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CONTINUOUS-TIME RANDOM MATCHING[1]

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We show the existence and properties of continuous-time independent random matching for a large population of agents whose matching intensities can be directed by type and depend on the current cross-sectional type distribution. The agents' type processes form a continuum of independent continuous-time Markov chains whose type changes can be caused by random mutation and random matching. Using the exact law of large numbers, we show how the cross-sectional distribution of agent types evolves deterministically, according to an explicit ordinary differential equation. The results provide the first mathematical foundation for a large literature on continuous-time search-based models of labor markets, money, and over-the-counter financial markets.

Keywords: Independent dynamic random matching, directed search, enduring partnerships, exact law of large numbers, continuous-time, random mutation.

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1. INTRODUCTION

Because of its tractability, continuous-time independent random matching among a continuum of agents is a popular modeling framework for a wide variety of applications in economics.[1]

It is commonly assumed that agents search continuously over time for trading partners, independently of each other. The intensity with which an agent of a given type contacts counterparties of another given type can be specified or determined endogenously. The origins of this general approach can be traced to monetary and labor-market models of the 1970s.[2] This approach was subsequently applied to models of over-the-counter financial markets, general macroeconomics, and other areas of economics. Throughout, the research literature has exploited the idea that independence should, by the law of large numbers, lead to a deterministic cross-sectional (population) distribution of agent types. This paper provides the first mathematical foundations for this general modeling approach and for the associated deterministic aggregate behavior.[3] In particular, we prove the existence of continuous-time random search models among a continuum of

[1]See, for example, Diamond (1993), Diamond and Yellen (1990), Hellwig (1976), Rupert, Schindler, and Wright (2001), Shi (1996), Shi (1997), Trejos and Wright (1995) and Zhou (1997) in monetary theory; Acemoglu and Shimer (1999), Battalio, Samuelson, and Huyck (2001), Flynn (2006), Hosios (1990), Kiyotaki and Lagos (2007), Mortensen (1982), Moscarini (2005), Mortensen and Pissarides (1994), Postel-Vinay and Robin (2002), Rogerson, Shimer, and Wright (2005), Shi and Wen (1999), and Shimer (2005) in labor economics; Duffie, Gârleanu, and Pedersen (2005), Hugonnier, Lester, and Weill (2018), Lagos and Rocheteau (2009), Lester, Rocheteau, and Weill (2015), Üslü (2018), Vayanos and Wang (2007), Weill (2008) in over-the-counter financial markets; Benaïm and Weibull (2003), Currarini, Jackson, and Pin (2009), and Hofbauera and Sandholmb (2007) in game theory; and Amador and Weill (2012), Börgers (1997), Hopkins (1999), and Duffie, Malamud, and Manso (2009) in social learning theory. The same sort of "ansatz" is also applied without mathematical foundations in the natural sciences, including genetics and biological molecular dynamics, as explained by Bomze (1983), Eigen (1971), and Schuster and Sigmund (1983).

[2]See, for example, Hellwig (1976) and Mortensen (1978).

[3]Prior results for a corresponding discrete-time model are considered in Duffie and Sun (2007), Duffie and Sun (2012) and Duffie, Qiao, and Sun (2018). See Section 5 of Duffie, Qiao, and Sun (2018) for a discussion of some related papers. In comparison with discrete-time models, the analysis of continuous-time random matching is substantially more challenging because of the need to capture the cumulative effect of recursive random matching and type changing over "infinitesimally small" time intervals.

agents for the first time, demonstrate key properties that have commonly been relied upon in the literature without prior rigorous foundations, and show new properties that link individual-level behavior in these models to population-level behavior.

Our basic model, to be formalized later, begins with an atomless measure space of agents and a finite set of agent types. We construct a joint agent-probability space on which the agents' type processes form a continuum of independent continuous-time Markov chains, respecting properties derived structurally from random type mutation over time, pair-wise random matching between agents, and random match-induced type changes. Using the exact law of large numbers, we show that the cross-sectional distribution p_t of agents' types at time t is deterministic and satisfies an explicit ordinary differential equation. We also show that there is an initializing cross-sectional distribution of types for which the population's cross-sectional type distribution p_t is constant over time.

A key primitive of the model is the matching intensity function θ, which specifies the intensity (conditional mean arrival rate) $\theta_{kl}(p_t)$ at which an individual agent of current type k at time t is matched to some agent of type l. This intensity is allowed to depend on the cross-sectional distribution p_t of agents' types, subject to minor technical conditions. This accommodates the "matching-function" approach that is popular in labor economics.[4] A second key primitive is the probability distribution ς_{kl} of the new type of a type-k agent that is induced by a match with a type-l agent. Finally, $\eta_{kl} \in [0, \infty)$ is a primitive specifying the intensity with which any type-k agent mutates on its own to type l. Mutation allows for random changes over time in an agent's preferences or productivity, among other type properties.

[4] For a survey, see Petrongolo and Pissarides (2001).

In many practical applications, for example in labor markets, once two agents are matched they may form a long-term relationship rather than immediately break up. For instance, when a worker and a firm meet, they may form a job match. At this point, the worker might stop or slow down searching for new jobs until he or she becomes unemployed again. This is a key aspect of the standard Diamond-Mortensen-Pissarides (DMP) model, as discussed by Diamond (1982). We incorporate enduring forms of matches in Appendix A.[5]

Appendix B provides illustrative applications, drawing from Diamond (1982) for the standard DMP model in labor economics and from Zhou (1997) for the Kiyotaki-Wright model in monetary economics. These examples are intended to show how easily our general model can be applied in various settings.

The proofs of the results on the exact law of large numbers and stationarity for a general continuous-time random matching model, as stated in Theorem A.1, are given in Appendix C.[6] The proofs of our existence results in Theorems 2.1 and A.2 require detailed constructions involving nonstandard analysis, and are thus provided in an on-line-only supplement, Duffie, Qiao, and Sun (2019). This online supplement begins with an overview of the basics of nonstandard analysis.[7] In our proofs, mutations, pairwise random matchings, random match-induced type changes, and randomly timed break-ups are generated recursively at successive infinitesimally spaced time periods. The final results, Theorems 2.1 and A.2, however, are provided in the form of standard measurable continuous-

[5]To this end, for any pair (k, l) of agent types, we introduce the probability ξ_{kl} that an enduring partnership is formed at the time of a match. If formed, this partnership ends at a time with arrival intensity ϑ_{kl}.

[6]Nonstandard analysis is not needed in the proofs of those results.

[7]For a more comprehensive introduction to nonstandard analysis, see the first three chapters of Loeb and Wolff (2015). Nonstandard analysis has been used to study continuous time stochastic processes such as Poisson process in Loeb (1975), and Brownian motion and Itô process in Anderson (1976), Perkins (1981), and Keisler (1984). For applications of nonstandard analysis to economics, see, for example, Brown and Robinson (1975), Khan (1974), Hammond (1999), Anderson and Raimondo (2008), and Stinchcombe (2011).

time stochastic processes that are defined on a standard probability space and on the usual real time line.

2. THE BASIC MODEL AND RESULTS

The set of agents is specified by an atomless measure space $(I, \mathcal{I}, \lambda)$. Without loss of generality, the total mass $\lambda(I)$ of agents is 1. The set of states of the world is given by a probability space[8] $(\Omega, \mathcal{F}, P)$. A key modeling concern is that for any continuum of random variables, independence and joint measurability with respect to the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ are in general not compatible. For applications such as ours, one can work instead with a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, which extends the usual product probability space while retaining the Fubini property, allowing a change in the order of iterated integrals.[9]

Let $S = \{1, 2, \ldots, K\}$ be a finite set of agent types, Δ be the set of probability measures on S (which can be viewed as the simplex in the Euclidean space $\mathbb{R}^K$), and $\mathbb{R}_+$ be the set of non-negative real numbers with its Borel σ-algebra $\mathcal{B}$. The parameters of the model are the initial cross-sectional distribution $p^0 \in \Delta$ of agents' types and, for any k and l in S:

(i) A mutation intensity $\eta_{kl} \in [0, \infty)$ for $k \neq l$ specifying the intensity at which any type-k agent mutates to type l.

(ii) A continuous matching intensity function $\theta_{kl} : \Delta \to \mathbb{R}_+$ specifying the

[8] We follow the convention that a probability space or other measure space such as $(I, \mathcal{I}, \lambda)$ or $(\Omega, \mathcal{F}, P)$ is countably additive.

[9] See Sun (2006) on the measurability issue (Proposition 2.1) and its resolution via a Fubini extension (Section 2). A probability space $(I \times \Omega, \mathcal{W}, Q)$ extending the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ is said to be a *Fubini extension* of $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ if for any real-valued Q-integrable function g on $(I \times \Omega, \mathcal{W})$, the functions $g_i = g(i, \cdot)$ and $g_\omega = g(\cdot, \omega)$ are integrable respectively on $(\Omega, \mathcal{F}, P)$ for λ-almost all $i \in I$ and on $(I, \mathcal{I}, \lambda)$ for P-almost all $\omega \in \Omega$; and if, moreover, $\int_\Omega g_i \, dP$ and $\int_I g_\omega \, d\lambda$ are integrable, respectively, on $(I, \mathcal{I}, \lambda)$ and on $(\Omega, \mathcal{F}, P)$, with $\int_{I \times \Omega} g \, dQ = \int_I \left(\int_\Omega g_i \, dP \right) d\lambda = \int_\Omega \left(\int_I g_\omega \, d\lambda \right) dP$. To reflect the fact that the probability space $(I \times \Omega, \mathcal{W}, Q)$ has $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ as its marginal spaces, as required by the Fubini property, this space is denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

intensity $\theta_{kl}(p)$ with which any type-k agent is matched to some agent of type l, if the current cross-sectional agent type distribution is $p \in \Delta$. This function satisfies the mass-balancing requirement $p_k\theta_{kl}(p) = p_l\theta_{lk}(p)$ that the total aggregate rate of matches of type-k agents to type l agents is of course equal to the aggregate rate of matches of type-l agents to type-k agents. We also require Lipschitz continuity[10] for the mapping $p_k\theta_{kl}(p)$ from Δ to $\mathbb{R}$, for each k and l in S.

(iii) The probability distribution $\varsigma_{kl} \in \Delta$ of the new type of a type-k agent that is induced by a match with a type-l agent. For expositional simplicity, we denote $\varsigma_{kl}(\{r\})$ as ς_{klr} or $\varsigma_{kl}(r)$.

The main solution objects of our model are, for any agent i, state ω, and time t, the agent's type $\alpha(i, \omega, t)$ and the agent's last partner $\varphi(i, \omega, t)$. As a matter of definition, if by time t agent i has never been matched, then $\varphi(i, \omega, t) = i$. Thus, $\varphi(i, \omega, 0) = i$. These solution objects form functions $\alpha : I \times \Omega \times \mathbb{R}_+ \to S$, and $\varphi : I \times \Omega \times \mathbb{R}_+ \to I$.

For agent i, we let $\alpha(i)$ denote her type process, and $\alpha(i, t)$ denote her type at time t. Our objective is to model all agents' type processes, as well as the random mutation, random matching, and matched-induced type changes, in a manner consistent with the given parameters.

Because the counting processes for the cumulative number of mutations and matches of any agent i have an intensity, all of $\alpha(i, t)$ and $\varphi(i, t)$ are piece-wise

[10] A mapping ψ from a subset X of an Euclidean space to another Euclidean space is said to be Lipschitz continuous if there is a positive real number C such that for any $x, x' \in X$, $\|\psi(x) - \psi(x')\| \leq C\|x - x'\|$, where $\|\cdot\|$ is the usual Euclidean norm. This Lipschitz continuity condition on $p_k\theta_{kl}(p)$ (which is weaker than the Lipschitz continuity of θ_{kl}) accommodates the general labor-market matching functions used in the standard DMP model. For this, see Subsection B.1 below. The Lipschitz continuity of $p_k\theta_{kl}(p)$ leads to a Lipschitz condition on the ordinary differential equation in Equation (2.2), which guarantees the uniqueness of the solution of the ordinary differential equation with a given initial condition; see Footnote 12 below.

constant in t. Without loss of generality, we can therefore take these processes to be right-continuous with left-limits (RCLL).[11]

Let N_{ikl} be the counting process for the number of matches by agent i, when of type k, to an agent of type l, and let $N_i = \sum_{k,l \in S} N_{ikl}$ be the counting process for the total number of matches by agent i. That is, $N_i(t)$ is the cumulative number of matches by agent i up to time t. The n-th matching time d_i^n of agent i is thus $\sup\{t \in \mathbb{R}_+ : N_i(t) < n\}$. At a finite matching time d_i^n, agent i is by definition matched to agent $\varphi(i, d_i^n)$. Let $\Theta_{kl}(t)$ be the cumulative total quantity of matches of agents of any given type k with agents of another given type l, by time t. That is, $\Theta_{kl}(\omega, t) = \int_I N_{ikl}(\omega, t)\, d\lambda(i)$.

Next, we define a mapping R from Δ to the space of $K \times K$ matrices by

$$R_{kr}(p) = \eta_{kr} + \sum_{l=1}^{K} \theta_{kl}(p)\varsigma_{klr} \text{ for } k \neq r, \quad \text{and } R_{kk}(p) = -\sum_{l \neq k} R_{kl}(p). \tag{2.1}$$

For a type-k agent and at a given cross-sectional distribution p of agent types, $R_{kr}(p)$ can be viewed as the intensity of transition to a type $r \neq k$ that stems from both mutation and match-induced type changing at a given cross-sectional type distribution p.

The ordinary differential equation (ODE)[12] defined on Δ by

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{x}(t)R(\mathbf{x}(t)) \tag{2.2}$$

[11]That is, for P-almost all $\omega \in \Omega$ and any $t \in \mathbb{R}_+$, there exists $\epsilon > 0$ such that $\alpha(i, \omega, \cdot)$ are constant on $(t - \epsilon, t)$ and $[t, t + \epsilon)$, and likewise for $\varphi(i, \omega, \cdot)$.

[12]We note that the r-th component of $pR(p)$,

$$\sum_{k \in S} p_k R_{kr}(p) = \sum_{k=1}^{K} \left[p_k \eta_{kr} + \sum_{l=1}^{K} p_k \theta_{kl}(p)\varsigma_{klr} \right],$$

is Lipschitz continuous, by the Lipschitz continuity of $p_k \theta_{kl}(p)$. Hence the ordinary differential equation in Equation (2.2) has a unique solution with a given initial condition, as noted in Footnote 10.

governs the evolution of the expected cross-sectional type distribution. We will use the exact law of large numbers to show conditions under which the cross-sectional type distribution is deterministic almost surely and therefore solves the same ODE (2.2).

We will also provide natural conditions under which the initial condition for the population cross-sectional distribution coincides with the initial probability distribution of each agent's type, in which case the path of each agent's type distribution coincides with the path of the cross-sectional type distribution.

For given parameters $(p^0, \eta, \theta, \varsigma)$, a continuous-time independent dynamical system $\mathbb{D}$ with random mutation, random matching, and match-induced type changes is defined by (α, φ) with the following properties:

1. The type $\alpha(i, \omega, t)$ is $(\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}$-measurable.[13]
2. The cross-sectional type distribution p_t at time t is defined by
$$p_{tk} = \lambda(\{i \in I : \alpha(i, t) = k\}),$$
with the specified initial condition $p_0 = p^0$. We let $\bar{p}_t = \mathbb{E}(p_t)$ be the expected cross-sectional type distribution at time t.
3. For each agent i, the type process $\alpha(i)$ of agent i is a continuous-time Markov chain[14] in S whose transition intensity at time t from any state k to any state $r \neq k$ is given by $R_{kr}(\bar{p}_t)$. Let $p_i(t)$ be agent i's type distribution at time t. Then, $\frac{dp_i(t)}{dt} = p_i(t) R(\bar{p}_t)$.[15]
4. The agents' stochastic type processes $\{\alpha_i : i \in I\}$ are pairwise independent. That is, for any $i, j \in I$ with $i \neq j$, α_i and α_j are independent.

[13] As usual, $(\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}$ denotes the product σ-algebra of $\mathcal{I} \boxtimes \mathcal{F}$ and $\mathcal{B}$.

[14] For the definition and properties of a continuous-time Markov chains with time-dependent transition intensities, see, for example, Stroock (2014).

[15] This ODE is known as the Kolmogorov forward equation. See, for instance, Stroock (2014). The matrix $R(\bar{p}_t)$ is known as the "generator" of the associated Markov chain.

5. When some agent i is matched to some agent j, agent j is also matched to agent i. That is, for any agent i, for P-almost all $\omega \in \Omega$, if the n-th matching time $d_i^n(\omega)$ is finite, then we have $\varphi(\varphi(i, d_i^n(\omega)), d_i^n(\omega)) = i$.

The following theorem presents the general existence and properties of a continuous-time independent dynamical system with random mutation, random matching and random type changing.

THEOREM 2.1 *For any given parameters $(p^0, \eta, \theta, \varsigma)$, there exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ on which is defined a continuous-time independent dynamical system $\mathbb{D}$ with these parameters such that:*

(1) The initial cross-sectional type distribution p_0 is p^0 with probability one. This can be achieved with an initial type process α_0 that is deterministic, or is i.i.d. across agents.[16]

(2) For P-almost all $\omega \in \Omega$, the realized cross-sectional type distribution $p_t(\omega)$ at any time t is equal to the expected cross-sectional type distribution $\bar{p}_t$, which solves Equation (2.2).

(3) For P-almost all $\omega \in \Omega$ and for any types k and l, at any time t the cumulative total quantity $\Theta_{kl}(\omega, t)$ of matches of agents of type k with agents of type l is equal to its expectation $\mathbb{E}(\Theta_{kl}(t))$ and grows at the rate $\bar{p}_{tk}\theta_{kl}(\bar{p}_t)$.

(4) For P-almost all $\omega \in \Omega$, the cross-sectional type process α_ω is a Markov chain in S with, at any time t, the same generator (transition intensity matrix) $R(\bar{p}_t)$.

(5) There exists a probability distribution p^* on S such that $p^* R(p^*) = 0$.

[16]This means that for any $i, j \in I$ with $i \neq j$, the random variables $\alpha(i, 0)$ and $\alpha(j, 0)$ are independent, and $\alpha(i, 0)$ has distribution p^0. In this case, all of the agents' type processes are Markov chains with the same finite-dimensional distributions. In this case, the cross-sectional type distribution evolves according to exactly the same dynamics as those governing the evolution of the probability distribution of each agent's type.

(6) *For any* $p^* \in \Delta$ *satisfying* $p^* R(p^*) = 0$*, the dynamical system* $\mathbb{D}$ *with parameters* $(p^*, \eta, \theta, \varsigma)$ *has* p^* *as a stationary type distribution. That is, with probability one the realized cross-sectional type distribution* p_t *is* p^* *at any time* t *and the transition intensity matrix* $R(\bar{p}_t)$ *is constant and equal to* $R(p^*)$.

Property (3) can be used to compute the volumes of specific sorts of transactions, such as financial trades, the velocity of circulation of money, quantities of job matches and layoffs, and so on. Property (4) implies, in principle, the ability to empirically recover the full stochastic evolution behavior of agents' life-time type processes (including all sample-path moments) by observing the cross-sectional distribution of sample paths of agents' types in the single given observed state of the world.

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APPENDICES

In these appendices, we first consider random matching with enduring partnerships, then present some illustrative applications, and finally provide a proof of Theorem A.1. A separate on-line-only supplement, Duffie, Qiao, and Sun (2019), provides proofs of our other results and some ancillary content.

APPENDIX A: RANDOM MATCHING WITH ENDURING PARTNERSHIPS

When a pair of agents stays together for some amount of time after matching, one needs to keep track of the types of the agents and their partners. For this purpose, we introduce a special symbol J to represent "no-match" and the notion of extended types. Let $\hat{S} = S \times (S \cup \{J\})$ be the set of extended types. An agent with an extended type of the form (k, l) has type $k \in S$ and is currently matched to some agent of type l in S. If an agent's extended type is of the form (k, J), then the agent is "unmatched." The space $\hat{\Delta}$ of extended type distributions is the set of probability distributions $\hat{p}$ on $\hat{S}$ such that the probability $\hat{p}_{kl}$ at (k, l) is the same as the probability $\hat{p}_{lk}$ at (l, k) for all k and l in S. A time is an element of $\mathbb{R}_+$, the set of non-negative real numbers, with its Borel σ-algebra $\mathcal{B}$.

The main objects of our model are $\alpha : I \times \Omega \times \mathbb{R}_+ \to S$, $\pi : I \times \Omega \times \mathbb{R}_+ \to I$, and $g : I \times \Omega \times \mathbb{R}_+ \to S \cup \{J\}$ specifying, for any agent i, state ω, and time t, the agent's type $\alpha(i, \omega, t)$, the agent's current partner $\pi(i, \omega, t)$, and the partner's type $g(i, \omega, t)$. As usual, let $\alpha(i)$ (or α_i) and $g(i)$ (or g_i) denote the type processes for agent i and her partners; let $\alpha(i, t)$ (or α_{it}) and $g(i, t)$ (or g_{it}) denote the random types of agent i and of the partner of agent i at time t, respectively; and let α_t and g_t denote the respective mappings $\alpha(\cdot, \cdot, t)$ and $g(\cdot, \cdot, t)$ on $I \times \Omega$. Our objective is to model the type processes α and g, as well as random matching between agents in a manner consistent with given parameters for independent random mutation, independent random matching among agents, independent random break-up for matched pairs, and independent random type changes at each matching and break-up.

The parameters of the model are the initial extended type distribution $\hat{p}^0 \in \hat{\Delta}$ and, for any k and l in S:

(i) The intensity $\eta_{kl} \in \mathbb{R}_+$ for $k \neq l$ with which any type k agent mutates to type l.

(ii) A matching intensity function $\theta_{kl} : \hat{\Delta} \to \mathbb{R}_+$, specifying the intensity $\theta_{kl}(\hat{p})$ with which any type-k agent is matched with a type-l agent, if the cross-sectional agent extended type distribution is $\hat{p} \in \hat{\Delta}$. This function is continuous and satisfies the mass-balancing requirement $\hat{p}_{kJ} \cdot \theta_{kl}(\hat{p}) = \hat{p}_{lJ} \cdot \theta_{lk}(\hat{p})$, that the total aggregate rate of matches of type-k agents to type l agents is of course equal to the aggregate rate of matches of type-l agents to type-k agents. We also require the functions $\hat{p}_{kJ}\theta_{kl}(\hat{p})$, $k, l \in S$ from $\hat{\Delta}$ to $\mathbb{R}$ to be Lipschitz continuous.[17]

(iii) The probability $\xi_{kl} \in [0, 1]$ that a match between a type-k agent and a type-l agent causes a long-term relationship between the two agents after their match, where $\xi_{kl} = \xi_{lk}$.

(iv) $\sigma_{kl} \in \mathcal{M}(S \times S)$ specifying the probability distribution of the new types of a type-k agent and a type-l agent who have been matched, conditional on the event that the match causes an enduring relationship between them, where $\sigma_{kl}\left((k', l')\right) = \sigma_{lk}\left((l', k')\right)$ for any $k', l' \in S$.

(v) $\varsigma_{kl} \in \mathcal{M}(S)$ specifying the probability distribution of the new type of a type-k agent who is matched with a type-l agent, conditional on the event that there is no enduring relationship (the match is dissolved immediately).

(vi) The intensity $\vartheta_{kl} \in [0, \infty)$ of break-up rate of an existing long term relationship between a type-k agent and a type-l agent, where $\vartheta_{kl} = \vartheta_{lk}$.

For simplicity, we assume that when an enduring match between a type-k agent and a type-l agent is eventually broken, these agents emerge with new types drawn independently from the probability distributions ς_{kl} and ς_{lk}, respectively.

Next, we define a mapping from the space $\hat{\Delta}$ of extended type distributions to the space of $K(K+1) \times K(K+1)$ matrices. For $l, l' \in S$, we denote $\delta_l(l') = 0$ for $l \neq l'$, and $\delta_l(l) = 1$. Though η_{kk} is not defined for $k \in S$, we interpret $\eta_{kk}\delta_l(l')$ to be zero for $l \neq l'$. For any $\hat{p} \in \hat{\Delta}$,

[17] As noted earlier in Footnote 10, this Lipschitz continuity condition, which is weaker than the Lipschitz continuity of θ_{kl}, accommodates general labor-market matching functions as in Subsection B.1 below. The Lipschitz continuity of $\hat{p}_{kJ}\theta_{kl}(\hat{p})$ leads to the Lipschitz condition on the ordinary differential equation in Equation (A.6), which guarantees the uniqueness of the solution of the ordinary differential equation for the cross-sectional extended type distributions.

and $k, l, k', l' \in S$, let

$$Q_{(k,l)(k',l')}(\hat{p}) = \eta_{kk'}\delta_l(l') + \eta_{ll'}\delta_k(k') \quad \text{if } (k,l) \neq (k',l'), \tag{A.1}$$

$$Q_{(k,l)(k',J)}(\hat{p}) = \vartheta_{kl}\varsigma_{kl}(k'), \tag{A.2}$$

$$Q_{(k,J)(k',l')}(\hat{p}) = \sum_{r=1}^{K} \theta_{kr}(\hat{p})\xi_{kr}\sigma_{kr}(k',l'), \tag{A.3}$$

$$Q_{(k,J)(k',J)}(\hat{p}) = \eta_{kk'} + \sum_{r=1}^{K} \theta_{kr}(\hat{p})(1-\xi_{kr})\varsigma_{kr}(k') \quad \text{if } k \neq k', \tag{A.4}$$

$$Q_{(k,r)(k,r)}(\hat{p}) = -\sum_{(s',r') \in \hat{S}\setminus\{(k,r)\}} Q_{(k,r)(s',r')}(\hat{p}) \quad \text{for any } r \in S \cup \{J\}. \tag{A.5}$$

For any $(k,l) \in \hat{S}$, it is easy to verify that the function $\sum_{(k',l')\in\hat{S}} \hat{p}_{k'l'} Q_{(k',l')(k,l)}(\hat{p})$ from $\hat{\Delta}$ to $\mathbb{R}$ is Lipschitz continuous, using the Lipschitz continuity of $\hat{p}_{kJ}\theta_{kl}(\hat{p})$.

For given parameters $(\hat{p}^0, \eta, \theta, \xi, \sigma, \varsigma, \vartheta)$, a continuous-time independent dynamical system $\hat{\mathbb{D}}$ with enduring partnerships, if it exists, is a triple (α, π, g) defined by the properties:

1. $\alpha(i,\omega,t)$ and $g(i,\omega,t)$ are $(\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}$-measurable. The stochastic processes α_i and g_i have sample paths that are right-continuous with left limits (RCLL), a standard regularity property of stochastic processes, found for example, in Protter (2005). For any $t \in \mathbb{R}_+$, $\pi(\cdot,\cdot,t)$ (also denoted by $\pi_t(\cdot,\cdot)$) is a random matching on $I \times \Omega$ in the sense that: (i) $\pi_t(\cdot,\cdot)$ is a measurable mapping from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F})$ to $(I, \mathcal{I})$ and (ii) for any $\omega \in \Omega$, the mapping $\pi_{\omega t}(\cdot) = \pi(\cdot, \omega, t)$ is an involution on I, that is, for any $i \in I$, $\pi_{\omega t}(\pi_{\omega t}(i)) = i$; and is measure-preserving, that is, for any $A \in \mathcal{I}$, $\lambda\left(\pi_{\omega t}^{-1}(A)\right) = \lambda(A)$. For any $i \in I$ and $t \in \mathbb{R}_+$,

$$g(i,\omega,t) = \begin{cases} \alpha(\pi(i,\omega,t)) & \text{if } \pi(i,\omega,t) \neq i \\ J & \text{if } \pi(i,\omega,t) = i \end{cases}$$

   for P-almost all $\omega \in \Omega$.

2. The cross-sectional extended type distribution $\hat{p}(t)$ at time t is defined by

$$\hat{p}_{kl}(t) = \lambda(\{i \in I : \alpha(i,t) = k, g(i,t) = l\}).$$

   Let $\check{p}(t)$ be the expected cross-sectional extended type distribution $\mathbb{E}(\hat{p}(t))$. For any agent $i \in I$, the extended type process $(\alpha(i), g(i))$ of agent i is a continuous-time Markov chain in $S \times (S \cup \{J\})$ whose generator (transition-intensity matrix) at time t is $Q(\check{p}(t))$.

3. For each $i, j \in I$ with $i \neq j$, if the probability for agents i, j to be matched at time zero

is zero, then the extended type processes (α_i, g_i) and (α_j, g_j) are independent.[18]

The exact law of large numbers (Theorem 2.16 of Sun (2006)) will be used to show that the cross-sectional type distribution $\hat{p}(t)$ is deterministic almost surely, and equal to its expectation $\check{p}(t)$, which is a solution of the following ordinary differential equation

$$\frac{d\check{p}(t)}{dt} = \check{p}(t)Q\left(\check{p}(t)\right), \qquad \check{p}(0) = \hat{p}^0. \tag{A.6}$$

We are now ready to state the properties of a continuous-time independent dynamical system with random mutation, random matching, and random match-induced type changing, and random break-up. Appendix C contains a proof of this result.

THEOREM A.1 *Let $(\hat{p}^0, \eta, \theta, \xi, \sigma, \varsigma, \vartheta)$ be the parameters for a continuous-time independent dynamical system $\hat{\mathbb{D}}$ with enduring partnerships. Then we have the following properties.*

(1) For P-almost all $\omega \in \Omega$, the realized cross-sectional extended type distribution $\hat{p}(\omega, t)$ at any time t is equal to the expected cross-sectional extended type distribution $\check{p}(t) = \mathbb{E}(\hat{p}(t))$, which satisfies Equation (A.6).

(2) For P-almost all $\omega \in \Omega$, the cross-sectional extended type process $(\alpha_\omega, g_\omega)$ is a continuous-time Markov chain with, at any time t, the transition intensity matrix $Q\left(\check{p}(t)\right)$.

(3) There exists a probability distribution $\hat{p}^*$ on $\hat{S}$ such that $\hat{p}^* Q\left(\hat{p}^*\right) = 0$.

(4) For any $\hat{p}^* \in \hat{\Delta}$ satisfying $\hat{p}^* Q\left(\hat{p}^*\right) = 0$, the dynamical system $\mathbb{D}$ with parameters $(\hat{p}^*, \eta, \theta, \xi, \sigma, \varsigma, \vartheta)$ has $\hat{p}^*$ as a stationary extended type distribution. That is, with probability one, the realized cross-sectional extended type distribution $\hat{p}(t)$ at any time t is $\hat{p}^*$, and all of the relevant Markov chains are time homogeneous with a constant transition intensity matrix $Q(\hat{p}^*)$.

Next, we state the general existence of a continuous-time independent dynamical system with random mutation, random matching, random type changing and random break-up.

THEOREM A.2 *For any given parameters $(\hat{p}^0, \eta, \theta, \xi, \sigma, \varsigma, \vartheta)$, there exists a Fubini extension on which is defined a continuous-time independent dynamical system $\hat{\mathbb{D}}$ with these parameters such that the initial cross-sectional extended type distribution is $\hat{p}^0$ with probability one. This*

[18]If agents i, j are matched at time zero with positive probability, then it would not be possible for (α_i, g_i) and (α_j, g_j) to be independent because the agents may break up jointly with a given intensity.

can be achieved by i.i.d. extended type processes $(\alpha(i), g(i))$, $i \in I$ *with initial extended type distribution being* $\hat{p}^0$ *for each* $i \in I$,[19] *or by an initial extended type process* (α_0, g_0) *that is deterministic.*[20]

A proof of this result is in our online supplement, Duffie, Qiao, and Sun (2019).

APPENDIX B: ILLUSTRATIVE APPLICATIONS

The appendix offers two illustrative applications, drawn respectively from labor economics and monetary theory.

B.1. *The DMP model in labor economics*

Our first example is taken from Diamond (1982). The agents are workers and firms. Each firm has a single job position. Our results for continuous-time random matching with enduring partnerships provide a foundation for the equilibrium employment rate as a result of job search with frictions.

The type space of the agents is $S = \{W, F\}$, where W and F represent workers and firms respectively. The sizes of the populations of workers and firms are L and K respectively.

Frictions in the labor market make it impossible for all the unemployed workers to find jobs instantaneously. The quantity of new job matches is governed by a continuously differentiable mapping $(U, V) \mapsto g(U, V)$. That is, the aggregate matching rate of unemployed workers and vacant jobs is $g(U, V)$, where U and V are the populations of unemployed workers and vacant firms respectively. Clearly, the population of employed workers is $L - U = K - V$, and $g(0, V) = g(U, 0) = 0$. When a firm and a worker meet, they form a (long term) job match with probability one. Furthermore, each matched job-worker pair faces a randomly timed separation at an exogenously specified intensity b.

In Diamond (1982), the total population size $L+K$ of workers and firms is not assumed to be one. In order to stay with our convention that the agent space has total mass one, we can rescale

[19]This means that for $i, j \in I$ with $i \neq j$, the Markov chains $(\alpha(i), g(i))$ and $(\alpha(j), g(j))$ are independent and have the same finite-dimensional distributions, and $(\alpha_0(i), g_0(i))$ has distribution $\hat{p}^0$. Note that this pairwise independence condition here removes the condition in Property (3) of the continuous-time independent dynamical system $\hat{\mathbb{D}}$ that "the probability that agents i and j are matched at time zero is zero."

[20]By Property (3) of the continuous-time independent dynamical system $\hat{\mathbb{D}}$, we know that for any agent $i \in I$, the Markov chains $(\alpha(i), g(i))$ and $(\alpha(j), g(j))$ are independent except when agent $j \in I$ is the initial partner of agent i.

without loss of generality. Viewed in terms of our model, the fraction of unemployed workers is $\hat{p}_{WJ} = U/(L+K)$ and the fraction of vacant firms is $\hat{p}_{FJ} = V/(L+K)$. The corresponding parameters are given as follows. There is no mutation in this model, so $\eta_{WF} = \eta_{FW} = 0$.

Matching occurs only between unemployed workers and firms with vacant jobs. For matching intensities, we define

$$\theta_{kl}(\hat{p}) = \begin{cases} \frac{g\left((L+K)\hat{p}_{WJ},\,(L+K)\hat{p}_{FJ}\right)}{(L+K)\hat{p}_{kJ}} & \text{if } (k,l) = (W,F) \text{ or } (F,W) \text{ and } \hat{p}_{kJ} > 0 \\ \frac{\partial g}{\partial U}\left(0,\,(L+K)\hat{p}_{FJ}\right) & \text{if } (k,l) = (W,F) \text{ and } \hat{p}_{WJ} = 0 \\ \frac{\partial g}{\partial V}\left((L+K)\hat{p}_{WJ},\,0\right) & \text{if } (k,l) = (F,W) \text{ and } \hat{p}_{FJ} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that θ_{WF} is continuous for $\hat{p}_{WJ} > 0$. Since $g\left(0,\,(L+K)\hat{p}_{FJ}\right) = 0$, it is clear that $\frac{g\left((L+K)\hat{p}_{WJ},\,(L+K)\hat{p}_{FJ}\right)}{(L+K)\hat{p}_{WJ}}$ goes to $\frac{\partial g}{\partial U}\left(0,\,(L+K)\hat{p}_{FJ}\right)$ when $\hat{p}_{WJ}$ goes to zero. Hence, θ_{WF} is also continuous when $\hat{p}_{WJ} = 0$. The continuity of θ_{FW} follows the same proof. It is easy to see that for any $\hat{p}_{WJ}$, $\hat{p}_{WJ}\theta_{WF}$ always equals $\frac{g\left((L+K)\hat{p}_{WJ},\,(L+K)\hat{p}_{FJ}\right)}{(L+K)}$. Since g is continuously differentiable, we know that $\hat{p}_{WJ}\theta_{WF}$ is Lipschitz continuous. Similarly, we have $\hat{p}_{FJ}\theta_{FW} = \frac{g\left((L+K)\hat{p}_{WJ},\,(L+K)\hat{p}_{FJ}\right)}{(L+K)}$. Hence, the mass-balancing requirement and the Lipschitz continuity condition are satisfied.

For enduring-relationship probabilities, we define for any $k, l \in S$,

$$\xi_{kl} = \begin{cases} 1 & \text{if } (k,l) = (W,F) \text{ or } (F,W) \\ 0 & \text{otherwise.} \end{cases}$$

The match-induced type-change probabilities are

$$\sigma_{kl}(k', l') = \delta_k(k')\delta_l(l')$$

and

$$\varsigma_{kl}(k') = \delta_k(k').$$

The mean separation rates are

$$\vartheta_{kl} = \begin{cases} b & \text{if } (k,l) = (W,F) \text{ or } (F,W) \\ 0 & \text{otherwise.} \end{cases}$$

Theorem A.1 (1) implies that with probability one, the realized cross-sectional extended type distribution $\hat{p}(t)$ is the same as the expected cross-sectional extended type distribution $\check{p}(t)$, which satisfies Equation (A.6). In the notation of this example, with probability one,

$$
\begin{aligned}
\frac{d\hat{p}_{WF}(t)}{dt} &= \sum_{(k,l)\in\hat{S}} \hat{p}_{kl}(t)\, Q_{(k,l)(W,F)}\,(\hat{p}(t)) \\
&= \hat{p}_{WJ}(t)\, Q_{(W,J)(W,F)}\,(\hat{p}(t)) + \hat{p}_{WF}(t)\, Q_{(W,F)(W,F)}\,(\hat{p}(t)) \\
&= \hat{p}_{WJ}(t)\, \theta_{WF}\,(\hat{p}(t)) - \hat{p}_{WF}(t)\, Q_{(W,F)(W,J)}\,(\hat{p}(t)) \\
&= \frac{g\Big((L+K)\hat{p}_{WJ}(t),\,(L+K)\hat{p}_{FJ}(t)\Big)}{(L+K)} - b\,\hat{p}_{WF}(t).
\end{aligned} \tag{B.1}
$$

By Equation (32) in Diamond (1982), let $f(E,L,K) = g(U,V) = g(L-E, K-E)$, where E is the size of the population of employed workers. It is then clear that with probability one, the realized quantity $E(t)$ of employed workers at any time t is $(L+K)\,\hat{p}_{WF}(t)$. Further, $L - E(t) = (L+K)\,\hat{p}_{WJ}(t)$ and $K - E(t) = (L+K)\,\hat{p}_{FJ}(t)$. Equation (B.1) can then be expressed as

$$
\frac{dE(t)}{dt} = f(E(t), L, K) - bE(t),
$$

which is Equation (22) in Diamond (1982).

B.2. *The Kiyotaki-Wright money model in continuous time*

Our second example is the continuous-time version of the Kiyotaki-Wright Model from Zhou (1997). The economy is populated by a continuum of infinitely-lived agents of unit total mass. Agents are from two regions, Home and Foreign. Let $s \in (0,1)$ be the size of Home population.

Within each of the two regions, there are equal proportions of agents with K different traits, for some $K \geq 3$. The trait space is $\{1,\ldots,K,1^*,\ldots,K^*\}$, where i denotes a home trait and i^* denotes a foreign trait. We shall adopt the convention that $(i^*)^* = i$.

There are K kinds of indivisible commodities in each region. The commodity space is also $\{1,\ldots,K,1^*,\ldots,K^*\}$. An agent with trait k derives utility only from consumption of commodity k or k^*. After he consumes commodity k, he is able to produce one and only one unit of commodity $k+1$ (mod K) costlessly, and can also store up to one unit of his production good costlessly. He can neither produce nor store other types of goods.

An agent of type k has random preferences between goods of types k and k^*. One can think of goods k and k^* as a pair of goods with different features over which a consumer's taste switches from time to time. Let l describe the preference state for an agent with type k, that in which he prefers his local consumption good k over k^*. Let n be the preference state in which he instead prefers the non-local consumption good k^*. The preference state process of each agent is a two-state Markov chain with constant transition intensities b_{ln} from l to n, and b_{nl} from n to l.

In addition to the commodities described above, there are two distinguishable fiat monies, objects with zero intrinsic worth, which we call the Home currency 0 and the Foreign currency 0^*. Each currency is indivisible and can be stored costlessly in amounts of up to one unit by any agent, provided that the agent is not carrying his own production good or the foreign currency. This implies that, at any date, the inventory of each agent consists of one unit of the Home currency, or one unit of the Foreign currency, or one unit of his production good, but does not include more than one of these three objects in total at any one time.

Agents meet pairwise randomly and break up immediately without forming long-term partnerships after the matching. Any agent's potential trading partners arrive at the event times of a Poisson process with parameter ν.

The type space S is the set of ordered tuples of the form (a, b, c), where

$$a \in \{1, \ldots, K, 1^*, \ldots, K^*\},\ b \in \{0, 1, \ldots, K, 0^*, 1^*, \ldots, K^*\},\ c \in \{l, n\}.$$

For example, an agent of type $(1, 2, l)$ is a trait-1 agent who holds one unit of the type-2 good and who prefers local goods.

An agent chooses a trading strategy to maximize his expected discounted utility, taking as given the strategies of other agents and the distribution of inventories. In Zhou (1997), the author focused on pure strategies that depend only on an agent's trait, preference state, and the objects that he and his counterparty have as inventories. Thus, the trading strategy of a trait-a agent with preference state c can be described simply by

$$\tau_{bb'}^{ac} = \begin{cases} 1 & \text{if he agrees to trade object } b \text{ for object } b' \\ 0 & \text{otherwise,} \end{cases}$$

where b and b' are in $\{0, 1, \ldots, K, 0^*, 1^*, \ldots, K^*\}$.

We can use our model of continuous time independent random matching in Section 2 to give a rigorous foundation for the matching model in Zhou (1997) by choosing suitable parameters $(\eta, \theta, \varsigma)$ governing random mutation, random matching, and match-induced type changing. The mutation intensities are

$$\eta_{(a_1,b_1,c_1)(a_2,b_2,c_2)} = \begin{cases} \delta_{a_1}(a_2)\delta_{b_1}(b_2)b_{ln} & \text{if } c_1 = l, c_2 = n \\ \delta_{a_1}(a_2)\delta_{b_1}(b_2)b_{nl} & \text{if } c_1 = n, c_2 = l \\ 0 & \text{otherwise.} \end{cases}$$

Since the agent's potential trading partners arrive at the event times of a Poisson process with parameter ν, the matching intensities are given by

$$\theta_{(a_1,b_1,c_1)(a_2,b_2,c_2)}(p) = \nu\, p_{(a_2,b_2,c_2)},$$

for a given cross-sectional agent type distribution $p \in \Delta$. This matching intensity function obviously satisfies the continuity, mass-balance, and Lipschitz continuity conditions.

Because the consumption traits of agents do not change, a matched agent cannot change to a type with a different trait. Suppose that agent i is of type (a_1, b_1, c_1) and is matched with agent j who has type (a_2, b_2, c_2). The probability that agent i changes type to (a_3, b_3, c_3) is $\varsigma_{(a_1,b_1,c_1)(a_2,b_2,c_2)}[(a_3, b_3, c_3)]$ in our notation. Because the consumption traits and preference of agents are not changed by matching, $\varsigma_{(a_1,b_1,c_1)(a_2,b_2,c_2)}[(a_3, b_3, c_3)] = 0$ if $(a_1, c_1) \neq (a_3, c_3)$.

If an agent of type (a_1, b_1, c_1) obtains one unit of good a_1 or a_1^*, then she will consume the good and produce one unit of good $a_1 + 1$. Thus, there is no agent with type (a_1, b_1, c_1) in the market whenever $b_1 \neq a_1 + 1$.

If $b_3 \neq b_1$, trade occurs between these two types of agents, so

$$\varsigma_{(a_1,b_1,c_1)(a_2,b_2,c_2)}[(a_1, b_3, c_1)] = \begin{cases} \tau^{a_1c_1}_{b_1b_2}\tau^{a_2c_2}_{b_2b_1}\delta_{b_2}(b_3) & \text{if } b_2 \neq a_1 \text{ and } b_2 \neq a_1^* \\ \tau^{a_1c_1}_{b_1b_2}\tau^{a_2c_2}_{b_2b_1}\delta_{a_1+1}(b_3) & \text{if } b_2 = a_1 \text{ or } a_1^*, \text{ and } b_1 \neq a_1 + 1 \\ 0 & \text{if } b_2 = a_1 \text{ or } a_1^*, \text{ and } b_1 = a_1 + 1. \end{cases}$$

If $b_3 = b_1$, we have

$$\varsigma_{(a_1,b_1,c_1)(a_2,b_2,c_2)}[(a_1, b_1, c_1)] = 1 - \sum_{b_3 \neq b_1} \varsigma_{(a_1,b_1,c_1)(a_2,b_2,c_2)}[(a_1, b_3, c_1)].$$

APPENDIX C: PROOF OF THEOREM A.1

For any $i \in I$, $\omega \in \Omega$ and $t \in \mathbb{R}_+$, let $\beta(i,\omega,t) = (\alpha(i,\omega,t),\, g(i,\omega,t))$ be the extended type of agent i with sample realization ω at time t. For any $t > t_1 > \cdots > t_n > 0$, $\Delta t > 0$, $(k,l),(k',l'),(k_1,l_1),\ldots,(k_n,l_n) \in \hat{S}$ with $(k,l) \neq (k',l')$, we have

$$\begin{aligned}
&\lambda \boxtimes P\left(\beta^{t+\Delta t}(i,\omega) = (k',l'), \beta^{t}(i,\omega) = (k,l), \beta^{t_1}(i,\omega) = (k_1,l_1), \ldots, \beta^{t_n}(i,\omega) = (k_n,l_n)\right) \\
&= \int_I P\left(\beta_i^{t+\Delta t}(\omega) = (k',l'), \beta_i^{t}(\omega) = (k,l), \beta_i^{t_1}(\omega) = (k_1,l_1), \ldots, \beta_i^{t_n}(\omega) = (k_n,l_n)\right) d\lambda \\
&= \int_I P\left(\beta_i^{t} = (k,l), \beta_i^{t_1} = (k_1,l_1), \ldots, \beta_i^{t_n} = (k_n,l_n)\right) \\
&\quad P\left(\beta_i^{t+\Delta t} = (k',l') \,\middle|\, \beta_i^{t} = (k,l), \beta_i^{t_1} = (k_1,l_1), \ldots, \beta_i^{t_n} = (k_n,l_n)\right) d\lambda \\
&= \int_I P\left(\beta_i^{t} = (k,l), \beta_i^{t_1} = (k_1,l_1), \ldots, \beta_i^{t_n} = (k_n,l_n)\right) P\left(\beta_i^{t+\Delta t} = (k',l') \,\middle|\, \beta_i^{t} = (k,l)\right) d\lambda.
\end{aligned}$$

Because the transition intensity matrix $Q(\check{p}_t)$ at any time t of the Markov chain β_i does not depend on $i \in I$, so does the conditional probability $P\left(\beta_i^{t+\Delta t} = (k',l') \,\middle|\, \beta_i^{t} = (k,l)\right)$. Let

$$P\left(\beta_i^{t+\Delta t} = (k',l') \,\middle|\, \beta_i^{t} = (k,l)\right) = Q_{(k,l)(k',l')}\left(\check{p}(t)\right)\Delta t + o(\Delta t).$$

Since $Q(\check{p}_t)$ is the transition intensity matrix, we know that $o(\Delta t)$ divided by Δt converges to zero whenever Δt goes to zero. Hence, we obtain that

$$\begin{aligned}
&\lambda \boxtimes P\left(\beta^{t+\Delta t}(i,\omega) = (k',l'), \beta^{t}(i,\omega) = (k,l), \beta^{t_1}(i,\omega) = (k_1,l_1), \ldots, \beta^{t_n}(i,\omega) = (k_n,l_n)\right) \\
= \;& \int_I P\left(\beta_i^{t} = (k,l), \beta_i^{t_1} = (k_1,l_1), \ldots, \beta_i^{t_n} = (k_n,l_n)\right) \left(Q_{(k,l)(k',l')}\left(\check{p}(t)\right)\Delta t + o(\Delta t)\right) d\lambda \\
= \;& [\lambda \boxtimes P(\mathcal{A})]\left[Q_{(k,l)(k',l')}\left(\check{p}(t)\right)\Delta t + o(\Delta t)\right],
\end{aligned}$$

where

$$\mathcal{A} = \{(i,\omega) :\, \beta^{t}(i,\omega) = (k,l), \beta^{t_1}(i,\omega) = (k_1,l_1), \ldots, \beta^{t_n}(i,\omega) = (k_n,l_n)\}.$$

Therefore, we have

$$\lambda \boxtimes P\left(\{(i,\omega) : \beta^{t+\Delta t}(i,\omega) = (k',l')\} \;\middle|\; \mathcal{A}\right) = Q_{(k,l)(k',l')}\left(\check{p}(t)\right)\Delta t + o(\Delta t). \tag{C.1}$$

Note that Equation (C.1) also holds in the case when $n = 0$, which means that

$$\lambda \boxtimes P\left(\{(i,\omega) : \beta^{t+\Delta t}(i,\omega) = (k',l')\} \;\middle|\; \beta^{t}(i,\omega) = (k,l)\right) = Q_{(k,l)(k',l')}\left(\check{p}(t)\right)\Delta t + o(\Delta t). \tag{C.2}$$

By Equations (C.1) and (C.2), we know that β, when viewed as a stochastic process with sample space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, is a Markov chain with time-t transition intensity matrix $Q\left(\check{p}(t)\right)$. By the Fubini property, it is clear that the distribution of β^t is $\check{p}(t)$. Therefore, $\check{p}(t)$

satisfies the ordinary differential equation (A.6).

Parts (1) and (2) of Theorem A.1: For any agent $i \in I$, let $B_i = \{j \in I : P(\pi_0(i) = j) > 0\}$, which is the set of agents $j \in I$ who could be matched with agent i with positive probability at the initial time. It is then clear that B_i is finite or countably infinite, and has λ-measure zero (because λ is atomless). By property (3) of the continuous-time independent dynamical system $\hat{\mathbb{D}}$ with enduring partnerships, we know that for any $j \notin B_i$, the extended type processes $\beta_i = (\alpha_i, g_i)$ and $\beta_j = (\alpha_j, g_j)$ are independent. Hence, the stochastic processes $\beta_i, i \in I$ satisfy the condition of essential pairwise independence in the sense that for each $i \in I$, β_i and β_j are independent for λ-almost all $j \in I$.

By the exact law of large numbers in Theorem 2.16 of Sun (2006), we know that for P-almost all $\omega \in \Omega$, the processes β_ω and β have the same finite-dimensional distributions in the sense that for any $0 \le t_1 \le \cdots \le t_n$, $(\beta_\omega^{t_1}, \ldots, \beta_\omega^{t_n})$ and $(\beta^{t_1}, \ldots, \beta^{t_n})$ (viewed as random vectors) have the same distribution. The finite-dimensional distributions of a process determines whether the process is a Markov chain and also its transition intensity matrix. Thus, for P-almost all $\omega \in \Omega$, β_ω is also a Markov chain with transition intensity matrix $Q(\check{p}(t))$ at time t. So Part (2) of Theorem A.1 is proven. We also know that for P-almost all $\omega \in \Omega$, β_ω^t and β^t have the same distribution at any time t, which implies that $\hat{p}(\omega, t) = \check{p}(t)$. Hence, Part (1) of Theorem A.1 is shown.

Part (3) of Theorem A.1: for any $\hat{p}, \hat{q} \in \hat{\Delta}$, define a vector $\hat{q}Q(\hat{p})$ in $\mathbb{R}^K \times \mathbb{R}^{K+1}$ by letting

$$[\hat{q}Q(\hat{p})]_{kl} = \sum_{(k',l') \in \hat{S}} \hat{q}_{k'l'} Q(\hat{p})_{(k'l')(kl)}.$$

Since $Q_{(kl)(k'l')}$ is continuous on $\hat{\Delta}$, and $\hat{\Delta}$ is compact, we can find a positive real number c such that $\left|cQ_{(kl)(k'l')}(\hat{p})\right| \le 1$ for any $\hat{p} \in \hat{\Delta}$, and $(k,l),(k',l') \in \hat{S}$.

It is easy to see that $\hat{p}Q(\hat{p}) = 0$ is equivalent to the statement that $f(\hat{p}) \triangleq \hat{p} + c\hat{p}Q(\hat{p})$ has a fixed point $\hat{p} = f(\hat{p})$.

Next we show that f is a function from $\hat{\Delta}$ to $\hat{\Delta}$. For this purpose, we need to show that the values of f are probabilities and satisfy some symmetry condition for $\hat{\Delta}$. Note that

$$\begin{aligned} f(\hat{p})_{kl} &= \hat{p}_{kl} + \sum_{(k',l') \in \hat{S}} c\hat{p}_{k'l'} Q(\hat{p})_{(k',l')(k,l)} \\ &= \left(1 + cQ(\hat{p})_{(k,l)(k,l)}\right)\hat{p}_{kl} + \sum_{(k',l') \neq (k,l)} c\hat{p}_{k'l'} Q(\hat{p})_{(k',l')(k,l)}. \end{aligned}$$

By the definition of $Q(\hat{p})$, we know that $Q(\hat{p})_{(k',l')(k,l)} \ge 0$ if $(k',l') \neq (k,l)$. The choice of c implies that $\left(1 + cQ(\hat{p})_{(k,l)(k,l)}\right) \ge 0$. Thus, $[f(\hat{p})]_{kl} \ge 0$ for any $(k,l) \in \hat{S}$.

It is also easy to see that

$$\begin{aligned}
\sum_{(k,l)\in\hat{S}} f(\hat{p})_{(k,l)} &= \sum_{(k,l)\in\hat{S}} \sum_{(k',l')\in\hat{S}} \left(\hat{p}_{k'l'} + c\hat{p}_{k'l'} Q(\hat{p})_{(k',l')(k,l)}\right) \\
&= \sum_{(k',l')\in\hat{S}} \sum_{(k,l)\in\hat{S}} \left(\hat{p}_{k'l'} + c\hat{p}_{k'l'} Q(\hat{p})_{(k',l')(k,l)}\right) \\
&= \sum_{(k',l')\in\hat{S}} \left(\hat{p}_{k'l'} + c\hat{p}_{k'l'} \sum_{(k,l)\in\hat{S}} Q(\hat{p})_{(k',l')(k,l)}\right) \\
&= \sum_{(k',l')\in\hat{S}} \hat{p}_{(k',l')} = 1,
\end{aligned}$$

where the last identity follows from the fact that $\sum_{(k,l)\in\hat{S}} Q(\hat{p})_{(k',l')(k,l)} = 0$. Hence, the values of f are probabilities.

Fix any $\hat{p} \in \hat{\Delta}$, and $k, l \in S$. For any $k_1, l_1 \in S$ with $(k_1, l_1) \neq (k, l)$, we have

$$\begin{aligned}
\hat{p}_{k_1 l_1} Q(\hat{p})_{(k_1,l_1)(k,l)} &= \hat{p}_{k_1 l_1} \left(\eta_{k_1 k}\delta_{l_1}(l) + \eta_{l_1 l}\delta_{k_1}(k)\right) \\
&= \hat{p}_{l_1 k_1} \left(\eta_{l_1 l}\delta_{k_1}(k) + \eta_{k_1 k}\delta_{l_1}(l)\right) \\
&= \hat{p}_{l_1 k_1} Q(\hat{p})_{(l_1,k_1)(l,k)}.
\end{aligned} \tag{C.3}$$

By the mass-balancing requirement, and the symmetries of ξ and σ, we can obtain that

$$\begin{aligned}
\sum_{k'=1}^{K} \hat{p}_{k'J} Q(\hat{p})_{(k',J)(k,l)} &= \sum_{k'=1}^{K} \hat{p}_{k'J} \sum_{l'=1}^{K} \theta_{k'l'}(\hat{p})\xi_{k'l'}\sigma_{k'l'}(k,l) \\
&= \sum_{k'=1}^{K} \sum_{l'=1}^{K} \hat{p}_{k'J}\theta_{k'l'}(\hat{p})\xi_{k'l'}\sigma_{k'l'}(k,l) \\
&= \sum_{l'=1}^{K} \hat{p}_{l'J} \sum_{k'=1}^{K} \theta_{l'k'}(\hat{p})\xi_{l'k'}\sigma_{l'k'}(l,k) \\
&= \sum_{k'=1}^{K} \hat{p}_{l'J} Q(\hat{p})_{(l',J)(l,k)}.
\end{aligned}$$

It follows from Equation (C.4) and the symmetry of ϑ that

$$\begin{aligned}
\hat{p}_{kl} Q(\hat{p})_{(k,l)(k,l)} &= -\sum_{(k',l')\in\hat{S}\setminus(k,l)} \hat{p}_{kl} Q(\hat{p})_{(k,l)(k',l')} \\
&= -\sum_{(k',l')\in S\times S\setminus(k,l)} \hat{p}_{kl} Q(\hat{p})_{(k,l)(k',l')} - \sum_{k'\in S} \hat{p}_{kl} Q(\hat{p})_{(k,l)(k',J)} \\
&= -\sum_{(l',k')\in S\times S\setminus(l,k)} \hat{p}_{lk} Q(\hat{p})_{(l,k)(l',k')} - \sum_{k'\in S} \hat{p}_{kl}\vartheta_{kl}\varsigma_{kl}(k') \\
&= -\sum_{(l',k')\in S\times S\setminus(l,k)} \hat{p}_{lk} Q(\hat{p})_{(l,k)(l',k')} - \hat{p}_{kl}\vartheta_{kl} \\
&= -\sum_{(l',k')\in S\times\hat{S}\setminus(l,k)} \hat{p}_{lk} Q(\hat{p})_{(l,k)(l',k')} \\
&= \hat{p}_{lk} Q(\hat{p})_{(l,k)(l,k)}.
\end{aligned}$$

The above equations imply that

$$
\begin{aligned}
f(\hat{p})_{kl} &= \hat{p}_{kl} + c \sum_{(k',l') \in \hat{S}} \hat{p}_{k'l'} Q(\hat{p})_{(k',l')(k,l)} \\
&= \hat{p}_{kl} + c \sum_{(k',l') \in S \times S \setminus (k,l)} \hat{p}_{k'l'} Q(\hat{p})_{(k',l')(k,l)} + c \sum_{k' \in S} \hat{p}_{k'J} Q(\hat{p})_{(k',J)(k,l)} + c\hat{p}_{kl} Q(\hat{p})_{(k,l)(k,l)} \\
&= \hat{p}_{lk} + c \sum_{(l',k') \in S \times S \setminus (l,k)} \hat{p}_{l'k'} Q(\hat{p})_{(l',k')(l,k)} + c \sum_{l' \in S} \hat{p}_{l'J} Q(\hat{p})_{(l',J)(l,k)} + c\hat{p}_{lk} Q(\hat{p})_{(l,k)(l,k)} \\
&= f(\hat{p})_{lk}
\end{aligned}
$$

Hence, f is a function from $\hat{\Delta}$ to $\hat{\Delta}$.

It is clear that f is continuous on $\hat{\Delta}$. By Kakutani's Fixed Point Theorem, there exists a $\hat{p}^* \in \hat{\Delta}$ such that $\hat{p}^* + c\hat{p}^* Q(\hat{p}^*) = \hat{p}^*$. Therefore, $\hat{p}^* Q(\hat{p}^*) = 0$.

Part (4) of Theorem A.1: Assume that $\hat{p}^*\, Q(\hat{p}^*) = 0$. Since the function $\hat{p} Q(\hat{p})$ is Lipschitz continuous in $\hat{p}$, the ordinary differential equation in Equation (A.6) with the initial condition $\check{p}(0) = \hat{p}^*$ must have a unique solution $\hat{p}^*$, and hence $\check{p}(t) = \hat{p}^*$ at any time t. Therefore, $\hat{p}^*$ is a stationary distribution.