Continuous Time Random Matching*

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Abstract

Models of labor markets, money, and over-the-counter financial markets are often based on continuous-time search by a continuum of agents. This paper presents a general formulation of continuous-time random matching and its properties, which provide the first supporting mathematical foundations for these search-based models. We allow matching intensities to be directed by types and to depend on the current cross-sectional type distribution, covering a wide range of existing applications. The agents' types form a continuum of independent continuous-time Markov chains. Agent-level type changes can be caused by random mutation, random matching, and random break-up. We show that the exact law of large numbers applies, and thus that the cross-sectional distribution of agent types evolves deterministically, according to an explicit ordinary differential equation. We also provide conditions for a stationary cross-sectional distribution of agents' types.

KEYWORDS: Independent dynamic random matching, directed search, enduring partnerships, exact law of large numbers, continuous-time, random mutation.

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1 Introduction

Because of its tractability, continuous-time independent random matching among a continuum of agents is a popular modeling framework for a wide variety of applications in economics.[1]

This literature commonly assumes that agents search continuously over time for trading partners, independently of each other. The intensity with which an agent of a given type contacts counterparties of another given type can be specified or determined endogenously. The origins of this general approach can be traced to monetary and labor-market models of the 1970s.[2] This approach was subsequently applied to models of over-the-counter financial markets, general macroeconomics, and other areas of economics. Throughout, the research literature has exploited the intuitive idea that independence should, by the law of large numbers, lead to a deterministic cross-sectional (population) distribution of agent types. While this is a natural approach, *none* of this literature has shown the existence and required properties of the underlying continuous-time dynamic matching models. This paper provides the first mathematical foundations for this general modeling approach and for the associated deterministic aggregate behavior. In particular, we prove the existence of continuous-time random search models among a continuum of agents for the first time, demonstrate key properties that have commonly been relied upon in the literature, and show new properties that link individual-level behavior in these models to population-level behavior.

Our basic model, to be formalized later, begins with an atomless measure space of agents and a finite set of agent types. We construct a joint agent-probability space on which the agents' type processes form a continuum of independent continuous-time Markov chains, respecting properties derived structurally from random type mutation over time, pair-wise random matching between agents, and random match-induced type changes. Using the exact law of large numbers, we show that the cross-sectional distribution p_t of agents' types at time t is deterministic and satisfies an explicit ordinary differential equation. We also show that there is an initializing cross-sectional distribution of types for which the population's cross-sectional

[1] See, for example, Diamond (1993), Diamond and Yellin (1990), Hellwig (1976), Rupert, Schindler, and Wright (2001), Shi (1996), Shi (1997), Trejos and Wright (1995) and Zhou (1997) in monetary theory; Acemoglu and Shimer (1999), Battalio, Samuelson and Huyck (2001), Flinn (2006), Hosios (1990), Kiyotaki and Lagos (2007), Mortensen (1982), Moscarini (2005), Mortensen and Pissarides (1994), Postel-Vinay and Robin (2002), Rogerson, Shimer and Wright (2005), Shi and Wen (1999), and Shimer (2005) in labor economics; Duffie, Gârleanu, and Pedersen (2005), Hugonnier, Lester and Weill (2018), Lagos and Rocheteau (2009), Lester, Rocheteau and Weill (2015), Üslü (2019), Vayanos and Wang (2007), Weill (2008) in over-the-counter financial markets; Benaïm and Weibull (2003), Currarini, Jackson and Pin (2009), and Hofbauera and Sandholmb (2007) in game theory; and Amador and Weill (2012), Börgers and Sarin (1997), Hopkins (1999), and Duffie, Malamud and Manso (2009) in social learning theory. The same sort of "ansatz" is also applied without mathematical foundations in the natural sciences, including genetics and biological molecular dynamics, as explained by Bomze (1983), Eigen (1971), and Schuster and Sigmund (1983).

[2] See, for example, Hellwig (1976) and Mortensen (1978).

type distribution p_t is constant over time.

A key primitive of the model is the matching intensity function θ, which specifies the conditional mean rate $\theta_{kl}(p_t)$ at which an individual agent of current type k at time t is matched to some agent of type l. This matching intensity is allowed to depend on the cross-sectional distribution p_t of agents' types, subject to minor technical conditions. This accommodates the "matching-function" approach that is popular in labor economics.[3] A second key primitive is the probability distribution ς_{kl} of the new type of a type-k agent that is induced by a match with a type-l agent. Finally, $\eta_{kl} \in [0, \infty)$ is a primitive specifying the intensity with which any type-k agent mutates on its own to type l. Mutation allows for random changes over time in an agent's preferences or productivity, among other type properties.

In many practical applications, for example in labor markets, once two agents are matched they may form a long-term relationship rather than immediately break up. For instance, when a worker and a firm meet, they may form a job match. At this point, the worker might stop or slow down searching for new jobs until he or she becomes unemployed again. This is a key aspect of the standard Diamond-Mortensen-Pissarides (DMP) model, as discussed by Diamond (1982). We incorporate enduring forms of matches in Appendix A.[4]

Appendix B provides illustrative applications, drawing from Diamond (1982) for the standard DMP model in labor economics and from Duffie, Gârleanu, and Pedersen (2005) for a typical model of over-the-counter financial markets. Like all earlier works on continuous-time search, these two papers describe the matching ideas and relevant results intuitively without showing the existence and properties of the underlying dynamic matching model. Our two illustrative applications are intended to show how easily our general models can provide the micro foundations for continuous-time search in various settings with or without enduring partnerships.

Proofs of the results on the exact law of large numbers and stationarity for a general continuous-time random matching model, as stated in Propositions A.1 and A.2, are given in Appendix C. In Appendix D, we briefly explain ideas of the proofs of our existence results in Theorems 2.1 and A.1. Given their complexity, detailed proofs of Theorems 2.1 and A.1 are provided in an on-line-only supplement, Duffie, Qiao and Sun (2020).

[3]For a survey, see Petrongolo and Pissarides (2001).

[4]To this end, for any pair (k, l) of agent types, we introduce the probability ξ_{kl} that an enduring partnership is formed at the time of a match. If formed, this partnership ends at a time with arrival intensity ϑ_{kl}.

2 The Basic Model and Results

The set of agents is specified by an atomless measure space $(I, \mathcal{I}, \lambda)$. Without loss of generality, the total mass $\lambda(I)$ of agents is 1. The set of states of the world is given by a probability space[5] $(\Omega, \mathcal{F}, P)$. A key modeling concern is that for any continuum of random variables, independence and joint measurability with respect to the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ are in general not compatible. For applications such as ours, one can work instead with a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, which extends the usual product probability space while retaining the Fubini property, allowing a change in the order of iterated integrals.[6]

Let $S = \{1, 2, \ldots, K\}$ be a finite set of agent types and Δ be the set of probability measures on S (which can be viewed as the simplex in the Euclidean space $\mathbb{R}^K$). The time domain $\mathbb{R}_+$ is the set of non-negative real numbers with its Borel σ-algebra $\mathcal{B}$. The parameters of the model are the initial cross-sectional distribution $p^0 \in \Delta$ of agents' types and, for any k and l in S:

(i) A mutation intensity $\eta_{kl} \in [0, \infty)$ for $k \neq l$ specifying the intensity at which any type-k agent mutates to type l.

(ii) A continuous matching intensity function $\theta_{kl} : \Delta \to \mathbb{R}_+$ specifying the intensity $\theta_{kl}(p)$ with which any type-k agent is matched to some agent of type l, whenever the current cross-sectional agent type distribution is $p \in \Delta$. This function satisfies the mass-balancing requirement $p_k \theta_{kl}(p) = p_l \theta_{lk}(p)$ that the total aggregate rate of matches of type-k agents to type l agents is of course equal to the aggregate rate of matches of type-l agents to type-k agents. We also require Lipschitz continuity[7] for the mapping $p_k \theta_{kl}(p)$ from Δ to $\mathbb{R}$, for each k and l in S.

[5] We follow the convention that a probability space or other measure space such as $(I, \mathcal{I}, \lambda)$ or $(\Omega, \mathcal{F}, P)$ is countably additive.

[6] See Doob (1937, 1953), Judd (1985) and Uhlig (1996) on the measurability issue associated with a continuum of independent random variables. We apply a resolution of this issue that is based on the Fubini extension in Section 2 of Sun (2006). A probability space $(I \times \Omega, \mathcal{W}, Q)$ extending the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ is said to be a *Fubini extension* of $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ if for any real-valued Q-integrable function g on $(I \times \Omega, \mathcal{W})$, the functions $g_i = g(i, \cdot)$ and $g_\omega = g(\cdot, \omega)$ are integrable respectively on $(\Omega, \mathcal{F}, P)$ for λ-almost all $i \in I$ and on $(I, \mathcal{I}, \lambda)$ for P-almost all $\omega \in \Omega$; and if, moreover, $\int_\Omega g_i \, dP$ and $\int_I g_\omega \, d\lambda$ are integrable, respectively, on $(I, \mathcal{I}, \lambda)$ and on $(\Omega, \mathcal{F}, P)$, with $\int_{I \times \Omega} g \, dQ = \int_I \left(\int_\Omega g_i \, dP \right) d\lambda = \int_\Omega \left(\int_I g_\omega \, d\lambda \right) dP$. To reflect the fact that the probability space $(I \times \Omega, \mathcal{W}, Q)$ has $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ as its marginal spaces, as required by the Fubini property, this space is denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

[7] A mapping ψ from a subset X of an Euclidean space to another Euclidean space is said to be Lipschitz continuous if there is a positive real number C such that for any $x, x' \in X$, $\|\psi(x) - \psi(x')\| \leq C\|x - x'\|$, where $\|\cdot\|$ is the usual Euclidean norm. This Lipschitz continuity condition on $p_k \theta_{kl}(p)$ (which is weaker than the Lipschitz continuity of θ_{kl}) accommodates the general labor-market matching functions used in the standard DMP model. For this, see Subsection B.1 below. The Lipschitz continuity of $p_k \theta_{kl}(p)$ leads to a Lipschitz condition on the ordinary differential equation in Equation (2.2), which guarantees the uniqueness of the solution of the ordinary differential equation with a given initial condition; see Footnote 9 below.

(iii) A probability distribution $\varsigma_{kl} \in \Delta$ of the new type of a type-k agent that is induced by a match with a type-l agent. For expositional simplicity, we denote $\varsigma_{kl}(\{r\})$ as ς_{klr} or $\varsigma_{kl}(r)$.

The main solution objects of our model are, for any agent i, state ω, and time t, the agent's type $\alpha(i, \omega, t)$ and the agent's last partner $\varphi(i, \omega, t)$. As a matter of definition, if by time t agent i has never been matched, then $\varphi(i, \omega, t) = i$. Thus, $\varphi(i, \omega, 0) = i$. These solution objects form functions $\alpha : I \times \Omega \times \mathbb{R}_+ \to S$ and $\varphi : I \times \Omega \times \mathbb{R}_+ \to I$.

For agent i, we let $\alpha(i)$ denote her type process and let $\alpha(i, t)$ denote her type at time t. Our objective is to model all agents' type processes, as well as the random mutation, random matching, and matched-induced type changes, in a manner consistent with the given parameters.

Because the counting processes for the cumulative number of mutations and matches of any agent i have an intensity, all of $\alpha(i, t)$ and $\varphi(i, t)$ are piece-wise constant in t. Without loss of generality, we can therefore take these processes to be right-continuous with left-limits (RCLL).[8]

Let N_{ikl} be the counting process for the number of matches by agent i, when of type k, to an agent of type l, and let $N_i = \sum_{k,l \in S} N_{ikl}$ be the counting process for the total number of matches by agent i. That is, $N_i(t)$ is the cumulative number of matches by agent i up to time t. The n-th matching time d_i^n of agent i is thus $\sup\{t \in \mathbb{R}_+ : N_i(t) < n\}$. At a finite matching time d_i^n, agent i is by definition matched to agent $\varphi(i, d_i^n)$. Let $\Theta_{kl}(t)$ be the cumulative total quantity of matches of agents of any given type k with agents of another given type l by time t. That is, $\Theta_{kl}(\omega, t) = \int_I N_{ikl}(\omega, t)\, d\lambda(i)$.

Next, we define a mapping R from Δ to the space of $K \times K$ matrices by

$$R_{kr}(p) = \eta_{kr} + \sum_{l=1}^{K} \theta_{kl}(p)\varsigma_{klr} \text{ for } k \neq r, \quad \text{and } R_{kk}(p) = -\sum_{l \neq k} R_{kl}(p). \tag{2.1}$$

For a type-k agent and at a given cross-sectional distribution p of agent types, $R_{kr}(p)$ can be viewed as the intensity of transition to a type $r \neq k$ that stems from both mutation and match-induced type changing.

[8] That is, for P-almost all $\omega \in \Omega$ and any $t \in \mathbb{R}_+$, there exists $\epsilon > 0$ such that $\alpha(i, \omega, \cdot)$ are constant on $(t - \epsilon, t)$ and $[t, t + \epsilon)$, and likewise for $\varphi(i, \omega, \cdot)$.

We will show that the ordinary differential equation (ODE)[9] defined on Δ by

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{x}(t)R(\mathbf{x}(t)) \tag{2.2}$$

governs the evolution of the expected cross-sectional type distribution. Further, we will use the exact law of large numbers to show conditions under which the cross-sectional type distribution is deterministic almost surely and therefore solves the same ODE (2.2).

We will also provide natural conditions under which the initial condition for the population cross-sectional distribution coincides with the initial probability distribution of each agent's type, in which case the path of each agent's type distribution coincides with the path of the cross-sectional type distribution.

For given parameters $(p^0, \eta, \theta, \varsigma)$, a continuous-time independent dynamical system $\mathbb{D}$ with random mutation, random matching, and match-induced type changes is defined by (α, φ) with the following properties:

1. The type $\alpha(i, \omega, t)$ is $(\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}$-measurable.[10]

2. The cross-sectional type distribution p_t at time t is defined by

$$p_{tk} = \lambda(\{i \in I : \alpha(i, t) = k\}),$$

   and has the specified initial condition $p_0 = p^0$.

3. For each agent i, the type process $\alpha(i)$ is a process in S whose transition intensity[11] at time t to any type r, given $\alpha(i, t) \neq r$ and cross-sectional type distribution p_t, is $R_{\alpha(i,t),r}(p_t)$.

4. The agents' stochastic type processes $\{\alpha_i : i \in I\}$ are pairwise independent. That is, for any $i, j \in I$ with $i \neq j$, α_i and α_j are independent.

5. When some agent i is matched to some agent j, agent j is also matched to agent i. That

[9] We note that the r-th component of $pR(p)$,

$$\sum_{k \in S} p_k R_{kr}(p) = \sum_{k=1}^{K} \left[p_k \eta_{kr} + \sum_{l=1}^{K} p_k \theta_{kl}(p) \varsigma_{klr} \right],$$

is Lipschitz continuous, by the Lipschitz continuity of $p_k \theta_{kl}(p)$. Hence the ordinary differential equation in Equation (2.2) has a unique solution with a given initial condition, as noted in Footnote 7.

[10] As usual, $(\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}$ denotes the product σ-algebra of $\mathcal{I} \boxtimes \mathcal{F}$ and $\mathcal{B}$.

[11] That is, the conditional probability of the event $(\alpha(i, t + \Delta t) = r)$ given $\alpha(i, t)$ and p_t, divided by Δt goes to $R_{\alpha(i,t),r}(p_t)$ as Δt goes to zero. Following Brémaud (1981), for each k and $r \neq k$, letting C_{ikr} denote the counting process for the cumulative number of transitions of agent i from type k to type r, a martingale M_{ikr} is defined by $M_{ikr}(t) = C_{ikr}(t) - \int_0^t 1_{\{\alpha(i,s)=k\}} R_{kr}(p_s)\, ds$.

is, for any agent i, for P-almost all $\omega \in \Omega$, if the n-th matching time $d_i^n(\omega)$ is finite, then we have $\varphi(\varphi(i, d_i^n(\omega)), d_i^n(\omega)) = i$.

The following theorem presents the general existence and properties of a continuous-time independent dynamical system with random mutation, random matching and random type changing.

Theorem 2.1. *For any given parameters $(p^0, \eta, \theta, \varsigma)$, there exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ on which is defined a continuous-time independent dynamical system $\mathbb{D}$ with these parameters such that:*

(1) For P-almost all $\omega \in \Omega$, the realized cross-sectional type distribution $p_t(\omega)$ at any time t is equal to the expected cross-sectional type distribution $\bar{p}_t$, which solves Equation (2.2).

(2) For each agent i, the type process $\alpha(i)$ is a continuous-time Markov chain[12] in S whose transition intensity at time t from any state k to any state $r \neq k$ is $R_{kr}(\bar{p}_t)$. The probability distribution $p_i(t)$ of the type of agent i at time t thus satisfies the ODE[13]

$$\frac{dp_i(t)}{dt} = p_i(t) R(\bar{p}_t).$$

(3) For P-almost all $\omega \in \Omega$ and for any types k and l, at any time t the cumulative total quantity $\Theta_{kl}(\omega, t)$ of matches of agents of type k with agents of type l is equal to its expectation $\mathbb{E}(\Theta_{kl}(t))$ and grows at the rate $\bar{p}_{tk}\theta_{kl}(\bar{p}_t)$.

(4) For P-almost all $\omega \in \Omega$, the cross-sectional type process α_ω is a Markov chain in S with, at any time t, the same generator (transition intensity matrix) $R(\bar{p}_t)$.

(5) There exists a probability distribution p^* on S such that $p^* R(p^*) = 0$.

(6) For any $p^* \in \Delta$ satisfying $p^* R(p^*) = 0$, the dynamical system $\mathbb{D}$ with parameters $(p^*, \eta, \theta, \varsigma)$ has p^* as a stationary type distribution. That is, with probability one the realized cross-sectional type distribution p_t is p^* at any time t and the transition intensity matrix $R(\bar{p}_t)$ is constant and equal to $R(p^*)$.

Property (3) can be used to compute the volumes of specific sorts of transactions, such as financial trades, the velocity of circulation of money, quantities of job matches and layoffs, and so on. Property (4) implies, in principle, the ability to empirically recover the full stochastic

[12]For the definition and properties of a continuous-time Markov chains with time-dependent transition intensities, see, for example, Stroock (2014).

[13]This ODE is known as the Kolmogorov forward equation. See, for instance, Stroock (2014). The matrix $R(\bar{p}_t)$ is known as the "generator" of the associated Markov chain.

evolution behavior of agents' life-time type processes (including all sample-path moments) by observing the cross-sectional distribution of sample paths of agents' types in the single given observed state of the world.

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APPENDICES

In these appendices, we first consider random matching with enduring partnerships, then present some illustrative applications, and provide proofs of Propositions A.1 and A.2. The proof of Theorem A.1 is complex. For the convenience of the reader, we provide in Appendix D a sketch of its proof. A separate on-line-only supplement, Duffie, Qiao and Sun (2020), provides detailed proofs of Theorem A.1 and of the remaining results in Section 2.

A Random Matching with Enduring Partnerships

When a pair of agents stays together for some amount of time after matching, one needs to keep track of the types of the agents and their partners. For this purpose, we introduce a special symbol J to represent "no-match" and the notion of extended types. Let $\hat{S} = S \times (S \cup \{J\})$ be the set of extended types. An agent with an extended type of the form (k, l) has type $k \in S$ and is currently matched to some agent of type l in S. If an agent's extended type is of the form (k, J), then the agent is "unmatched." The space $\hat{\Delta}$ of extended type distributions is the set of probability distributions $\hat{p}$ on $\hat{S}$ such that the probability $\hat{p}_{kl}$ at (k, l) is the same as the probability $\hat{p}_{lk}$ at (l, k) for all k and l in S. A time is an element of $\mathbb{R}_+$, the set of non-negative real numbers, with its Borel σ-algebra $\mathcal{B}$.

The main objects of our model are $\alpha : I \times \Omega \times \mathbb{R}_+ \to S$, $\pi : I \times \Omega \times \mathbb{R}_+ \to I$, and $g : I \times \Omega \times \mathbb{R}_+ \to S \cup \{J\}$ specifying, for any agent i, state ω, and time t, the agent's type $\alpha(i, \omega, t)$, the agent's current partner $\pi(i, \omega, t)$, and the partner's type $g(i, \omega, t)$. As usual, let $\alpha(i)$ (or α_i) and $g(i)$ (or g_i) denote the type processes for agent i and her partners; let $\alpha(i, t)$ (or α_{it}) and $g(i, t)$ (or g_{it}) denote the random types of agent i and of the partner of agent i at time t, respectively; and let α_t and g_t denote the respective mappings $\alpha(\cdot, \cdot, t)$ and $g(\cdot, \cdot, t)$ on $I \times \Omega$. Our objective is to model the type processes α and g, as well as random matching between agents in a manner consistent with given parameters for independent random mutation, independent random matching among agents, independent random break-up for matched pairs, and independent random type changes at each matching and break-up.

The parameters of the model are the initial extended type distribution $\hat{p}^0 \in \hat{\Delta}$ and, for any k and l in S:

(i) The mutation intensity $\eta_{kl} \in \mathbb{R}_+$ for $k \neq l$ with which any type k agent mutates to type l.

(ii) A matching intensity function $\theta_{kl} : \hat{\Delta} \to \mathbb{R}_+$, specifying the intensity $\theta_{kl}(\hat{p})$ with which any type-k agent is matched with a type-l agent, if the cross-sectional agent extended type distribution is $\hat{p} \in \hat{\Delta}$. This function is continuous and satisfies the mass-balancing requirement $\hat{p}_{kJ} \cdot \theta_{kl}(\hat{p}) = \hat{p}_{lJ} \cdot \theta_{lk}(\hat{p})$, that the total aggregate rate of matches of type-k agents to type l agents is of course equal to the aggregate rate of matches of type-l agents to type-k agents. We also require the functions $\hat{p}_{kJ}\theta_{kl}(\hat{p})$, $k, l \in S$ from $\hat{\Delta}$ to $\mathbb{R}$ to be Lipschitz continuous.[14]

(iii) The enduring probability $\xi_{kl} \in [0, 1]$ that a match between a type-k agent and a type-l agent causes a long-term relationship between the two agents after their match, where $\xi_{kl} = \xi_{lk}$.

(iv) $\sigma_{kl} \in \mathcal{M}(S \times S)$ specifying the probability distribution of the new types of a type-k agent and a type-l agent who have been matched, conditional on the event that the match causes an enduring relationship between them, where $\sigma_{kl}\left((k', l')\right) = \sigma_{lk}\left((l', k')\right)$ for any $k', l' \in S$.

(v) $\varsigma_{kl} \in \mathcal{M}(S)$ specifying the probability distribution of the new type of a type-k agent who is matched with a type-l agent, conditional on the event that there is no enduring relationship (the match is dissolved immediately).

(vi) The break-up intensity $\vartheta_{kl} \in [0, \infty)$, which is the mean rate at which an existing long-term relationship between a type-k agent and a type-l agent is broken, where $\vartheta_{kl} = \vartheta_{lk}$.

For simplicity, we assume that when an enduring match between a type-k agent and a type-l agent is eventually broken, these agents emerge with new types drawn independently with the probability distributions ς_{kl} and ς_{lk}, respectively.

[14]As noted earlier in Footnote 7, this Lipschitz continuity condition, which is weaker than the Lipschitz continuity of θ_{kl}, accommodates general labor-market matching functions as in Subsection B.1 below. The Lipschitz continuity of $\hat{p}_{kJ}\theta_{kl}(\hat{p})$ leads to the Lipschitz condition on the ordinary differential equation in Equation (A.1), which guarantees the uniqueness of the solution of the ordinary differential equation for the cross-sectional extended type distributions.

Table 1: Transition intensities

from \ to	(k',l')	(k',J)
(k,l)	$\eta_{kk'}\,\delta_l(l')+\eta_{ll'}\,\delta_k(k')$ Case 1: mutation	$\vartheta_{kl}\,\varsigma_{kl}(k')$ Case 2: break up
(k,J)	$\sum_{r=1}^{K}\theta_{kr}(\hat{p})\,\xi_{kr}\,\sigma_{kr}(k',l')$ Case 3: enduring matching	$\eta_{kk'}+\sum_{r=1}^{K}\theta_{kr}(\hat{p})(1-\xi_{kr})\varsigma_{kr}(k')$ Case 4: mutation, or matching without enduring

Note that $\delta_k(k')$ in Case 1 is one for $k=k'$ and zero for $k\neq k'$. Given the parameters above and any extended type distribution $\hat{p}$, it is easy to derive the desired transition intensities listed in Table 1. For example, in Case 2, an agent with extended type (k,l) may break up with the partner and become an agent with type (k',J), the corresponding transition intensity is $\vartheta_{kl}\varsigma_{kl}(k')$. Let $Q(\hat{p})$ be the transition matrix described in the table above. For any $(k,l)\in\hat{S}$, it is easy to verify that the function $\sum_{(k',l')\in\hat{S}}\hat{p}_{k'l'}Q_{(k',l')(k,l)}(\hat{p})$ from $\hat{\Delta}$ to $\mathbb{R}$ is Lipschitz continuous, using the Lipschitz continuity of $\hat{p}_{kJ}\theta_{kl}(\hat{p})$.

For given parameters $(\hat{p}^0,\eta,\theta,\xi,\sigma,\varsigma,\vartheta)$, a continuous-time independent dynamical system $\hat{\mathbb{D}}$ with enduring partnerships, if it exists, is a triple (α,π,g) defined by the properties:

1. $\alpha(i,\omega,t)$ and $g(i,\omega,t)$ are $(\mathcal{I}\boxtimes\mathcal{F})\otimes\mathcal{B}$-measurable. The stochastic processes α_i and g_i are right-continuous with left limits (RCLL), a standard regularity property of stochastic processes, found for example, in Protter (2005). For any $t\in\mathbb{R}_+$, $\pi(\cdot,\cdot,t)$ (also denoted by $\pi_t(\cdot,\cdot)$) is a random matching on $I\times\Omega$ in the sense that: (i) $\pi_t(\cdot,\cdot)$ is a measurable mapping from $(I\times\Omega,\mathcal{I}\boxtimes\mathcal{F})$ to $(I,\mathcal{I})$ and (ii) for any $\omega\in\Omega$, the mapping $\pi_{\omega t}(\cdot)=\pi(\cdot,\omega,t)$ is an involution on I, that is, for any $i\in I$, $\pi_{\omega t}(\pi_{\omega t}(i))=i$; and is measure-preserving, that is, for any $A\in\mathcal{I}$, $\lambda\left(\pi_{\omega t}^{-1}(A)\right)=\lambda(A)$. For any $i\in I$ and $t\in\mathbb{R}_+$,

$$g(i,\omega,t)=\begin{cases}\alpha(\pi(i,\omega,t),\omega,t) & \text{if } \pi(i,\omega,t)\neq i\\ J & \text{if } \pi(i,\omega,t)=i\end{cases}$$

for P-almost all $\omega \in \Omega$.

2. The cross-sectional extended type distribution $\hat{p}(t)$ at time t is defined by

$$\hat{p}_{kl}(t) = \lambda(\{i \in I : \alpha(i,t) = k, g(i,t) = l\}).$$

   Let $\check{p}(t)$ be the expected cross-sectional extended type distribution $\mathbb{E}\left(\hat{p}(t)\right)$. For any agent $i \in I$, the extended type process $(\alpha(i), g(i))$ of agent i is a continuous-time Markov chain in $S \times (S \cup \{J\})$ whose generator (transition-intensity matrix) at time t is $Q\left(\check{p}(t)\right)$.

3. For each $i, j \in I$ with $i \neq j$, if agents i, j are not matched at time zero, then the extended type processes (α_i, g_i) and (α_j, g_j) are independent.[15]

The exact law of large numbers (Theorem 2.16 of Sun (2006)) will be used to show that the cross-sectional type distribution $\hat{p}(t)$ is deterministic almost surely, and equal to its expectation $\check{p}(t)$, which is a solution of the following ordinary differential equation

$$\frac{d\check{p}(t)}{dt} = \check{p}(t)Q\left(\check{p}(t)\right), \qquad \check{p}(0) = \hat{p}^0. \tag{A.1}$$

We are now ready to state the general existence of a continuous-time independent dynamical system with random mutation, random matching, random type changing and random break-up.

Theorem A.1. *For any given parameters $(\hat{p}^0, \eta, \theta, \xi, \sigma, \varsigma, \vartheta)$, there exists a Fubini extension on which is defined a continuous-time independent dynamical system $\hat{\mathbb{D}}$ with these parameters such that the initial cross-sectional extended type distribution is $\hat{p}^0$ with probability one.*

A proof of this result is in our online supplement, Duffie, Qiao and Sun (2020). For the convenience of the reader, we also provide a sketch for the proof in Appendix D.

In the next two propositions, we state the properties of a continuous-time independent dynamical system with random mutation, random matching, and random match-induced type changing, and random break-up. Appendix C contains the proofs of both results.

Using the exact law of large numbers, Part (1) of the next proposition shows that the cross-sectional distribution of agent types evolves deterministically, according to Equation (A.1). Part (2) implies, in principle, the ability to empirically recover the full stochastic evolution behavior of agents' life-time type processes (including all sample-path moments) by observing the cross-sectional distribution of sample paths of agents' types in the single given observed state of the world.

[15] If agents i, j are matched at time zero, then it would not be possible for (α_i, g_i) and (α_j, g_j) to be independent because the agents may break up jointly with a given intensity.

Proposition A.1. *Let $(\hat{p}^0, \eta, \theta, \xi, \sigma, \varsigma, \vartheta)$ be the parameters for a continuous-time independent dynamical system $\hat{\mathbb{D}}$ with enduring partnerships. Then we have the following properties.*

(1) For P-almost all $\omega \in \Omega$, the realized cross-sectional extended type distribution $\hat{p}(\omega, t)$ at any time t is equal to the expected cross-sectional extended type distribution $\check{p}(t) = \mathbb{E}(\hat{p}(t))$, which satisfies Equation (A.1).

(2) For P-almost all $\omega \in \Omega$, the cross-sectional extended type process $(\alpha_\omega, g_\omega)$ is a continuous-time Markov chain with, at any time t, the transition intensity matrix $Q(\check{p}(t))$.

The following proposition shows the general existence of a stationary cross-sectional distribution of agents' types.

Proposition A.2. *(1) There exists a probability distribution $\hat{p}^*$ on $\hat{S}$ such that $\hat{p}^* Q(\hat{p}^*) = 0$.*

(2) For any $\hat{p}^* \in \hat{\Delta}$ satisfying $\hat{p}^* Q(\hat{p}^*) = 0$, a continuous-time independent dynamical system $\mathbb{D}$ with parameters $(\hat{p}^*, \eta, \theta, \xi, \sigma, \varsigma, \vartheta)$ has $\hat{p}^*$ as a stationary extended type distribution. That is, with probability one, the realized cross-sectional extended type distribution $\hat{p}(t)$ at any time t is $\hat{p}^*$, and all of the relevant Markov chains are time homogeneous with a constant transition intensity matrix $Q(\hat{p}^*)$.

B Illustrative Applications

This appendix provides two illustrative applications of our results, providing mathematical foundations for typical models of labor and financial markets.

B.1 The Diamond (1982) model of labor markets

In the literature on continuous-time search with enduring partnerships, often used to study labor markets, a common assumption is that unmatched agents (unemployed workers and vacant positions) have matching opportunities described by independent Poisson processes. Already matched agents break up with specified intensities. Our illustrative application here is the model of Diamond (1982).

The agents are workers and firms. Each firm has a single job position. Our results for continuous-time random matching with enduring partnerships provide a foundation for the equilibrium employment rate as a result of job search with frictions.

The type space of the agents is $S = \{W, F\}$, where W and F represent workers and firms respectively. The sizes of the populations of workers and firms are L and K respectively.

Frictions in the labor market make it impossible for all the unemployed workers to find jobs instantaneously. The quantity of new job matches is governed by a continuously differentiable mapping $(U,V) \mapsto g(U,V)$. That is, the aggregate matching rate of unemployed workers and vacant jobs is $g(U,V)$, where U and V are the populations of unemployed workers and vacant firms respectively. Clearly, the population of employed workers is $L-U=K-V$, and $g(0,V)=g(U,0)=0$. When a firm and a worker meet, they form a (long term) job match with probability one. Furthermore, each matched job-worker pair faces a randomly timed separation at an exogenously specified intensity b.

In Diamond (1982), the total population size $L+K$ of workers and firms is not assumed to be one. In order to stay with our convention that the agent space has total mass one, we can rescale without loss of generality. Viewed in terms of our model, the fraction of unemployed workers is $\hat{p}_{WJ} = U/(L+K)$ and the fraction of vacant firms is $\hat{p}_{FJ} = V/(L+K)$. The corresponding parameters are given as follows. There is no mutation in this model, so $\eta_{WF} = \eta_{FW} = 0$.

Matching occurs only between unemployed workers and firms with vacant jobs. For matching intensities, we define

$$\theta_{kl}(\hat{p}) = \begin{cases} \frac{g\left((L+K)\hat{p}_{WJ},(L+K)\hat{p}_{FJ}\right)}{(L+K)\hat{p}_{kJ}} & \text{if } (k,l)=(W,F) \text{ or } (F,W) \text{ and } \hat{p}_{kJ}>0 \\ \frac{\partial g}{\partial U}\left(0,\,(L+K)\hat{p}_{FJ}\right) & \text{if } (k,l)=(W,F) \text{ and } \hat{p}_{WJ}=0 \\ \frac{\partial g}{\partial V}\left((L+K)\hat{p}_{WJ},\,0\right) & \text{if } (k,l)=(F,W) \text{ and } \hat{p}_{FJ}=0 \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that θ_{WF} is continuous for $\hat{p}_{WJ}>0$. Since $g\left(0,\,(L+K)\hat{p}_{FJ}\right)=0$, it is clear that

$$\lim_{\hat{p}_{WJ}\to 0} \frac{g\left((L+K)\hat{p}_{WJ},\,(L+K)\hat{p}_{FJ}\right)}{(L+K)\hat{p}_{WJ}} = \frac{\partial g}{\partial U}\left(0,\,(L+K)\hat{p}_{FJ}\right).$$

Hence, θ_{WF} is also continuous when $\hat{p}_{WJ}=0$. The continuity of θ_{FW} follows the same proof. It is easy to see that for any $\hat{p}_{WJ}$,

$$\hat{p}_{WJ}\theta_{WF} = \frac{g\left((L+K)\hat{p}_{WJ},\,(L+K)\hat{p}_{FJ}\right)}{(L+K)}.$$

Since g is continuously differentiable, we know that $\hat{p}_{WJ}\theta_{WF}$ is Lipschitz continuous. Similarly, we have

$$\hat{p}_{FJ}\theta_{FW} = \frac{g\left((L+K)\hat{p}_{WJ},\,(L+K)\hat{p}_{FJ}\right)}{(L+K)}.$$

Hence, the mass-balancing requirement and the Lipschitz continuity condition are satisfied.

For enduring-relationship probabilities, we define for any $k, l \in S$,

$$\xi_{kl} = \begin{cases} 1 & \text{if } (k,l) = (W,F) \text{ or } (F,W) \\ 0 & \text{otherwise.} \end{cases}$$

The match-induced type-change probabilities are

$$\sigma_{kl}(k', l') = \delta_k(k')\delta_l(l')$$

and

$$\varsigma_{kl}(k') = \delta_k(k').$$

The mean separation rates are

$$\vartheta_{kl} = \begin{cases} b & \text{if } (k,l) = (W,F) \text{ or } (F,W) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition A.1 (1) implies that with probability one, the realized cross-sectional extended type distribution $\hat{p}(t)$ is the same as the expected cross-sectional extended type distribution $\check{p}(t)$, which satisfies Equation (A.1). In the notation of this example, with probability one,

$$\begin{aligned} \frac{d\hat{p}_{WF}(t)}{dt} &= \sum_{(k,l)\in\hat{S}} \hat{p}_{kl}(t)\, Q_{(k,l)(W,F)}\left(\hat{p}(t)\right) \qquad \text{(B.1)} \\ &= \hat{p}_{WJ}(t)\, Q_{(W,J)(W,F)}\left(\hat{p}(t)\right) + \hat{p}_{WF}(t)\, Q_{(W,F)(W,F)}\left(\hat{p}(t)\right) \\ &= \hat{p}_{WJ}(t)\, \theta_{WF}\left(\hat{p}(t)\right) - \hat{p}_{WF}(t)\, Q_{(W,F)(W,J)}\left(\hat{p}(t)\right) \\ &= \frac{g\left((L+K)\hat{p}_{WJ}(t),\, (L+K)\hat{p}_{FJ}(t)\right)}{(L+K)} - b\,\hat{p}_{WF}(t). \end{aligned}$$

By Equation (32) in Diamond (1982), we can let

$$f(E, L, K) = g(U, V) = g(L - E, K - E),$$

where E is the size of the population of employed workers. It is then clear that with probability one, the realized quantity $E(t)$ of employed workers at any time t is $(L+K)\,\hat{p}_{WF}(t)$. Further, $L - E(t) = (L+K)\,\hat{p}_{WJ}(t)$ and $K - E(t) = (L+K)\,\hat{p}_{FJ}(t)$. Equation (B.1) can then be expressed as

$$\frac{dE(t)}{dt} = f(E(t), L, K) - bE(t), \qquad \text{(B.2)}$$

which is Equation (22) in Diamond (1982).[16]

B.2 Over-the-counter financial markets

Our second illustrative application is the model of over-the-counder financial markets of Duffie, Gârleanu, and Pedersen (2005). There are two classes of agents, investors and marketmakers. Each agent consumes a single nonstorable consumption good that is used as a numéraire. In Duffie, Gârleanu, and Pedersen (2005), the masses of investors and marketmakers are each 1. In order to stay within the simplex Δ of agent masses stipulated by our basic model, without loss of generality we rescale these two type masses to 1/2.

Investors can hold 0 or 1 unit of the asset. A fraction s of investors are initially endowed with 1 unit of the asset. An investor is characterized as an asset owner or non-owner, and also by an intrinsic preference for ownership that is high (h) or low (l). A low type switches from low to high with intensity λ_u, and switches back with intensity λ_d.

The type space is thus $S = \{ho, hn, lo, ln, m\}$, where the letters "$h$" and "$l$" designate the investor's intrinsic preference, "o" and "n" indicate whether the investor owns the asset or not, and "m" indicates a marketmaker. Marketmakers never change their type. When a high-preference non-owner meets an low-preference owner, they endogenously choose to trade the asset, generating a change of types for each. Other investor-to-investor matches generate no trade, thus no type changes. Trades generated by contact with a marketmaker will be characterized shortly.

Investors meet by independent random search, as follows. At the successive event times of a Poisson process with some intensity parameter λ, an investor contacts another investor chosen at random, uniformly from the entire investor population. Thus, letting

$$\mu_k(t) = \frac{p_k(t)}{p_{ho}(t) + p_{hn}(t) + p_{lo}(t) + p_{ln}(t)} = 2p_k(t)$$

denote the relative fraction of investors (among all the investors) of type k at time t, the intensity with which any given investor contacts an investor of type k is $\lambda\mu_k(t)$. In Duffie, Gârleanu, and Pedersen (2005), contact is directional, in the sense that the event of a type k investor contacting a type r investor is distinguished from the event of a type r investor contacting a type k investor. Thus the total meeting intensity for specific type-k investor with some type-r investor is $\theta_{kr}(p(t)) = 2\lambda\mu_r(t) = 4\lambda p_r(t)$. This directional-contact formulation implies that the derived matching intensity function θ automatically satisfies the mass-balance

[16] Since f and b are the matching and mean separation rates respectively, Equation (B.2) was claimed intuitively in Diamond (1982) without showing the existence and properties of the underlying dynamic matching model. Our results in Appendix A provide a rigorous micro foundation for the continuous time search ideas in Diamond (1982), and prove the existence of the desired dynamic matching model and the validity of Equation (B.2).

condition. Directional contact also allows in principle for a difference in the terms of trade in the asset bargaining outcome, depending on which of a pair contacts the other, but that difference plays no role here.

Each investor also contacts some randomly drawn marketmaker at the event times of a Poisson process with a fixed intensity of ρ. When a type-hn investor meets a marketmaker, the community of marketmakers may be experiencing an excess of buyer contacts relative to seller contacts. Marketmakers are able to instantly lay off their trades in the inter-dealer market, but do not absorb excess order flow into their own accounts. In that case, each marketmaker rations trade by randomizing whether it quotes a price that is acceptable to the contacting investor. Specifically, at contact with a marketmaker, a type-hn investor trades successfully with probability $\frac{\min(p_{hn}, p_{lo})}{p_{hn}}$ (and becomes a type-ho investor), which implies that the intensity for her successful search with a marketmaker is $\rho\frac{\min(p_{hn}, p_{lo})}{p_{hn}}$. Similarly, the intensity for a type-lo investor to search successfully for a marketmaker to trade is $\rho\frac{\min(p_{hn}, p_{lo})}{p_{lo}}$.

Viewed in terms of our model in Section 2, the corresponding parameters are given as follows. The type mutation intensities are, for any k and $r \in S$ such that $k \neq r$,

$$\eta_{kr} = \begin{cases} \lambda_u & \text{if } (k,r) = (lo, ho) \text{ or } (ln, hn) \\ \lambda_d & \text{if } (k,r) = (ho, lo) \text{ or } (hn, ln) \\ 0 & \text{otherwise.} \end{cases}$$

For matching intensities, we have for any k and $r \in S$,

$$\theta_{kr}(p) = \begin{cases} 4\lambda p_r & \text{if } k, r \in \{ho, lo, hn, ln\} \\ \rho\frac{\min(p_{hn}, p_{lo})}{p_{hn}} & \text{if } k = hn \text{ and } r = m \\ \rho\frac{\min(p_{hn}, p_{lo})}{p_{lo}} & \text{if } k = lo \text{ and } r = m \\ 2\rho\min(p_{hn}, p_{lo}) & \text{if } k = m \text{ and } r \in \{hn, lo\} \\ 0 & \text{otherwise.} \end{cases}$$

It is obvious that the mass-balancing requirements for θ as well as the Lipschitz continuity for the mapping $p_k\theta_{kr}(p)$ are satisfied. When a type-hn investor meets a type-lo investor or finds successfully a marketmaker to trade, the type-hn investor, having purchased the asset, becomes a type-ho investor. Likewise the type-lo investor becomes a type-ln investor. Thus, we have for any $k, r, k' \in S$,

$$\varsigma_{kr}(k') = \begin{cases} \delta_{ho}(k') & \text{if } k = hn \text{ and } r \in \{lo, m\} \\ \delta_{ln}(k') & \text{if } k = lo \text{ and } r \in \{hn, m\} \\ \delta_k(k') & \text{otherwise.} \end{cases}$$

Then we can derive the following transition intensities as listed in Equation (2.1):

$$\begin{aligned} R_{(ho)(lo)}(p) &= \lambda_d \\ R_{(hn)(lo)}(p) &= 0 \\ R_{(lo)(lo)}(p) &= -\lambda_u - 4\lambda p_{hn} - \frac{\rho \min(p_{hn}, p_{lo})}{p_{lo}} \\ R_{(ln)(lo)}(p) &= 0 \\ R_{(m)(lo)}(p) &= 0. \end{aligned}$$

The evolution of the fraction of type-lo agents is thus governed by (with probability one)

$$\begin{aligned} \frac{dp_{lo}(t)}{dt} &= p_{ho}(t) R_{(ho)(lo)}\left(p(t)\right) + p_{lo}(t) R_{(lo)(lo)}\left(p(t)\right) \\ &= \lambda_d \, p_{ho}(t) - \left(\lambda_u + 4\lambda \, p_{hn}(t) + \frac{\rho \min(p_{hn}(t), p_{lo}(t))}{p_{lo}(t)}\right) p_{lo}(t) \\ &= \lambda_d \, p_{ho}(t) - \lambda_u \, p_{lo}(t) - 4\lambda \, p_{hn} \, p_{lo}(t) - \rho \, \min(p_{hn}(t), p_{lo}(t)). \end{aligned}$$

Because $\mu_k(t) = 2p_k(t)$,

$$\frac{d\mu_{lo}(t)}{dt} = \lambda_d \, \mu_{ho}(t) - \lambda_u \, \mu_{lo}(t) - 2\lambda \, \mu_{hn}(t) \, \mu_{lo}(t) - \rho \, \min(\mu_{hn}(t), \mu_{lo}(t)), \tag{B.3}$$

which is Equation (3) of Duffie, Gârleanu, and Pedersen (2005). The remaining population-distribution evolution equations of Duffie, Gârleanu, and Pedersen (2005) follow similarly. Similar to the remark in Footnote 16, while Equation (B.3) was claimed in Duffie, Gârleanu, and Pedersen (2005) intuitively without showing the existence and properties of the underlying dynamic matching model, the verbal description of continuous time matching as in Duffie, Gârleanu, and Pedersen (2005) can be formulated and proved by our model and results in Section 2, which lead to a rigorous derivation of Equation (B.3).

C Proofs of Propositions A.1 and A.2

C.1 Proof of Proposition A.1

For any $i \in I$, $\omega \in \Omega$ and $t \in \mathbb{R}_+$, let $\beta(i,\omega,t) = (\alpha(i,\omega,t),\, g(i,\omega,t))$ be the extended type of agent i with sample realization ω at time t. For any $t > t_1 > \cdots > t_n > 0$, $\Delta t > 0$, $(k,l), (k',l'), (k_1,l_1), \ldots, (k_n,l_n) \in \hat{S}$ with $(k,l) \neq (k',l')$, we have

$$
\begin{aligned}
&\lambda \boxtimes P\left(\beta^{t+\Delta t}(i,\omega) = (k',l'), \beta^t(i,\omega) = (k,l), \beta^{t_1}(i,\omega) = (k_1,l_1), \ldots, \beta^{t_n}(i,\omega) = (k_n,l_n)\right) \\
&= \int_I P\left(\beta_i^{t+\Delta t}(\omega) = (k',l'), \beta_i^t(\omega) = (k,l), \beta_i^{t_1}(\omega) = (k_1,l_1), \ldots, \beta_i^{t_n}(\omega) = (k_n,l_n)\right) d\lambda \\
&= \int_I P\left(\beta_i^t = (k,l), \beta_i^{t_1} = (k_1,l_1), \ldots, \beta_i^{t_n} = (k_n,l_n)\right) \\
&\quad P\left(\beta_i^{t+\Delta t} = (k',l') \,\middle|\, \beta_i^t = (k,l), \beta_i^{t_1} = (k_1,l_1), \ldots, \beta_i^{t_n} = (k_n,l_n)\right) d\lambda \\
&= \int_I P\left(\beta_i^t = (k,l), \beta_i^{t_1} = (k_1,l_1), \ldots, \beta_i^{t_n} = (k_n,l_n)\right) P\left(\beta_i^{t+\Delta t} = (k',l') \,\middle|\, \beta_i^t = (k,l)\right) d\lambda.
\end{aligned}
$$

Because the transition intensity matrix $Q(\check{p}_t)$ at any time t of the Markov chain β_i does not depend on $i \in I$, so does the conditional probability $P\left(\beta_i^{t+\Delta t} = (k',l') \,\middle|\, \beta_i^t = (k,l)\right)$. Let

$$
P\left(\beta_i^{t+\Delta t} = (k',l') \,\middle|\, \beta_i^t = (k,l)\right) = Q_{(k,l)(k',l')}\left(\check{p}(t)\right) \Delta t + o(\Delta t).
$$

Since $Q(\check{p}_t)$ is the transition intensity matrix, we know that $o(\Delta t)$ divided by Δt converges to zero whenever Δt goes to zero. Hence, we obtain that

$$
\begin{aligned}
&\lambda \boxtimes P\left(\beta^{t+\Delta t}(i,\omega) = (k',l'), \beta^t(i,\omega) = (k,l), \beta^{t_1}(i,\omega) = (k_1,l_1), \ldots, \beta^{t_n}(i,\omega) = (k_n,l_n)\right) \\
= \;& \int_I P\left(\beta_i^t = (k,l), \beta_i^{t_1} = (k_1,l_1), \ldots, \beta_i^{t_n} = (k_n,l_n)\right) \left(Q_{(k,l)(k',l')}\left(\check{p}(t)\right) \Delta t + o(\Delta t)\right) d\lambda \\
= \;& [\lambda \boxtimes P(\mathcal{A})] \left[Q_{(k,l)(k',l')}\left(\check{p}(t)\right) \Delta t + o(\Delta t)\right],
\end{aligned}
$$

where

$$
\mathcal{A} = \{(i,\omega) :\ \beta^t(i,\omega) = (k,l), \beta^{t_1}(i,\omega) = (k_1,l_1), \ldots, \beta^{t_n}(i,\omega) = (k_n,l_n)\}.
$$

Therefore, we have

$$
\lambda \boxtimes P\left(\{(i,\omega) : \beta^{t+\Delta t}(i,\omega) = (k',l')\} \;\middle|\; \mathcal{A}\right) = Q_{(k,l)(k',l')}\left(\check{p}(t)\right) \Delta t + o(\Delta t). \qquad \text{(C.1)}
$$

Note that Equation (C.1) also holds in the case when $n = 0$, which means that

$$
\lambda \boxtimes P\left(\{(i,\omega) : \beta^{t+\Delta t}(i,\omega) = (k',l')\} \;\middle|\; \beta^t(i,\omega) = (k,l)\right) = Q_{(k,l)(k',l')}\left(\check{p}(t)\right) \Delta t + o(\Delta t). \quad \text{(C.2)}
$$

By Equations (C.1) and (C.2), we know that β, when viewed as a stochastic process with sample space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, is a Markov chain with time-t transition intensity matrix $Q(\check{p}(t))$. By the Fubini property, it is clear that the distribution of β^t is $\check{p}(t)$. Therefore, $\check{p}(t)$ satisfies the ordinary differential equation (A.1).

For any agent $i \in I$, let $B_i = \{j \in I : P(\pi_0(i) = j) > 0\}$, which is the set of agents $j \in I$ who could be matched with agent i with positive probability at the initial time. It is then clear that B_i is finite or countably infinite, and has λ-measure zero (because λ is atomless). By property (3) of the continuous-time independent dynamical system $\hat{\mathbb{D}}$ with enduring partnerships, we know that for any $j \notin B_i$, the extended type processes $\beta_i = (\alpha_i, g_i)$ and $\beta_j = (\alpha_j, g_j)$ are independent. Hence, the stochastic processes $\beta_i, i \in I$ satisfy the condition of essential pairwise independence in the sense that for each $i \in I$, β_i and β_j are independent for λ-almost all $j \in I$.

By the exact law of large numbers in Theorem 2.16 of Sun (2006), we know that for P-almost all $\omega \in \Omega$, the processes β_ω and β have the same finite-dimensional distributions in the sense that for any $0 \leq t_1 \leq \cdots \leq t_n$, $(\beta_\omega^{t_1}, \ldots, \beta_\omega^{t_n})$ and $(\beta^{t_1}, \ldots, \beta^{t_n})$ (viewed as random vectors) have the same distribution. The finite-dimensional distributions of a process determines whether the process is a Markov chain and also its transition intensity matrix. Thus, for P-almost all $\omega \in \Omega$, β_ω is also a Markov chain with transition intensity matrix $Q(\check{p}(t))$ at time t. So Part (2) of Proposition A.1 is proven. We also know that for P-almost all $\omega \in \Omega$, β_ω^t and β^t have the same distribution at any time t, which implies that $\hat{p}(\omega, t) = \check{p}(t)$. Hence, Part (1) of Proposition A.1 is shown.

C.2 Proof of Proposition A.2

Part (1) of Proposition A.2: for any $\hat{p}, \hat{q} \in \hat{\Delta}$, define a vector $\hat{q}Q(\hat{p})$ in $\mathbb{R}^K \times \mathbb{R}^{K+1}$ by letting

$$[\hat{q}Q(\hat{p})]_{kl} = \sum_{(k',l') \in \hat{S}} \hat{q}_{k'l'} Q(\hat{p})_{(k'l')(kl)}.$$

Since $Q_{(kl)(k'l')}$ is continuous on $\hat{\Delta}$, and $\hat{\Delta}$ is compact, we can find a positive real number c such that $\left|cQ_{(kl)(k'l')}(\hat{p})\right| \leq 1$ for any $\hat{p} \in \hat{\Delta}$, and $(k,l), (k',l') \in \hat{S}$.

It is easy to see that $\hat{p}Q(\hat{p}) = 0$ is equivalent to the statement that $f(\hat{p}) \triangleq \hat{p} + c\hat{p}Q(\hat{p})$ has a fixed point $\hat{p} = f(\hat{p})$.

Next we show that f is a function from $\hat{\Delta}$ to $\hat{\Delta}$. For this purpose, we need to show that the values of f are probabilities and satisfy some symmetry condition for $\hat{\Delta}$. Note that

$$f(\hat{p})_{kl} = \hat{p}_{kl} + \sum_{(k',l') \in \hat{S}} c\hat{p}_{k'l'} Q(\hat{p})_{(k',l')(k,l)}$$

$$= \left(1 + cQ(\hat{p})_{(k,l)(k,l)}\right)\hat{p}_{kl} + \sum_{(k',l')\neq(k,l)} c\hat{p}_{k'l'}Q(\hat{p})_{(k',l')(k,l)}.$$

By the definition of $Q(\hat{p})$, we know that $Q(\hat{p})_{(k',l')(k,l)} \geq 0$ if $(k',l') \neq (k,l)$. The choice of c implies that $\left(1 + cQ(\hat{p})_{(k,l)(k,l)}\right) \geq 0$. Thus, $[f(\hat{p})]_{kl} \geq 0$ for any $(k,l) \in \hat{S}$.

It is also easy to see that

$$\begin{aligned}
\sum_{(k,l)\in\hat{S}} f(\hat{p})_{(k,l)} &= \sum_{(k,l)\in\hat{S}} \sum_{(k',l')\in\hat{S}} \left(\hat{p}_{k'l'} + c\hat{p}_{k'l'}Q(\hat{p})_{(k',l')(k,l)}\right) \\
&= \sum_{(k',l')\in\hat{S}} \sum_{(k,l)\in\hat{S}} \left(\hat{p}_{k'l'} + c\hat{p}_{k'l'}Q(\hat{p})_{(k',l')(k,l)}\right) \\
&= \sum_{(k',l')\in\hat{S}} \left(\hat{p}_{k'l'} + c\hat{p}_{k'l'} \sum_{(k,l)\in\hat{S}} Q(\hat{p})_{(k',l')(k,l)}\right) \\
&= \sum_{(k',l')\in\hat{S}} \hat{p}_{(k',l')} = 1,
\end{aligned}$$

where the last identity follows from the fact that $\sum_{(k,l)\in\hat{S}} Q(\hat{p})_{(k',l')(k,l)} = 0$. Hence, the values of f are probabilities.

Fix any $\hat{p} \in \hat{\Delta}$, and $k,l \in S$. For any $k_1, l_1 \in S$ with $(k_1,l_1) \neq (k,l)$, we have

$$\begin{aligned}
\hat{p}_{k_1l_1}Q(\hat{p})_{(k_1,l_1)(k,l)} &= \hat{p}_{k_1l_1}\left(\eta_{k_1k}\delta_{l_1}(l) + \eta_{l_1l}\delta_{k_1}(k)\right) \\
&= \hat{p}_{l_1k_1}\left(\eta_{l_1l}\delta_{k_1}(k) + \eta_{k_1k}\delta_{l_1}(l)\right) \\
&= \hat{p}_{l_1k_1}Q(\hat{p})_{(l_1,k_1)(l,k)}.
\end{aligned} \tag{C.3}$$

By the mass-balancing requirement, and the symmetries of ξ and σ, we can obtain that

$$\begin{aligned}
\sum_{k'=1}^{K} \hat{p}_{k'J}Q(\hat{p})_{(k',J)(k,l)} &= \sum_{k'=1}^{K} \hat{p}_{k'J} \sum_{l'=1}^{K} \theta_{k'l'}(\hat{p})\xi_{k'l'}\sigma_{k'l'}(k,l) \\
&= \sum_{k'=1}^{K}\sum_{l'=1}^{K} \hat{p}_{k'J}\theta_{k'l'}(\hat{p})\xi_{k'l'}\sigma_{k'l'}(k,l) \\
&= \sum_{l'=1}^{K} \hat{p}_{l'J} \sum_{k'=1}^{K} \theta_{l'k'}(\hat{p})\xi_{l'k'}\sigma_{l'k'}(l,k) \\
&= \sum_{k'=1}^{K} \hat{p}_{l'J}Q(\hat{p})_{(l',J)(l,k)}.
\end{aligned}$$

It follows from Equation (C.4) and the symmetry of ϑ that

$$\begin{aligned}
\hat{p}_{kl}Q(\hat{p})_{(k,l)(k,l)} &= -\sum_{(k',l')\in\hat{S}\setminus(k,l)} \hat{p}_{kl}Q(\hat{p})_{(k,l)(k',l')} \\
&= -\sum_{(k',l')\in S\times S\setminus(k,l)} \hat{p}_{kl}Q(\hat{p})_{(k,l)(k',l')} - \sum_{k'\in S}\hat{p}_{kl}Q(\hat{p})_{(k,l)(k',J)} \\
&= -\sum_{(l',k')\in S\times S\setminus(l,k)} \hat{p}_{lk}Q(\hat{p})_{(l,k)(l',k')} - \sum_{k'\in S}\hat{p}_{kl}\vartheta_{kl}\varsigma_{kl}(k') \\
&= -\sum_{(l',k')\in S\times S\setminus(l,k)} \hat{p}_{lk}Q(\hat{p})_{(l,k)(l',k')} - \hat{p}_{kl}\vartheta_{kl} \\
&= -\sum_{(l',k')\in S\times \hat{S}\setminus(l,k)} \hat{p}_{lk}Q(\hat{p})_{(l,k)(l',k')} \\
&= \hat{p}_{lk}Q(\hat{p})_{(l,k)(l,k)}.
\end{aligned}$$

The above equations imply that

$$\begin{aligned}
f(\hat{p})_{kl} &= \hat{p}_{kl} + c\sum_{(k',l')\in\hat{S}} \hat{p}_{k'l'}Q(\hat{p})_{(k',l')(k,l)} \\
&= \hat{p}_{kl} + c\sum_{(k',l')\in S\times S\setminus(k,l)} \hat{p}_{k'l'}Q(\hat{p})_{(k',l')(k,l)} + c\sum_{k'\in S}\hat{p}_{k'J}Q(\hat{p})_{(k',J)(k,l)} + c\hat{p}_{kl}Q(\hat{p})_{(k,l)(k,l)} \\
&= \hat{p}_{lk} + c\sum_{(l',k')\in S\times S\setminus(l,k)} \hat{p}_{l'k'}Q(\hat{p})_{(l',k')(l,k)} + c\sum_{l'\in S}\hat{p}_{l'J}Q(\hat{p})_{(l',J)(l,k)} + c\hat{p}_{lk}Q(\hat{p})_{(l,k)(l,k)} \\
&= f(\hat{p})_{lk}
\end{aligned}$$

Hence, f is a function from $\hat{\Delta}$ to $\hat{\Delta}$.

It is clear that f is continuous on $\hat{\Delta}$. By Kakutani's Fixed Point Theorem, there exists a $\hat{p}^* \in \hat{\Delta}$ such that $\hat{p}^* + c\hat{p}^*Q(\hat{p}^*) = \hat{p}^*$. Therefore, $\hat{p}^*Q(\hat{p}^*) = 0$.

Part (2) of Proposition A.2: Assume that $\hat{p}^*\,Q(\hat{p}^*) = 0$. Since the function $\hat{p}Q(\hat{p})$ is Lipschitz continuous in $\hat{p}$, the ordinary differential equation in Equation (A.1) with the initial condition $\check{p}(0) = \hat{p}^*$ must have a unique solution $\hat{p}^*$, and hence $\check{p}(t) = \hat{p}^*$ at any time t. Therefore, $\hat{p}^*$ is a stationary distribution.

D Sketch for the Proof of Theorem A.1

In this section, we briefly explain the ideas for proving Theorem A.1.[17] The detailed proof of Theorem A.1 is provided in an on-line-only supplement, Duffie, Qiao and Sun (2020).

The popular idea that a continuum of agents search continuously over time for trading

[17] Note that the existence result in Theorem 2.1 is a special case of Theorem A.1 with all the enduring probabilities being zero.

partners independently of each other is to model the situation that a large but finite number of agents are matched with small probabilities at small time intervals without central coordinations. Though such an idea of continuous-time independent random matching is widely used intuitively, there has been no underlying mathematical model representing the idea. The main purpose of this paper is to provide the first mathematical foundation for continuous-time independent random matching.

We take the following approach for constructing a general model of continuous-time independent random matching. First, we analyze a finite-period dynamic random matching model with finitely many agents in Appendices E.1, E.2 and E.3 of the online supplement. Then, using techniques in nonstandard analysis,[18] we transform in Appendix E.4 such a finite model to a limit model with infinitely many agents who are matched at each infinitesimal time interval with infinitesimal probabilities, which leads to continuous-time independent random matching.

It is difficult to work with a general dynamic random matching model with finitely many agents. Since agent i being matched to agent j implies agent j being matched to agent i, random matching with finitely many agents must induce correlations. Such kind of individual level correlations also happens when a pair of matched agents breaks up with some probability. A source of systemic correlations for all the agents come from the matching probabilities at each time period, since such matching probabilities depend on the underlying cross-sectional type distribution of agents which is random in the finite-agent setting. In order to obtain meaningful approximate results, we provide delicate estimations in Appendix E.3 for the cumulative effect of the correlations and randomness across multiple time periods. The advantage of working with a limit model of continuous-time independent random matching is that such kind of correlations and randomness disappears completely, which is convenient for applications to economic models in various areas.[19]

In Appendix E.1, we construct a static random matching model with $\hat{M}$ agents given general matching parameters $\{q_{kl}\}_{k,l\in S}$, where $\hat{M}$ is a finite positive even integer. Since we allow enduring partnerships in our dynamic matching model, we need to consider a static

[18]For a comprehensive introduction to nonstandard analysis, see the first three chapters of Loeb and Wolff (2015). Nonstandard analysis has been used to study continuous time stochastic processes such as Poisson process in Loeb (1975), Brownian motion and Itô process in Anderson (1976), Perkins (1981), and Keisler (1984), and Markov processes in Duanmu, Rosenthal and Weiss (2018). For applications of nonstandard analysis to economics, see, for example, Brown and Robinson (1975), Khan (1974), Hammond (1999), and Anderson and Raimondo (2008). In the proof of Theorem A.1 in Appendix E, nonstandard analysis is only used in Appendix E.4.

[19]Economic models with many agents are also very popular in situations without the consideration of random matching. For some recent developments, see McLean and Postlewaite (2002, 2004), Kalai (2004), Yannelis (2009), Acemoglu and Jensen (2015), Bierbrauer and Hellwig (2015), Hammond (2015), He and Yannelis (2016), Khan et al. (2017), Hellwig (2019) and Anderson et al. (2020) among others.

random matching model with some agents being matched initially, which will be used in the matching step of each time period. We assume that only single (i.e., initially unmatched) agents will participate in the static matching considered here. Let I_k be the set of single agents with type k, and $|I_k|$ the number of agents in I_k. We first randomly choose a subset A_{kl} in I_k to be matched with type-l agents such that the number of agents in A_{kl} (denoted by $|A_{kl}|$) is taken to be the largest even integer less than or equal to $|I_k|q_{kl}$. Then, the probability of a type-k agent to be matched with a type-l agent is $|A_{kl}|/|I_k|$. As mentioned above, the matching outcome is correlated due to the finiteness of the agent space. Another difficulty is that the matching probability $|A_{kl}|/|I_k|$ is, in general, only an approximation of the matching parameter q_{kl} (note that q_{kl} can be quite arbitrary). Since the static random matching model will be used in the matching step of our dynamic matching model, the matching parameters/probabilities will go to zero as the time length of each period goes to zero and the number of agents goes to infinity. In Lemma 1, we provide delicate estimations of the (joint) matching probabilities to control the cumulative effect in the dynamic matching model.[20]

In Appendix E.2, we develop a finite-agent dynamic matching model with $\hat{M}$ agents and M^2 periods with M being a positive integer. The time length of each period is $\frac{1}{M}$. In each period, there are three steps. The first step is the mutation step, agents (single or matched) change their types independently. The second step is the matching step, only single agents take part in the static random matching described in Appendix E.1. The third step is the type changing with break-up step, at which agents who were just matched in the last step may enter into a long-term partnership with a given probability, and then experience type changes according to specific type-changing probabilities. At this step, agents who have been matched for more than one step may break up with a given probability, and change their types according to some type-changing probabilities if they indeed break up.

In Appendix E.3, we present some properties of the finite-agent dynamic matching model constructed in Appendix E.2, which will be used for the continuous time model in Appendix E.4. In particular, Lemma E.4 provides a careful estimate on the difference between the (conditional) matching probabilities and the corresponding matching parameters. Lemmas E.6 an E.7 show that the finite-agent dynamic matching model is approximately Markovian and independent when the number of agents and number of periods are large enough. As mentioned above, the proofs of those lemmas are difficult since we need to estimate very carefully about the cumulative effect of the correlations and randomness across multiple time periods.

[20]Lemma 7 in Duffie, Qiao and Sun (2018) considered a static random matching model with infinitely many agents. When such a result is interpreted in the large finite setting, it means that the *difference* between the matching probability $|A_{kl}|/|I_k|$ and the matching parameter q_{kl} is small. In the setting of this paper, we need to consider the case that both the matching probability $|A_{kl}|/|I_k|$ and the matching parameter q_{kl} are small, which cannot be covered by the idea used in Lemma 7 of Duffie, Qiao and Sun (2018).

In Appendix E.4, we turn to the study of the continuous time model of independent random matching. By the Transfer Principle in nonstandard analysis, the finite-agent dynamic matching model and its properties can be recast in the setting with a hyperfinite number of agents and time periods. By rounding off some infinitesimals, we obtain a continuous time random matching model with a standard atomless probability space as the agent space. Then, we need to show that the continuous time random matching model as constructed satisfies all the requirements stated in Appendix A. First, the Markov property and independence of the continuous time model are proved, based on the approximate results in the finite-agent model. It is difficult to show that the transition intensities are the same as listed in Table 1 above. In order to do this, we need to provide very subtle estimates on various cumulative effects of infinitely many rare events for each of the four cases over infinitely many periods with infinitesimal time length.[21]

[21] A discrete-time model of random matching with enduring partnerships is considered in Duffie, Qiao and Sun (2018); see also Section 5 of Duffie, Qiao and Sun (2018) for a discussion of some related papers on more specialized models of random matching in the static and discrete-time settings. Unlike the delicate analysis associated with continuous-time random matching in this paper, there is no need to consider the cumulative effect of multiple periods in the discrete-time model.