} = \hat{X}_i^m, F^m) = 0$, then the inequality above is trivially satisfied. For the step of random matching and type changing with break up, we can derive

$$P_0(\hat{X}_i^{m+\Delta m} = \hat{X}_i^m | F^m) \geq P_0(\hat{X}_i^{m+\Delta m-1} = \hat{X}_i^m | F^m) \left(1 - \frac{K\bar{q}^{e(m+\Delta m)}}{M}\right) \tag{23}$$

and

$$P_0(\hat{X}_i^{m+\Delta m} = \hat{X}_i^m | F^m) \geq P_0(\hat{X}_i^{m+\Delta m-1} = \hat{X}_i^m | F^m) \left(1 - \frac{K\bar{\vartheta}^{e(m+\Delta m)}}{M}\right) \tag{24}$$

respectively. By combining Inequalities (22), (23) and (24), we can derive

$$\begin{aligned}
& P_0(\hat{X}_i^{m+\Delta m} = \hat{X}_i^m | F^m) \\
\geq \;& P_0(\hat{X}_i^{m+\Delta m-1} = \hat{X}_i^m | F^m) \left(1 - \frac{K\bar{\eta}^{e(m+\Delta m)}}{M}\right)^2 \left(1 - \frac{K\bar{q}^{e(m+\Delta m)}}{M}\right) \left(1 - \frac{K\bar{\vartheta}^{e(m+\Delta m)}}{M}\right) \\
\geq \;& P_0(\hat{X}_i^m = \hat{X}_i^m | F^m) \Pi_{m'=m+1}^{m+\Delta m} \left(1 - \frac{K\bar{\eta}^{e(m')}}{M}\right)^2 \left(1 - \frac{K\bar{q}^{e(m')}}{M}\right) \left(1 - \frac{K\bar{\vartheta}^{e(m')}}{M}\right) \\
\geq \;& \left(1 - \frac{K\bar{\eta}^{e(m+\Delta m)}}{M}\right)^{2\Delta m} \left(1 - \frac{K\bar{q}^{e(m+\Delta m)}}{M}\right)^{\Delta m} \left(1 - \frac{K\bar{\vartheta}^{e(m+\Delta m)}}{M}\right)^{\Delta m} \\
\simeq \;& e^{-\frac{K\Delta m\left(2\bar{\eta}^{e(m+\Delta m)} + \bar{q}^{e(m+\Delta m)} + \bar{\vartheta}^{e(m+\Delta m)}\right)}{M}} .
\end{aligned} \tag{25}$$

7.5.6 Proof of Lemma 5

For any $(k,l,r) \in \tilde{S}$,

$$\begin{aligned}
& \left|\mathbb{E}\left(\tilde{\rho}_{klr}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}_{klr}^{m}\right)\right| \\
= & \left|\mathbb{E}\left(\frac{1}{\hat{M}}\sum_{i\in I}\mathbf{1}_{klr}\left(\tilde{\beta}_i^{m+\Delta m}\right)\right) - \mathbb{E}\left(\frac{1}{\hat{M}}\sum_{i\in I}\mathbf{1}_{klr}\left(\tilde{\beta}_i^{m}\right)\right)\right| \\
\leq & \frac{1}{\hat{M}}\sum_{i\in I}\mathbb{E}\left|\mathbf{1}_{klr}\left(\tilde{\beta}_i^{m+\Delta m}\right) - \mathbf{1}_{klr}\left(\tilde{\beta}_i^{m}\right)\right| \\
\leq & \frac{1}{\hat{M}}\sum_{i\in I}P_0\left(\hat{X}_i^{m+\Delta m} > \hat{X}_i^{m}\right).
\end{aligned}$$

By Lemma 4,

$$P_0\left(\hat{X}_i^{m+\Delta m} > \hat{X}_i^{m}\right) \lesssim 1 - e^{-\frac{K\Delta m\left(2\bar{\eta}^{e(m+\Delta m)}+\bar{q}^{e(m+\Delta m)}+\bar{\vartheta}^{e(m+\Delta m)}\right)}{M}}.$$

Therefore,

$$\left|\mathbb{E}\left(\tilde{\rho}_{klr}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}_{klr}^{m}\right)\right| \lesssim 1 - e^{-\frac{K\Delta m\left(2\bar{\eta}^{e(m+\Delta m)}+\bar{q}^{e(m+\Delta m)}+\bar{\vartheta}^{e(m+\Delta m)}\right)}{M}},$$

which implies

$$\|\mathbb{E}\left(\tilde{\rho}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}^{m}\right)\|_\infty \lesssim 1 - e^{-\frac{K\Delta m\left(2\bar{\eta}^{e(m+\Delta m)}+\bar{q}^{e(m+\Delta m)}+\bar{\vartheta}^{e(m+\Delta m)}\right)}{M}}.$$

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