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Price Operators: Extensions, Potentials, and the Markov Valuation of Securities

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ABSTRACT

This paper provides properties of *price operators*, functions that map the payoff of a contingent claim to its market value as a function of the state of the economy. First we provide conditions for a norm–preserving arbitrage–free extension of an arbitrage–free price operator from the space of actually marketed assets to the space of all possible assets. This can be useful for the characterization of equilibrium in settings of asymmetric information or sequential trade. Then we show, in the multiperiod setting, that the market value of a security may be treated as the potential of its dividend, and show several properties that derive from this characterization. Finally, we demonstrate the existence of an "eigen–probability measure" on the state space under which the mean current value of any security is its discounted mean future payoff. The fixed discount factor is the spectral radius of the valuation operator, the reciprocal of the smallest possible fixed rate of return on any security.

I thank Mark Garman for bringing my attention to the connection between price operators and semigroups, and Kai Lai Chung for teaching me what little I know of the connection between Markov processes and potentials. This is a draft. Comments are invited. Any errors are my own.

Price Operators: Extensions, Potentials, and the Markov Valuation of Securities

1. Introduction

A *price operator* is a function f mapping a vector space M of assets of some sort into a vector space L of market values. We treat N as a vector space of possible assets, $A \subset N$ as the subset of actually traded assets, and $M = span(A)$ as the *marketed subspace*, achieved by linear combinations of traded assets. Typically L is a vector space of functions on a set Ξ representing the state of the economy. For an asset $m \in M$, the function $f(m) \in L$ states the market value of m as a function of the state, $[f(m)](\xi)$ in state ξ. By the usual linearity of market valuation, a price operator is linear. A price operator f is *arbitrage-free* if positive: $f(m)$ is positive whenever m is positive. We will first state conditions under which an arbitrage-free price operator f on the marketed subspace M can be extended to an arbitrage-free price operator on the entire space N of possible assets. This, in turn, permits one to represent f, for example, as a conditional expectation operator. This extension may be useful for such purposes as characterizing price operators as an evolution family or semi-group of operators in a multi-period setting [7], and in generalizing the connection between optimality and absence of arbitrage found by Kreps [13] to a multi-period or rational expectations framework. We will apply the extension to a Markovian setting in the latter part of the paper to show that the market value of a security is the potential of its dividend under the valuation semi-group. Making this connection allows one to apply a large body of results from potential theory to the problem of asset pricing. For example, the Complete Maximum Principle implies that if Security A is worth more than Security B in states of the economy for which B pays dividends, then A is worth more than B in every state. The Resolvent Equation gives a direct connection between the time-rate of preference and security price in a Lucas [14] or Merton [15] style economy. We also see that the reciprocal of the spectral radius of the valuation operator is the smallest possible rate of return under which the current value of a security may be treated as the expected future discounted payoff of the security, under some probability assessments.

The remainder of the paper is divided into two parts. The next section studies the operator extension problem. Section 2.1 reviews several classical results on linear extensions; Section 2.2 presents the basic price operator extension problem; the main extension results are Theorems 3 and 4 of Section 2.3. Section 3 applies the results of Section 2 to the connections between asset pricing in a multi-period setting and potential theory. The direct connection is made in Section 3.1. The role in this connection played by Markov processes is outlined in Section 3.2. Finally, Section 3.3 shows how one can price assets merely by discounted expected payoffs under a spectral radius discount rate and under expectations given by "eigenprobabilities," a kind of steady-state Arrow-Debreu prices. Many readers, particularly those interested mainly in Markovian security valuation, may wish to proceed directly to Section 3.

2. Price Operator Extensions

2.1. Background

We first review a classical extension result commonly known as the Hahn-Banach Theorem. For convenience, we temporarily let L denote the vector space R, the real-line. We let M denote a vector subspace of a vector space N. A map $p : M \to L$ is *sublinear* if $p(x + y) \le p(x) + p(y)$ for all x and y in M, and $p(\alpha x) = \alpha p(x)$ for all $\alpha > 0$ and x in M. A function $F : N \to L$ is an *extension* of a function $f : M \to L$ if $F(x) = f(x)$ for all x in M.

THEOREM (HAHN-BANACH). *Suppose f is an L-valued linear form on M and p is a sublinear L-valued form on N such that $f(x) \le p(x)$ for all x in M. Then f has a linear extension $F : N \to L$ such that $F(x) \le p(x)$ for all x in M.*

As a corollary, any continuous linear functional on a vector subspace of a locally convex space, for example a normed space, has a continuous linear extension to the whole space. If M is ordered by the positive cone N_+ and the original functional $f : M \to L$ is positive on $M_+ = M \cap N_+$, we usually want a positive extension F, meaning $F(x) \ge 0$ for all x in N_+. A simple condition is given by the well-known *Krein-Rutman Theorem:*

THEOREM (KREIN-RUTMAN). *Suppose N is a locally convex space and $M \cap int(N_+)$ is not empty. If f is positive, linear, and continuous on M then f has a positive continuous linear extension $F : M \to L$. If N is normed, the extension is norm-preserving.*

The condition that N_+ has interior is strong, but can be weakened by applying the following definition.

Subspace Positive Intersection Property (SPIP). Suppose M is a vector subspace of an ordered vector space N. Then (M, N) has SPIP if, for each x in N, there exists y in M such that $x + y \geq 0$ if and only if there exists z in M such that $x + z \leq 0$.

LEMMA. *If $M \cap int(N_+) \neq \emptyset$ then (M, N) has SPIP.*

The proof is simple. We can then weaken the Krein–Rutman conditions as follows.

THEOREM (MONOTONE EXTENSION). *Suppose, for any ordered vector space N and vector subspace M of N, that (M, N) has the subspace positive intersection property. Then any positive linear L-valued form on M has a positive linear extension $F : N \to L$.*

A proof may be found in Day [6].

2.2. Operator Extensions

Now we attempt generalizations of the results in the last section for f a linear functional ($L = R$) to f a linear operator mapping a subspace M of a vector space N into a general vector space L. For most applications in a Markovian setting, $L = N$. For example, let N denote a vector space of real-valued functions on a set Ω. A security, in a two-period setting is identified with a function x in N, where $x(\omega)$ denotes the *payoff* of the security in state ω at the next period. If A is the subset of securities actually available for trade and there is free formation of portfolios (linear combinations) of securities, then $M = span(A)$ is the *marketed subspace* of portfolios, a vector subspace. Let Ξ denote the set of "current" states of the world; in a Markovian setting we take $\Xi = \Omega$. The market value $f(x)$ of portfolio x is $[f(x)](\xi)$ in current state $\xi \in \Xi$. We treat $f(x)$, then, as an element of a space L of real-valued functions on Ξ, and f as an operator mapping M into L. By the linearity of market valuation, f is linear. There are (at least) two particular reasons we may want a positive linear extension of f to the full space N of possible securities. First, one may want to draw conclusions similar to Kreps' [13] on the relationship between the extension of price functionals, "viability" (in his language, the existence of optimal choices), and absence of arbitrage. Second, we may wish to obtain a representation of f as a "conditional expectation operator," along the lines of Harrison and Kreps [10]. Duffie and Garman [7], for example, apply the results of this paper to an intertemporal security market setting, showing that lack of arbitrage implies that the valuation operators between successive dates compose and extend to an evolution family of operators, or a semigroup in the stationary case. In a rational expectations economy, the image space L could be viewed as the the space of functions on Ξ measurable with respect to the sigma–algebra generated by the join of all "private information" sigma–algebras.

Of course, in the setting we present, one can always obtain a real–valued linear extension of $[f(\cdot)](\xi)$ separately for each "state" ξ, but there is no reason to believe that the resulting "extension" F is "measurable", in the sense that $[F(x)](\cdot)$ is a measurable function on Ξ, which is a key property for most purposes.

The existence of a positive operator extension of f depends on property of L or N known as *order completeness.* In fact, order completeness is both *necessary and sufficient* under other weak regularity conditions. An ordered vector space L is *order complete* if it has the following property: [Let $A \subset L$ be a subset such that there exists $x \in L$ with $a \leq x$ for all a in A. Then A has a *least upper bound*, a vector $y \in L$ such that $a \leq y$ for all a in A and no other vector $z \leq y$ with this property.]

The *lineal closure* of a subset A of a vector space L is the set of x in L such that there exists y in A with $\{\alpha x + (1-\alpha)y, \alpha \in (0,1]\} \subset A$. A subset A of L is *lineally closed* provided A and the lineal closure of A coincide. Of course, if a subset of a topological vector space is closed, it is lineally closed. (But not necessarily the converse!) We will stipulate in some of the following results that N_+ or L_+ is lineally closed. This is certainly the case for any foreseen application.

A vector space L has the *Hahn–Banach extension property* if the Hahn–Banach Theorem stated in Section 2.1 (for $L = R$) is true for any vector space N and vector subspace $M \subset N$. A vector space L has the *Monotone Extension Property* if it obeys the Monotone Extension Theorem (stated in Section 2.1 for $L = R$.)

THEOREM. *Suppose L is an ordered vector space whose positive cone is lineally closed. Then the following conditions are equivalent.*

(i) *L is order complete*

(ii) *L has the Hahn–Banach Extension property*

(iii) *L has the Monotone extension property.*

This statement is highly simplified from a similar result of Day [6], his Theorem VI, 3.1. and tells us that even under the favorable "subspace positive intersection property" there is no guarantee of a positive linear extension of the market valuation operator unless L is order complete. This is strong. For example, the space $C_b(\Omega)$ of bounded continuous functions on a normal topological space Ω is order complete if and only if Ω is extremally disconnected (each open subset of Ω has open closure). In effect, Ω must be discrete. For $(M, \mathcal{M}, \mu)$ a σ–finite measure space, however, all of the spaces $L^q(\mu)$, $1 \leq q \leq \infty$, are order complete. The σ–finite restriction is weak: $(M, \mathcal{M}, \mu)$ is *σ–finite* provided M is the union of a countable set of its measurable subsets, each of finite measure. In particular, any Euclidean space is σ–finite (Lebesgue measure), as is any finite measure space for example, a probability space. We will be focusing on $L^\infty(\mu)$ since that is a natural space for Markov processes [11]. This is the space of essentially bounded measurable real-valued functions on $(M, \mathcal{M}, \mu)$. The *σ–finite* restriction can be weakened to *localizable*, as defined by Schaefer [18, p. 157].

2.3. Main Extension Results

An ordered vector space L is a *lattice* if every set $\{x, y\}$ of two elements x and y of L has a least upper bound, denoted $x \vee y$. For example, the spaces $C_b(\Omega)$ and $L^q(\mu)$, $\quad 1 \leq q \leq \infty$ described in Section 2.2 are lattices, with $f \vee g \equiv \{max\{f(\omega), g(\omega)\},\ \omega \in \Omega\}$. The space $C^1[0,1]$ of differentiable functions on $[0,1]$ is not a lattice because the least upper bound of two differentiable functions need not be differentiable.

A Banach space L that is a lattice is a *Banach lattice* provided the function $x \mapsto x^+ \equiv x \vee 0$ is continuous. The function spaces $C_b(\Omega)$ (supremum norm) and $L^q(\mu)$ are examples of Banach lattices. A Banach lattice L with the property $\| x \vee y \| = \| x \| \vee \| y \|$ for all x and y in L_+ is an *abstract M-space*, or *AM*–space. If the unit ball of L has a least upper bound, say e, then e is the *unit* of L. An example of an *AM*–space with a unit is $L^\infty(\mu)$; there are many other examples, for instance $C_b(\Omega)$. The unit of both $L^\infty(\mu)$ and $C_b(\Omega)$ is the constant unity function. The following famous result will soon be used:

THEOREM 1. *If L is an AM–space with unit then L is isomorphic with the space $C(K)$ of continuous functions on a compact topological space K. If, in addition, L is order complete, then K is extremally disconnected.*

This result, due more or less to Kakutani [18, pp. 104–108], is the basis of the following result. First we note that a vector subspace of a lattice is a *sublattice* if a lattice under the same ordering.

THEOREM 2 (SCHAEFER). *Suppose M is a closed sublattice of a Banach lattice N, and L is an order complete AM-space with unit. Then any positive linear operator $f : M \to L$ has a norm preserving positive linear extension $F : N \to L$.*

For a multiperiod application, the norm preserving property of this extension is important, and we will strive for it in the following.[1] It is implicit in the statement of Schaefer's result, and a useful property, that every positive linear operator on a Banach lattice into a normed lattice is continuous, or equivalently, of finite norm. See [18, p. 84]. Unfortunately, and this is critical, it is extremely restrictive to assume that the marketed subspace M of the space N of all possible securities is itself a lattice. That is, if portfolios x and y are in M, there is no reason to believe that the portfolios with payoffs $(x \vee y)(\omega) = max\{x(\omega), y(\omega)\}, \quad \omega \in \Omega$, is also marketed. A sufficient condition is that all options and compound options are available for trade [12]. Thus, all of our remaining results are not for M *not* a sublattice.

A vector subspace M of an ordered vector space N *majorizes* N provided, for each $n \in N$ there exists $m \in M$ such that $m \geq n$. For example, if $N = L^\infty(\mu)$, it is necessary and sufficient for M to majorize N that there exists $x \in M$ and a scalar $k > 0$ such that $x \geq k$. Because majorization implies the subspace positive intersection property, the following result is a corollary to the Theorem of Section 2.2, but the proof is illustrative.

PROPOSITION 1. *Suppose L is an order complete vector space with a lineally closed positive cone. If N is an ordered vector space majorized by a vector subspace M, then any positive linear operator $f : M \to L$ has a positive linear extension.*

PROOF: Let $p : N \to L$ denote the function defined by

$$p(x) = inf\{f(y) : x \leq y, \quad y \in M\}, \quad x \in N.$$

[1] We recall that the norm $\| f \|$ of an operator $f : M \to L$ on a normed space $(M, \| \cdot \|_M)$ into a normed space $(L, \| \cdot \|_L)$ is $sup\{\| f(m) \|_L ; \| m \|_M \leq 1\}$.

Then p is sublinear and the Hahn–Banach extension property applies because L is order complete (See Section 2.2). Thus f has a p-dominated linear extension $F : N \to L$. Because, for any $x \in N_+$, $F(-x) \leq p(-x) = 0$, F is a positive linear extension. ∎

There is not a hint of a norm preserving argument to be gotten from this proof. The following result has thus been developed. This result heavily exploits the properties of the Markov setting, $N = L$, and the properties of an order complete AM-space with unit. A special case is $N = L = L^\infty(\mu)$, which is precisely the setting most suited to the security valuation problem in a multiperiod setting.

THEOREM 3. *Suppose L is an order complete AM-space with unit, M is a vector subspace of L, and f is a continuous positive linear operator on M into L. Then f has a positive norm preserving linear extension $F : L \to L$.*

PROOF: First, by Theorem 1 of this section we can treat L as the space $C(K)$ of continuous functions on a compact extremally disconnected topological space K. [The proof proceeds identically without this transformation for the case $L = L^\infty(\mu)$, μ a σ-finite measure, which is the principal application.]

Let $p : L \to L$ denote the sublinear form:

$$p(x) = \| f \| \, x^+, \quad x \in L.$$

[That $p(\alpha x) = \alpha p(x)$ for $\alpha > 0$ is obvious. Since $(x + y)^+ \leq x^+ + y^+$, sublinearity is then trivial.] Now $f(x) \leq p(x)$ for $x \in M$ follows by positivity of f and the definition of p. Thus, since L is order complete, f has a p-dominated linear extension $F : L \to L$ by the theorem of Section 2.2. The extension is positive since, for any $x \in L_+$, $F(-x) \leq p(-x) = 0$. The extension is norm-preserving since, for any $x \in L$,

$$\begin{aligned}
\| F(x) \| &= \| F(x^+) - F(x^-) \| \\
&\leq max\{\| F(x^+) \|, \| F(x^-) \|\} \\
&\leq max\{\| p(x^+) \|, \| p(x^-) \|\} \\
&= max\{\| f \| \| x^+ \|, \| f \| \| x^- \|\} \\
&= \| f \| \; max\, \{\| x^+ \|, \| x^- \|\} \\
&= \| f \| \| x \| .
\end{aligned}$$

This completes the proof. ∎

One of the conditions of Theorem 3 is that f is continuous. We note that this is automatic if M is a closed sublattice, as noted earlier, or under the following condition.

LEMMA 1. *Suppose L is an AM-space with unit majorized by a vector subspace M. If f is a positive linear operator on a vector subspace M of L into a normed ordered vector space Y then f is continuous.*

PROOF: Let $y \in M$ be such that $y \geq e$, where e is the order unit of L. Then, for any $x \in M$ of norm less than or equal to unity, $\| f(x) \| \leq \| f(y) \|$ by positivity of f. ∎

This is actually a special case of a much more general result. See Schaefer [17, p. 230]. Of course a more general result is obtained under the subspace positive interaction property, but the proof is less clear.

LEMMA 2. *Suppose N is a Banach Lattice and M is a vector subspace of L such that (M, N) has the subspace positive intersection property. If f is a positive linear operator on M into a normed vector lattice L, then f is continuous.*

PROOF: By the Theorem of Section 2.2, f has a positive linear extension $F : N \to L$. But any such operator F is continuous since N is a Banach Lattice. Thus f is continuous. ∎

We also have an extension result for the case $N \neq L$. An operator P between normed spaces is *contractive* if continuous and of norm less than or equal to unity. An operator P is a *projection* if the composition $P^2 = P \circ P$ is equal to P. We will make use of the following technical result.

PROPOSITION 2. *Suppose L is an order complete AM-space with unit e and L_0 is a closed vector sublattice of L containing e. Then there exists a positive contractive projection $P : L \to L_0$ whose range is L_0.*

This Proposition is Corollary 2, p. 110 of Schaefer [18]. We then have our second extension result.

THEOREM 4. *Suppose L and N are AM-spaces with unit, the former order complete, and f is a positive linear operator into L on a vector subspace M of N that majorizes N. Then f is continuous and has a positive linear norm preserving extension $F : N \to L$.*

PROOF: As earlier, we can identify L with the space $C(K)$ of continuous functions on a compact extremally disconnected space K. Let $\ell^\infty(K)$ denote the space of bounded sequences on K. Because K is extremally disconnected, we can treat $C(K)$ as a closed vector sublattice of $\ell^\infty(K)$ with the constant unity function as its unit. By the previous proposition, there is a contractive projection $P : \ell^\infty(K) \to C(K)$.

For each $t \in K$, the linear functional $q : M \to \ell^\infty(K)$ defined by

$$q(x) = [f(x)](t), \quad x \in M$$

is positive, and thus by Lemma 1 continuous of norm no greater than $\| f \|$. Because M majorizes N, we know $M \cap int(N_+)$ is not empty. Thus, by the usual Krein–Reitman Theorem (Section 2.1), q has a norm-preserving positive linear extension $Q_t : N \to R$. The operator $Q : N \to \ell^\infty(K)$ defined by

$$[Qx](t) = Q_t x, \quad t \in K,$$

is positive, linear, and of norm $\| f \|$. The composition $F = P \circ Q$ is thus a positive linear norm preserving extension of f. ∎

The conditions on N can be weakened, it seems. There is then a possible result on strictly positive extensions of strictly positive operators, following the ideas in Kreps [13] and Duffie–Huang [8].

3. Prices and Potentials

We now apply the extension results of the last section to show that the market value of a security in a multiperiod setting may be treated as the potential of its dividend under an extension of the single-period valuation operator. There may be a folklore concerning this result, although I am not aware of a reference. At least in the context of a deterministic growth economy, the idea might be fairly common knowledge. Aside from the intuition afforded by connecting prices with the physical phenomenon of potentials, there is a body of results from potential theory that may be of interest to asset pricing theorists. We will take a small sample. A major post-war occupation of probabilists has been the drawing of parallels between Markov processes and potential theory. This is evident, for example, in the work of Blumenthal, Chung, Doob, Dynkin, Getoor, Hunt, and Meyer, to name a few. Of course it is this parallel, combined with an assumption of Markovian uncertainty in market models such as those of Merton [15], Lucas [14], Brock [3,4], Breeden [2], and Cox, Ingersoll, and Ross [5], that creates the link between potentials and prices shown here. We will ignore probability theory for the moment, however, and illustrate that link directly. Then we show several applications of potential theory to asset pricing. Finally, we bring in Markov properties.

3.1. The Price–Potential Equivalence

Let L be a vector space of "*assets*". A *security* is identified with its dividend, a vector d in L. Typically, L is a space of real-valued functions on a "state" space Z, and $d(z)$ is the dividend paid in state $z \in Z$. Similarly, the market value of a security is a vector p in L. In the framework suggested, $p(z)$ is the "price" of the security in state z. We call (p, d) a *price–dividend pair.* Although there is no formal requirement to do so, we will imagine an infinite-horizon setting in which the market value p and dividend d of a security are the same functions of the state at all times. Of course, by adding time to the state space, we can convert to this interpretation even in time-dependent settings. For simplicity, we take the convention that market values are pre-dividend, so that $p(z) - d(z)$ is the value in state z of a claim to p in the following period, which depends of course on the state in the following period. Let $\mathcal{A}$ index the set of all securities and let $M = span\ (\{p_a, a \in \mathcal{A}\})$ denote the *marketed subspace* of L. With post dividend trading, that is, any asset $m \in M$ can be created as a portfolio $\alpha \in R^N$ of N securities in the form $m = \sum_{n=1}^{N} \alpha_n p_n$, where $\{1, \ldots, N\}$ indexes some finite subset of $\mathcal{A}$. Let $V : M \to L$ denote the *valuation operator*, which maps any marketed asset $m \in M$ to the required investment, or $V(m) = \sum_{n=1}^{N} \alpha_n (p_n - d_n)$, where $m = \sum_{n=1}^{N} \alpha_n p_n$. In other words, $[V(m)](z)$ is the market value of asset m in state z. Of course V is a linear operator. Although there is some room for generalization, we take the asset space L to be the space L^∞ of bounded measurable functions on the state space $(Z, \mathcal{Z})$, treating functions equal almost everywhere as identical. We assume the underlying measure space is σ-finite. A Euclidean state space is an example.

The valuation operator V is *arbitrage-free* if positive, that is, if any asset whose payoff is positive in every future state requires a positive investment in every current state. The valuation operator V is *strictly contractive* if $\| V \| < 1$, meaning that the maximum possible payoff of any portfolio is strictly greater in magnitude than the required investment. A sufficient (but far from necessary) condition is the existence of a scalar $\epsilon > 0$ such that, for any price–dividend pair (p, d), we have $p \geq d \geq \epsilon$. If $P : L \to L$ is a positive operator, the *potential operator associated with* P is the operator $G = \sum_{n=0}^{\infty} P^n$, where P^n is the

n-th power of P. [For example, $P^2 \equiv P(Pf)$.] The function Gf is the *potential* of any $f \in L$.

PROPOSITION. *Suppose the market valuation operator V is arbitrage-free and strictly contractive. Then V has a positive strictly contractive linear extension $P : L \to L$. The market value $p \in L$ of any security is the associated potential Gd of its dividend $d \in L$.*

PROOF: Apply Theorem 3 of the previous section for the existence of a positive strictly contractive linear extension $P : L \to L$. Then, for any price–dividend pair (p, d), we have $p - d = Pp$, implying

$$\begin{aligned} p &= Pp + d = P(Pp + d) + d = \cdots \\ &= lim_{N\to\infty}\left(P^N p + \sum_{n=1}^{N} P^n d\right) \\ &= Gd, \end{aligned}$$

since $P^N p \to 0$ in norm by the fact that $\| P \| < 1$. ∎

The proof is nothing more than the usual Neumann series expansion of $(I - P)^{-1}$.

An example of how potential theory may be applied to asset pricing is given by the *Complete Maximum Principle.*

THEOREM (COMPLETE MAXIMUM PRINCIPLE). *Let G denote a bounded potential operator associated with a positive contractive operator P. For any g in L_+, h in L, and positive scalar k, if $k + [Gg](z) \geq [Gh](z)$ for all z such that $h(z) > 0$, then $k + Gg \geq Gh$.*

PROOF: Since G is bounded, the associated kernel is proper in the sense of Meyer [16], p. 173. A positive function $f \in L$ is *excessive* if $Pf \leq f$. We note that $k1_Z + Gg$ is excessive, an easy lemma to prove. Then Theorem T27 of Meyer [16], p. 184, applies. ∎

COROLLARY. *Suppose the market valuation operator is positive and strictly contractive. Let (p, d) and (q, f) be price–dividend pairs in L, with f positive. If k is a positive scalar such that $q(z) + k \geq p(z)$ whenever $d(z) > 0$, then $q + k \geq p$.*

The Corollary states that security A has a greater market value than security B in any state of the economy provided the market value of A is greater than that of B in any state in which the dividend of B is strictly positive. More generally, an excess in value by the indicated constant k is preserved. The connection between prices and potentials also applies in continuous-time under continuity assumptions. For a continuous-time setting in which the price-dividend pair (p, f) is invariant in time, under continuity assumptions we would have the relationship

$$p = \int_0^T V_t f \, dt + V_T p, \quad T \geq 0, \tag{1}$$

where $\{V_t\}$ is the family of operators that map future to current market values. Relation (1) was first suggested by Garman [9]. That is, $V_t : L \to L$ is the operator that assigns a current market value $V_t f$ to an asset that pays $f(z)$ in state z at time t in the future. As argued in Duffie and Garman [7], the family $\{V_t\}$ is a *semigroup*, meaning $V_r = V_t V_s$ whenever $s + t = r$. In that case, we take G to be the potential of the semigroup $\{V_t\}$ of valuation operators, or

$$Gf = \int_0^\infty V_t f dt, \quad f \in L.$$

In order to establish (1), of course, one requires, in addition to the absence of arbitrage, strong continuity conditions that are not required in discrete time.

3.2. Price Operators and Markov Processes

Whether in discrete or continuous time settings, as soon as one has positive contractive price operators, there is also a Markov (or sub-Markov) state process $\hat{X}$ under which the current market value of a security is the total infinite horizon expected dividends of the security. In discrete-time for example, if (p, d) is a price-dividend pair, we will have

$$p(z) = E_z \left[\sum_{t=0}^{\infty} d(\hat{X}_t) \right], \quad z \in Z,$$

where E_z denotes expectation for $\hat{X}_0 = z$. The discounting effect of security pricing is incorporated within the "killing rate" of the sub-Markov process $\hat{X}$. For discrete-time, one can simply define $\hat{X}$ to be the Z-valued process with probability transition function Q on the state space Z defined by $Q_z(B) = [V(1_B)](z)$ for all B in Z, where V is a norm preserving positive linear extension of the valuation operator. That is, $Q_z(B)$ is the conditional probability that $\hat{X}_{t+1} \in B$ given that $\hat{X}_t = z$. [See, for example, Dynkin [1] for details.] In continuous-time, the semi-group of extended valuation operators $\{V_t\}$ is associated in the same way with some Markov process $\hat{X}$ under regularity conditions. See, for example, Dynkin [1], Chapter 2. In that case, $p(z) = E_z\left(\int_0^\infty d(\hat{X}_t)dt\right)$.

We have shown that one can begin with security valuation and obtain a corresponding sub-Markov process $\hat{X}$ under which the market value of a security is the expected total dividends to be paid. A likely equilibrium foundation for this type of *Markov pricing* is an underlying Markov process X. For example, in a discrete-time Markov economy, let P be the sub-Markov transition operator associated with an underlying Z-valued state process X and let $\rho \in (0,1)$ denote the discount rate of the single or representative agent, whose marginal utility for consumption is given by a strictly positive function $v \in L$, or $v(z) = \frac{\partial}{\partial c}u(c(z), z)$, where $c : Z \to R_+$ denotes the aggregate consumption function and $u : R \times Z \to R$ denotes the time and state additive differentiable utility function of the agent. Let $A : L \to L$ denote the operator mapping any $p \in L$ to $vp = \{v(z)p(z), z \in Z\}$. Then the valuation operator is given by $V = A^{-1}\rho PA$. See, for example, Lucas [14], who calls this the *Stochastic Euler Equation.* However V is not generally strictly contractive unless v is bounded away from zero. But ρP is strictly contractive, has some bounded potential operator G_ρ, and the market value of a security with dividend $d \in L$ is $p = A^{-1}G_\rho Ad$.

The same approach applies in continuous time. Let $\{P_t\}$ denote the semigroup of the underlying Z-valued Markov process X. Utility is given by

$$U(c) = E\left(\int_0^\infty e^{-\rho t}u(c_t, X_t)dt\right).$$

Let G_ρ denote the *ρ-potential* of $\{P_t\}$, meaning $G_\rho \equiv \int_0^\infty e^{-\rho t}P_t dt$. See Meyer [16], pp. 187–201 and Dynkin [1], Chapter 2. Again the equilibrium market value, relative to the consumption numeraire, of a security paying dividend $d \in L$ in a single or representative agent equilibrium can be nothing other than $p = A^{-1}G_\rho Ad$. More explicitly,

$$p(z) = E_z\left[\int_0^\infty e^{-\rho t}v(X_t)d(X_t)dt\right],$$

where E_z denotes expectation for starting point $X_0 = z \in Z$. If A^{-1} is bounded, that is, if marginal utility for aggregate consumption is bounded away from zero across states, then $A^{-1}G_\rho A$ is bounded and we can "design" a new sub-Markov process $\hat{X}$ with the same state space Z under which the value of a security paying dividend $d \in L$ is merely $kE(\int_0^\infty d(\hat{X}_t)dt)$, where k is a scaling constant which we can take to be $\| A^{-1} \| \| A \|$. This

$e^{-\rho t}$

is the sub–Markov process $\hat{X}$ defined by the contraction semi–group $\{\frac{1}{k}A^{-1}P_tA\}$. That this is indeed a contraction semi-group can be checked from the definition given by Dynkin [1, p.22]. That there corresponds a sub–Markov transition function can be confirmed from Theorem 2.1 (p.51) of Dynkin [1], provided $\{P_t\}$ is itself well–behaved in the sense of that theorem. Finally, the existence of a corresponding sub-Markov process $\hat{X}$ is given by Theorem 3.2, p. 85 of Dynkin [1], again assuming the underlying process X satisfies minimal regularity conditions.

For any scalars $\rho > 0$ and $\gamma > 0$, we have the *resolvent equation* (for continuous–time):

$$G_\rho = G_\gamma + (\gamma - \rho)G_\gamma G_\rho.$$

This allows us to deduce, for instance, how the rate of time discount in the preferences of the agent affects the market value of securities. For example, if the agent's discount rate changes from γ to ρ, the value of a security claiming dividend d changes from $p = A^{-1}G_\gamma Ad$ to $\hat{p} = p + (\gamma - \rho)A^{-1}G_\gamma G_\rho Ad$. The corresponding discrete–time resolvent equation can be deduced from Meyer [16], p. 201. We could also think of ρ as the "killing rate" of the underlying Markov process. The probability of survival of the agent over an interval $[0, T]$ is then $e^{-\rho T}$.

3.3. Eigen–probability–prices

The curious title of this section comes from a connection between the *spectral radius* of an (extended) valuation operator $V : L \to L$ and financial rates of return. The relation $\lambda f = Vf$, for some scalar $\lambda \neq 0$, and non–zero f in L means that f is an *eigenfunction* and λ is an *eigenvalue* for V. (See, for example, the appendix of Schaefer [17].) In our setting, this equation implies that one can "invest" λf and receive f in the next period, for a fixed return of $\frac{1}{\lambda}$, regardless of the current state. The supremum of the set of absolute values of eigenvalues is the *spectral radius* $r(V)$ of V. In finite–dimensional cases, or when V is *compact* [17, p. 266], or even more generally [17, Theorem 3.4], the positivity of V implies that there is an eigenvalue $\lambda = r(V) \geq 0$ with a corresponding eigenfunction $f_\lambda \geq 0$. In the finite–dimensional case, this is known as the Frobenius-Perron Theorem, and if V is strictly positive, then $\lambda = r(V) > 0$ and $f_\lambda > 0$ is strictly positive and unique. We note that the largest eigenvalue corresponds to the smallest possible fixed rate of return. For example, in the Lucas model explained earlier, it is short work to show, if marginal utility for consumption is bounded away from zero, that the maximal eigenvalue is the discount rate ρ and that $f_\rho = \{\frac{1}{v(z)}, z \in Z\}$ is the corresponding eigenfunction. Obviously, a security whose payoff is the reciprocal of marginal utility is "valuable," and a small rate-of-return will induce an agent to hold it.

There is also an interesting relationship between the spectral radius of the valuation operator and "expected returns" under an "eigenmeasure."

THEOREM. *Suppose $V : L^\infty \to L^\infty$ is a positive linear operator. Then there is a probability measure Q on the state space $(Z, \mathcal{Z})$ such that*

$$E^Q(Vf) = r(V)E^Q(f)$$

for all f in L^∞, where the positive scalar $r(V)$ is the spectral radius of V.

PROOF: Let V' denote the adjoint of V. By the Corollary to Appendix Theorem 2.6 of Schaefer [17], V' has a positive eigenvector $\nu \neq 0$ with eigenvalue $r(V)$. Since the dual of L^∞ is the space of measures on $(Z, \mathcal{Z})$, we can treat ν as the product of a strictly positive constant k and a probability measure Q on $(Z, \mathcal{Z})$. By the definition of the adjoint operator and an eigenvector, we have the desired result. ∎

The Theorem says nothing more than that there is a way to assign probabilities to states, the "stationary Arrow-Debreu prices," under which the mean future value of any security is a fixed multiple of its mean current value. This multiple is the supremum of all fixed rates of return on assets. If one insists on valuing a security by taking the expected value of its discounted payoffs, using a fixed discount rate and a fixed set of probability assessments for all securities, then the discount rate $r(V)$ and the probability assessments given by Q seem an obvious choice.

COROLLARY. *Suppose the market valuation operator is arbitrage-free and strictly contractive. Then there exists an i.i.d. Markov state process X under which, for any price-dividend pair (p, d),*

$$E(p(X_1)) = E\left[\sum_{k=1}^{\infty} \delta^{k-1} d(X_k)\right]. \tag{2}$$

The discount factor $\delta < 1$ can be taken to be the spectral radius of an extension of the market valuation operator.

PROOF: Let δ be the spectral radius of a positive norm-preserving extension V of the valuation operator. Let Q denote the probability measure on the state space $(Z, \mathcal{Z})$ defined by the preceding Theorem. Let X be an *i.i.d.* Z-valued Markov process with transition probability Q. By the theorem, for any price-dividend pair (p, d), we have $E^Q(Vp) = \delta E^Q(p)$, which implies

$$\begin{aligned}
E^Q(p-d) &= \delta E^Q(p) \\
&= \delta(E^Q(p-d) + E^Q(d)) \\
&= \delta(E^Q(V_p) + E^Q(d)) \\
&= \delta^2 E^Q(p) + \delta E^Q(d) \\
&= \cdots \\
&= \lim_{N\to\infty} \left[\delta^N E^Q(p) + \sum_{n=1}^{N-1} \delta^n E^Q(d) \right] \\
&= \sum_{n=1}^{\infty} \delta^n E^Q(d),
\end{aligned}$$

since $\delta^N \to 0$. Thus, $E^Q(p) = \sum_{n=0}^{\infty} \delta^n E^Q(d)$. But this is equivalent to (2) since X has transition Q and is i.i.d. The expectation in (2) is given in the usual way, under the measure determined by Q on the product space $Z \times Z \times \cdots$ of sample paths with the σ-algebra generated by all measurable cylinder sets. ∎

As a direct result of this corollary we have, for any price-dividend pair (p, d),

$$E^Q(p) = \frac{1}{1-\delta} E^Q(d), \tag{3}$$

where Q is the transition probability of the constructed Markov process X.

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