Abstract

We show the existence of independent random matching among a continuum of agents in discrete-time dynamic systems for which matching probabilities can be “directed.” That is, the probability with which an agent is matched to agents of a given type need not be equal to the fraction of that type in the matched population. We prove the existence of a continuum of independent discrete-time Markov processes, one for each agent, that incorporates the effects of random mutation, random matching with directed probabilities, and match-induced random type changes. The empirical type evolution of such a discrete-time dynamic system is also determined via an exact law of large numbers. The results provide a mathematical foundation for many previously studied search-based models of labor markets, money, and financial markets.

*This work was initiated in July 2013 while the first author visited the National University of Singapore.
†Graduate School of Business, Stanford University, Stanford, CA 94305-5015, and National Bureau of Economic Research. e-mail: duffie@stanford.edu
‡Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Singapore 119076. e-mail: a0086330@nus.edu.sg
§Department of Economics, National University of Singapore, 1 Arts Link, Singapore 117570. e-mail: yn-sun@nus.edu.sg
1 Introduction

Models of economies with independent random matching among a continuum population are extensive.\(^1\) Previous work ([13] and [14]) provides a construction of discrete-time Markov independent dynamical systems with random mutation, and with type changes induced by pair-wise random matching. It was assumed in this prior work that when a given agent is matched, the paired agent is drawn with the uniform distribution from the population of other agents that are matched. A consequence is that the probability that the paired agent is of a given type is equal to the fraction of that type in the matched population.

Here, we extend to the case of “directed” random matching, in that the probability that an agent of type\( k \) is matched to an agent of type\( l \) is of the form \( \theta_{kl}p_l \), where \( p_l \) is the fraction of agents that are of type\( l \) and \( \theta_{kl} \) is the per-capita rate with which a type\( k \) meets a type\( l \) agent. This matching rate \( \theta_{kl} \) is a parameter of the model that can vary with\( k \) and\( l \).\(^2\)

We extend further by allowing the matching parameter \( \theta_{kl}(t) \) to depend on the time period\( t \), or on the current population proportions, in that \( \theta_{kl}(t) = f(p_k(t), p_l(t)) \), subject to regularity conditions on the function\( f \). This incorporates the “matching function” approach that has frequently been applied in the labor literature. (See [37].)

Economic agents have natural motives for focusing their search for counterparts toward those types of counterparts that offer greater gains from trade, or toward those types that are less costly to find. Directed random matching is indeed motivated by many applications, including [32] in monetary theory, [22] and [21] in labor economics, and [12] in financial economics. In [32], a pair of agents of the same type are relatively more likely to be matched than is a pair of agents of different types. In [22] and [21], the matching functions are non-linear with respect to the underlying populations of different types. In [27], an equilibrium can be non-steady-state and a general strategy is time-dependent; and thus the parameter in the type changing step will be time-dependent. In [12], agents in a financial market direct their search based on the relative informativeness of different types of trading counterparts.

This is a small sample of the research that has relied by assumption on the existence of a dynamic random matching model with directed search probabilities or with non-linear matching functions. The results of this paper provide for the existence of such models.

The remainder of the paper is organized as follows. In Section 2, we construct a static model and prove the corresponding existence result. In Section 3, we construct a dynamical

---

\(^1\)See, for example, [18], [19], [33], [45] in general equilibrium theory; [2], [3], [5], [17], [23] in game theory; [8], [20], [24], [28], [29], [39], [42] in monetary economics; [7], [25], [35], [36], [38] in labor economics; and [10], [11], [43], [44] in financial economics.

\(^2\)In the case of random full matching with matching probability equal to the fraction of the type in the entire population, the parameter \( \theta_{kl} \) is identically equal to one.
system with random mutation, random matching and type changing with directed search probabilities and time-dependent parameters. This section includes the existence results, exact law of large numbers and stationarity of the dynamical system. In Section 4, we present a few examples of applications. Proofs are found in Section 5.

2 Preliminaries

For the entire paper we fix an atomless probability space \((I, \mathcal{I}, \lambda)\) representing the space of agents and a sample probability space \((\Omega, \mathcal{F}, P)\) representing the states of the world, and we let \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)\) be a Fubini extension\(^3\) of the usual product probability space. This Fubini extension includes a sufficiently rich collection of measurable sets to allow applications of the exact law of large numbers that we shall need.

We begin with a static model of directed random matching, and then a dynamic model that incorporates random changes over time in agents’ types that are caused by matching and mutation.

3 The static model

Let \(S\) be a finite or countably infinite agent type space and \(\alpha : I \to S\) be a measurable type function, mapping individual agents to their types. For any \(k \in S\), we let \(p_k = \lambda(\{i : \alpha(i) = k\}\) denote the fraction of agents that are of type \(k\). We can view \((p_k)_{k \in S}\) as an element of the space \(\Delta\) of probability measures on \(S\).

A function \(\theta : S \times S \to \mathbb{R}_+\) is a matching rate function for the type distribution \(p\) if \(\theta_{kl} = \theta_{lk}\) for any \(k\) and \(l\) in \(S\), and if \(\sum_{l \in S} p_l \theta_{kl} \leq 1\) for each \(k \in S\). The matching rate \(\theta_{kl}\) specifies the “per-capita” rate of matching of agents of type \(k\) with agents of type \(l\), in the sense that \(q_{kl} = p_l \theta_{kl}\) is the probability that a given agent of type \(k\) is matched to an agent of type \(l\). Thus, \(q_k = 1 - \sum_{l \in S} p_l \theta_{kl}\) is the associated probability of no matching for an agent of type \(k\).

A mapping \(\pi\) from \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)\) to \(I\) is defined to be a random matching if it satisfies the two conditions:

(i) For each \(\omega \in \Omega\), \(\pi_\omega\) is a bijection between \(I\) and itself. Letting \(B^\omega = \{i \in I : \pi_\omega(i) = i\}\) denote the agents that not matched by \(\pi_\omega\) to a distinct agent, the restriction of \(\pi_\omega\) to

\(^3\)A formal definition of Fubini extension was introduced by [40]. A probability space \((I \times \Omega, \mathcal{W}, Q)\) extending the usual product space \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)\) is said to be a Fubini extension of \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)\) if for any real-valued \(Q\)-integrable function \(g\) on \((I \times \Omega, \mathcal{W})\), the functions \(g_i = g(i, \cdot)\) and \(g_\omega = g(\cdot, \omega)\) are integrable respectively on \((\Omega, \mathcal{F}, P)\) for \(\lambda\)-almost all \(i \in I\) and on \((I, \mathcal{I}, \lambda)\) for \(P\)-almost all \(\omega \in \Omega\); and if, moreover, \(\int_I g_i \, dP\) and \(\int_\Omega g_\omega \, d\lambda\) are integrable, respectively, on \((I, \mathcal{I}, \lambda)\) and on \((\Omega, \mathcal{F}, P)\), with \(\int_{I \times \Omega} g \, dQ = \int_I (\int_\Omega g_i \, dP) \, d\lambda = \int_\Omega (\int_I g_\omega \, d\lambda) \, dP\). To reflect the fact that the probability space \((I \times \Omega, W, Q)\) has \((I, \mathcal{I}, \lambda)\) and \((\Omega, \mathcal{F}, P)\) as its marginal spaces, as required by the Fubini property, it is denoted by \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)\).
I \setminus B^\omega is one-to-one and satisfies \( \pi_\omega(\pi_\omega(i)) = i \).

(ii) Letting \( J \) denote the event of no matching, an \( I \boxtimes \mathcal{F} \)-measurable type assignment function \( g \) for \( \pi \) is defined by

\[
g(i, \omega) = \begin{cases} 
\alpha(\pi(i, \omega)), & i \notin B^\omega \\
J, & i \in B^\omega.
\end{cases}
\]

We say that a random matching \( \pi \) with type assignment function \( g \) has parameters \((p, \theta)\) if, for \( \lambda \)-almost every agent \( i \in I \) of type \( k \), we have \( P(g_i = J) = q_k \) and \( P(g_i = l) = q_{kl} \), where \( g_i \) denotes the random variable \( g(i, \cdot) \).

The “partial matching” for a finite type space \( S \) as defined by [13] is a special case for which the matching rates are defined by

\[
\theta_{kl} = \frac{(1 - q_k)(1 - q_l)}{\sum_{r \in S} p_r (1 - q_r)}.
\]

The following is a direct application of the exact law of large numbers. We say that \( \pi \) is pairwise independent in types if its type assignment function \( g \) is essentially pairwise independent.\(^4\)

**Proposition 1** Let \( \pi \) be a random matching with type assignment function \( g \) and parameters \((p, \theta)\). If \( \pi \) is pairwise independent in types then, for \( P \)-almost every \( \omega \in \Omega \):

(i) \( \lambda(\{i \in I : \alpha(i) = k, g_\omega(i) = J\}) = p_k q_k \).

(ii) For any \((k, l) \in S^2\), \( \lambda(\{i : \alpha(i) = k, g_\omega(i) = l\}) = p_k q_{kl} = p_k \theta_{kl} p_l \).

The following theorem shows the existence of independent random matching with directed probability and with general parameters.

**Theorem 1** For any type distribution \( p \) on \( S \) and any matching rate \( \theta \), there exists a random matching \( \pi \) with parameters \((p, \theta)\) that is essentially pairwise independent in types.

**Example 1.** For a model of search-based labor markets, one could suppose that firms and workers are characterized by their types. A common modeling device in search-based models of labor markets is a matching function \( m_{kl} : [0, 1] \times [0, 1] \to [0, 1] \), specifying that the quantity

\[
\lambda \times \lambda(\{(i, j) \in I^2 : \alpha(i) = k, \alpha(j) = l, \pi(i) = j\})
\]

\(^4\)An \( I \boxtimes \mathcal{F} \)-measurable process \( f \) from \( I \times \Omega \) to a complete separable metric space \( X \) is said to be essentially pairwise independent if for \( \lambda \)-almost all \( i \in I \), the random variables \( f_i \) and \( f_j \) are independent for \( \lambda \)-almost all \( j \in I \). Two random variables \( \phi \) and \( \psi \) from \((\Omega, \mathcal{F}, P)\) to \( X \) are said to be independent if the \( \sigma \)-algebras \( \sigma(\phi) \) and \( \sigma(\psi) \) generated respectively by \( \phi \) and \( \psi \) are independent. The proof of Proposition 1 will be given in Section 6.1 of the appendix.
of matches between agents of type $k$ and agents of type $l$ is $P$-almost surely equal to $m_{kl}(p_k, p_l)$. Whenever $p_k$ and $p_l$ are non-zero, the associated matching rate function $\theta_{kl}$ is defined by

$$\theta_{kl} = \frac{m_{kl}(p_k, p_l)}{p_k p_l}.$$  

See [37] for a survey of the matching-function approach. A common parametric specification is the Cobb-Douglas matching function, for which

$$m_{kl}(p_k, p_l) = A_{kl} p_j^{\beta_{(k,l)}} p_l^{\beta_{(l,k)}},$$

for parameters $\beta : S \times S \to (0, 1)$, and a non-negative scaling parameter $A_{kl}$. Matching functions can allow the probabilities of matching to be directed and to depend on an endogenously determined cross-sectional distribution of types.

\section{Dynamic directed random matching}

In this section, we consider a dynamical system with random mutation, random matching with directed probabilities, and match-induced random type changes. We also allow for time-dependent parameters. We first define such a dynamical system. Then we formulate the key property of being Markovian and conditional independence in types. We then prove existence and an exact law of large numbers for such a dynamical system. For time-independent parameters and with finitely many types, we also demonstrate and characterize stationarity.

\subsection{Definitions for dynamic random matching}

This sub-section defines a discrete-time random process for agent types with the property that at each integer time period $n \geq 1$, agents first experience a random mutation and then a random matching with directed probability. Finally, any pair of matched agents are randomly assigned new types whose probabilities depend on the prior types of the two agents in a manner to be defined.

A random type function is a measurable mapping from $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ to $S$. The initial random type function $\alpha^0$ is assumed to be essentially pairwise independent with a cross-sectional type distribution $p^0$ defined by $p^0_i = \lambda(\{i : \alpha^0_i = k\})$.

We will characterize a dynamical system with, at each period $n$, a random type function $h^n$, assigning to agent $i$ the type $h^n_i$ after mutation but before matching. At period $n$ after matching, the types of agents are likewise specified by a random type function $\alpha^n$. A key objective is to show the existence and properties of a random type process $(h, \alpha) = \{(h^1, \alpha^1), (h^2, \alpha^2), \ldots\}$ that respects specified properties and parameters for mutation, directed random matching, and match-induced random type change.
At period $n$, before random matching, any agent of type $k$ mutates so as to become an agent of type $l$ with some specified probability $b_{kl}^n$, where for each $k$, $(b_{k1}^n, b_{k2}^n, \ldots)$ is in $\Delta$. We thus require that for $\lambda$ almost-every agent $i$,

$$P(h_i^n = l \mid \alpha_i^{n-1} = k) = b_{kl}^n.$$  \hfill (1)

For $n \geq 1$ and for each $(k, l) \in S^2$, let $\theta_{kl}^n$ be a continuous function on $\Delta$ into $\mathbb{R}_+$ with the property that, for all $k$ and all $p$ in $\Delta$,

$$\sum_{l \in S} \theta_{kl}^n(p) p_l \leq 1.$$  

An agent of type $k$ is matched at period $n$ to an agent with type $l$ at the per-capita matching rate $\theta_{kl}^n(\hat{p}^n)$, where $\hat{p}^n$ is the type distribution of $h^n$, defined by $\hat{p}_k^n = \lambda(\{i : h_i^n = k\})$. When an agent of type $k$ is matched at time $n$ to an agent of type $l$, the agent of type $k$ becomes an agent of type $r$ with a specified probability $\nu_{kl}^n(r)$.

The parameters of the model are $(p^0, b, \theta, \nu)$.

At each period $n$, agents are to be matched according to a random matching $\pi^n$ with a type assignment function $g^n$ that respects the property that for every type $k$ and $\lambda$-almost every agent $i$,

$$P(g_i^n = l \mid h_i^n = k) = \hat{q}_{kl}^n = \hat{\theta}_{kl}^n(\hat{p}^n)\hat{p}_l^n,$$  \hfill (2)

and

$$P(g_i^n = J \mid h_i^n = k) = \hat{q}_k^n = 1 - \sum_{l \in S} \hat{q}_{kl}^n.$$  \hfill (3)

We also require that the type function $\alpha_i^n$ after match-induced type changes satisfies, for $\lambda$-almost all agent $i \in I$,

$$P(\alpha_i^n = r \mid h_i^n = k, g_i^n = J) = \delta_k(r)$$  \hfill (4)

and

$$P(\alpha_i^n = r \mid h_i^n = k, g_i^n = l) = \nu_{kl}^n(r),$$  \hfill (5)

where $\delta_k(r)$ is one if $r = k$, and zero otherwise.

For any given model parameters $(p^0, b, \theta, \nu)$, by induction in the period $n$, we will prove the existence of $(h, \alpha, \pi, g)$, determining agent types with random mutation, random matching with directed probability, and match-induced type changing, respecting the definitional transition probabilities (1)-(4)-(5) and matching probabilities (2). In this case, we say that $\mathbb{D} = (h, \alpha, \pi, g)$ is a dynamical system with parameters $(p^0, b, \theta, \nu)$. Under the assumption that random mutation, matching, and match-induced type changes are essentially pairwise independent, an application of the exact law of large numbers implies that the cross sectional type distributions are almost surely deterministic, a property that is frequently used in applications.
4.2 Markov conditional independence in types

In this section we define Markovian and cross-sectional independence properties for a dynamical system $D = (h, \alpha, \pi, g)$. The idea of the property is that each agent’s type process is Markovian and, moreover, that the random mutation, random matching, and match-induced type changes that occur in any period $n$ are probabilistically independent across almost all agents.

We say that $D$ has random mutation that is Markovian and conditionally independent in types if, for $\lambda$-almost all $i \in I$ and $\lambda$-almost all $j \in I$,

$$P(h^n_i = k, h^n_j = l | \alpha^n_i = \alpha^{n-1}_i; \alpha^n_j = \alpha^{n-1}_j) = P(h^n_i = k | \alpha^{n-1}_i)P(h^n_j = l | \alpha^{n-1}_j),$$

for every period $n$ and for all types $k$ and $l$ in $S$.

We say that $D$ has random matching that is Markovian and conditionally independent in types if, for $\lambda$-almost all $i \in I$ and $\lambda$-almost all $j \in I$,

$$P(g^n_i = c, g^n_j = d | \alpha^{n-1}_i = \alpha^{n-1}_j; h^n_i = h^n_j) = P(g^n_i = c | h^n_i)P(g^n_j = d | h^n_j),$$

for every period $n$ and for all $c$ and $d$ in $S \cup \{J\}$.

We say that $D$ has match-induced random type change that is Markovian and conditionally independent in types if for $\lambda$-almost all $i \in I$, and $\lambda$-almost all $j \in I$,

$$P(\alpha^n_i = c, \alpha^n_j = d | \alpha^{n-1}_i = \alpha^{n-1}_j; h^n_i, g^n_i; \alpha^{n-1}_j, h^n_j, g^n_j) = P(\alpha^n_i = c | \alpha^{n-1}_i; h^n_i, g^n_i)P(\alpha^n_j = d | \alpha^{n-1}_j, h^n_j, g^n_j),$$

for every period $n$ and for all $k$ and $l$ in $S$.

Finally, we say that $D$ is Markovian and conditionally independent in types if its random mutation, random matching, and match-induced type change is Markovian and independent in types.

4.3 The existence of Markovian and conditionally independent dynamical systems

Our main result is the following.

**Theorem 2** For any parameters $(p^0, b, \theta, \nu)$, there exists a dynamical system $D = (h, \alpha, \pi, g)$ with these parameters that is Markovian and conditionally independent in types.

In the next proposition, we show that the properties for an agent type process given in the previous Theorem can be obtained for an agent space $(I, \mathcal{I}, \lambda)$ that is an extension of the classical Lebesgue unit interval $(L, \mathcal{L}, \eta)$. That is, we can take $I = L = [0, 1]$ with a $\sigma$-algebra $\mathcal{I}$ that contains the Lebesgue $\sigma$-algebra $\mathcal{L}$, and so that the restriction of $\lambda$ to $\mathcal{L}$ is the Lebesgue measure $\eta$. We also obtain as a corollary existence results for random matchings with directed probability in the static case.
Proposition 2  Fixing any parameters \((p^0, b, \theta, \nu)\), there exists a Fubini extension \((I \times \Omega, I \times F, \lambda \times \mathcal{P})\) such that:

1. The agent space \((I, I, \lambda)\) is an extension of the Lebesgue unit interval \((L, L, \eta)\).
2. There is defined on the Fubini extension a dynamical system that is Markovian and conditionally independent in types with the parameters \((p^0, b, \theta, \nu)\).

Next, by restricting the dynamic model of Proposition 2 to the first period without random mutation, we obtain the following existence result for independent random matching with directed probability.

Corollary 1  For any type distribution \(p\) on \(S\) and per-capita directed matching rates \(\theta = (\theta_{kl})\) for \(p\), there exists a Fubini extension \((I \times \Omega, I \times F, \lambda \times \mathcal{P})\) such that

1. The agent space \((I, I, \lambda)\) is an extension of the Lebesgue unit interval \((L, L, \eta)\).
2. There is defined on the Fubini extension a random matching \(\pi\) from \((I \times \Omega, I \times F, \lambda \times \mathcal{P})\) to \(I\) with type distribution \(p\) and with per-capital matching rate \(\theta\).

4.4 Exact law of large numbers and stationarity

We now define a sequence \(\Gamma^n\) of mappings from \(\Delta\) to \(\Delta\) such that, for each \(p = (p_1, \ldots, p_k, \ldots)\) in \(\Delta\),

\[
\Gamma^n_r(p_1, \ldots, p_k, \ldots) = q^n_r \sum_{l \in S} p_l b^n_{lr} + \sum_{k, l \in S} \bar{q}^{n}_{kl} \nu^n_{kl}(r) \bar{p}_k,
\]

where \(\bar{p}_k = \sum_{k \in S} p_l b^n_{lk}, \bar{q}^n_{kl} = \theta^n_{kl}(\bar{p}) \bar{p}_l\) and \(\bar{q}^n_{kl} = 1 - \sum_{l \in S} \bar{q}^n_{kl}\).

The following proposition provides an exact law of large numbers for agent type processes allowing for random mutation, random matching with directed probability, and match-induced random type changing that is Markov conditionally independent in types. The proposition also gives a recursive calculation of the the cross-sectional type distribution \(p^n\).

Proposition 3  If \(\mathbb{D} = (h, \alpha, \pi, g)\) is a dynamical system with parameters \((p^0, b, \theta, \nu)\) that is Markovian and conditionally independent in types, then:

1. For each time \(n \geq 1\), the expectation \(\bar{p}^n = E(p^n)\) of the cross-sectional type distribution is given by

\[
\bar{p}^n_r = \Gamma^n_r(\bar{p}^{n-1}) = q^n_r \sum_{l \in S} \bar{p}_l^{n-1} b^n_{lr} + \sum_{k, l \in S} \bar{q}^{n}_{kl} \nu^n_{kl}(r) \bar{p}_k^n,
\]

where \(\bar{p}^n_k = \sum_{l \in S} b^n_{lk} \bar{p}_l^{n-1}, \bar{q}^n_{kl} = \theta^n_{kl}(\bar{p}^n) \bar{p}_l^n\) and \(\bar{q}^n_k = 1 - \sum_{l \in S} \bar{q}^n_{kl}\).
(2) For \( \lambda \)-almost all \( i \in I \), \( \{\alpha^n_i\}_{n=0}^{\infty} \) is a Markov chain with transition matrix \( z^n \) at time \( n - 1 \) defined by
\[
z^n_{kl} = q^n_{kl} + \sum_{r,j \in S} \nu^n_{lj}(l) b^n_{kj} q^n_{rj}.
\]

(3) For \( \lambda \)-almost all \( i \in I \) and \( \lambda \)-almost all \( j \in I \), the Markov chains \( \{\alpha^n_i\}_{n=0}^{\infty} \) and \( \{\alpha^n_j\}_{n=0}^{\infty} \) are independent.

(4) For \( P \)-almost all \( \omega \in \Omega \), the cross-sectional type process \( \{\alpha^n_\omega\}_{n=0}^{\infty} \) is a Markov chain with transition matrix \( z^n \) at time \( n - 1 \).

(5) For \( P \)-almost all \( \omega \in \Omega \), at each time period \( n \geq 1 \), the realized cross-sectional type distribution after random mutation \( \lambda(h^n_\omega)^{-1} \) is \( \tilde{p}^n \) and the realized cross-sectional type distribution at the end of the period \( n \), \( p^n(\omega) = \lambda(\alpha^n_\omega)^{-1} \), is equal to its expectation \( \bar{p}^n \).

With time-independent parameters and finitely many types, we also have the following proposition, providing for a stationary measure of types.

**Proposition 4** Suppose that the parameters \( (b, \theta, \nu) \) are time independent and the number \( |S| \) of types is finite. Then there exists a probability distribution \( p^* \) on \( S \) such that there is a dynamical system \( D = (h, \alpha, \pi, g) \) with parameters \( (p^*, b, \theta, \nu) \) that is Markovian and conditionally independent in types, and stationary. In particular, for every \( n \geq 1 \), the realized cross-sectional type distribution \( p^n \) at time \( n \) is \( p^* \) \( P \)-almost surely and the probability transition matrix \( z^n \) is equal to \( z^1 \). Further, for \( \lambda \)-almost every agent \( i \), the initial type \( \alpha^0_i \) of agent \( i \) can be chosen to have the same probability distribution \( p^* \) established in this proposition, and in this case each agent’s type process is also stationary with distribution \( p^* \). That is, under these conditions, for every period \( n \) and for \( \lambda \)-almost every \( i \), the probability distribution of \( \alpha^n_i \) is \( p^* \).

## 5 Applications to Dynamic Random Matching

In this section, we illustrate dynamic random matching through a couple of examples that provide a mathematical foundation for existing models in monetary economics.

**Example 2:** This example comes from Kehoe, Kiyotaki and Wright [27]. In [27], time is discrete and continues forever. There are three indivisible goods, labeled 1, 2, and 3. There is a continuum of agents of unit total mass. A given type of agents consumes good \( k \) and can store one unit of good \( l \), for some \( l \neq k \). This type is denoted \((k, l)\). The economy is thus

---

5For a given sample realization \( \omega \in \Omega \), \( \{\alpha^n_\omega\}_{n=0}^{\infty} \) is defined on the agent space \((I, I, \lambda)\), which is a probability space itself. Thus, \( \{\alpha^n_\omega\}_{n=0}^{\infty} \) can be viewed as a discrete-time process.
populated by agents with 6 different types. To keep track of agents consuming good $k$, we call those agents having trait $k$. It is assumed in [27] that there are equal proportions of agents with the three different traits.

In each period $n$, every agent randomly matches with some other agent. When matched, two agents decide whether or not to trade. If there is no trade between the matched pair, they keep their goods. If there is a trade, and if the agent who consumes good $k$ gets good $k$ from the other, then that agent immediately consumes good $k$ and produces one unit of good $k + 1$ (modulo 3), so that his type becomes $(k, k + 1)$ (modulo 3, as needed). If there is a trade and an agent with trait $k$ gets good $l$ for $l \neq k$, then his type becomes $(k, l)$.

In Section 6 of [27], Kehoe, Kiyotaki and Wright considered equilibria in which the strategies vary with time. Suppose that $(s_1(n), s_2(n), s_3(n))$ is a time-dependent mixed strategy at time $n$, where $s_k(n)$ is the probability that an agent with trait $k$ trades good $k + 1$ for $k + 2$.

We can compute the probability $P^m_{(k_1, k_2)}(k_3)$ that an agent with type $(k_1, k_2)$ trades for good $k_3$. Viewed in our model, the corresponding parameters are $b^m_{(k_1, l_1)(k_2, l_2)} = \delta_{k_1}(k_2)\delta_{l_1}(l_2)$ and $\theta^m_{(k_1, l_1)(k_2, l_2)} = 1$. Because the consumption traits of agents do not change and there are no agents with type $(k, k)$, we only need to consider the cases

$$
\nu^m_{(k_1, k_1+1)(k_2, l_2)}(k_1, k_1 + 1) = \begin{cases} 
1 & \text{if } l_2 \neq k_1 + 2 \\
1 - P^m_{(k_1, k_1+1)}(l_2)P^m_{(k_2, l_2)}(k_1 + 1) & \text{if } l_2 = k_1 + 2
\end{cases}
$$

$$
\nu^m_{(k_1, k_1+2)(k_2, l_2)}(k_1, k_1 + 1) = \begin{cases} 
0 & \text{if } l_2 = k_1 + 2 \\
P^m_{(k_1, k_1+2)}(l_2)P^m_{(k_2, l_2)}(k_1 + 2) & \text{otherwise}.
\end{cases}
$$

Thus, the parameter $\nu$ in the step of match-induced type change is time-dependent.

**Example 3:** This example is from Matsuyama, Kiyotaki and Matsui [32]. The economy is populated by a continuum of infinitely-lived agents of unit total mass. Agents are from two regions, Home and Foreign. Let $p \in (0, 1)$ be the size of Home population. Let $\beta \in (0, 1)$. There are $K$ ($K \geq 3$) kinds of indivisible commodities. Within each economy, there are equal proportions of agents with $K$ traits. An agent with trait $k$ derives utility only from consumption of commodity $k$. After he consumes commodity $k$, he is able to produce one and only one unit of commodity $k + 1$ (mod $K$) costlessly, and he also knows how to store his production good costlessly up to one unit; he can neither produce nor store other types of goods.

In addition to the commodities described above, there are two distinguishable fiat monies, objects with zero intrinsic worth, which we call the Home currency and the Foreign currency.

---

6In [27], agents are of three types, but the meaning of type is different from our paper, so we use “trait” instead of type.
It is assumed that each currency is indivisible, and can be stored costlessly up to one unit by every agent if he does not carry his production good or the other currency. This implies that, at any date, the inventory of each agent contains either one unit of the Home currency, one unit of the Foreign currency, or one unit of his production good, but no more than one object at the same time.

Let $\beta \in (0, 1)$. In each period $n$, for a Home agent, the probability that he matches with a Home agent is $p$, the probability that he matches with a Foreign agent is $\beta(1 - p)$ and the probability that he does not match with anybody is $(1 - \beta)(1 - p)$; similarly for a Foreign agent, the probability that he matches with a Home agent is $\beta p$, the probability that he matches with a Foreign agent is $(1 - p)$ and the probability that he does not match with anybody is $(1 - \beta)p$.

For any agent, the type is $(a, b, c)$, where $a = H$ or $F$, $b = 1, \ldots, K$ and $c = g, h, f$, where $H$ represents Home, $F$ represents Foreign, $g$ represents good, $h$ represents Home currency and $f$ represents Foreign currency.

An agent chooses a trade strategy to maximize his expected discounted utility, taking as given the strategies of other agents and the distribution of inventories. In [32], Matsuyama, Kiyotaki and Matsui focused on pure strategies which only depend on his nationality and the objects he and his opponent have in inventory. Thus, the Home agent’s trade strategy can be described simply as

$$\tau^H_{ab} = \begin{cases} 1 & \text{if he agrees to trade object } a \text{ for object } b, \\ 0 & \text{otherwise}, \end{cases}$$

where $a, b = g, h, f$; the Foreign agent’s trade strategy can be described simply as

$$\tau^F_{ab} = \begin{cases} 1 & \text{if he agrees to trade object } a \text{ for object } b, \\ 0 & \text{otherwise}, \end{cases}$$

where $a, b = g, h, f$.

Viewed in our model, the corresponding parameters are $b^n_{(a_1, b_1, c_1)(a_2, b_2, c_2)} = \delta_{a_1}(a_2)\delta_{b_1}(b_2)\delta_{c_1}(c_2)$, $\theta^n_{(a_1, b_1, c_1)(a_2, b_2, c_2)} = 1_{b_1=b_2} + \beta 1_{b_1 \neq b_2}$. Given the strategies $\tau^H$ and $\tau^F$, we only need to consider the following cases

$$\nu^n_{(a_1, b_1, g)(a_2, b_2, c_2)}(a_1, b_1, h) = \begin{cases} \tau^g_{a_1} \tau^h_{a_2} & \text{if } b_2 \equiv b_1 + 1 \pmod{K} \text{ and } c_2 = h, \\ 0 & \text{otherwise}. \end{cases}$$

$$\nu^n_{(a_1, b_1, g)(a_2, b_2, c_2)}(a_1, b_1, f) = \begin{cases} \tau^f_{a_1} \tau^g_{a_2} & \text{if } b_2 \equiv b_1 + 1 \pmod{K} \text{ and } c_2 = f, \\ 0 & \text{otherwise}. \end{cases}$$

$$\nu^n_{(a_1, b_1, h)(a_2, b_2, c_2)}(a_1, b_1, g) = \begin{cases} \tau^g_{a_1} \tau^h_{a_2} & \text{if } b_2 \equiv b_1 - 1 \pmod{K} \text{ and } c_2 = g, \\ 0 & \text{otherwise}. \end{cases}$$
\[\nu^n_{(a_1,b_1,h)(a_2,b_2,c_2)}(a_1,b_1,f) = \begin{cases} \tau_{a_1} f & c_2 = f. \\ 0 & \text{otherwise.} \end{cases}\]

\[\nu^n_{(a_1,b_1,f)(a_2,b_2,c_2)}(a_1,b_1,g) = \begin{cases} \tau_{a_1} a_2 g & \text{if } b_2 \equiv b_1 - 1 \pmod{K} \text{ and } c_2 = g. \\ 0 & \text{otherwise.} \end{cases}\]

\[\nu^n_{(a_1,b_1,f)(a_2,b_2,c_2)}(a_1,b_1,h) = \begin{cases} \tau_{a_1} a_2 h & \text{if } c_2 = h. \\ 0 & \text{otherwise.} \end{cases}\]

6 Appendix

6.1 Exact law of large numbers

The following general version of the exact law of large numbers from [40] is stated as a lemma here for the convenience of the reader.\footnote{Part (2) of the lemma is part of Theorem 2.8 in [40]. That theorem actually shows that the statement in Part (2) here is equivalent to the condition of essential pairwise independence. While Parts (1) and (3) of the lemma are special cases of Part (2), they are stated respectively in Corollary 2.9 and Theorem 2.12 of [40].}

**Lemma 1** Let \( f \) be a measurable process from a Fubini extension \((I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)\) to a complete separable metric space \( X \). Assume that the random variables \( f_i \) are essentially pairwise independent in the sense that for \( \lambda \)-almost all \( i \in I \), the random variables \( f_i \) and \( f_j \) are independent for \( \lambda \)-almost all \( j \in I \).

1. For \( P \)-almost all \( \omega \in \Omega \), the sample distribution \( \lambda f^{-1}_\omega \) of the sample function \( f_\omega \) is the same as the distribution \( (\lambda \boxtimes P)f^{-1} \) of the process.\footnote{Here, \((\lambda \boxtimes P)f^{-1}\) is the distribution \( \nu \) on \( X \) such that \( \nu(B) = (\lambda \boxtimes P)(f^{-1}(B)) \) for any Borel set \( B \) in \( X \); \( \lambda f^{-1}_\omega \) is defined similarly.}

2. For any \( A \in \mathcal{I} \) with \( \lambda(A) > 0 \), let \( f^A \) be the restriction of \( f \) to \( A \times \Omega \), \( \lambda^A \) and \( \lambda^A \boxtimes P \) the probability measures rescaled from the restrictions \( \lambda \) and \( \lambda \boxtimes P \) to \( \{D \in \mathcal{I} : D \subseteq A\} \) and \( \{C \in \mathcal{I} \boxtimes \mathcal{F} : C \subseteq A \times \Omega\} \) respectively. Then for \( P \)-almost all \( \omega \in \Omega \), the sample distribution \( \lambda^A(f^A)^{-1}_\omega \) of the sample function \( (f^A)_\omega \) is the same as the distribution of \( (\lambda^A \boxtimes P)(f^A)^{-1} \) of the process \( f^A \).

3. If there is a distribution \( \mu \) on \( X \) such that for \( \lambda \)-almost all \( i \in I \), the random variable \( f_i \) has distribution \( \mu \), then the sample function \( f_\omega \) (or \( (f^A)_\omega \)) also has distribution \( \mu \) for \( P \)-almost all \( \omega \in \Omega \).

By viewing a discrete-time stochastic process taking values in \( X \) as a random variable taking values in \( X^\infty \), Lemma 1 implies the following exact law of large numbers for a continuum of discrete-time stochastic processes, which is formally stated in Theorem 2.16 in [40].
Define $\mu$ for $A$ matchings on $\times$ product space $\lambda$ and $\theta$. Let $\bar{\rho}$ integers. This covers the case of finitely many types as a special case, by assuming that $n \in \Omega$, the empirical process $f_\omega = \{f_n^\omega\}_{n=0}^\infty$ has the same finite-dimensional distributions as that of $f = \{f_n\}_{n=0}^\infty$, i.e. $(f_0^\omega, \ldots, f_n^\omega)$ and $(f_0^\omega, \ldots, f_n)$ have the same distribution for any $n \geq 0$.

Proof of Proposition 1: Let $I_k = \{i \in I : \alpha(i) = k\}$. The result follows from Lemma 1 by applying the exact law of large numbers to the process $g^{I_k} = g|I_k \times \Omega$ on the rescaled probability space. 

6.2 Proof of Theorem 1

In the rest of this paper, we only work with the case that the type space $S$ is the set of positive integers. This covers the case of finitely many types as a special case, by assuming that $p_l = 0$ and $\theta_{kl} = 0$ for all $l > K$ and $k \in S$, for some finite integer $K$.

Let $(I, \mathcal{I}_0, \lambda_0)$ be the agent space, where $I = \{1, \ldots, \hat{M}\}$, $\mathcal{I}_0$ is the internal power set on $I$, $\lambda_0$ is the internal counting probability measure on $\mathcal{I}_0$, $\hat{M}$ is an unlimited hyperfinite integer in $\ast \mathbb{N}_\infty$. Let $\hat{K}$ be an unlimited hyperfinite integer, $\hat{S} = \{1, 2, \ldots, \hat{K}\}$ and $\hat{S} = S \cup \{J\}$. Assume that $K \hat{K} \leq \hat{M}$.

Let $\hat{\alpha}$ be an internal function from $I$ to $\hat{S}$ such that $\lambda(\hat{\alpha}(i) = k) \simeq p_k$ for any $k \in S$ and $\lambda_0\{i : \hat{\alpha}(i) = k\} \geq \frac{1}{M}$ for any $k \in \hat{S}$. Let $\hat{p}_k = \lambda_0\{\hat{\alpha}(i) = k\}$ for $k \in \hat{S}$, and $\{\hat{\theta}_{kl}\}_{k,l \in \ast \mathbb{N}}$ be the transfer of $\{{\theta}_{kl}\}_{k,l \in \mathbb{N}}$ and $\hat{\theta}_{kl} = \hat{\theta}_{kl}\hat{p}_l$. For each $k \leq \hat{K}$, define an internal probability $\rho_k$ on $\hat{S}$ such that $\rho_k(l) = \hat{q}_{kl}$ for $l \in \hat{S}$ and $\rho_k(J) = 1 - \sum_{l=1}^{\hat{K}} \hat{q}_{kl}$. Let $\Omega_0 = \hat{S}^I$ be the internal set of all the internal functions from $I$ to $\hat{S}$. Let $\mu_0$ be the internal product probability $\Pi_{i \in I} \rho_{\hat{\alpha}(i)}$ on $\Omega_0$. Let $I_k = \{i \in I : \hat{\alpha}(i) = k\}$ for $k \in \hat{S}$. Let $\Omega_1 = \{A_1 \times \cdots \times A_{K+2} : A_k \subseteq I$ and $A_k$ is internal, where $1 \leq k \leq K^2\}$. For each $\omega_0 \in \Omega_0$, $k, l \in S$, let $\bar{A}_{kl}^{\omega_0} = \{i \in I_k : \omega_0(i) = l\}$. For $k \neq l$, let $C_{kl}^{\omega_0} = \{A : A \subseteq A_{kl}^{\omega_0}$ and $|A| = \min\{|A_{kl}^{\omega_0}|, |A_{lk}^{\omega_0}|\}\}$. For $k = l$, let $C_{kk}^{\omega_0} = \{A_{kk}^{\omega_0}\} \setminus \{i \in A_{kk}^{\omega_0} : A_{kk}^{\omega_0}$ is odd and $C_{kk}^{\omega_0} = \{A_{kk}^{\omega_0} :$ if $A_{kk}^{\omega_0}$ is even. Let $C_{kl}^{\omega_0}$ be the product space $\Pi_{k,l \in \hat{S}} C_{kl}^{\omega_0}$. For any $A^{\omega_0} \in C_{kl}^{\omega_0}$, let $B_{kl}^{\omega_0} = I_k \cup_{i \in A_{kl}^{\omega_0}} A_{kl}^{\omega_0}$. Let $B_{kl}^{\omega_0} = \cup_{k=1}^{\hat{K}} B_{kl}^{\omega_0}$. Define $\mu_1$ be the internal product probability on $\Omega_0 \times \Omega_1$ by letting $\mu_1(\omega_0, A) = \mu_0(\omega_0) \times \mu_{\omega_0}(A)$ for $\omega_0 \in \Omega_0, A \in \Omega_1$, where $\mu_{\omega_0}$ is the internal counting probability on $C_{kl}^{\omega_0}$, and $\mu_{\omega_0}(A) = 0$ for $A \notin C_{kl}^{\omega_0}$.

For each $k \in \hat{S}, \omega_0 \in \Omega_0, A^{\omega_0} \in C_{kl}^{\omega_0}$, let $\Omega_{kk}^{\omega_0, A^{\omega_0}}$ be the internal set of all the internal full matchings on $A_{kk}^{\omega_0}$. Let $\mu_{kk}^{\omega_0, A^{\omega_0}}$ be the internal counting probability on $\Omega_{kk}^{\omega_0, A^{\omega_0}}$. For $k,l \in \hat{S}$
such that \( k < l \), let \( \Omega_{kl}^{\alpha_0, A_{\alpha_0}} \) be the internal set of all the internal bijections from \( A_{kl}^{\alpha_0} \) to \( A_{lk}^{\alpha_0} \). Let \( \mu_{kl}^{\alpha_0, A_{\alpha_0}} \) be the internal counting probability on \( A_{kl}^{\alpha_0} \). Let \( \Omega_2 \) be the internal set of all the internal bijections \( \phi \) from \( I \) to \( I \) such that \( \phi(\phi(i)) = i \).

For \( \omega_0 \in \Omega_0 \) and \( A_{\alpha_0} \in C_{\alpha_0} \), let \( \Omega_{\omega_0, A_{\alpha_0}} \) be the set of \( \phi \in \Omega_2 \), with

(i) \( \{ i \in I : \phi(i) = i \} = B_k^{\alpha_0} \) for each \( k \in \bar{S} \);
(ii) the restriction \( \phi|_{A_{\alpha_0}} \in \Omega_{\omega_0, A_{\alpha_0}} \) for \( k \in \bar{S} \);
(iii) for \( k, l \in \bar{S} \) such that \( k < l \), \( \phi|_{A_{\alpha_0}} \in \Omega_{\omega_0, A_{\alpha_0}} \).

Define an internal probability \( \mu_{\omega_0, A_{\alpha_0}} \) on \( \Omega_2 \) such that

(i) \( \mu_{\omega_0, A_{\alpha_0}}(\phi) = 0 \) if \( \phi \notin \Omega_{\omega_0, A_{\alpha_0}} \);
(ii) for \( \phi \in \Omega_{\omega_0, A_{\alpha_0}} \), \( \mu_{\omega_0, A_{\alpha_0}}(\phi) = \prod_{k \leq l} \mu_{kl}^{\omega_0, A_{\alpha_0}}(\phi|_{A_{\alpha_0}}) \).

For \( \omega_0 \in \Omega_0 \), \( A \notin C_{\omega_0} \), let \( \mu_{\omega_0, A} \) be the internal counting probability on \( \Omega_2 \).

Define an internal probability measure \( P_0 \) on \( \Omega = \Omega_0 \times \Omega_1 \times \Omega_2 \) by letting \( P_0((\omega_0, A, \omega_2)) = \mu_1(\omega_0, A) \times \mu_2^{\omega_0, A}(\omega_2) \). For \( (i, \omega) \in I \times \Omega_1 \), let \( \pi(i, (\omega_0, A, \omega_2)) = \omega_2(i) \). We will need to check \( \pi \) to satisfy the required properties.

By the construction of \( \pi \), it is easy to verify properties (i) and (ii) of a random matching with directed probability. Let \( \hat{g}(i, \omega) = \begin{cases} \hat{\alpha}(\pi(i, \omega)) & \text{if } \pi(i, \omega) \neq i \\ J & \text{if } \pi(i, \omega) = i \end{cases} \). For \( i \in I \) and \( k \in \bar{S} \), let \( P_0(\hat{g}_i = k) \) be the probability that agent \( i \) with type \( \hat{\alpha}(i) \) matches with a type-\( k \) agent in the matching model considered above. Similarly, for \( i, j \in I \) and \( k, l \in \bar{S} \), let \( P_0(\hat{g}_i = k, \hat{g}_j = l) \) be the probability that agent \( i \) with type \( \hat{\alpha}(i) \) matches with a type-\( k \) agent and agent \( j \) with type \( \hat{\alpha}(j) \) matches with a type-\( l \) agent.

To prove property (iii), we need the following lemma, which provides an estimation on some relevant probabilities.

**Lemma 2** Suppose \( i \neq j \), \( \hat{\alpha}(i) = k_1 \) and \( \hat{\alpha}(j) = k_3 \), where \( k_1, k_3 \in \bar{S} \). For \( k_2, k_4 \in \bar{S} \), if \( \hat{p}_{k_1} \hat{q}_{k_2, k_1} > \hat{M}^{\frac{1}{2}} \), then \( P_0(\hat{g}(i) = k_2) \simeq \hat{q}_{k_1, k_2} \); if, in addition, \( \hat{p}_{k_3} \hat{q}_{k_2, k_3} > \hat{M}^{\frac{1}{2}} \), we will also have \( P_0(\hat{g}(i) = k_2, \hat{g}(j) = k_4) \simeq \hat{q}_{k_1, k_2} \hat{q}_{k_3, k_4} \).

**Proof:** By the above construction, it is obvious that \( P_0(\hat{g}(i) = k_2) \leq \hat{q}_{k_1, k_2} \) and \( P_0(\hat{g}(i) = k_2, \hat{g}(j) = k_4) \leq \hat{q}_{k_1, k_2} \hat{q}_{k_3, k_4} \). For any \( k, l \in \bar{S} \), the construction of \( \mu_0 \) implies that \( \text{var}(\frac{\hat{A}_{kl}^{\alpha_0}}{M}) \leq \frac{1}{M} \).

By Chebyshev’s inequality, we have

\[
P_0 \left( \left| \frac{\hat{A}_{kl}^{\alpha_0}}{M} - \hat{p}_k \hat{q}_{kl} \right| \geq \frac{1}{M^{\frac{1}{2}}} \right) \leq \frac{1}{M^{\frac{1}{2}}}.\]
Let $F_N = \{\omega_0 : \exists k, l \in \bar{S} \text{ such that } |\bar{A}^{\omega_0}_{k_{l}}| - \hat{p}_k \hat{q}_{kl} \geq \frac{1}{M^2}\}$. We then obtain that $P_0(F_N) \leq \frac{K^2}{M^2} \simeq 0$. For $\omega_0 \notin F_N$ and $k, l \in \bar{S}$, we have $\left|\frac{|\bar{A}^{\omega_0}_{kl}|}{M} - \hat{p}_k \hat{q}_{kl}\right| < \frac{1}{M^2}$.

It is easy to see that

$$P_0(\hat{g}(i) = k_2) = \int_{\{\omega_0(i) = k_2\}} \int_{\{A^{\omega_0} \in C^{\omega_0}\}} 1_{A^{\omega_0}_{k_{1}k_{2}}} (i) d\mu^\omega (A^{\omega_0}) d\mu_0(\omega_0) \quad (6)$$

If $k_1 = k_2$, then we have

$$P_0(\hat{g}(i) = k_2) = \int_{\{\omega_0(i) = k_1, |\bar{A}^{\omega_0}_{k_{1}k_{1}}| \text{ is even}\}} 1_{A^{\omega_0}_{k_{1}k_{1}}} (i) d\mu_0(\omega_0)$$
$$+ \int_{\{\omega_0(i) = k_1, |\bar{A}^{\omega_0}_{k_{1}k_{1}}| \text{ is odd}\}} \frac{|\bar{A}^{\omega_0}_{k_{1}k_{1}}| - 1}{|\bar{A}^{\omega_0}_{k_{1}k_{1}}|} 1_{A^{\omega_0}_{k_{1}k_{1}}} (i) d\mu_0(\omega_0)$$
$$\simeq \int_{\{\omega_0(i) = k_1, \omega_0 \notin F_N\}} 1_{A^{\omega_0}_{k_{1}k_{1}}} (i) d\mu_0(\omega_0) \simeq \hat{q}_{k_1 k_1},$$

where we use the fact that $i \in A^{\omega_0}_{k_{1}k_{1}}$ when $\hat{a}(i) = k_1$ and $\omega_0(i) = k_1$.

Now assume that $k_1 \neq k_2$ and $\hat{p}_k \hat{q}_{k_{1}k_{2}} > M^{-\frac{1}{2}}$. Then, we can obtain that $\frac{1}{M^2 \hat{p}_k \hat{q}_{k_{1}k_{2}}} < M^{-\frac{1}{2}} \simeq 0$, and

$$P_0(\hat{g}(i) = k_2) = \int_{\{\omega_0(i) = k_2\}} \frac{|A^{\omega_0}_{k_{1}k_{2}}| - 1}{\min(|A^{\omega_0}_{k_{1}k_{2}}|, |A^{\omega_0}_{k_{2}k_{1}}|) - 1} \mu_0(\omega_0)$$
$$= \int_{\{\omega_0(i) = k_2\}} \frac{\min(|A^{\omega_0}_{k_{1}k_{2}}|, |A^{\omega_0}_{k_{2}k_{1}}|)}{\min(|A^{\omega_0}_{k_{1}k_{2}}|, |A^{\omega_0}_{k_{2}k_{1}}|)} \mu_0(\omega_0)$$
$$\simeq \int_{\{\omega_0(i) = k_2, \omega_0 \notin F_N\}} \frac{\min(|A^{\omega_0}_{k_{1}k_{2}}|, |A^{\omega_0}_{k_{2}k_{1}}|)}{|A^{\omega_0}_{k_{1}k_{2}}|} \mu_0(\omega_0)$$
$$\geq \frac{\hat{p}_k \hat{q}_{k_{1}k_{2}} - \frac{1}{M^2}}{\hat{p}_k \hat{q}_{k_{1}k_{2}} + \frac{1}{M^2}} \mu_0(\omega_0) \simeq \hat{q}_{k_{1}k_{2}}, \quad (7)$$

where $\binom{n}{a}$ represents the binomial number for negative integers $a$ and $b$. Note that for $\omega_0 \notin F_N$, we have $\hat{p}_k \hat{q}_{k_{1}k_{2}} - \frac{1}{M^2} < \frac{|A^{\omega_0}_{k_{1}k_{2}}|}{M} < \hat{p}_k \hat{q}_{k_{1}k_{2}} + \frac{1}{M^2}$, which implies the last inequality in the above displayed formula. Since $P_0(\hat{g}(i) = k_2) \leq \hat{q}_{k_1 k_2}$, we have $P_0(\hat{g}(i) = k_2) \simeq \hat{q}_{k_1 k_2}$.

Next, similar to equation (6), we can obtain that

$$P_0(\hat{g}(i) = k_2, \hat{g}(j) = k_4) = \int_{\{\omega_0(i) = k_2, \omega_0(j) = k_4\}} \int_{\{A^{\omega_0} \in C^{\omega_0}\}} 1_{A^{\omega_0}_{k_1 k_2}} (i) 1_{A^{\omega_0}_{k_3 k_4}} (j) d\mu^\omega (A^{\omega_0}) d\mu_0(\omega_0).$$

If $(k_1, k_2) = (k_3, k_4)$, $k_1 \neq k_2$, and $\hat{p}_k \hat{q}_{k_1 k_2} > M^{-\frac{1}{2}}$, then we can obtain as in equation
(7) that
\[ P_0(\hat{g}(i) = k_2, \hat{g}(j) = k_4) = \int_{\{\omega_0(i)=k_2, \omega_0(j)=k_4\}} \frac{|A_{k_1 k_2}^{\omega_0}|-2}{\min(|A_{k_1 k_2}^{\omega_0}|, |A_{k_2 k_1}^{\omega_0}|)} d\mu_0(\omega_0) \]
\[ = \int_{\{\omega_0(i)=k_2, \omega_0(j)=k_4\}} \frac{\min(|A_{k_1 k_2}^{\omega_0}|, |A_{k_2 k_1}^{\omega_0}|) \min(|A_{k_1 k_2}^{\omega_0}|, |A_{k_2 k_1}^{\omega_0}| - 1)}{|A_{k_1 k_2}^{\omega_0}|} d\mu_0(\omega_0) \]
\[ \geq \int_{\{\omega_0(i)=k_2, \omega_0(j)=k_2, \omega_0 \notin F_N\}} \frac{(\hat{p}_{k_1} \hat{q}_{k_1 k_2} - \frac{1}{M}) (\hat{p}_{k_1} \hat{q}_{k_1 k_2} - \frac{1}{M} \frac{1}{M} - \frac{1}{M})}{(\hat{p}_{k_1} \hat{q}_{k_1 k_2} + \frac{1}{M}) (\hat{p}_{k_1} \hat{q}_{k_1 k_2} + \frac{1}{M} \frac{1}{M} - \frac{1}{M})} d\mu_0(\omega_0) \simeq \hat{q}_{k_1 k_2} \hat{q}_{k_3 k_4}. \]

Since \( P_0(\hat{g}(i) = k_2, \hat{g}(j) = k_4) \leq \hat{q}_{k_1 k_2} \hat{q}_{k_3 k_4} \), we have \( P_0(\hat{g}(i) = k_2, \hat{g}(j) = k_4) \simeq \hat{q}_{k_1 k_2} \hat{q}_{k_3 k_4} \).

Next, consider the case when \( (k_1, k_2) = (k_3, k_4) \) and \( k_1 = k_2 \), which means that \( k_1 = k_2 = k_3 = k_4 \). We have
\[ P_0(\hat{g}(i) = k_2, \hat{g}(j) = k_4) \]
\[ = \int_{\{\omega_0(i)=k_1, \omega_0(j)=k_1, |A_{k_1 k_2}^{\omega_0}| \text{ is even}\}} 1_{A_{k_1 k_2}^{\omega_0}}(i) 1_{A_{k_1 k_2}^{\omega_0}}(j) d\mu_0(\omega_0) \]
\[ + \int_{\{\omega_0(i)=k_1, \omega_0(j)=k_1, |A_{k_1 k_2}^{\omega_0}| \text{ is odd}\}} \frac{|A_{k_1 k_2}^{\omega_0}|-2}{|A_{k_1 k_2}^{\omega_0}|} 1_{A_{k_1 k_2}^{\omega_0}}(i) 1_{A_{k_1 k_2}^{\omega_0}}(j) d\mu_0(\omega_0) \]
\[ \simeq \int_{\{\omega_0(i)=k_1, \omega_0(j)=k_1, \omega_0 \notin F_N\}} 1_{A_{k_1 k_2}^{\omega_0}}(i) 1_{A_{k_1 k_2}^{\omega_0}}(j) d\mu_0(\omega_0) \simeq \hat{q}_{k_1 k_1}, \]
where we use the fact that \( i, j \in A_{k_1 k_1}^{\omega_0} \) when \( \hat{\alpha}(i) = \hat{\alpha}(j) = k_1 \) and \( \omega_0(i) = \omega_0(j) = k_1 \).

Finally, assume that \( (k_1, k_2) \neq (k_3, k_4) \), \( \hat{p}_{k_1} \hat{q}_{k_1 k_2} > M^{-\frac{1}{4}} \), and \( \hat{p}_{k_3} \hat{q}_{k_3 k_4} > M^{-\frac{1}{4}} \). By using a similar computation as in equation (7), we have
\[ P_0(\hat{g}(i) = k_2, \hat{g}(j) = k_4) \]
\[ = \int_{\{\omega_0(i)=k_2, \omega_0(j)=k_4\}} \frac{|A_{k_1 k_2}^{\omega_0}|-1}{\min(|A_{k_1 k_2}^{\omega_0}|, |A_{k_2 k_1}^{\omega_0}|)} \frac{|A_{k_1 k_2}^{\omega_0}|-1}{\min(|A_{k_1 k_2}^{\omega_0}|, |A_{k_2 k_1}^{\omega_0}|)} d\mu_0(\omega_0) \]
\[ = \int_{\{\omega_0(i)=k_2, \omega_0(j)=k_4\}} \frac{\min(|A_{k_1 k_2}^{\omega_0}|, |A_{k_2 k_1}^{\omega_0}|) \min(|A_{k_1 k_2}^{\omega_0}|, |A_{k_2 k_1}^{\omega_0}|)}{|A_{k_1 k_2}^{\omega_0}|} d\mu_0(\omega_0) \]
\[ \geq \int_{\{\omega_0(i)=k_2, \omega_0(j)=k_3, \omega_0 \notin F_N\}} \frac{\hat{p}_{k_1} \hat{q}_{k_1 k_2} - M^{-\frac{1}{4}}}{\hat{p}_{k_1} \hat{q}_{k_1 k_2} + M^{-\frac{1}{4}}} \frac{\hat{p}_{k_1} \hat{q}_{k_1 k_2} - M^{-\frac{1}{4}}}{\hat{p}_{k_1} \hat{q}_{k_1 k_2} + M^{-\frac{1}{4}}} d\mu_0(\omega_0) \simeq \hat{q}_{k_1 k_2} \hat{q}_{k_3 k_4}. \]

By the fact that \( P_0(\hat{g}(i) = k_2, \hat{g}(j) = k_4) \leq \hat{q}_{k_1 k_2} \hat{q}_{k_3 k_4} \), we can obtain that \( P_0(\hat{g}(i) = k_2, \hat{g}(j) = k_4) \simeq \hat{q}_{k_1 k_2} \hat{q}_{k_3 k_4} \).

To finish the proof of Theorem 1, suppose that \( \hat{\alpha}(i) = k_1 \) and \( \hat{\alpha}(j) = k_3 \), if \( \hat{p}_{k_1} \hat{q}_{k_1 k_2} > \frac{1}{M^\frac{1}{4}} \), by Lemma 2,
\[ P_0(\hat{g}(i) = k_2) \simeq \hat{q}_{k_1 k_2}(\hat{p}). \]
If \( \hat{p}_k \hat{q}_{k_1 k_2} \leq \frac{1}{M_k} \), it is easy to prove \( \hat{q}_{k_2 k_1} \leq \frac{1}{M_{k_2}} \) since \( \hat{p}_k \geq \frac{1}{M_k} \). Note that \( 0 \leq P_0(\hat{g}(i) = k_2) \leq \hat{q}_{k_1 k_2}(\hat{p}) \), then

\[
P_0(\hat{g}(i) = k_2) \simeq \hat{q}_{k_1 k_2}(\hat{p})
\]

for any \( i \in I \). We can prove

\[
P_0(\hat{g}(i) = k_2, \hat{g}(j) = k_4) \simeq \hat{q}_{k_1 k_2} \hat{q}_{k_3 k_4}
\]

for any \( i, j \in I \) in the same way. Then

\[
P_0(\hat{g}(i) = k_2, \hat{g}(j) = k_4) \simeq P_0(\hat{g}(i) = k_2)P_0(\hat{g}(j) = k_4)
\]

for any \( i, j \in I \).

Let

\[
g(i, \omega) = \begin{cases} \hat{\alpha}(\pi(i, \omega)) & \text{if } \pi(i, \omega) \neq i \text{ and } \hat{\alpha}(\pi(i, \omega)) \in S \\ J & \text{otherwise.} \end{cases}
\]

Note that \( g(i, \omega) = \hat{g}(i, \omega) \) on \( \{(i, \omega) : \hat{g}(i, \omega) \in S\} \), if \( \hat{\alpha}(i) = k_1 \) and \( \hat{\alpha}(j) = k_3 \), \( P(g_i = k_2) \simeq P_0(\hat{g}_i = k_2) \simeq q_{k_1 k_2} \) for \( k_1, k_2 \in S \) and

\[
P(g_i = k_2, g_j = k_4) = P(g_i = k_2)P(g_j = k_4)
\]

for \( k_2, k_4 \in S \). The case that involves the no-matching type \( J \) follows as well. Then the property (iii) is satisfied and \( \pi \) is independent in types.

### 6.3 Proof of Theorem 2

What we need to do is to construct a sequence of internal transition probabilities and a hyperfinite sequence of internal type functions. Since we need to consider random mutation, random matching and random type changing at each time period, three internal measurable spaces with internal transition probabilities will be constructed at each time period.

Let \( T_0 \) be the hyperfinite discrete time line \( \{n\}_{n=0}^M \) and \( (I, I_0, \lambda_0) \) be the agent space, where \( I = \{1, \ldots, M\} \), \( I_0 \) is the internal power set on \( I \), \( \lambda_0 \) is the internal counting probability measure on \( I_0 \), \( M \) and \( \hat{M} \) are unlimited hyperfinite numbers in \( \star \mathbb{N}_\infty \) such that \( M \geq \hat{K}^2 \) and \( \hat{M} \geq M^M \).

Before we construct the dynamical system, we need the following lemma:

**Lemma 3** There exist non-negative parameters \( \{b^n_{kl}\}_{k,l,}\in \check{S}, n \in T_0^* \), \( \{\hat{\theta}_n^{kl}\}_{k,l,}\in \check{S}, n \in T_0^* \) and \( \{\nu^n_{kl}(r)\}_{k,l,r}\in \check{S}, n \in T_0^* \) such that \( b^n_{kl} \simeq \star b^n_{kl} \), \( \hat{\theta}_n^{kl}(p) \simeq \star \theta_n^{kl}(p) \) and \( \nu^n_{kl}(r) \simeq \star \nu^n_{kl}(r) \) for \( k, l, r \in S, n \in \mathbb{N} \) and

\[
\sum_{l=1}^{K} b^n_{kl} = \sum_{l=1}^{K} \nu^n_{kl}(r) = 1 \text{ for } k, l, r \in \check{S}, n \in \mathbb{N}.
\]
Proof: Let \( \tilde{b}_n^{kl} = *b_n^{kl} + \frac{2K}{M}1_{\{l=1\}} \) and \( \tilde{\nu}_n^{kl}(r) = *\nu_n^{kl}(r) + \frac{1}{M}1_{\{r=1\}} \). Then define

\[
\hat{b}_n^{kl} = \begin{cases} 
\frac{\hat{b}_n^{kl}}{\sum_{l'=1}^K \hat{b}_n^{kl'}} - \frac{K-1}{M} & \text{if } l = 1 \\
\frac{\hat{b}_n^{kl}}{\sum_{r'=1}^M \hat{\nu}_n^{kl'}} + \frac{1}{M} & \text{otherwise},
\end{cases}
\]

\( \theta_n^{kl}(p) = (*\theta_n^{kl})(p) \) and \( \tilde{\nu}_n^{kl}(r) = \frac{\tilde{\nu}_n^{kl}(r)}{\sum_{l'=1}^K \tilde{\nu}_n^{kl'}} \) for \( n \in T_0 \) and \( p \in \Delta^3 \). Then \{\( \hat{b}_n^{kl} \}_{k,l \in \bar{\mathbb{S}}, n \in T_0} \) and \{\( \hat{\nu}_n^{kl}(r) \)\}_{k,l,r \in \bar{\mathbb{S}}, n \in T_0} \) satisfy all the condition above.

Suppose that the construction for the dynamical system \( \mathbb{D} \) has been done up to time period \( n - 1 \). Thus, \((\{\Omega_m, \mathcal{F}_m, Q_m\})_{m=1}^{3n-3} \) and \((\alpha^m)_{m=0}^{3n-3} \) have been constructed, where each \( \Omega_m \) is a hyperfinite internal set with its internal power set \( \mathcal{F}_m \), \( Q_m \) an internal transition probability from \( \Omega^{m-1} \) to \( \Omega_m \), and \( \alpha^m \) an internal type function from \( I \times \Omega^{3l} \) to the type space \( \bar{\mathbb{S}} \).

Here, \( \Omega^m = \prod_{j=1}^{3n-1} \Omega_j \), and \( \{\omega_j\}_{j=1}^{m} \) will also be denoted by \( \omega^m \) when there is no confusion. Denote the internal product transition probability \( Q_1 \otimes Q_2 \otimes \cdots \otimes Q_m \) by \( Q^m \), and \( \otimes_{j=1}^{m} \mathcal{F}_j \) by \( \mathcal{F}^m \). Then \( Q^m \) is the internal product of the internal transition probability \( Q_m \) with the internal probability measure \( Q^{m-1} \). Suppose that \( \lambda \boxtimes Q^{3n-3}(\bar{\alpha}^{n-1} \in \mathbb{S}) = 1 \), where \( \bar{\alpha}^{n-1} \) is the type function at the end of time \( n - 1 \).

We shall now consider the constructions for time \( n \). We first work with the random mutation step. Let \( \Omega_{3n-2} = \bar{\mathbb{S}}^I \) (the space of all internal functions from \( I \) to \( \bar{\mathbb{S}} \)) with its internal power set \( \mathcal{F}_{3n-2} \).

For each \( i \in I \), \( \omega^{3n-3} \in \Omega^{3n-3} \), there is \( k \in \bar{\mathbb{S}} \) such that \( \alpha^{n-1}(i, \omega^{3n-3}) = k \); define a probability measure \( \gamma^{3n-3}_i \) on \( \bar{\mathbb{S}} \) by letting \( \gamma^{3n-3}_i(\{l\}) = \hat{b}_n^{kl} \) for each \( l \in \bar{\mathbb{S}} \). Define an internal probability measure \( Q_{3n-2}^{\omega^{3n-3}} \) on \( (\bar{\mathbb{S}}^I, \mathcal{F}_{3n-2}) \) to be the internal product measure \( \prod_{i \in I} \gamma^{3n-3}_i \). Let \( \hat{h}^n : (I \times \prod_{m=1}^{3n-2} \Omega_m) \rightarrow \bar{\mathbb{S}} \) be such that \( \hat{h}^n(i, \omega^{3n-2}) = \omega_{3n-2}(i) \). Let \( \tilde{p}_{\omega}^{3n-2} \) the random cross-sectional type distribution induced by \( h^n_{\omega} \).

Next, we consider the step of random matching. Let \((\Omega_{3n-1}, \mathcal{F}_{3n-1}) \) be the internal sample measurable space \( (\bar{\mathbb{S}}, \bar{\mathcal{F}}_0) \). For any given \( \omega^{3n-2} \in \Omega^{3n-2} \), the type function is \( h^n_{\omega^{3n-2}}(\cdot) \), denoted by \( \hat{\alpha} \) for short. Let \( Q_{3n-1}^{\omega^{3n-2}} \) be the internal probability measure corresponding to the internal probability measure \( \tilde{\mu}^0(\cdot) \) with initial distribution \( \alpha \) in the static model with matching probability \( \theta_n^{kl}(\tilde{p}_i^{3n-2}) \tilde{p}_i^{3n-2} \) for a type \( k \) agent to meet a type \( l \) agent.

Finally, we consider the step of random type changing for matched agents. Let \( \Omega_{3n} = \bar{\mathbb{S}}^I \) with its internal power set \( \mathcal{F}_{3n} \); each point \( \omega_{3n} \in \Omega_{3n} \) is an internal function from \( I \) to \( \bar{\mathbb{S}} \). For \( k, l \in \bar{\mathbb{S}}, \) \( \nu_n^{kl} \) is an internal distribution on \( \bar{\mathbb{S}} \) and \( \tilde{\nu}_n^{kl}(r) \) the probability for a type-\( k \) agent to change to a type-\( r \) agent when the type-\( k \) agent meets a type-\( l \) agent at step \( n \).

Define a new type function \( \hat{\alpha} : (I \times \Omega^{3n}) \rightarrow \bar{\mathbb{S}} \) by letting \( \hat{\alpha}^n(i, \omega^{3n}) = \omega_{3n}(i) \).

Fix \( \omega^{3n-1} \in \Omega^{3n-1} \). For each \( i \in I \), (1) if \( \omega_{3n-1}(i) = i \) (\( i \) is not matched at time \( n \)), let \( \tau_i^{\omega^{3n-1}} \) be the probability measure on the type space \( \bar{\mathbb{S}} \) that gives probability one to
the type \( \hat{h}^n(i, \omega^{3n-2}) \) and zero for the rest; (2) if \( \omega^{3n-1}(i) \neq i \) (i is matched at time \( n \)), \( h^n(i, \omega^{3n-2}) = k \) and \( h^n(\omega^{3n-1}(i), \omega^{3n-2}) = l \), let \( \tau_i^{3n-1} \) be the distribution \( \nu_k^l \) on \( S \). Define an internal probability measure \( Q^{3n-1}_{\omega^n} \) on \( \bar{S}^n \) to be the internal product measure \( \prod_{i\in I} \tau_i^{3n-1} \).

By induction, we can construct a hyperfinite sequence \( \{\Omega_m, \mathcal{F}_m, Q_m\}_{m=1}^M \) of internal transition probabilities and a hyperfinite sequence \( \{\alpha^l\}_{l=0}^M \) of internal type functions.

Let \( (I \times \Omega^M, I_0 \otimes \mathcal{F}^M, \lambda_0 \otimes Q^M) \) be the internal product probability space of \( (I, I_0, \lambda_0) \) and \( (\Omega^M, \mathcal{F}^M, Q^M) \). Let \( (I \times \Omega^M, I \boxtimes \mathcal{F}, \lambda \boxtimes P) \) be the Loeb space of the internal product. For simplicity, let \( \Omega^M \) be denoted by \( \Omega \).

To prove Theorem 2, we first introduce some notations for convenience. We separate one period into three steps: the first step contains mutation, the second step contains random matching and the third step contains type changing.

If \( m = 3n - 1 \), then the \( m \)th step is the random matching step, denote \( \hat{g}_k^3(\hat{p}) = \theta_k^3(\hat{p})\hat{p}_t \).

For each period \( n \), we have \( \hat{p}_k^3 \geq \frac{1}{M^3} \), and hence at the beginning of the matching step \( \mathbb{E} \hat{p}_k^{3n-1} \geq \frac{1}{M} \). By Chebyshev’s inequality,

\[
Q^{3n-1}(\hat{p}_k^{3n-1} - \mathbb{E}\hat{p}_k^{3n-1}) \leq \frac{1}{M^{3n-1}},
\]

then \( \hat{p}_k^{3n-1} \geq \frac{1}{M^{3n-1}} \) \( P \)-almost surely. To connect with the notation in the proof of Theorem 1, we use \( P_0^{h_n^{3n-2}} \) to represent \( Q^{3n-1} \). The following is a simple corollary to Lemma 2.

**Corollary 3** For \( P \)-almost all \( \omega \),

\[
P_0^{h_n^{3n-2}}(\hat{g}_i^n = k_2) \simeq q_{k_2k_1}^{3n-1}(\hat{p}_k^{3n-1}(\omega))
\]

and

\[
P_0^{h_n^{3n-2}}(\hat{g}_i^n = k_2, \hat{g}_j^n = k_4) \simeq q_{k_2k_1}^{3n-1}(\hat{p}_k^{3n-1}(\omega))q_{k_4k_3}^{3n-1}(\hat{p}_k^{3n-1}(\omega)),
\]

where \( h_n^{3n-2}(i) = k_1 \) and \( h_n^{3n-2}(j) = k_2 \).

**Proof:** If \( \hat{p}_k^{3n-1} q_{k_2k_1}^{3n-1} > \frac{1}{M^{3n-1}} \), then by Lemma 2,

\[
P_0^{h_n^{3n-2}}(\hat{g}_i^n = k_2) \simeq q_{k_2k_1}^{3n-1}(\hat{p}_k^{3n-1}(\omega)).
\]

If \( \hat{p}_k^{3n-1} q_{k_2k_1}^{3n-1} \leq \frac{1}{M^{3n-1}} \), it is easy to prove \( q_{k_2k_1}^{3n-1} \leq \frac{1}{M^{3n-1}} \) since \( \hat{p}_k^{3n-1} \geq \frac{1}{M^{3n-1}} \), note that 0 \( \leq P_0^{h_n^{3n-2}}(\hat{g}_i^n = k_2) \leq q_{k_2k_1}^{3n-1}(\hat{p}_k^{3n-1}(\omega)) \), then

\[
P_0^{h_n^{3n-2}}(\hat{g}_i^n = k_2, \hat{g}_j^n = k_4) \simeq q_{k_2k_1}^{3n-1}(\hat{p}_k^{3n-1}(\omega))q_{k_4k_3}^{3n-1}(\hat{p}_k^{3n-1}(\omega)).
\]

We can prove the second part in the same way. ■
To finish the proof of Theorem 2, suppose that the Markov conditional independence are satisfied up to step \( m \), it remains to check the Markov conditional independence for step \( m + 1 \). For notational simplicity, we use \( P_0 \) to denote the relevant internal measure \( Q^m \) at the particular step.

For \( m = 3n - 2 \), we can obtain that

\[
P_0(\hat{h}_i^n = k_2, \hat{h}_j^n = k_4; \hat{\alpha}_i^n = k_1, \ldots, \hat{\alpha}_i^0; \hat{\alpha}_j^{n-1} = k_3, \ldots, \hat{\alpha}_j^0) = \int_D P_0^{\omega_3n-3}(\hat{h}_i^n = k_2, \hat{h}_j^n = k_4) dP^{3n-2}(\omega)
\]

\[
= \int_D P_0^{\omega_3n-3}(\hat{h}_i^n = k_2) P_0^{\omega_3n-3}(\hat{h}_j^n = k_4) dP^{3n-2}(\omega)
\]

\[
= \int_D b^n_{k_1k_2b^n_{k_3k_4}} dP^{3n-2}(\omega)
\]

\[
= P_0(D) b^n_{k_1k_2b^n_{k_3k_4}}
\]

where \( D = \{ \omega : \hat{\alpha}_i^{n-1} = k_1, \ldots, \hat{\alpha}_i^0; \hat{\alpha}_j^{n-1} = k_3, \ldots, \hat{\alpha}_j^0 \} \). Thus

\[
P_0(\hat{h}_i^n = k_2, \hat{h}_j^n = k_4 | \hat{\alpha}_i^{n-1} = k_1, \ldots, \hat{\alpha}_i^0; \hat{\alpha}_j^{n-1} = k_3, \ldots, \hat{\alpha}_j^0) = b^n_{k_1k_2b^n_{k_3k_4}}.
\]

We can prove \( P_0(\hat{h}_i^n = k_2 | \hat{\alpha}_i^{n-1} = k_1) = b^n_{k_1k_2} \) in the same way. Therefore

\[
P_0(\hat{h}_i^n = k_2, \hat{h}_j^n = k_4 | \hat{\alpha}_i^{n-1} = k_1, \ldots, \hat{\alpha}_i^0; \hat{\alpha}_j^{n-1} = k_3, \ldots, \hat{\alpha}_j^0) = P_0(\hat{h}_i^n = k_2 | \hat{\alpha}_i^{n-1} = k_1) P_0(\hat{h}_j^n = k_4 | \hat{\alpha}_j^{n-1} = k_3).
\]

If \( m = 3n - 1 \), the exact law of large numbers implies that

\[
\hat{p}^m(\omega) \simeq E(\hat{p}^m(\omega)) \triangleq \hat{p}^m \text{ } P\text{-almost surely.}
\]

Note that

\[
P_0(\hat{g}_i^n = k_2, \hat{g}_j^n = k_4, \hat{h}_i^n = k_1, \hat{\alpha}_i^{n-1}, \ldots, \hat{\alpha}_i^0, \hat{h}_j^n = k_3, \hat{\alpha}_j^{n-1}, \ldots, \hat{\alpha}_j^0) = \int_D P_0^{\omega_3n-2}(\hat{g}_i^n = k_2, \hat{g}_j^n = k_4) dP^{3n-1}(\omega)
\]

where \( D = \{ \omega : \hat{h}_i^n = k_1, \hat{\alpha}_i^{n-1}, \ldots, \hat{\alpha}_i^0, \hat{h}_j^n = k_3, \hat{\alpha}_j^{n-1}, \ldots, \hat{\alpha}_j^0 \} \). By Corollary 3, we have

\[
P_0^{\omega_3n-2}(\hat{g}_i^n = k_2, \hat{g}_j^n = k_4) \simeq \hat{q}^{3n-1}(\hat{p}^{3n-1}(\omega)) \hat{q}^{3n-1}(\hat{p}^{3n-1}(\omega)) \text{ } P\text{-almost surely,
\]

and hence

\[
P_0(\hat{g}_i^n = k_1, \hat{g}_j^n = k_3, \hat{h}_i^n = k_2, \hat{\alpha}_i^{n-1}, \ldots, \hat{\alpha}_i^0; \hat{h}_j^n = k_4, \hat{\alpha}_j^{n-1}, \ldots, \hat{\alpha}_j^0) \simeq \int_D \hat{q}^{3n-1}(\hat{p}^{3n-1}(\omega)) \hat{q}^{3n-1}(\hat{p}^{3n-1}(\omega)) dP^m(\omega)
\]

\[
\simeq P_0(D) \hat{q}^{3n-1}(\hat{p}^{3n-1}(\omega)) \hat{q}^{3n-1}(\hat{p}^{3n-1}(\omega)). \tag{8}
\]
We can prove that $P_0(g^n_0 = k_2, h^n_0 = k_1) = \int_{D'} P_0^n h^{n-2}(g^n_i = k_2) dP^m(\omega)$, where $D' = \{h^n_i = k_2\}$, by Corollary 1, $P_0^n h^{n-2}(g^n_i = k_2) \approx q^{3n-1}(\bar{p}^{3n-1}(\omega)) P$-almost surely, then

$$P_0(g^n_i = k_2, h^n_i = k_1) \approx \int_{D'} q^{3n-1} k_2 (\bar{p}^{3n-1}(\omega)) \rho P_0(D) \approx P_0(D) q^{3n-1} k_2 (\bar{p}^{3n-1}(\omega)).$$

(9)

Combine equations (8) and (9) together, we have

$$P_0(g^n_i = k_2, g^n_j = k_4) h^n_i = k_1, \alpha_i^{n-1}, \ldots, \alpha_i^0, g^n_j = 0, j_i = k_3, \hat{\alpha}_j^{n-1}, \ldots, \hat{\alpha}_j^0 \rangle \approx q^{3n-1}(\bar{p}^{3n-1} k_2) \rho P_0(D) q^{3n-1} k_2 (\bar{p}^{3n-1}(\omega)).$$

If $m = 3n$, we have

$$P_0(\alpha_i^n = k_3, \alpha_j^n = k_6, g^n_i = k_2, h^n_i = k_1, \alpha_i^{n-1}, \ldots, \alpha_i^0, g^n_j = 0, j_i = k_5, \hat{\alpha}_j^n = k_3, \hat{\alpha}_j^{n-1}, \ldots, \hat{\alpha}_j^0)$$

$$= \int D P_0 h^{n-2} g^{n-1} (\alpha_i^n = k_3, g^n_i = k_6) dP^{3n-1}(\omega)$$

$$= \int D P_0 h^{n-2} g^{n-1} (\alpha_i^n = k_3) P_0 h^{n-2} (\alpha_j^n = k_6) dP^{3n-1}(\omega)$$

$$= \int D \nu_{i,k}^{n-2} (k_3) \nu_{i,k}^{n-2} (k_6) dP^{3n-1}(\omega)$$

$$= P_0(D) \nu_{i,k}^{n-2} (k_3) \nu_{i,k}^{n-2} (k_6)$$

where $D = \{g^n_i = k_2, h^n_i = k_1, \alpha_i^{n-1}, \ldots, \alpha_i^0, g^n_j = 0, j_i = k_5, \hat{\alpha}_j^n = k_3, \hat{\alpha}_j^{n-1}, \ldots, \hat{\alpha}_j^0\}$. Hence

$$P_0(\alpha_i^n = k_3, \alpha_j^n = k_6, g_i^n = k_2, h_i^n = k_1, \alpha_i^{n-1}, \ldots, \alpha_i^0, g_j^n = 0, h_j^n = k_5, \hat{\alpha}_j^n = k_3, \hat{\alpha}_j^{n-1}, \ldots, \hat{\alpha}_j^0) = \nu_{i,k}^{n-2} (k_3) \nu_{i,k}^{n-2} (k_6).$$

We can prove $P_0(\alpha_i^n = k_3, \alpha_j^n = k_6, g_i^n = k_2, h_i^n = k_1, \alpha_i^{n-1}, \ldots, \alpha_i^0, g_j^n = 0, h_j^n = k_5, \hat{\alpha}_j^n = k_3, \hat{\alpha}_j^{n-1}, \ldots, \hat{\alpha}_j^0) = \nu_{i,k}^{n-2} (k_3) \nu_{i,k}^{n-2} (k_6)$ in the same way. Then

$$P_0(\alpha_i^n = k_3, \alpha_j^n = k_6, g_i^n = k_2, h_i^n = k_1) = \nu_{i,k}^{n-2} (k_3) \nu_{i,k}^{n-2} (k_6)$$

Thus the proof is finished.

### 6.4 Proof of Proposition 2 and Corollary 1

In this appendix, the unit interval $[0,1]$ will have a different notation in a different context. Recall that $(L, \mathcal{L}, \eta)$ is the Lebesgue unit interval, where $\eta$ is the Lebesgue measure defined on the Lebesgue $\sigma$-algebra $\mathcal{L}$.

Note that the agent space used in the proof of Theorem 2 is a hyperfinite Loeb counting probability space. Using the usual ultrapower construction as in [31], the hyperfinite index set
of agents can be viewed as an equivalence class of a sequence of finite sets with elements in natural numbers, and thus this index set of agents has the cardinality of the continuum.\(^9\)

The purpose of Proposition 2 in this paper is to show that one can find some extension of the Lebesgue unit interval as the agent space so that the associated version of Theorem 2 still holds.

Fix a set \(\hat{I}\) with cardinality of the continuum as in Proposition 2.\(^{10}\) The following lemma is a strengthened version of Lemma 2 in [26]; see also Lemma 419I of Fremlin [15] and Lemma 3 in [41].\(^{11}\) The proof given below is a slight modification of the proof of Lemma 2 in [26].

**Lemma 4** There is a disjoint family \(C = \{C_i : i \in \hat{I}\}\) of subsets of \(L = [0, 1]\) such that \(\bigcup_{i \in \hat{I}} C_i = L\), and for each \(i \in \hat{I}\), \(C_i\) has the cardinality of the continuum, \(\eta_*(C_i) = 0\) and \(\eta^*(C_i) = 1\), where \(\eta_*\) and \(\eta^*\) are, respectively, the inner and outer measures of the Lebesgue measure \(\eta\).

**Proof.** Let \(c\) be the cardinality of the continuum. As usual in set theory, \(c\) can be viewed as the set of all ordinals below the cardinality of the continuum. Let \(\mathcal{H}\) be the family of closed subsets of \(L = [0, 1]\) with positive Lebesgue measure. Then, the cardinality of \(\mathcal{H}\) is \(c\), and hence the cardinality of \(\mathcal{H} \times c\) is \(c\) as well. Enumerate the elements in \(\mathcal{H} \times c\) as a transfinite sequence \(\{(F_\xi, \alpha_\xi)\}_{\xi < c}\), where \(\xi\) is an ordinal.

Define a transfinite sequence \(\{x_\xi\}_{\xi < c}\) by transfinite induction as follows. Suppose that for an ordinal \(\xi < c\), \(\{x_\beta\}_{\beta < \xi}\) is defined. Note that the set of elements \(\{x_\beta\}_{\beta < \xi}\) has cardinality strictly less than the continuum. Since \(F_\xi\) has the cardinality of the continuum, one can take any \(x_\xi\) from the nonempty set \(F_\xi \setminus \{x_\beta\}_{\beta < \xi}\). One can continue this procedure to define the whole transfinite sequence \(\{x_\xi\}_{\xi < c}\). Note that the elements in the transfinite sequence \(\{x_\xi\}_{\xi < c}\) are all distinct.

For each ordinal \(\alpha < c\), let \(A_\alpha\) be the set of all the \(x_\xi\) with \(\xi < c\) and \(\alpha_\xi = \alpha\), that is, \(A_\alpha = \{x_\xi : \xi < c, \alpha_\xi = \alpha\}\). It is clear that the sets \(A_\alpha, \alpha < c\) are disjoint.

Next, fix an ordinal \(\alpha < c\). Since \(\{(F_\xi, \alpha_\xi)\}_{\xi < c}\) enumerates the elements in \(\mathcal{H} \times c\), the set \(\{F_\xi : \xi < c, \alpha_\xi = \alpha\}\) equals \(\mathcal{H}\), which has cardinality of the continuum. Thus, \(\{\xi : \xi < c, \alpha_\xi = \alpha\}\) has the cardinality of the continuum. Since the elements in \(\{x_\xi\}_{\xi < c}\) are all distinct, the set \(A_\alpha = \{x_\xi : \xi < c, \alpha_\xi = \alpha\}\) has the cardinality of the continuum as well.

Suppose that the inner measure \(\eta_*(L \setminus A_\alpha)\) is positive. Then there is a Lebesgue measurable subset \(E\) of \(L \setminus A_\alpha\) with \(\eta(E) > 0\), which implies the existence of \(F \in \mathcal{H}\) with

---

\(^9\)The notation for the agent space in the proof of Theorem 3.1 in [13] is \((I, \mathcal{I}, \lambda)\), which is replaced by the notation \((\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})\) in Theorem 2 here. The notation \((\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})\) will be used below for a different purpose.

\(^{10}\)Note that we replace the corresponding notation \((K, K, \kappa)\) used in the Appendix of [41] by \((\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})\) in this paper. The reason is that the notation \(K\) has been used earlier as the number of agent types.

\(^{11}\)The original version of Lemma 2 of [26] and Lemma 419I of Fremlin [15] requires neither that each \(C_i\) has cardinality of the continuum nor \(\bigcup_{i \in \hat{I}} C_i = L\).
\( F \subseteq E \subseteq L \setminus A_{\alpha} \). Let \( \xi < c \) be the unique ordinal such that \( F_{\xi} = F \) and \( \alpha_{\xi} = \alpha \). By the definitions of \( x_{\xi} \) and \( A_{\alpha} \), we have \( x_{\xi} \in F_{\xi} \) and \( x_{\xi} \in A_{\alpha} \), and hence \( x_{\xi} \in F \cap A_{\alpha} \). However, \( F \subseteq L \setminus A_{\alpha} \), which means that \( F \cap A_{\alpha} = \emptyset \). This is a contradiction. Hence, \( \eta_{*}(L \setminus A_{\alpha}) = 0 \), which means that the outer measure \( \eta^*(A_{\alpha}) = 1 \). It is clear that \( 0 \leq \eta_{*}(A_{\alpha}) \leq \eta_{*}(L \setminus A_{\alpha + 1}) = 0 \). Therefore \( \eta_{*}(A_{\alpha}) = 0 \).

Finally, since \( \hat{I} \) has the cardinality of the continuum, there is a bijection \( \hat{\xi} \) between \( \hat{I} \) and \( c \). For each \( \hat{i} \in \hat{I} \), let \( C_{\hat{i}} = A_{\xi(\hat{i})} \). Let \( B = L \setminus \bigcup_{\hat{i} \in \hat{I}} C_{\hat{i}} \). Since the cardinality of \( B \) is at most the cardinality of the continuum, we can redistribute at most one point of \( B \) into each \( C_{\hat{i}} \) in the family \( \mathcal{C} = \{ C_{\hat{i}} : \hat{i} \in \hat{I} \} \). The rest is clear.

Kakutani [26] provided a non-separable extension of the Lebesgue unit interval by adding subsets of the unit interval directly. As in the Appendix of [41], we follow some constructions used in the proof of Lemma 521P(b) of [16], which allows one to work with Fubini extensions in a more transparent way. The spirit of the Lebesgue extension itself is similar in the constructions used in [26] and here. Define a subset \( C \) of \( L \times \hat{I} \) by letting \( C = \{ (l, \hat{i}) \in L \times \hat{I} : l \in C_{\hat{i}}, \hat{i} \in \hat{I} \} \). Let \( (L \times \hat{I}, \mathcal{L} \otimes \hat{\mathcal{L}}, \eta \otimes \hat{\lambda}) \) be the usual product probability space. For any \( \mathcal{L} \otimes \hat{\mathcal{L}} \)-measurable set \( U \) that contains \( C \), \( C_{\hat{i}} \subseteq U_{\hat{i}} \) for each \( \hat{i} \in \hat{I} \), where \( U_{\hat{i}} = \{ l \in L : (l, \hat{i}) \in U \} \) is the \( \hat{i} \)-section of \( U \). The Fubini property of \( \eta \otimes \hat{\lambda} \) implies that for \( \hat{\lambda} \)-almost all \( \hat{i} \in \hat{I} \), \( U_{\hat{i}} \) is \( \mathcal{L} \)-measurable, which means that \( \eta(U_{\hat{i}}) = 1 \) (since \( \eta^*(C_{\hat{i}}) = 1 \)). Since \( \eta \otimes \hat{\lambda}(U) = \int_{\hat{I}} \eta(U_{\hat{i}}) d\hat{\lambda} \), we have \( \eta \otimes \hat{\lambda}(U) = 1 \). Therefore, the \( \eta \otimes \hat{\lambda} \)-outer measure of \( C \) is one.

Since the \( \eta \otimes \hat{\lambda} \)-outer measure of \( C \) is one, the method in [9] (see p. 69) can be used to extend \( \eta \otimes \hat{\lambda} \) to a measure \( \gamma \) on the \( \sigma \)-algebra \( \mathcal{U} \) generated by the set \( C \) and the sets in \( \mathcal{L} \otimes \hat{\mathcal{L}} \) with \( \gamma(C) = 1 \). It is easy to see that \( \mathcal{U} = \left\{ (U^{1} \cap C) \cup (U^{2} \setminus C) : U^{1}, U^{2} \in \mathcal{L} \otimes \hat{\mathcal{L}} \right\} \), and that \( \gamma[(U^{1} \cap C) \cup (U^{2} \setminus C)] = \eta \otimes \hat{\lambda}(U^{1}) \) for any measurable sets \( U^{1}, U^{2} \in \mathcal{L} \otimes \hat{\mathcal{L}} \). Let \( \mathcal{T} \) be the \( \sigma \)-algebra \( \{ U \cap C : U \in \mathcal{L} \otimes \hat{\mathcal{L}} \} \), which is the collection of all the measurable subsets of \( C \) in \( \mathcal{U} \). The restriction of \( \gamma \) to \( (C, \mathcal{T}) \) is still denoted by \( \gamma \). Then, \( \gamma(U \cap C) = \eta \otimes \hat{\lambda}(U) \), for every measurable set \( U \in \mathcal{L} \otimes \hat{\mathcal{L}} \). Note that \( (L \times \hat{I}, \mathcal{U}, \gamma) \) is an extension of \( (L \times \hat{I}, \mathcal{L} \otimes \hat{\mathcal{L}}, \eta \otimes \hat{\lambda}) \).

Consider the projection mapping \( p^{L} : L \times \hat{I} \to L \) with \( p^{L}(l, \hat{i}) = l \). Let \( \psi \) be the restriction of \( p^{L} \) to \( C \). Since the family \( C \) is a partition of \( L = [0, 1] \), \( \psi \) is a bijection between \( C \) and \( L \). It is obvious that \( p^{L} \) is a measure-preserving mapping from \( (L \times \hat{I}, \mathcal{L} \otimes \hat{\mathcal{L}}, \eta \otimes \hat{\lambda}) \) to \( (L, \mathcal{L}, \eta) \) in the sense that for any \( B \in \mathcal{L} \), \( (p^{L})^{-1}(B) \in \mathcal{L} \otimes \hat{\mathcal{L}} \) and \( \eta \otimes \hat{\lambda}[(p^{L})^{-1}(B)] = \eta(B) \); and thus \( p^{L} \) is a measure-preserving mapping from \( (L \times \hat{I}, \mathcal{U}, \gamma) \) to \( (L, \mathcal{L}, \eta) \). Since \( \gamma(C) = 1 \), \( \psi \) is a measure-preserving mapping from \( (C, \mathcal{T}, \gamma) \) to \( (L, \mathcal{L}, \eta) \), that is, \( \gamma \circ \psi^{-1}(B) = \eta(B) \) for any \( B \in \mathcal{L} \).

To introduce one more measure structure on the unit interval \([0,1] \), we shall also denote it by \( I \). Let \( \mathcal{I} \) be the \( \sigma \)-algebra \( \{ S \subseteq I : \psi^{-1}(S) \in \mathcal{T} \} \). Define a set function \( \lambda \) on \( \mathcal{I} \) by
letting \( \lambda(S) = \gamma[\psi^{-1}(S)] \) for each \( S \in \mathcal{I} \). Since \( \psi \) is a bijection, \( \lambda \) is a well-defined probability measure on \((I, \mathcal{I})\). Moreover, \( \psi \) is also an isomorphism from \((C, \mathcal{T}, \gamma)\) to \((I, \mathcal{I}, \lambda)\). Since \( \psi \) is a measure-preserving mapping from \((C, \mathcal{T}, \gamma)\) to \((L, \mathcal{L}, \eta)\), it is obvious that \((I, \mathcal{I}, \lambda)\) is an extension of the Lebesgue unit interval \((L, \mathcal{L}, \eta)\).

We shall now follow the procedure used in the proof of Proposition 2 in [41] to construct a Fubini extension based on the probability spaces \((I, \mathcal{I}, \lambda)\) as defined above, and \((\Omega, \mathcal{F}, P)\) as in our Theorem 2 here.

First, consider the usual product space \((L \times \hat{I} \times \Omega, \mathcal{L} \otimes (\hat{I} \otimes \mathcal{F}), \eta \otimes (\hat{\lambda} \otimes P))\) of the Lebesgue unit interval \((L, \mathcal{L}, \eta)\) with the Fubini extension \((\hat{I} \times \Omega, \hat{I} \otimes \mathcal{F}, \hat{\lambda} \otimes P)\). The following lemma is shown in Step 1 of the proof of Proposition 2 in [41].

**Lemma 5** The probability space \((L \times \hat{I} \times \Omega, \mathcal{L} \otimes (\hat{I} \otimes \mathcal{F}), \eta \otimes (\hat{\lambda} \otimes P))\) is a Fubini extension of the usual triple product space \(((L \times \hat{I}) \times \Omega, (\mathcal{L} \otimes \hat{I}) \otimes \mathcal{F}, (\eta \otimes \hat{\lambda}) \otimes P)\).

Next, as shown in Step 2 of the proof of Proposition 2 in [41], the set \(C \times \Omega\) has \(\eta \otimes (\hat{\lambda} \otimes P)\)-outer measure one. Based on the Fubini extension \((L \times \hat{I} \times \Omega, \mathcal{L} \otimes (\hat{I} \otimes \mathcal{F}), \eta \otimes (\hat{\lambda} \otimes P))\), we can construct a measure structure on \(C \times \Omega\) as follows. Let \(\mathcal{E} = \{D \cap (C \times \Omega) : D \in \mathcal{L} \otimes (\hat{I} \otimes \mathcal{F})\}\) (which is a \(\sigma\)-algebra on \(C \times \Omega\)), and \(\tau\) be the set function on \(\mathcal{E}\) defined by \(\tau(D \cap (C \times \Omega)) = \eta \otimes (\hat{\lambda} \otimes P)(D)\) for any measurable set \(D\) in \(\mathcal{L} \otimes (\hat{I} \otimes \mathcal{F})\). Then, \(\tau\) is a well-defined probability measure on \((C \times \Omega, \mathcal{E})\) since the \(\eta \otimes (\hat{\lambda} \otimes P)\)-outer measure of \(C \times \Omega\) is one. The result in the following lemma is shown in Step 2 of the proof of Proposition 2 in [41].

**Lemma 6** The probability space \((C \times \Omega, \mathcal{E}, \tau)\) is a Fubini extension of the usual product probability space \((C \times \Omega, \mathcal{T} \otimes \mathcal{F}, \gamma \otimes P)\).

Let \(\Psi\) be the mapping \((\psi, \text{Id}_\Omega)\) from \(C \times \Omega\) to \(I \times \Omega\), where \(\text{Id}_\Omega\) is the identity map on \(\Omega\). That is, for each \((l, i) \in C, \omega \in \Omega\), \(\Psi((l, i), \omega) = (\psi, \text{Id}_\Omega)((l, i), \omega) = (\psi(l, i), \omega)\). Since \(\psi\) is a bijection from \(C\) to \(I\), \(\Psi\) is a bijection from \(C \times \Omega\) to \(I \times \Omega\). Let \(\mathcal{W} = \{H \subseteq I \times \Omega : \Psi^{-1}(H) \in \mathcal{E}\}\); then \(\mathcal{W}\) is a \(\sigma\)-algebra of subsets of \(I \times \Omega\). Define a probability measure \(\rho\) on \(\mathcal{W}\) by letting \(\rho(H) = \tau[\Psi^{-1}(H)]\) for any \(H \in \mathcal{W}\). Therefore, \(\Psi\) is an isomorphism from the probability space \((C \times \Omega, \mathcal{E}, \tau)\) to the probability space \((I \times \Omega, \mathcal{W}, \rho)\). The following lemma is shown in Step 3 of the proof of Proposition 2 in [41].

**Lemma 7** The probability space \((I \times \Omega, \mathcal{W}, \rho)\) is a Fubini extension of the usual product probability space \((I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)\).

---

12 We replace here the notation “\(\nu\)” used in the Appendix of [41] with “\(\tau\)” here, because “\(\nu\)” has been used earlier here for match-induced type-change probabilities.
Since \((I \times \Omega, W, \rho)\) is a Fubini extension, we shall follow the usual notation to denote 
\((I \times \Omega, W, \rho)\) by \((I \times \Omega, I \otimes F, \lambda \otimes P)\).

Now, define a mapping \(\varphi\) from \(I\) to \(\hat{I}\) by letting \(\varphi(i) = \hat{i}\) if \(i \in C_i\). Since the family 
\(C = \{C_i : \hat{i} \in \hat{I}\}\) is a partition of \(I = [0, 1]\), \(\varphi\) is well-defined.

**Lemma 8** The following properties of \(\varphi\) hold.

1. The mapping \(\varphi\) is measure preserving from \((I, I, \lambda)\) to \((\hat{I}, \hat{I}, \hat{\lambda})\), in the sense that for any 
\(A \in \hat{I}\), \(\varphi^{-1}(A)\) is measurable in \(I\) with \(\lambda[\varphi^{-1}(A)] = \hat{\lambda}(A)\).

2. Let \(\Phi\) be the mapping \((\varphi, \text{Id}_\Omega)\) from \(I \times \Omega\) to \(\hat{I} \times \Omega\), that is, \(\Phi(i, \omega) = (\varphi(i), \omega) = (\varphi(i), \omega)\) for any \((i, \omega) \in I \times \Omega\). Then \(\Phi\) is measure preserving from \((I \times \Omega, I \otimes F, \lambda \otimes P)\) to 
\((\hat{I} \times \Omega, \hat{I} \otimes F, \hat{\lambda} \otimes P)\) in the sense that for any \(V \in \hat{I} \otimes F\), \(\Phi^{-1}(V)\) is measurable in 
\(I \otimes F\) with \((\lambda \otimes P)[\Phi^{-1}(V)] = (\hat{\lambda} \otimes P)(V)\).

**Proof.** Property (1) obviously follows from (2) by considering those sets \(V\) in the form of 
\(A \times \Omega\) for \(A \in \hat{I}\). Thus, we only need to prove (2). Consider the projection mapping \(p^{I \times \Omega} : \)
\(L \times \hat{I} \times \Omega \to \hat{I} \times \Omega\) with \(p^{I \times \Omega}(i, \hat{i}, \omega) = (\hat{i}, \omega)\). Let \(\Psi_1\) be the restriction of \(p^{I \times \Omega}\) to \(C \times \Omega\).

Fix any \((i, \omega) \in I \times \Omega\). There is a unique \(\hat{i} \in \hat{I}\) such that \(i \in C_i\). Thus, \(\varphi(i) = \hat{i}\), and 
\((i, \hat{i}) \in C\) by the definition of \(C\). We also have \(\psi(i, \hat{i}) = i\), \(\psi^{-1}(i) = (i, \hat{i})\), and \(\Psi^{-1}(i, \omega) = ((i, \hat{i}), \omega)\). Note that \(\Psi^{-1}\) is a well-defined mapping from \(I \times \Omega\) to \(C \times \Omega\) since \(\Psi\) is a bijection from \(C \times \Omega\) to \(I \times \Omega\). Hence, we have 

\[\Psi_1[\Psi^{-1}(i, \omega)] = (\hat{i}, \omega) = (\varphi(i), \omega) = \Phi(i, \omega)\]

Therefore \(\Phi\) is the composition mapping \(\Psi_1[\Psi^{-1}]\).

Fix any \(V \in \hat{I} \otimes F\). We have \(\Phi^{-1}(V) = \Psi[\Psi_1^{-1}(V)]\). By the definition of \(\Psi_1\), we obtain 
that \(\Psi_1^{-1}(V) = (L \times V) \cap (C \times \Omega)\), which is obviously measurable in \(F\). For simplicity, we denote 
the set \(\Psi_1^{-1}(V)\) by \(E\). It follows from the definition of \(\tau\) that 
\(\tau(E) = \eta \otimes (\hat{\lambda} \otimes P)(L \times V) = (\hat{\lambda} \otimes P)(V)\). Since \(\Psi\) is an isomorphism from the probability space 
\((C \times \Omega, E, \tau)\) to the probability space \((I \times \Omega, W, \rho)\), we know that \(\Psi(E)\) is measurable in \(W\) and 
\(\rho[\Psi(E)] = \tau(E) = (\hat{\lambda} \otimes P)(V)\). It is clear that \(\Psi(E) = \Phi^{-1}(V)\). Therefore, \(\Phi^{-1}(V)\) is measurable in \(W\) with 
\(\rho[\Phi^{-1}(V)] = (\hat{\lambda} \otimes P)(V)\). The rest follows from the fact that \((I \times \Omega, W, \rho)\) is denoted by 
\((I \times \Omega, I \otimes F, \lambda \otimes P)\).

For notational convenience, we let \(\hat{D}\) denote the dynamical system with random mutation, 
random matching with directed probability and type changing that is Markov conditionally independent in types with parameters \((p^0, \theta, q, \nu)\), as presented in Theorem 2. For \(\hat{D}\), we add 
a hat to the relevant type functions, random mutation functions, and random assignments of
types for the matched agents. Let \( \hat{\alpha}^0 : \hat{I} \to S = \mathbb{N} \) be an initial \( \hat{D} \)-measurable type function with distribution \( p^0 \) on \( S \).

For each time period \( n \geq 1 \), \( \hat{h}^n \) is a random mutation function from \( (\hat{I} \times \Omega, \hat{D} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P) \) to \( S \) such that for each agent \( \hat{i} \in \hat{I} \), and for any types \( k, l \in S \),

\[
P(\hat{h}^n_{\hat{i}} = l \mid \hat{\alpha}^n_{\hat{i}} = k) = b_{kl}^n.
\]  

The expected cross-sectional type distribution immediately after random mutation \( \bar{p}^n \) follows from the recursive formula in part (1) of Proposition 3.

The random matching at time \( n \) is described by a function \( \hat{\pi}^n \) from \( \hat{I} \times \Omega \) to \( \hat{I} \cup \{J\} \) such that

1. For any \( \omega \in \Omega \), \( \hat{\pi}^n_\omega(\cdot) \) is a full matching on \( \hat{I} - (\hat{\pi}^n_\omega)^{-1}(\{J\}) \). For simplicity, the set \( \hat{I} - (\hat{\pi}^n_\omega)^{-1}(\{J\}) \) will be denoted by \( \hat{H}^n_\omega \).

2. \( \hat{g}^n \) is a \( \hat{D} \boxtimes \mathcal{F} \)-measurable mapping from \( \hat{I} \times \Omega \) to \( S \cup \{J\} \) with \( \hat{g}^n(\hat{i}, \omega) = \hat{h}^n(\hat{\pi}^n(\hat{i}, \omega), \omega) \), where we assume that \( \hat{h}^n(J, \omega) = J \) for any \( \omega \in \Omega \).

3. For each agent \( \hat{i} \in \hat{I} \) and for any types \( k, l \in S \),

\[
P(\hat{g}^n_{\hat{i}} = l \mid \hat{h}^n_{\hat{i}} = k) = \theta_{kl}^n \bar{p}^n_{\hat{l}},
\]

\[
P(\hat{g}^n_{\hat{i}} = J \mid \hat{h}^n_{\hat{i}} = k) = 1 - \sum_{l=1}^{\infty} \theta_{kl}^n \bar{p}^n_{\hat{l}}
\]  

A random assignment of types for the matched agents at time \( n \) is a function \( \hat{\alpha}^n \) from \( (\hat{I} \times \Omega, \hat{D} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P) \) to \( S \) such that for each agent \( \hat{i} \in \hat{I} \),

\[
P(\hat{\alpha}^n_{\hat{i}} = r \mid \hat{h}^n_{\hat{i}} = k, \hat{g}^n_{\hat{i}} = J) = \delta_k^r,
\]

\[
P(\hat{\alpha}^n_{\hat{i}} = r \mid \hat{h}^n_{\hat{i}} = k, \hat{g}^n_{\hat{i}} = l) = \nu_{kl}^n(r).
\]

**Proof of Proposition 2:** Based on the dynamical system \( \hat{D} \) on the Fubini extension \( (\hat{I} \times \Omega, \hat{D} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P) \), we shall now define, inductively, a new dynamical system \( D \) on the Fubini extension \( (I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P) \).

We first fix some bijections between the \( \hat{i} \)-sections of the set \( C \). For any \( \hat{i}, \hat{i}' \in \hat{I} \) with \( \hat{i} \neq \hat{i}' \), let \( \Theta^{\hat{i}, \hat{i}'} \) be a bijection from \( C_{\hat{i}} \) to \( C_{\hat{i}'} \), and \( \Theta^{\hat{i}', \hat{i}} \) be the inverse mapping of \( \Theta^{\hat{i}, \hat{i}'} \). This is possible since both \( C_{\hat{i}} \) and \( C_{\hat{i}'} \) have cardinality of the continuum, as noted in Lemma 4.

Let \( \alpha^0 \) be the mapping \( \hat{\alpha}^0(\varphi) \) from \( I \) to \( S \). By the measure preserving property of \( \varphi \) in Lemma 8, we know that \( \alpha^0 \) is \( \mathcal{I} \)-measurable type function with distribution \( p^0 \) on \( S \).
For each time period \( n \geq 1 \), let \( h^n \) and \( \alpha^n \) be the respective mappings \( \hat{h}^n(\Phi) \) and \( \hat{\alpha}^n(\Phi) \) from \( I \times \Omega \) to \( S \). Define a mapping \( \pi^n \) from \( I \times \Omega \) to \( I \cup \{J\} \) such that for each \((i, \omega) \in I \times \Omega\),

\[
\pi^n(i, \omega) = \begin{cases} 
J & \text{if } \hat{\pi}^n_\omega(\varphi(i)) = J, \\
\Theta \varphi(i), \hat{\pi}^n_\omega(\varphi(i)) & \text{if } \hat{\pi}^n_\omega(\varphi(i)) \neq J.
\end{cases}
\]

When \( \hat{\pi}^n_\omega(\varphi(i)) \neq J \), \( \hat{\pi}^n_\omega \) defines a full matching on \( \hat{H}^n_\omega = \hat{I} - (\hat{\pi}^n_\omega)^{-1}(\{J\}) \), which implies that \( \hat{\pi}^n_\omega(\varphi(i)) \neq \varphi(i) \). Hence, \( \pi^n \) is a well-defined mapping from \( I \times \Omega \) to \( I \cup \{J\} \).

Since \( \Phi \) is measure-preserving and \( h^n \) is a measurable mapping from \( (I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P) \) to \( S \), \( h^n \) is \( \mathcal{I} \otimes \mathcal{F} \)-measurable. By the definitions of \( h^n \) and \( \alpha^n \), it is obvious that for each \( i \in I \),

\[
h^n_i = \hat{h}^n_{\varphi(i)} \quad \text{and} \quad \alpha^n_i = \hat{\alpha}^n_{\varphi(i)},
\]

which, together with equation (10), implies that

\[
P \left( h^n_i = l \mid \alpha^n_i - 1 = k \right) = P \left( \hat{h}^n_{\varphi(i)} = l \mid \hat{\alpha}^n_{\varphi(i)} = k \right) = b^n_{kl}.
\]

Next, we consider the property of \( \pi^n \).

1. Fix any \( \omega \in \Omega \). Let \( H^n_\omega = I - (\pi^n_\omega)^{-1}(\{J\}) \); then \( H^n_\omega = \varphi^{-1}(\hat{H}^n_\omega) \). Pick any \( i \in H^n_\omega \) and denote \( \pi^n_\omega(i) \) by \( j \). Then, \( \varphi(i) \in \hat{H}^n_\omega \). The definition of \( \pi^n \) implies that \( j = \Theta \varphi(i), \hat{\pi}^n_\omega(\varphi(i)) \). Since \( \Theta \varphi(i), \hat{\pi}^n_\omega(\varphi(i)) \) is a bijection between \( C_\varphi(i) \) and \( \hat{C}_\varphi(i) \), it follows that \( \varphi(j) = \varphi(\pi^n_\omega(i)) = \hat{\pi}^n_\omega(\varphi(i)) \) by the definition of \( \varphi \). Thus, \( j = \Theta \varphi(i), \varphi(j)(i) \). Since the inverse of \( \Theta \varphi(i), \varphi(j) \) is \( \Theta \varphi(j), \varphi(i) \), we know that \( \Theta \varphi(j), \varphi(i)(j) = i \). By the full matching property of \( \hat{\pi}^n_\omega \), \( \varphi(j) \neq \varphi(i) \), \( \varphi(j) \in \hat{H}^n_\omega \) and \( \hat{\pi}^n_\omega(\varphi(j)) = \varphi(i) \). Hence, we have \( j \neq i \), and

\[
\pi^n_\omega(j) = \Theta \varphi(i), \hat{\pi}^n_\omega(\varphi(i))(j) = \Theta \varphi(j), \varphi(i)(j) = i.
\]

This means that the composition of \( \pi^n_\omega \) with itself on \( H^n_\omega \) is the identity mapping on \( H^n_\omega \), which also implies that \( \pi^n_\omega \) is a bijection on \( H^n_\omega \). Therefore \( \pi^n_\omega \) is a full matching on \( H^n_\omega = I - (\pi^n_\omega)^{-1}(\{J\}) \).

2. Extending \( h^n \) so that \( h^n(J, \omega) = J \) for any \( \omega \in \Omega \), we define \( g^n : I \times \Omega \to S \cup \{J\} \) by

\[
g^n(i, \omega) = h^n(\pi^n(i, \omega), \omega).
\]

Denote \( \varphi(J) = J \). As noted in the above paragraph, for any fixed \( \omega \in \Omega \), \( \varphi(\pi^n(i, \omega)) = \hat{\pi}^n_\omega(\varphi(i)) \) for \( i \in H^n_\omega \). When \( i \notin H^n_\omega \), we have \( \varphi(i) \notin \hat{H}^n_\omega \), and \( \pi^n_\omega(i) = J \), \( \hat{\pi}^n_\omega(\varphi(i)) = J \). Therefore, \( \varphi(\pi^n_\omega(i)) = \hat{\pi}^n_\omega(\varphi(i)) \) for any \( i \in I \). Then,

\[
g^n(i, \omega) = \hat{h}^n(\varphi(\pi^n(i, \omega)), \omega) = \hat{h}^n(\hat{\pi}^n(\varphi(i), \omega), \omega) = \hat{g}^n(\varphi(i), \omega) = \hat{g}^n(\Phi)(i, \omega).
\]

Hence, the measure-preserving property of \( \Phi \) implies that \( g^n \) is \( \mathcal{I} \otimes \mathcal{F} \)-measurable. The above equation also means that

\[
g^n_i(\cdot) = \hat{g}^n(\varphi(i))(\cdot), \quad i \in I.
\]
3. Equations (11), (13) and (15) imply that for each agent $i \in I$,
\[
P\left(g^n_i = l \mid h^n_i = k\right) = P\left(\hat{g}^n_{\varphi(i)} = J \mid \hat{h}^n_{\varphi(i)} = k\right) = \theta^n_{k,l} P^n_i,
\]
\[
P\left(g^n_i = J \mid h^n_i = k\right) = P\left(\hat{g}^n_{\varphi(i)} = l \mid \hat{h}^n_{\varphi(i)} = k\right) = 1 - \sum_{l=1}^{\infty} \theta^n_{k,l} P^n_i. \tag{16}
\]

Now, we consider the type-changing function $\alpha^n$ for the matched agents. Since $\Phi$ is
 measurable-preserving and $\hat{\alpha}^n$ is a measurable mapping from $(\hat{I} \times \Omega, \hat{\mathcal{I}} \otimes \mathcal{F}, \hat{\lambda} \otimes P)$ to $S$, $\alpha^n$ is
 $\mathcal{I} \otimes \mathcal{F}$-measurable. Equations (12), (13) and (15) imply that for each agent $i \in I$,
\[
P\left(\alpha^n_i = r \mid h^n_i = k, g^n_i = J\right) = P\left(\hat{\alpha}^n_{\varphi(i)} = r \mid \hat{h}^n_{\varphi(i)} = k, \hat{g}^n_{\varphi(i)} = J\right) = \delta^r_k,
\]
\[
P\left(\alpha^n_i = r \mid h^n_i = k, g^n_i = l\right) = P\left(\hat{\alpha}^n_{\varphi(i)} = r \mid \hat{h}^n_{\varphi(i)} = k, \hat{g}^n_{\varphi(i)} = l\right) = \nu^r_k(n). \tag{17}
\]

Therefore, $\mathcal{D}$ is a dynamical system with random mutation, random matching with
directed probability and type changing and with the parameters $(p^0, b, \theta, \nu)$.

It remains to check the Markov conditional independence for $\mathcal{D}$. Since the dynamical
system $\hat{\mathcal{D}}$ is Markov conditionally independent in types, for each $n \geq 1$, there is a set $\hat{I}' \in \hat{\mathcal{I}}$
with $\hat{\lambda}(\hat{I}') = 1$, and for each $\hat{i} \in \hat{I}'$, there exists a set $\hat{E}_{\hat{i}} \in \hat{\mathcal{I}}$ with $\hat{\lambda}(\hat{E}_{\hat{i}}) = 1$, with the following
properties being satisfied for any $\hat{i} \in \hat{I}'$ and any $\hat{j} \in \hat{E}_{\hat{i}}$:

1. For all types $k, l \in S$,
\[
P\left(\hat{h}^n_i = k, \hat{h}^n_j = l \mid \hat{\alpha}^n_i, \ldots, \hat{\alpha}^{n-1}_i, \hat{\alpha}^n_j, \ldots, \hat{\alpha}^{n-1}_j\right) = P\left(\hat{h}^n_i = k \mid \hat{\alpha}^{n-1}_i\right) P\left(\hat{h}^n_j = l \mid \hat{\alpha}^{n-1}_j\right). \tag{18}
\]

2. For all types $c, d \in S \cup \{J\}$,
\[
P\left(\hat{g}^n_i = c, \hat{g}^n_j = d \mid \hat{\alpha}^n_i, \ldots, \hat{\alpha}^{n-1}_i, \hat{h}^n_i, \hat{\alpha}^n_j, \ldots, \hat{\alpha}^{n-1}_j, \hat{h}^n_j\right) = P\left(\hat{g}^n_i = c \mid \hat{h}^n_i\right) P\left(\hat{g}^n_j = d \mid \hat{h}^n_j\right). \tag{19}
\]

3. For all types $k, l \in S$,
\[
P\left(\hat{\alpha}^n_i = k, \hat{\alpha}^n_j = l \mid \hat{\alpha}^n_i, \ldots, \hat{\alpha}^{n-1}_i, \hat{h}^n_i, \hat{\alpha}^n_j, \ldots, \hat{\alpha}^{n-1}_j, \hat{h}^n_j, \hat{g}^n_i\right) = P\left(\hat{\alpha}^n_i = k \mid \hat{h}^n_i, \hat{g}^n_i\right) P\left(\hat{\alpha}^n_j = l \mid \hat{h}^n_j, \hat{g}^n_j\right). \tag{20}
\]

Let $I' = \varphi^{-1}(\hat{I}')$. For any $i \in I'$, let $E_i = \varphi^{-1}(\hat{E}_{\varphi(i)})$. Since $\varphi$ is measure-preserving,
$\lambda(I') = \lambda(E_i) = 1$. Fix any $i \in I'$, and any $j \in E_i$. Denote $\varphi(i)$ by $\hat{i}$ and $\varphi(j)$ by $\hat{j}$. Then, it is
obvious that $\hat{i} \in \hat{I}'$ and $\hat{j} \in \hat{E}_{\hat{i}}$.  


By equations (13) and (15), we can rewrite equations (18), (19) and (20) as follows. For all types \( k, l \in S \),
\[
P(h_i^n = k, h_j^n = l \mid \alpha_i^0, \ldots, \alpha_i^{n-1}, \alpha_j^0, \ldots, \alpha_j^{n-1})
\]
\[
= P(\hat{h}_i^n = k, \hat{h}_j^n = l \mid \hat{\alpha}_i^0, \ldots, \hat{\alpha}_i^{n-1}, \hat{\alpha}_j^0, \ldots, \hat{\alpha}_j^{n-1})
\]
\[
= P(\hat{h}_i^n = k \mid \hat{\alpha}_i^{n-1})P(h_j^n = l \mid \hat{\alpha}_j^{n-1})
\]
\[
= P(h_i^n = k \mid \alpha_i^{n-1})P(h_j^n = l \mid \alpha_j^{n-1}).
\] (21)

For all types \( c, d \in S \cup \{J\} \),
\[
P(g_i^n = c, g_j^n = d \mid \alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n, \alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n)
\]
\[
= P(\hat{g}_i^n = c, \hat{g}_j^n = d \mid \hat{\alpha}_i^0, \ldots, \hat{\alpha}_i^{n-1}, \hat{h}_i^n, \hat{\alpha}_j^0, \ldots, \hat{\alpha}_j^{n-1}, \hat{h}_j^n)
\]
\[
= P(\hat{g}_i^n = c \mid \hat{h}_i^n)P(\hat{g}_j^n = d \mid \hat{h}_j^n)
\]
\[
= P(g_i^n = c \mid h_i^n)P(g_j^n = d \mid h_j^n).
\] (22)

For all types \( k, l \in S \),
\[
P(\alpha_i^n = k, \alpha_j^n = l \mid \alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n, g_i^n, \alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n, g_j^n)
\]
\[
= P(\hat{\alpha}_i^n = k, \hat{\alpha}_j^n = l \mid \hat{\alpha}_i^0, \ldots, \hat{\alpha}_i^{n-1}, \hat{h}_i^n, \hat{g}_i^n, \hat{\alpha}_j^0, \ldots, \hat{\alpha}_j^{n-1}, \hat{h}_j^n, \hat{g}_j^n)
\]
\[
= P(\hat{\alpha}_i^n = k \mid \hat{h}_i^n, \hat{g}_i^n)P(\hat{\alpha}_j^n = l \mid \hat{h}_j^n, \hat{g}_j^n)
\]
\[
= P(\alpha_i^n = k \mid h_i^n, g_i^n)P(\alpha_j^n = l \mid h_j^n, g_j^n).
\] (23)

Therefore the dynamical system \( \mathcal{D} \) is Markov conditionally independent in types. \( \blacksquare \)

**Proof of Corollary 1**: In the proof of Proposition 2, take the initial type distribution \( p^0 \) to be \( p \). Assume that there is no genuine random mutation in the sense that \( b_{kl}^1 = \delta_{kl} \) for all \( k, l \in S \). Then, it is clear that \( \tilde{p}_k^1 = p_k \) for any \( k \in S \). Consider the random matching \( \pi^1 \) in period one.

Fix an agent \( i \) with \( \alpha^0(i) = k \). Then equation (14) implies that \( P(h_i^1 = k) = 1 \). By equation (16),
\[
P(g_i^1 = l) = \theta_{kl}^1 p_l, \quad P(g_i^1 = J) = 1 - \sum_{l=1}^{\infty} \theta_{kl}^1 p_l.
\] (24)

Similarly, equation (22) implies that the process \( g^1 \) is essentially pairwise independent. By taking the type function \( \alpha \) to be \( \alpha^0 \), the matching function \( \pi \) to be \( \pi^1 \), and the associated process \( g \) to be \( g^1 \), the corollary holds. \( \blacksquare \)

### 6.5 Proof of Propositions 3 and 4

Before proving Theorem 3, we need to prove a few lemmas. The first lemma shows how to compute the expected cross-sectional type distributions \( \mathbf{\bar{\pi}}^n \) and \( \mathbf{\bar{p}}^n \).
Lemma 9  (1) For each \( n \geq 1 \), \( \tilde{\rho}^n = \Gamma^n(\tilde{\rho}^{n-1}) \).

(2) For each \( n \geq 1 \), the expected cross-sectional type distribution \( \tilde{\rho}^n \) immediately after random mutation at time \( n \), satisfies \( \tilde{\rho}^n = \sum_{l=1}^{\infty} b^n_{lk} \tilde{\rho}^{n-1}_l = \sum_{l=1}^{\infty} b^n_{kl} \Gamma^{n-1}_l(p^0) \).

Proof. Note that

\[
\tilde{p}^n_k = \int_I P(h^n_i = k) \, d\lambda(i) = \int_I \sum_{l=1}^{\infty} P(h^n_i = k, \alpha_i^{n-1} = l) \, d\lambda(i)
\]

\[
= \int_I \sum_{l=1}^{\infty} P(h^n_i = k \mid \alpha_i^{n-1} = l)P(\alpha_i^{n-1} = l) \, d\lambda(i)
\]

\[
= \sum_{l=1}^{\infty} \int_I b^n_{lk}P(\alpha_i^{n-1} = l) \, d\lambda(i) = \sum_{l=1}^{\infty} b^n_{lk} \tilde{\rho}^{n-1}_l.
\]

Then, we can express \( \tilde{\rho}^n \) in terms of \( \tilde{p}^n \) as

\[
\tilde{\rho}^n_r = \int_I P(\alpha_i^n = r) \, d\lambda(i)
\]

\[
= \int_I \sum_{k=1}^{\infty} \left[ P(\alpha_i^n = r, h^n_i = k, g^n_i = J) + \sum_{l=1}^{\infty} P(\alpha_i^n = r, h^n_i = k, g^n_i = l) \right] \, d\lambda(i)
\]

\[
= \int_I \sum_{k=1}^{\infty} \left[ \sum_{l=1}^{\infty} P(\alpha_i^n = r \mid h^n_i = k, g^n_i = J)P(g^n_i = J \mid h^n_i = k)P(h^n_i = k)
\]

\[
+ \sum_{l=1}^{\infty} P(\alpha_i^n = r \mid h^n_i = k, g^n_i = l)P(g^n_i = l \mid h^n_i = k)P(h^n_i = k) \right] \, d\lambda(i)
\]

\[
= q^n_i \sum_{l=1}^{\infty} p_l b^n_{lr} + \sum_{k,l=1}^{\infty} q^n_{kl} \mu^n_{ld}(r) \tilde{p}_k.
\]

By combining equations (25) and (26), it is easy to see that \( \tilde{\rho}^n = \Gamma(\tilde{\rho}^{n-1}) \). \( \blacksquare \)

The following lemma shows the Markov property of the agents’ type processes.

Lemma 10 Suppose the dynamical system \( \mathbb{D} \) is Markov conditionally independent in types. Then, for \( \lambda \)-almost all \( i \in I \), the type process for agent \( i \), \( \{\alpha_i^n\}_{n=0}^{\infty} \), is a Markov chain with transition matrix \( z^n \) at time \( n - 1 \).

Proof. Fix \( n \geq 1 \). Equation (??) implies that for \( \lambda \)-almost all \( i \in I \), \( \lambda \)-almost all \( j \in I \),

\[
P(h^n_i = k_n, h^n_j \in S \mid \alpha_i^0 = k_0, \ldots, \alpha_i^{n-1} = k_{n-1}; \alpha_j^0 \in S, \ldots, \alpha_j^{n-1} \in S)
\]

\[
= P(h^n_i = k_n \mid \alpha_i^{n-1} = k_{n-1})P(h^n_j \in S \mid \alpha_j^{n-1}),
\]

holds for any \( (k_0, \ldots, k_n) \in S^{n+1} \). Thus, for \( \lambda \)-almost all \( i \in I \),

\[
P(h^n_i = k \mid \alpha_i^0, \ldots, \alpha_i^{n-1}) = P(h^n_i = k \mid \alpha_i^{n-1})
\]

(28)
holds for any \( k \in S \). By grouping countably many null sets together, we know that for \( \lambda \)-almost all \( i \in I \), equation (28) holds for all \( k \in S \) and \( n \geq 1 \).

Similarly, equations (??) and (??) imply that for \( \lambda \)-almost all \( i \in I \),

\[
P(g_n^i = c \mid \alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n) = P(g_i^n = c \mid h_i^n)
\]

\[
P(\alpha_i^n = k \mid \alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n, g_i^n) = P(\alpha_i^n = k \mid h_i^n, g_i^n)
\]

(29)

hold for all \( k \in S \), \( c \in S \cup \{J\} \) and \( n \geq 1 \). A simple computation shows that for \( \lambda \)-almost all \( i \in I \), \( P(\alpha_i^n = k \mid \alpha_i^0, \ldots, \alpha_i^{n-1}) = P(\alpha_i^n = k \mid \alpha_i^{n-1}) \) for all \( k \in S \) and \( n \geq 1 \). Hence, for \( \lambda \)-almost all \( i \in I \), agent \( i \)'s type process \( \{\alpha_i^n\}_{n=0}^{\infty} \) is a Markov chain; it is also easy to see that the transition matrix \( z^n \) from time \( n - 1 \) to time \( n \) is

\[
z_{kl}^n = \sum_{r=1}^{\infty} \sum_{c \in S \cup \{J\}} \nu_{rj}(l) \nu_{kl} \nu_{jr} \nu_{j}^n
\]

(31)

Now, for each \( n \geq 1 \), we view each \( \alpha^n \) as a random variable on \( I \times \Omega \). Then \( \{\alpha^n\}_{n=0}^{\infty} \) is a discrete-time stochastic process.

**Lemma 11** Assume that the dynamical system \( \mathbb{D} \) is Markov conditionally independent in types. Then, \( \{\alpha^n\}_{n=0}^{\infty} \) is also a Markov chain with transition matrix \( z^n \) at time \( n - 1 \).

**Proof.** We can compute the transition matrix of \( \{\alpha^n\}_{n=0}^{\infty} \) at time \( n - 1 \) as follows. For any \( k, l \in S \), we have

\[
(\lambda \boxtimes P)(\alpha^n = l, \alpha^{n-1} = k) = \int_I P(\alpha_i^n = l \mid \alpha_i^{n-1} = k)P(\alpha_i^{n-1} = k) \ d\lambda(i)
\]

\[
= \int_I z_{kl}^n \ d\lambda(i)
\]

\[
= z_{kl}^n \cdot (\lambda \boxtimes P)(\alpha^{n-1} = k),
\]

(32)

which implies that \( (\lambda \boxtimes P)(\alpha^n = l \mid \alpha^{n-1} = k) = z_{kl}^n \).

Next, for any \( n \geq 1 \), and for any \( (a^0, \ldots, a^{n-2}) \in S^{n-1} \), we have

\[
(\lambda \boxtimes P)(\alpha^n = l, \alpha^{n-2} = (a^0, \ldots, a^{n-2}), \alpha^{n-1} = k, \alpha^n = l) = \int_I P(\alpha_i^n = l \mid \alpha_i^{n-1} = k)P(\alpha_i^{n-1} = k) \ d\lambda(i)
\]

\[
= \int_I P(\alpha_i^n = l \mid \alpha_i^{n-1} = k)P((\alpha_i^0, \ldots, \alpha_i^{n-2}) = (a^0, \ldots, a^{n-2}), \alpha_i^{n-1} = k, \alpha_i^n = l) \ d\lambda(i)
\]

\[
= \int_I P(\alpha_i^n = l \mid \alpha_i^{n-1} = k)P((\alpha_i^0, \ldots, \alpha_i^{n-2}) = (a^0, \ldots, a^{n-2}), \alpha_i^{n-1} = k) \ d\lambda(i)
\]

\[
= z_{kl}^n \cdot (\lambda \boxtimes P)((a^0, \ldots, a^{n-2}) = (a^0, \ldots, a^{n-2}), \alpha^{n-1} = k),
\]

(33)
which implies that \((\lambda \otimes P)(\alpha^n = l \mid (\alpha^0, \ldots, \alpha^{n-2}) = (a^0, \ldots, a^{n-2}), \alpha^{n-1} = k) = z^n_{kl}\). Hence the discrete-time process \(\{\alpha^n\}_{n=0}^{\infty}\) is indeed a Markov chain with transition matrix \(z^n\) at time \(n - 1\).

To prove that the agents’ type processes are essentially pairwise independent in Lemma 13 below, we need the following elementary lemma.

**Lemma 12** Let \(\phi_m\) be a random variable from \((\Omega, \mathcal{F}, P)\) to a finite space \(A_m\), for \(m = 1, 2, 3, 4\). If the random variables \(\phi_3\) and \(\phi_4\) are independent, and if, for all \(a_1 \in A_1\) and \(a_2 \in A_2\),

\[
P(\phi_1 = a_1, \phi_2 = a_2 \mid \phi_3, \phi_4) = P(\phi_1 = a_1 \mid \phi_3)P(\phi_2 = a_2 \mid \phi_4),
\]

then the two pairs of random variables \((\phi_1, \phi_3)\) and \((\phi_2, \phi_4)\) are independent.

**Proof.** For any \(a_m \in A_m\), \(m = 1, 2, 3, 4\), we have

\[
P(\phi_1 = a_1, \phi_2 = a_2, \phi_3 = a_3, \phi_4 = a_4) \]
\[
= P(\phi_1 = a_1, \phi_2 = a_2 \mid \phi_3 = a_3, \phi_4 = a_4)P(\phi_3 = a_3, \phi_4 = a_4) \]
\[
= P(\phi_1 = a_1 \mid \phi_3 = a_3)P(\phi_2 = a_2 \mid \phi_4 = a_4)P(\phi_3 = a_3)P(\phi_4 = a_4) \]
\[
= P(\phi_1 = a_1, \phi_3 = a_3)P(\phi_2 = a_2, \phi_4 = a_4). \quad (35)
\]

Hence, the pairs \((\phi_1, \phi_3)\) and \((\phi_2, \phi_4)\) are independent.

The following lemma is useful for applying the exact law of large numbers in Corollary 2 to Markov chains.

**Lemma 13** Assume that the dynamical system \(\mathbb{D}\) is Markov conditionally independent in types. Then, the Markov chains \(\{\alpha_i^n\}_{n=0}^{\infty}, i \in I\), are essentially pairwise independent. In addition, the processes \(h^n\) and \(g^n\) are also essentially pairwise independent for each \(n \geq 1\).

**Proof.** Let \(E\) be the set of all \((i, j) \in I \times I\) such that equations (??), (??) and (??) hold for all \(n \geq 1\). Then, by grouping countably many null sets together, we obtain that for \(\lambda\)-almost all \(i \in I\), \(\lambda\)-almost all \(j \in I\), \((i, j) \in E\), i.e., for \(\lambda\)-almost all \(i \in I\), \(\lambda(E_i) = \lambda(\{j \in I : (i, j) \in E\}) = 1\).

We can use induction to prove that for any fixed \((i, j) \in E, (\alpha_i^0, \ldots, \alpha_i^n)\) and \((\alpha_j^0, \ldots, \alpha_j^n)\) are independent, so are the pairs \(h_i^n\) and \(h_j^n\), and \(g_i^n\) and \(g_j^n\). This is obvious for \(n = 0\). Suppose that it is true for the case \(n - 1\), i.e., \((\alpha_i^0, \ldots, \alpha_i^{n-1})\) and \((\alpha_j^0, \ldots, \alpha_j^{n-1})\) are independent, so are the pairs \(h_i^{n-1}\) and \(h_j^{n-1}\), and \(g_i^{n-1}\) and \(g_j^{n-1}\). Then, the Markov conditional independence condition and Lemma 12 imply that \((\alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n)\) and \((\alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n)\) are independent, so are the pairs \((\alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n, g_i^n)\) and \((\alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n, g_j^n)\), and \((\alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n, g_i^n, \alpha_i^n)\) and \((\alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n, g_j^n, \alpha_j^n)\)
and \((\alpha_0^j, \ldots, \alpha_{n-1}^j, h^n_j, g^n_j, \alpha_n^j)\). Hence, the random vectors \((\alpha_0^i, \ldots, \alpha_n^i)\) and \((\alpha_0^j, \ldots, \alpha_n^j)\) are independent for all \(n \geq 0\), which means that the Markov chains \(\{\alpha_n^i\}_{n=0}^\infty\) and \(\{\alpha_n^j\}_{n=0}^\infty\) are independent. It is also clear that for each \(n \geq 1\), the random variables \(h^n_i\) and \(h^n_j\) are independent, so are \(g^n_i\) and \(g^n_j\). The desired result follows. 

**Proof of Theorem 3:** Properties (1), (2), and (3) of the theorem are shown in Lemmas 9, 10, and 13 respectively.

By the exact law of large numbers in Corollary 2, we know that for \(P\)-almost all \(\omega \in \Omega\), \((\alpha_0^i, \ldots, \alpha_n^i)\) and \((\alpha_0^j, \ldots, \alpha_n^j)\) (viewed as random vectors) have the same distribution for all \(n \geq 1\). Since, as noted in Lemma 11, \(\{\alpha_n^\omega\}_{n=0}^\infty\) is a Markov chain with transition matrix \(z^n\) at time \(n - 1\), so is \(\{\alpha_n^\omega\}_{n=0}^\infty\) for \(P\)-almost all \(\omega \in \Omega\). Thus (4) is shown.

Since the processes \(h^n\) and \(g^n\) are essentially pairwise independent as shown in Lemma 13, the exact law of large numbers in Lemma 1 implies that at time period \(n\), for \(P\)-almost all \(\omega \in \Omega\), the realized cross-sectional distribution after the random mutation, \(p^n(\omega) = \lambda(h^n_\omega)^{-1}\) is the expected cross-sectional distribution \(\bar{p}^n\), and the realized cross-sectional distribution at the end of period \(n\), \(p^n(\omega) = \lambda(\alpha^n_\omega)^{-1}\) is the expected cross-sectional distribution \(\bar{p}^n\). Thus, (5) is shown.

**Proof of Theorem 4:**

Note that \(\Gamma\) is a continuous function from \(\Delta\) to itself. Hence, Brower’s Fixed Point Theorem implies that \(\Gamma\) has a fixed point \(p^*\). In this case, \(\bar{p}^n = p^*\), \(z^n_{kl} = z^1_{kl}\) for all \(n \geq 1\). Hence the Markov chains \(\{\alpha_n^i\}_{n=0}^\infty\) for \(\lambda\)-almost all \(i \in I\), \(\{\alpha_n^\omega\}_{n=0}^\infty\), \(\{\alpha_n^\omega\}_{n=0}^\infty\) for \(P\)-almost all \(\omega \in \Omega\) are time-homogeneous.

**References**


