Robust Benchmark Design*

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Abstract

Recent scandals over the manipulation of LIBOR and foreign exchange benchmarks have spurred policy discussions of the appropriate design of financial benchmarks. We solve a version of the problem faced by a financial benchmark administrator. Acting as a mechanism designer, the benchmark administrator constructs a "fixing," meaning an estimator of a market value or reference rate based on transactions or other submission data. The data are generated by agents whose profits depend on the realization of the estimator (the benchmark fixing). Agents can misreport, or trade at distorted prices, in order to manipulate the fixing. We characterize the best linear unbiased benchmark fixing.

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1 Introduction

This paper solves a version of the problem faced by a financial benchmark administrator. Acting as a mechanism designer, the benchmark administrator constructs a “fixing,” meaning an estimator of a market value or reference rate based on transactions or other submission data. The data are generated by agents whose profits depend on the realization of the fixing. Agents may misreport, or trade at distorted prices, in order to manipulate the fixing. We characterize optimal transactions weights for benchmark fixings, assuming that the mechanism designer cannot use transfers.

The London Interbank Offered Rate (LIBOR) is arguably the single most important benchmark used in financial markets. Literally millions of different financial contracts, including interest rate swaps, futures, options, variable rate loans, and mortgages, have payments that are contractually dependent on LIBOR. The aggregate outstanding amount of LIBOR-linked contracts has been estimated at over \$300 trillion (Hou and Skeie, 2013). LIBOR and related reference rates such as EURIBOR and TIBOR also serve an important price discovery function, as benchmarks for evaluating investment performance and as indicators of current conditions in credit and interest-rate markets. Given the important role of these interbank offering rate (IBOR) benchmarks in financial markets, reports that they have been systematically manipulated have triggered a regulatory reform process. Similar concerns have recently been raised over manipulation of foreign exchange and commodity benchmarks.[1]

LIBOR reflects the reference rate at which large banks indicate they can borrow short-term wholesale funds on an unsecured basis in the interbank market. Each day, in each major currency and for each of a range of key maturities, LIBOR is currently reported as a trimmed average of the rates reported by a panel of banks to the benchmark administrator. (For details, see, for example, Hou and Skeie, 2013.) Importantly, these bank submissions are for hypothetical loans; they need not be based on actual market transactions.[2] Investigations have revealed purposeful misreporting of these rates. Two rather different incentives for manipulation have been identified. The first, dramatically exacerbated by the recent financial crisis, was to improve market perceptions of a submitting bank's creditworthiness, by understating the rate at which the bank could borrow. (The reports of each individually named bank are revealed to the market.) The second incentive was to profit from LIBOR-linked positions held by the bank. For example, in a typical email uncovered by investigators, a trader at a reporting bank wrote to the LIBOR rate submitter: “For Monday we are very long 3m

[1] See Financial Stability Board (2014).

[2] Each bank submits an answer to the question: “At what rate could you borrow funds, were you to do so by asking for and then accepting inter-bank offers in a reasonable market size just prior to 11 am?”

cash here in NY and would like setting to be as low as possible...thanks".[3] This second form of manipulation, revealed by investigators to have been active over many years, is the main subject of this paper.

Manipulation has been reported across a range of financial market benchmarks, including those for term swap rates (ISDAFix), foreign exchange rates, and commodity prices. Benchmark manipulation has also been a concern in certain goods markets, such as those for pharmaceuticals.[4]

The Financial Stability Board is leading an ongoing global process to overhaul key reference rate and foreign currency benchmarks with a view to improving their robustness to manipulation. A key principle of IOSCO (2013) is that fixings of key benchmarks should be "anchored" in actual market transactions or executable quotations.[5]

This paper has a relatively narrow and theoretical focus. Under restrictive conditions, we focus on the optimal design of a transactions-based weighting scheme. In order to illustrate the problem that we study, we ask the reader to imagine the following abstract situation. An econometrician is choosing an efficient estimator of an unknown parameter. Data are generated by strategic agents whose utilities depend on the realized outcome of the estimator. Thus, the chosen estimator influences the data generating process. This game-theoretic component must be considered in the design of the estimator.

Our model features a benchmark administrator, which is a mechanism designer. The agents are traders, for example banks or individuals within banks or other trading firms. The mechanism designer observes the transactions generated by the anonymous agents. The data generated by each transaction consist only of the price and size (the notional amount of the transaction). Whether or not manipulated, the transactions prices are noisy signals of the fundamental value. For non-manipulated transactions, noise arises from market microstructure effects, as explained by Aït-Sahalia and Yu (2009), and also from asynchronous reporting. For example, the main WMR benchmark for currency exchange rates on a given day are based on transactions that occur within 30 seconds of 4:00pm London time, and based on trades that occur at bids or asks with uncertain price impacts. In an over-the-counter market, moreover, each pair of transacting counterparties is generally unaware of the prices at which other pairs of counterparties are negotiating trades at around the same time. The benchmark administrator is restricted to a benchmark that is linear with respect to transactions prices, with

[3] December 14, 2006, Trader in New York to Submitter; source: Malloch and Mamorsky (2013). Another example: "We have another big fixing tom[orrow] and with the market move I was hoping we could set the 1M and 3M Libors as high as possible".

[4] See for example Gencarelli (2002).

[5] See also Wheatley, 2012; BIS, 2013; Market Participants Group on Reference Rate Reform, 2014. In order to weaken the incentive to under-report funding costs (the first incentive mentioned above) it has been suggested that the bank-level reports be made public with a three-month lag.

weighting coefficients that can depend on the size of the transaction. A common benchmark used in equity and bond markets is the "volume weighted average price" (VWAP), for which the weight on a given transaction price is proportional to the size of the transaction. As we shall see, a VWAP benchmark is approximated, with a large number of transactions, within the family of fixing designs that our modeled benchmark administrator can consider.

Agents have private information about their exposures to the benchmark, and observe private signals of the fundamental value of the benchmark asset. If an agent decides to trade according to the signal received, there is no manipulation. However, the agent can choose to manipulate, effecting a transaction with an artificially inflated or reduced price in order to gain from the associated distortion of the benchmark. Manipulation is assumed to be costly for agents. For example, in order to cause an upward distortion in the benchmark, a trader would need to buy the underlying asset at a price above its fair market value. In order to manipulate the price downward, the agent would need to sell the asset at a price below its true value. Either way, by trading at a distorted price, the agent suffers a loss. The idea here is that the agent has pre-existing contracts (for example swaps) that can be settled at market values linked to the benchmark. On a large pre-existing swap position, for instance, the agent may be able to generate a profit from distorting the price of the underlying benchmark asset that exceeds the cost of the distortion.

This suggests the benefit of avoiding benchmarks whose underlying asset market is thinly traded relative to the market for financial instruments that are contractually linked to the benchmark. In the case of LIBOR, unfortunately, the volume of transactions in the underlying market for interbank loans that determines LIBOR is tiny by comparison with the volume of swap contracts that are contractually settled on LIBOR. This situation magnifies the incentive to manipulate LIBOR. Even the optimal fixing design may admit a significant potential for manipulation.

In addition to choice of the benchmark asset and the fixing design, regulators can implement a range of governance and compliance safeguards, raising the cost of manipulation, consistent with the suggestions of Wheatley (2012) and the IOSCO (2013). Our setting allows for an extra cost for trading at a price away from the fair value, associated with the risk of detection of manipulation by the authorities, and resulting penalties or loss of reputation.

Crucially, we assume that the mechanism designer cannot use transfers. In particular, fines or litigation damages, forms of negative transfer, are not available as a tool of the benchmark administrator. Coulter and Shapiro (2013) propose a model based on a "whistle-blower" mechanism that implements truthful reporting by heavy reliance on both positive and negative transfers. They assume that the private information of any bank is observed by at least two other banks. We avoid making such specialized assumptions.

Our work falls into a growing literature on mechanism design without transfers. The techniques we use are reminiscent of those used to study direct revelation mechanisms and, to some degree, principal-agent models. There are, however, essential differences. Due to the absence of transfers and without access to a single-crossing condition, truthful reporting is usually not implementable. Thus, in particular, we cannot rely on the Revelation Principle, forcing us to develop new solution techniques. The objective function is not typical. Our mechanism designer is minimizing the mean squared error of the estimator (benchmark).

Our main findings are the following. First, even if truthful reporting is implementable, it is not necessarily optimal from the viewpoint of overall efficiency, considering the potential for reporting distortions. Typically, an optimal benchmark will allow for a nonzero probability of manipulation. Second, a robust benchmark must put nearly zero weight on small transactions. This is intuitive, and stems from the fact that it is cheap for agents to make small manipulated transactions. For instance, Scheck and Gross (2013) describe a strategy said to be used by oil traders to manipulate the daily oil price benchmark published by Platt's: "Offer to sell a small amount at a loss to drive down published oil prices, then snap up shiploads at the lower price." Third, although the weight is always nondecreasing in the size of a transaction, the optimal benchmark assigns almost equal weight to all large transactions. This is in order to avoid overweighing manipulated transactions made by agents with particularly strong incentives to manipulate. Under conditions, our main result characterizes the exact shape of the optimal weighing function.

We do not analyze estimators that assign different weights to transactions based on the transactions prices themselves (that is, nonlinear estimators). This extension is an obvious next step. For example, some benchmarks such as LIBOR dampen or eliminate the influence of prices that are outliers.

The remainder of the paper is organized as follows. Section 2 introduces the primitives of the model and the solution concept. Section 3 offers some preliminary analysis in preparation for a treatment of the problem in Section 4. Section 5 concludes and discusses extensions and future research. Most proofs are relegated to appendices.

2 The baseline model

A mechanism designer (benchmark administrator) will estimate an uncertain variable Y, which can be viewed as the "true" market value of an asset. To this end, she designs a benchmark fixing, which is an estimator $\hat{Y}$ that can depend on the transaction data $\left\{(\hat{X}_i,\, \hat{s}_i)\right\}_{i=1}^{n}$ generated by a fixed set $\{1, 2, \ldots, n\}$ of agents. Here, $\hat{X}_i$ is the price and $\hat{s}_i$ is the quantity of the transaction of agent i. The size $\hat{s}_i$ of each transaction is restricted to $[0, \bar{s}]$, a technical simplification that could be motivated as a risk limit imposed by a market regulator or by an agent's available capital. The price $\hat{X}_i$ is a noisy or manipulated signal of Y, in a sense to be defined. Agents are strategic: they have preferences, to be explained, over their respective transactions and over the benchmark $\hat{Y}$. The sensitivity of a given agent's utility to $\hat{Y}$ is known only to that agent. The agents do not collude.

We describe in detail the problem of the benchmark administrator and the agents. Further interpretation of our assumptions is postponed to the end of the section.

2.1 The problem of the benchmark administrator

The benchmark administrator minimizes the mean squared error $\mathbb{E}\left[(Y-\hat{Y})^2\right]$ of the benchmark fixing $\hat{Y}$, which is restricted to a linear estimator of the form

$$\hat{Y} = \sum_{i=1}^{n} f(\hat{s}_i)\hat{X}_i,$$

where $f : [0, \bar{s}] \to \mathbb{R}^+$ is a transaction weighing function to be chosen. In particular, the weight placed on a given transaction depends only on its size, and not on its price or on the identities of the agents. We do not require that the weights sum to one, but we do require the estimator to be unbiased. We will later provide distributional conditions under which unbiasedness is equivalent to the condition that the weights sum to one in expectation, that is,

$$\mathbb{E}\left[\sum_{i=1}^{n} f(\hat{s}_i)\right] = 1.$$

Finally, we impose a technical regularity condition, restricting the chosen weighting function to $\mathcal{F} = \{f : [0, \bar{s}] \to \mathbb{R}^+ : f \text{ is continuous and piecewise } C_a^1\}$, where C_a^1 is the set of functions that have an absolutely continuous derivative.[6] That is, f must be continuous, and there must exist a finite partition of $[0, \bar{s}]$ into intervals such that, on the interiors of each of these intervals, f has an absolutely continuous derivative.

We summarize the problem of the administrator as

$$\inf_{f\in\mathcal{F}} \; \mathbb{E}\left[\left(Y - \sum_{i=1}^{n} f(\hat{s}_i)\hat{X}_i\right)^2\right] \quad \text{subject to} \quad \mathbb{E}\left[\sum_{i=1}^{n} f(\hat{s}_i)\right] = 1. \tag{$\mathcal{P}$}$$

Since $\mathcal{F}$ is an infinite-dimensional space that is not compact, existence of a solution is not a trivial issue.

[6]This is a smaller set than C^1, the space of continuously differentiable functions. Absolute continuity of the derivative is the minimal assumption that allows us to use the second derivative as a control variable in the optimal control problem that we will consider.

2.2 The problem of the agents

Agent i privately observes his type $(R_i,\, X_i,\, s_i)$, where $(X_i,\, s_i)$ is interpreted as the naturally preferred transaction (price and quantity), before considering the incentive to manipulate, and where R_i is the agent's profit exposure to the benchmark. Specifically, the agent's payoff includes a profit component $R_i\hat{Y}$. (Here, R_i can be negative). The agent chooses a transaction $(\hat{X}_i,\, \hat{s}_i)$ that can be different from $(X_i,\, s_i)$. This substitution, however, induces a cost $\gamma\hat{s}_i|X_i - \hat{X}_i|$ that is proportional to the size of the transaction and to the deviation of the price from the specified level X_i, where $\gamma > 0$ is a fixed parameter. We denote $z_i = \hat{X}_i - X_i$ and assume that the manipulation magnitude $|z_i|$ cannot exceed some maximal level $\bar{z}$, which can be thought of as an exogenous detection threshold. In total, given a weighting function f, the payoff of the agent choosing $(\hat{X}_i,\, \hat{s}_i) \neq (X_i,\, s_i)$ is $R_i f(\hat{s}_i) z_i - \gamma\hat{s}_i|z_i|$. Given the additivity of the benchmark across transactions, each agent can ignore the contribution of any of the other transactions chosen by that agent to the distortion-related profit $R_i\hat{Y}$.

Without loss of generality,[7] we normalize to zero the payoff from the truthful reporting choice $(\hat{X}_i,\, \hat{s}_i) = (X_i,\, s_i))$. We can summarize the problem of agent i as

$$\max_{z_i \in [-\bar{z},\, \bar{z}],\, \hat{s}_i \in [0,\, \bar{s}]} \; [R_i f(\hat{s}_i) z_i - \gamma\hat{s}_i|z_i|]\, \mathbf{1}_{\{z_i \neq 0\}}, \tag{$\mathcal{A}$}$$

where we assume that the agent chooses not to manipulate when he is indifferent.[8] The problem has a solution because f is continuous.

2.3 The distribution of data

The types $\{(R_i,\, X_i,\, s_i)\}_{i=1}^{n}$ are drawn in the following way. First, Y is drawn from some distribution with zero mean and a finite variance σ_Y^2. Then, a pair $(\epsilon_i,\, s_i)$ is drawn for every agent, *i.i.d.* across agents and independently of Y, from some joint distribution. We assume that $\mathbb{E}\,(\epsilon_i \,|\, s_i) = 0$, and that $\operatorname{var}(\epsilon_i|\, s_i) = v(s_i)$ for some C^2 function $v : [0,\, \bar{s}] \to \mathbb{R}^{++}$. The marginal distribution of s_i is given by a cumulative distribution function (cdf) G with a continuous density g that is strictly positive on $[0,\, \bar{s}]$. We define X_i to be $X_i = Y + \epsilon_i$. This implies that every agent observes a noisy and unbiased signal of Y. Finally, R_i are *i.i.d.* across agents and independent of everything else, with a distribution given by a cdf $\tilde{H}$ on $[-\bar{R},\, \bar{R}]$. We allow the case of $\bar{R} = \infty$.

We let $\kappa(s_i) = \operatorname{var}(X_i \,|\, s_i)^{-1} = \left(\sigma_2^Y + v(s_i)\right)^{-1}$, the precision of X_i conditional on the size s_i. We assume that this (unmanipulated) price precision is increasing with the size of

[7] In the sense that we can always rescale R_i and γ.

[8] This tie-breaking assumption in the direction favored by the mechanism designer is standard.

the transaction, as implicitly supported by volume-weighted-average-price (VWAP) schemes often used to report representative prices in financial markets.[9] As a technical and natural assumption, we also assume that as the size of a transaction increases the "marginal gain" in precision is diminishing, in the following sense.

Assumption 1. *The function* $\kappa : [0, \bar{s}] \to \mathbb{R}^{++}$ *is nondecreasing and concave.*

This is of course consistent with the simplest case of constant precision.

We next make an assumption with the effect that a manipulated transaction (with maximal allowed manipulation level) has a smaller precision that any unmanipulated transaction.

Assumption 2. $\sigma_\epsilon^2 + \bar{z}^2 > v(0)$, *where* $\sigma_\epsilon^2 = \int_0^{\bar{s}} v(s)g(s)\, ds$.

Finally, we assume that the probability distribution of the incentive R_i is symmetric around zero, and that bigger incentives to manipulate are relatively less likely to occur than smaller incentives, in the following sense.

Assumption 3. *The cdf* $\tilde{H}$ *of* R_i *has a finite variance and a piecewise* C^1 *density* $\tilde{h}$ *that is symmetric around zero and strictly decreasing on* $(0, \bar{R})$.

Examples of distributions that satisfy Assumption 3 include normal and Laplace ("double exponential") distributions. Given the symmetry of $\tilde{H}$, we can define a cdf H on $[0, \bar{R}]$ such that

$$\tilde{H}(R) = \begin{cases} \frac{1}{2} - \frac{1}{2}H(-R) & \text{if } R < 0 \\ \frac{1}{2} + \frac{1}{2}H(R) & \text{if } R \geq 0 \end{cases}.$$

That is, H is the distribution of R_i conditional on $R_i \geq 0$. We let h denote the density of H.

2.4 Comments on assumptions

For tractability, we have restricted attention to estimators that are linear with respect to price, with weights depending only on the sizes of the respective transactions. Our formulation approximates, as a special case, the common volume-weighted-average-price (VWAP) benchmark, which has *relative* size weights

$$\frac{\hat{s}_i}{\sum_{j=1}^{n} \hat{s}_j}.$$

For large n, the VWAP is approximately of the form that we study.

[9] See, for example, Berkowitz *et al.* (1988).

The problem faced by each agent is stylized. We aim to capture some of a manipulator's key incentives. The agent's type $(X_i,\, s_i)$ can be interpreted as the transaction that the agent would make, given current market conditions, to fulfill her usual "legitimate" business needs. For example, such a trade could be the result of a natural speculative, market making, or hedging motive. The assumption that each agent can make only one transaction is not essential, and could be relaxed. Formally, we could justify it by saying that all of the transactions made by a single agent are first aggregated and only then enter the estimator, as for one of the currently proposed approaches for fixing LIBOR.

For simplicity, we have also assumed that the size of a price manipulation is bounded by $\bar{z}$. Alternatively, we could assume that there is a convex cost function $\psi(|z|)$ that represents, for example, an increasing probability of detection. Formally, in our setting $\psi(|z|) = c\mathbf{1}_{\{z \notin [-\bar{z},\, \bar{z}]\}}$ for some large $c > 0$. The results depend mainly, in this regard, on Assumption 2, which essentially guarantees that the manipulation levels chosen by agents are high enough that manipulated transactions are less precise signals of price than unmanipulated transactions.

The cost of manipulation reflects the losses that the agent incurs when trading away from market prices in order to manipulate the fixing. We take a partial-equilibrium approach, avoiding for the purpose of tractability a general endogenous model of unmanipulated trades. The particular functional form (assumed mainly for tractability) can be further justifed by an alternative interpretation of the nature of manipulations. Namely, imagine that agents can submit "shill trades", that is, make fictitious transactions at distorted prices and reimburse each other using side payments. Then $\hat{s}_i|X_i - \hat{X}_i|$ corresponds precisely to the side payment that must be made.[10]

Finally, R_i can be thought of as the position that the agent holds in assets whose prices depend on the level of the benchmark. If the agent holds positions such as options whose market values are nonlinear with respect to the benchmark, we can view R_i as the so-called "delta" (first-order) approximation of the sensitivity to the benchmark of the position's mark-to-market value. The assumption that R_i is symmetric around zero is, in effect, a belief by the benchmark administrator that upward and downward manipulative incentives are similar, other than with respect to their signs.

3 Preliminary analysis

In this section we more formally state the problem of the benchmark administrator, provide some basic properties of an optimal benchmark, and present solutions to some preliminary

[10] This assumption is that the cost is linear in size. We can view γ as a per-dollar cost of using an illegal transfer channel, for example resulting from the possibility of detection and punishment.

cases that provide intuition as well as elements on which to build when solving the general case.

3.1 Solution without manipulation

We first solve the problem assuming that, regardless of the weighting function f, agents do not manipulate. The law of iterated expectation implies that

$$\mathbb{E}\,(\hat{Y}) \;=\; \mathbb{E}\left[\sum_{i=1}^{n} f(s_i)\right]\mathbb{E}\,(Y). \tag{3.1}$$

Thus, $\hat{Y}$ is unbiased if and only if $\mathbb{E}\left[\sum_{i=1}^{n} f(s_i)\right] = 1$. It follows that

$$\mathbb{E}\left[(Y-\hat{Y})^2\right] = -\frac{\sigma_Y^2}{n} + \sum_{i=1}^{n}\mathbb{E}\left[f^2(s_i)(\sigma_Y^2 + v(s_i))\right].$$

Using the symmetry assumption, we can now formulate the problem of the benchmark administrator as

$$\inf_{f\in\mathcal{F}} \int_0^{\bar{s}} f^2(s)(\sigma_Y^2 + v(s))g(s)\,ds \quad \text{subject to} \quad \int_0^{\bar{s}} f(s)g(s)\,ds = \frac{1}{n}.$$

Proposition 1. *Absent manipulation, the weighing function that solves problem $\mathcal{P}$ is given by*

$$f^{\star}(s) = \frac{1}{n}\frac{\eta}{\sigma_Y^2 + v(s)},$$

where

$$\eta = \left(\mathbb{E}\left[\frac{1}{\sigma_Y^2 + v(s_1)}\right]\right)^{-1}.$$

The proof is skipped. This problem can be viewed as special case of generalized least squares (heteroskedastic) estimation. The benchmark administrator's optimal weights are proportional to the precision of each price observation conditional on its size. There is an obvious extension to the case of general covariance structure on the observation "noise" $(\epsilon_1,\ldots,\epsilon)$. The function $f^{\star}(\,\cdot\,)$ is just a rescaling of $\kappa(\,\cdot\,)$, and thus nondecreasing and concave.

3.2 Solution with full manipulation

We turn next to the case in which every agent is assumed to manipulate.[11] Because, assuming manipulation, the size of a transaction is not informative about the variance of ϵ_i, and given

[11] Formally, we assume that agents solve the problem ($\mathcal{A}$) after replacing the indicator $\mathbf{1}_{\{z_i\neq 0\}}$ with 1.

the agent's optimal magnitude of manipulation, $|z_i| = \bar{z}$, we obtain the following result.

Proposition 2. *If agents are certain to manipulate, the optimal weighing function is* $f^\star(s) = 1/n$.

Again, we skip the easy proof. The result simply says that it is optimal to put equal weight on every transaction because, from the viewpoint of the benchmark administrator, every manipulated transaction is an equally precise signal of Y.

3.3 Incentives to manipulate

Having solved the two extreme cases with no manipulation and complete manipulation, we turn back to the problem $\mathcal{A}$ facing an agent. By symmetry, we may concentrate on the event that $R_i \geq 0$. Generically, an agent with type R_i will manipulate if and only if there is some $s \in [0,\, \bar{s}]$ such that $R_i f(s) > \gamma s$. Thus, if an agent with type R_i chooses to manipulate, then all agents with types $R > R_i$ will also manipulate. Similarly, if an agent with type R_i chooses not to manipulate, all agents with types $R < R_i$ will also choose not to manipulate. It follows that with every function f we may associate a unique threshold R_f such that types above R_f manipulate, and types below R_f do not. This easy observation leads to the following result.

Proposition 3. *It is possible to implement an outcome with no manipulations if and only if* $\bar{R} \leq n\gamma\mathbb{E}(s_1)$. *If the benchmark administrator is further constrained to implement non-manipulation, the optimal weighing function is given by* $f^\star(s) = \gamma\bar{R}^{-1}s$ *on* $[0,\, s_0]$ *and*

$$f^\star(s) = \frac{1}{n}\frac{\eta}{\sigma_Y^2 + v(s)}, \quad s \in [s_0,\, \bar{s}],$$

where $\eta = 2\gamma s_0 \bar{R}^{-1}\left(\sigma_Y^2 + v(s_0)\right)$, *and where* s_0 *is chosen to satisfy the constraint*

$$\int_0^{\bar{s}} f^\star(s) g(s)\, ds = \frac{1}{n}.$$

Proof. We sketch the proof. The remaining details are easy. By the above characterization, it is possible to implement truthful reporting if and only if, for every $s \in [0,\, \bar{s}]$, we have $\bar{R}f(s) \leq \gamma s$. Because the administrator is constrained by $\int_0^{\bar{s}} f(s)g(s)\, ds = 1/n$, it is necessary that

$$\frac{1}{n} \leq \frac{\gamma}{\bar{R}} \int_0^{\bar{s}} s g(s)\, ds.$$

This condition is also sufficient. If this condition holds, we can obtain the optimal weighing function by applying basic results from optimal control theory.

□

Although the result implies that implementing truthful reporting may sometimes be possible, $\bar{R}$ may be very large in practice, so the necessary (and sufficient) condition will typically be violated. Indeed, as mentioned in the Introduction, the underlying asset market for LIBOR is thinly traded relative to the market for instruments that determine the incentives to manipulate. The following example shows that it need not be optimal for the benchmark administrator to induce truthful reporting with certainty, even in the case when it is possible.

Example 4. *Suppose that $\gamma = 1$, $n = 10$, $\bar{R} = 5$, $\sigma_Y^2 = 1$, $v(s) = 1$ for every s, $\bar{z} = 1$, g is the uniform density on $[0, 1]$, and $\tilde{h}$ is the symmetric triangular density on $[-5, 5]$.*[12] *Then f is feasible and implements truthful reporting if and only if $f(s) = s/5$. The value of the administrator's objective function is $\frac{2}{75} \approx 0.0267$. Consider an alternative function f_α with $f_\alpha(s) = \alpha s$ for $s \in [0, s_0^\alpha]$, and $f_\alpha(s) = \alpha s_0^\alpha$ for all $s \in [s_0^\alpha, \bar{s}]$, where $\alpha \geq \frac{1}{5}$ and s_0^α is chosen such that the constraint in problem $\mathcal{P}$ holds. In words, f_α is linear up to s_0^α, and flat afterwards. Agents with $|R_i| \geq \frac{1}{\alpha}$ will manipulate, and in this particular case they will always choose $\hat{s}_i = s_0^\alpha$. The value of the administrator's objective function is strictly below $\frac{2}{75}$ for all α between 0.2 and 0.9. Thus, it is optimal to allow manipulation. At the optimal $\alpha^\star$ (equal to approximately 0.3) the objective function is approximately 0.0243, and the unconditional probability of manipulation is about 0.12. Perhaps somewhat surprisingly, $f_{\alpha^\star}$ is not the optimal function overall, as we will see later.*

The main consequence of Proposition 3 from the point of view of further analysis is that we cannot rely on the Revelation Principle. Using the language of Direct Revelation Mechanisms, the reports of agents will typically differ from their true types regardless of the function f that the mechanism designer chooses. In particular, we must determine how the function f influences the mapping from true types into reports made by optimizing agents.

3.4 Basic properties of optimal benchmarks

In this subsection we make a few important steps toward the solution of the benchmark administrator's problem. First, given the assumptions in Section 2, we can see that agents who decide to manipulate will choose $|z_i| = \bar{z}$ with probability one. Moreover, from the viewpoint of the benchmark administrator, $z_i = \bar{z}$ and $z_i = -\bar{z}$ are equally likely, even conditional on $\hat{s}_i$.[13] Therefore, equation (3.1) still holds if we replace s_i by $\hat{s}_i$, that is, forcing

[12] A careful reader will notice that this distribution does not satisfy Assumption 3. This does not matter in this example.

[13] Formally, if the function f is chosen in such a way that some agents who decide to manipulate are indifferent between choosing different levels of $\hat{s}_i$, we could specify the best responses of agents in

the estimator $\hat{Y}$ to be unbiased is equivalent to the requirement that $\mathbb{E}\left[\sum_{i=1}^{n} f(\hat{s}_i)\right] = 1$. We denote by $\Psi_f(\cdot)$ the cdf of the distribution of the transaction size $\hat{s}_i$ conditional on the transaction being manipulated. This distribution is a consequence of the agent's optimal manipulation in response to f, and will be determined later. From an application of the law of iterated expectations and arguments from subsection 3.3,

$$\mathbb{E}\left[(Y-\hat{Y})^2\right] = \sum_{i=1}^{n} \int_0^{\bar{s}} f^2(\hat{s}_i)\, dQ(\hat{s}_i) - \frac{\sigma_Y^2}{n},$$

where

$$dQ(s) = (\sigma_Y^2 + v(s))H(R_f)g(s)\, ds + (\sigma_X^2 + \bar{z}^2)(1 - H(R_f))\, d\Psi_f(s)$$

and where $\sigma_X^2 = \sigma_Y^2 + \sigma_\epsilon^2$ and $\sigma_\epsilon^2 = \int_0^{\bar{s}} v(s)g(s)\, ds$. (The integral with respect to Ψ_f is a Riemann-Stieltjes integral.) The displayed equation simply states that if $|R_i| \leq R_f$ (which happens with probability $H(R_f)$), then transaction i is unmanipulated, $\hat{s}_i = s_i$, $\hat{X}_i$ has variance $\sigma_Y^2 + v(s_i)$, and $\hat{s}_i$ is distributed according to g. On the other hand, if transaction i is manipulated (which happens with probability $(1 - H(R_f))$), then $\hat{s}_i$ is uninformative about s_i and $\hat{X}_i$ has variance $\sigma_X^2 + \bar{z}^2$ from the viewpoint of the benchmark administrator.

Our main task in this subsection is to determine $\Psi_f(\,\cdot\,)$ for each admissible $f \in \mathcal{F}$. This is complicated by the fact that f need not be well behaved. For example, f is not necessarily differentiable or even concave. The next two lemmas overcome these hurdles.

Lemma 5. *For any function f_0 that is feasible for problem $\mathcal{P}$, there exists a nondecreasing feasible function $\bar{f}$ that yields a (weakly) lower value of the program $\mathcal{P}$.*

Proof. See Appendix A. □

Lemma 6. *For any function f_0 that is feasible for problem $\mathcal{P}$, there exists a concave feasible function $\bar{f}$ that yields a (weakly) lower value of the program $\mathcal{P}$.*

Proof. See Appendix A. □

The proofs are technical and thus relegated to the Appendix but the intuition behind the results is straightforward and instructive. Suppose that a feasible function f_0 is not nondecreasing or not concave. Then we can find an interval $[s_0,\, s_1] \subset [0,\, \bar{s}]$ such that there is no manipulation within this interval. Absent manipulation, however, we saw in Proposition 1 that the optimal weight is proportional to $\kappa(\cdot)$, which by Assumption 1 is nondecreasing and

such a way that $\hat{s}_i$ would be informative of the sign of z_i. Because the probability of an agent being indifferent between multiple levels of $\hat{s}_i$ is zero for every f, we can ignore this issue.

concave. Thus, we can modify f_0 in that interval so as to retain feasibility but improve the value of the program $\mathcal{P}$.

Given Lemmas 5 and 6, from this point we restrict attention without loss of generality to weighting functions in the set

$$\mathcal{F}_c = \{f \in \mathcal{F} : f \text{ is nondecreasing and concave}\}.$$

The concavity of f implies that we can use first-order conditions to solve the agent's manipulation problem. Nevertheless, f is not necessarily differentiable, so we use "subdifferential" calculus.[14] We denote by $\partial f(s_0)$ the subdifferential of f at the point s_0. We have established the following result.

Lemma 7. *For f in $\mathcal{F}_c$, an agent with type R_i manipulates if and only if $|R_i| \geq R_f$, where*

$$R_f = \max\left\{R \in [0,\, \bar{R}] : \, Rf(s) \leq \gamma s, \; s \in [0,\, \bar{s}]\right\}.$$

If agent i manipulates, then she chooses $\hat{s}_i$ if and only if

$$\frac{\gamma}{R_i} \in \partial f(\hat{s}_i).$$

Proof. The first claim follows from the arguments made in subsection 3.3. A function $f \in \mathcal{F}_c$ is subdifferentiable at any point $s \in (0,\, \bar{s})$ because f is concave, and the existence of a subdifferential at 0 and $\bar{s}$ follows from $R_i f(s) \leq \gamma s$, and the fact that f is nondecreasing. Thus $\hat{s}_i$ is a global maximum of $R_i f(s) - \gamma s$ if and only if $0 \in \partial(R_i f(\hat{s}_i) - \gamma \hat{s}_i)$. □

If f is actually differentiable at s, the condition for optimality boils down to the usual first-order condition $R_i f'(s) = \gamma$.

[14] A note on terminology: usually the terms "subderivative" and "subdifferential" refer to convex rather than concave functions. However, we still use these terms to avoid using awkward phrases like "superderivative".

We can now characterize $\Psi_f(\,\cdot\,)$ for $f \in \mathcal{F}_c$. For any $s \in [0,\, \bar{s}]$,

$$\begin{aligned}
\Psi_f(s) &= \mathbb{P}\left(\hat{s}_i \le s \;\middle|\; |R_i| \ge R_f\right) \\
&= \mathbb{P}\left(\partial f(\hat{s}_i) \ge \partial f(s) \;\middle|\; R_i \ge R_f\right) \\
&= \mathbb{P}\left(\frac{\gamma}{R_i} \ge \partial f(s) \;\middle|\; R_i \ge R_f\right) \\
&= \mathbb{P}\left(R_i \le \frac{\gamma}{f'(s^+)} \;\middle|\; R_i \ge R_f\right) \\
&= \frac{H\left(\frac{\gamma}{f'(s^+)}\right) - H(R_f)}{1 - H(R_f)},
\end{aligned}$$

where "$\ge$" applied to sets should be understood as the strong set order, and where $f'(s^+)$ denotes the right derivative of f at s. (We define $f'(\bar{s}^+)$ to be 0.) Because the right derivative of a concave function is a right-continuous and nonincreasing function, $\Psi_f(\,\cdot\,)$ is a well defined cdf. We note that discontinuities in f' correspond to atoms in the distribution of manipulated transaction sizes.

We are ready to restate the problem of the benchmark administrator as an optimal control problem. For clarity of exposition, we henceforth assume that $\bar{R} = +\infty$, so that implementing truthful reporting is never possible.[15] Using an approach familiar from principal-agent models, we address the best way, given a target R, for the administrator to implement an outcome in which exactly those types with $|R_i| \ge R$ choose to manipulate. The problem of the administrator is now

$$\inf_{f \in \mathcal{F}_c} \quad \int_0^{\bar{s}} f^2(s) \left[(\sigma_Y^2 + v(s))H(R)\, dG(s) + (\sigma_X^2 + \bar{z}^2)\, dH\left(\frac{\gamma}{f'(s^+)}\right)\right] \tag{$\mathcal{P}(R)$}$$

$$\text{subject to } f(0) = 0,\; f'(0^+) = \frac{\gamma}{R},$$

$$\int_0^{\bar{s}} f(s) \left[H(R)\, dG(s) + dH\left(\frac{\gamma}{f'(s^+)}\right)\right] = \frac{1}{n}. \tag{3.2}$$

The three constraints $f(0) = 0$, $f'(0^+) = \gamma/R$, and $f \in \mathcal{F}_c$ together guarantee that

$$f(s) \le \frac{\gamma}{R} s, \quad s \in [0,\, \bar{s}].$$

The necessity of $f(0) = 0$ is obvious, and $f'(0^+) = \frac{\gamma}{R}$ is necessary to implement an outcome in which exactly those types with $|R_i| \ge R$ choose to manipulate.[16] Solving $\mathcal{P}(R)$ is an

[15] This is practically without loss of generality because we can specify the cdf H to put arbitrary small probability mass on R_i above some finite $\bar{R}$.

[16] Note the emphasis on the word "exactly". It is possible that the best way to implement an outcome in which types below some R do not manipulate is to make sure that types below some

intermediate step toward solving $\mathcal{P}$. In the final step, we determine the optimal threshold R.

4 Solution in the general case

In this section we present a partial solution to problem $\mathcal{P}(R)$. Describing the exact solution in complete generality seems difficult. Therefore, we first present a partial characterization in the general case, and then give a complete characterization for the case of constant noise variance $v(s)$, or uniform distribution of unmanipulated transaction sizes (constant $g(s)$).

As the proofs of results in this section are technical, they are relegated to Appendix B. Our approach to solving problem $\mathcal{P}(R)$ for a fixed R is that of optimal control theory. This optimal control problem has three state variables: the value of the function f, the value of the first derivative f', and an auxiliary state variable corresponding to the isoperimetric constraint (3.2). The second derivative of f serves as a control variable. Because f is only piecewise C^1, we need to allow for jumps in the state variable f'. Thus, we have additional controls that determine the points at which f' jumps and the magnitudes of the jumps. We apply a general version of the Maximum Principle that allows for jumps in the state variables. Because the Hamiltonian for problem $\mathcal{P}(R)$ is not convex, we are unable to apply any of the standard sufficiency results from optimal control theory. We thus prove existence of a solution directly from Weierstrass' Theorem. To this end, we approximate the non-compact space $\mathcal{F}_c$ by an increasing sequence of smaller compact spaces, and show that a corresponding sequence of solutions converges to a function that solves the problem in the original space.

The following second-order ordinary differential equation (ODE), parametrized by (R, η), will play a key role in the results. Consider

$$f''(s) = \frac{\left[2f(s)\left(\sigma_Y^2 + v(s)\right) - \eta\right] H(R)g(s) - 2\gamma\left(\sigma_X^2 + \bar{z}^2\right) h\left(\frac{\gamma}{f'(s)}\right)}{\left[2f(s)\left(\sigma_X^2 + \bar{z}^2\right) - \eta\right]\left(-h'\left(\frac{\gamma}{f'(s)}\right)\right)\frac{\gamma^2}{(f'(s))^3}}. \tag{$\diamondsuit$}$$

Further, we define

$$\hat{R} = \max\left\{R \geq 0 : \frac{\gamma H(R)}{R}\mathbb{E}\left(s_1\right) + \frac{\gamma(1 - H(R))}{R}\bar{s} \geq \frac{1}{n}\right\}$$

as the highest level of R that is implementable (in the sense that there exists a function f that is feasible for $\mathcal{P}(R)$).

$R^\star > R$ do not manipulate. Since we optimize over all R in the final step, this formulation is without loss of generality.

Proposition 8. *For any fixed $R \in (0, \hat{R})$, the solution $f^\star$ to problem $\mathcal{P}(R)$ exists and has the following properties. There exists $s_0 \in (0, \bar{s})$ and $\eta > 0$ such that $f^\star(s) = \frac{\gamma}{R}s$ for all $s \in [0, s_0]$; in the interval $(s_0, \bar{s}]$, $f^\star$ is C^1 and there exists a partition of $(s_0, \bar{s}]$ into a finite number of intervals such that $f^\star$ is either affine or satisfies the ODE ($\diamondsuit$). Moreover, in the last of these intervals, $f^\star$ satisfies the ODE ($\diamondsuit$) and $\lim_{s\nearrow\bar{s}} f'(s) = 0$.*

Proof. See Appendix B. □

This characterization is not sharp because, in full generality (that is, without some restriction on the function g or v), the term $\left[2f(s)\left(\sigma_Y^2 + v(s)\right) - \eta\right] H(R)g(s)$ is difficult to control. In particular, we cannot guarantee that the solution to the ODE ($\diamondsuit$) is concave. An inspection of the proof shows that the affine parts may only appear in places where the solution to the ODE ($\diamondsuit$) fails to be concave. Nevertheless, the optimal function is continuously differentiable in $(s_0, \bar{s})$, that is, the connections between the affine and strictly concave parts must be smooth. With some additional restrictions, we can rule out the affine parts and pin down the optimal function precisely.

Proposition 9. *Fix $R \in (0, \hat{R})$, and suppose that either (i) $v(s)$ is constant, or (ii) $g(s)$ is constant. Then the solution $f^\star$ to the problem $\mathcal{P}(R)$ is given by $f^\star(s) = \frac{\gamma}{R}s$ for $s \in [0, s_0]$, and by the solution to the ODE ($\diamondsuit$) on $[s_0, \bar{s}]$ with initial conditions $f(s_0) = \frac{\gamma s_0}{R}$ and $f'(s_0) = \frac{\gamma}{R}$, where the constant η is chosen so as to satisfy the terminal condition $(f^\star)'(\bar{s}) = 0$, and s_0 is chosen so that the constraint (3.2) holds. In particular, $f^\star$ is continuously differentiable on the entire domain.*

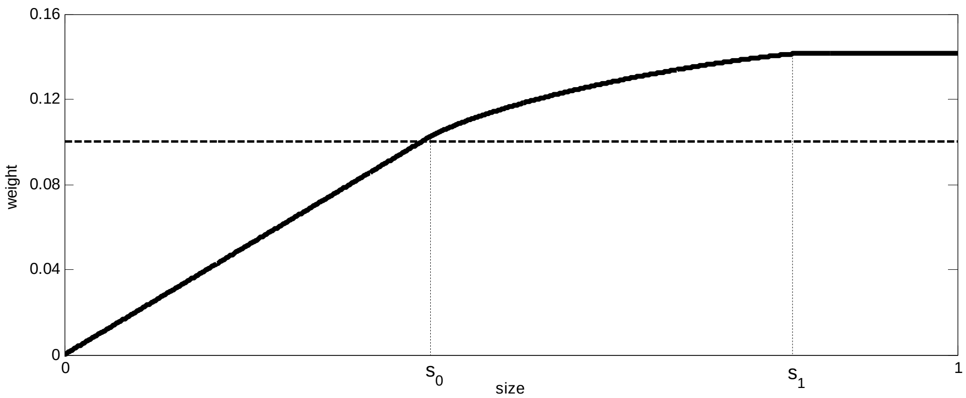
Proof. See Appendix B. □

Propositions 8 and 9 indicate that the optimal benchmark provides an incentive for "smoothing out" manipulations, preventing them from "bunching" around a given transaction size. This is perhaps somewhat surprising. The manipulated transactions have the same precisions, as signals of Y, and yet it is optimal to attach different weights to them. In particular this shows that the functions considered in Example 4 are not optimal. The intuition behind this result is quite clear and depends on the assumption that the cdf H is a (strictly) convex function.[17] Another robust finding is that the optimal weighing func-

[17] Notice that locally the term $f^2(s)(\sigma_Y^2 + v(s))H(R)g(s)$ does not depend on whether there is a jump in f' or not. However, the term $f^2(s)(\sigma_X^2 + \bar{z}^2)\, dH\left(\frac{\gamma}{f'_+(t)}\right)$ is sensitive to jumps in f'. After a transformation using integration by parts, we can show that the problem of choosing f' locally boils down to maximizing a concave functional, so the optimal thing to do is to minimize variation of f' (think about a risk-averse consumer smoothing out consumption).

Fig. 4.1: Optimal weighing function for Example 10

(the dotted line is the optimal solution in the absence of manipulations)



tion becomes flat as the transaction size increases. This is true regardless of the shape of v. Intuitively, assigning too much weight to very large transactions is suboptimal because it induces agents with high incentives to manipulate to choose large transaction sizes (resulting in overweighing such transactions in the estimator).

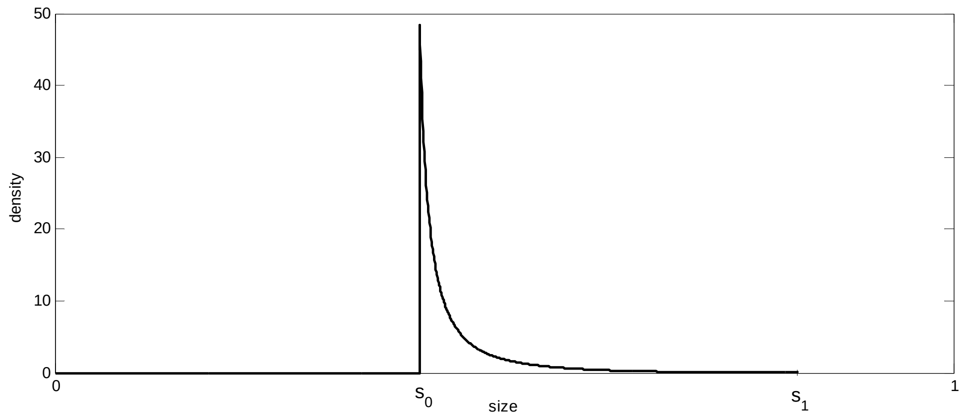
For the case of constant v, the optimal $f^\star$ will often be flat after some threshold transaction size $s_1 < \bar{s}$ (see Example 10). This is consistent with $f^\star$ being the solution to the ODE ($\diamondsuit$) because (as the proof in Appendix B formally demonstrates) we can interpret 0/0 in ODE ($\diamondsuit$) as 0. We then have $2f^\star(s)(\sigma_Y^2 + v) = \eta$ in $[s_1, \bar{s}]$. Since $f^\star$ is C^1, manipulations cease gradually as s approaches s_1 (that is, there is no atom of manipulations at s_1).

Finally, a general feature of an optimal benchmark is that $f^\star(s)$ coincides with $\frac{\gamma}{R}s$ for small transaction sizes. In other words, the constraint that the cutoff type $R_{f^\star}$ prefers to avoid manipulation is binding. As a consequence, robust benchmarks put small weights on small transactions.

Example 10. *To illustrate the above points we consider a numerical example. We take the same parameters as in Example 4, with the exception that* $h(x) = \frac{1}{2}\exp(-\frac{1}{2}x)$ *and* $R = 4$. *The unconditional probability of manipulation is around* 0.135. *The optimal weighing function is depicted in Figure 4.1. The function is smooth* (C^1)*, but the first derivative is changing rapidly close to* $s_0 \approx 0.40$ *and* $s_1 \approx 0.83$. *All of the manipulated transactions are in the interval* $[s_0, s_1]$. *As can be seen in Figure 4.2, manipulations are in fact highly concentrated around* s_0.

We conclude this subsection with a short discussion of the intuition for the ODE ($\diamondsuit$). We

Fig. 4.2: Density of manipulations



can rewrite the ODE as

$$\underbrace{\left[2f(s)\left(\sigma_X^2+\bar{z}^2\right)-\eta\right]\, dH\left(\frac{\gamma}{f'(s)}\right)}_{I_h} + \underbrace{\left[2f(s)\left(\sigma_Y^2+v(s)\right)-\eta\right] H(R)g(s)}_{I_g}$$
$$= \frac{d}{ds}\underbrace{\left[\left(2f(s)\left(\sigma_X^2+\bar{z}^2\right)-\eta\right) h\left(\frac{\gamma}{f'(s)}\right)\frac{\gamma}{f'(s)}\right]}_{I_a}.$$

The term I_h is zero (that is, $2f(s)\left(\sigma_X^2+\bar{z}^2\right)=\eta$) when the weight is chosen optimally from the point of view of manipulated transactions. This factor is multiplied by the density of sizes corresponding to manipulated transactions. In this case the term I_a is also zero because

$$h\left(\frac{\gamma}{f'(s)}\right)\frac{\gamma}{f'(s)}=0$$

whenever f is constant. On the other hand, the term I_g is zero (that is, $2f(s)\left(\sigma_Y^2+v(s)\right)=\eta$) when the weight is chosen optimally from the point of view of unmanipulated transactions. This factor is multiplied by the density of sizes corresponding to unmanipulated transactions. Ideally the benchmark administrator would like to set both of the terms I_h and I_g to zero, but this is of course impossible because $\sigma_X^2+\bar{z}^2>\sigma_Y^2+v(s)$. The administrator thus faces a tradeoff. Either she puts insufficient weight on unmanipulated transactions, which are relatively precise signals of the fundamental value, or she puts too much weight on manipulated transactions, which are relatively noisy signals of the fundamental value Y.

In balancing these two effects, the administrator takes into account the term I_a. Having chosen f, she knows that the types in $[R_f,\infty)$ will manipulate. By controlling f', she can control the sizes of the transactions at which consecutive types R_i in $[R_f, \infty)$ submit manipulations. We note that, roughly speaking, the type $R_i = \gamma/f'(s)$ chooses size s. The term $\gamma/f'(s)$ starts at R_f when $s = 0$, and ends at $+\infty$ when $s = \bar{s}$. The lower is f', the lower is the resulting probability mass of "remaining" manipulated transactions. When f' reaches 0, manipulations stop. In short, the term I_a accounts for the fact that when the benchmark administrator chooses $f(s)$ at s, she takes into account the effect of the speed with which the slope changes on the remaining mass of manipulated transactions.

4.1 Choosing the optimal R

Having characterized the solution to problem $\mathcal{P}(R)$ for a fixed manipulation threshold R, one can solve the original problem $\mathcal{P}$ by choosing an optimal threshold $R^\star$. This simply involves computing the optimal weighing function $f^\star$ for every $R < \hat{R}$, evaluating the objective function, and finding the maximum over all R, achieved at some $R^\star$ (the optimum is attained, by Berge's Theorem). This can be done numerically. The tradeoff is clear and the intuition doesn't go beyond that developed in Example 4. We have no theoretical results to offer at this time but it should be clear from the findings of the previous subsection that one should not expect to obtain $R^\star$ analytically. In particular, even with additional assumptions, we are not able to solve the ODE ($\diamondsuit$) analytically.

Example 11. *With the parametric assumptions of Example 10, it turns out that* $R^\star \approx 2.4$ *achieves the minimum for the objective* $\mathcal{P}$*. Figure 4.3 presents the optimal weighing function for* $R = 1$*,* $R = 2.4$*,* $R = 4$ *and* $R = 5$*. The ex-ante probabilities of manipulations under these target levels are* 0.61, 0.30, 0.14 *and* 0.08*, respectively. The corresponding densities of sizes of manipulated transactions are depicted in Figure 4.4.*

5 Conclusions and future research

We developed a simple model for the design of robust benchmark fixings in settings for which incentives to manipulate the benchmark arise from a profit motive related to investment positions that are valued according to the benchmark. We have restricted attention to fixings that are given by a size-dependent weighted average price, an important limitation. We characterize the optimal weight for each size of transaction. We showed that an optimal benchmark fixing must in general allow some amount of manipulation, puts very small weight on small transactions, and nearly equal weight on large transactions.

Fig. 4.3: Optimal weighing functions for Example 11

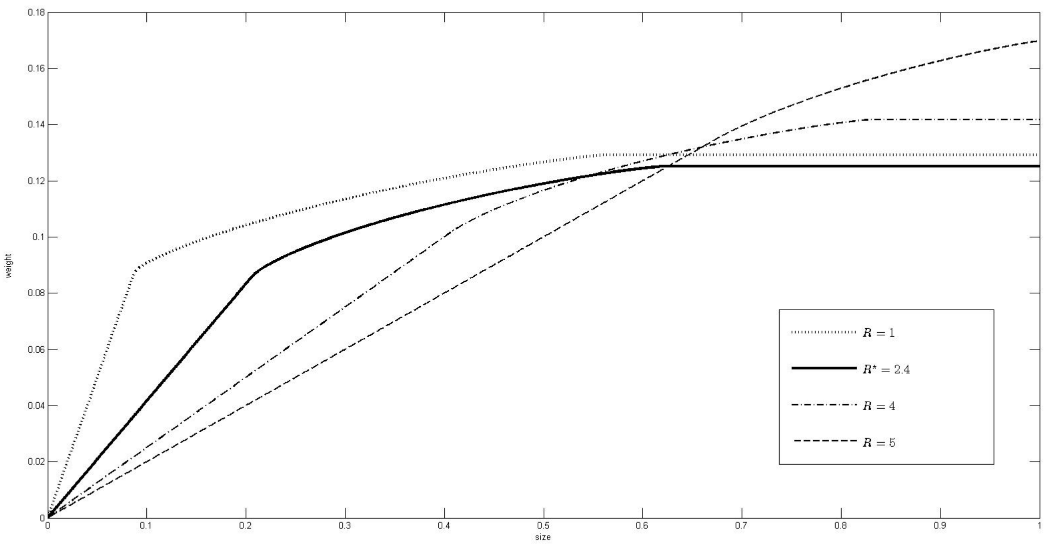
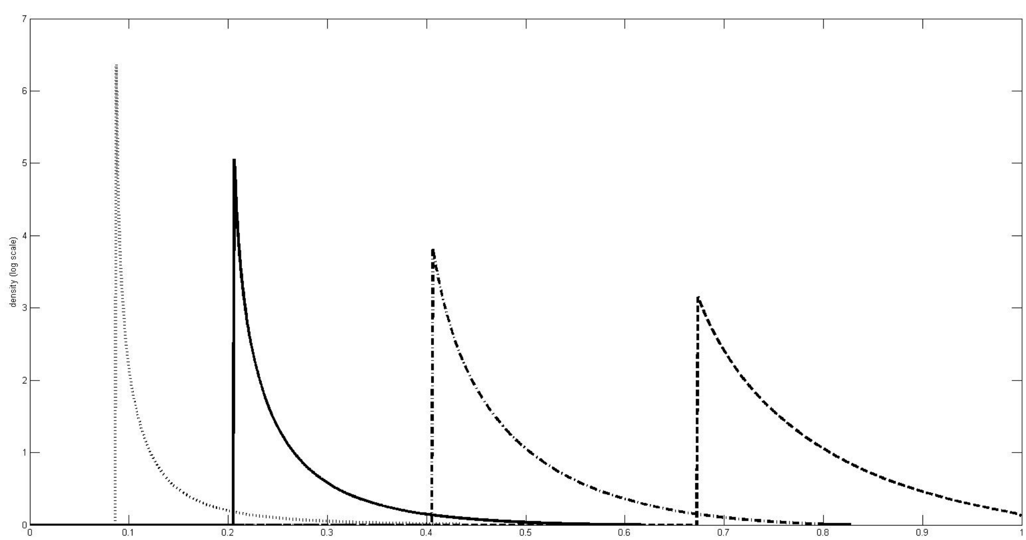


Fig. 4.4: Densities of manipulations (log scale)



The most important next step would be to allow weights that depend on the prices of transactions. The easiest example of this would be too exclude "outlier" prices. A slightly more sophisticated method would be to compute, for every transaction, the posterior probability that the transaction is manipulated, and to use this information to construct weights.

We have ignored collusion throughout.

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A Proofs for section 3

Proof of Lemma 5

Take a feasible function f_0 and suppose it is not nondecreasing (if it is, just set $\bar{f} = f_0$). Then there exist s_0 and s_1 such that $s_0 < s_1$, but $f_0(s_0) > f_0(s_1)$. Without loss of generality we can assume (making the interval smaller if necessary and using continuity of f_0) that f_0 is strictly decreasing in $[s_0, s_1]$. This implies that (increasing s_0 slightly if necessary) there are no manipulations in $[s_0, s_1]$, and this will continue to be true for any function f that is is non-increasing in this interval. It is easy to see that we can construct a non-increasing, continuous and piecewise C^2 function $\hat{f}$ on $[s_0, s_1]$ with the following properties: $\hat{f}(s_0) = f_0(s_0)$, $\hat{f}(s_1) = f_0(s_1)$, $\int_{s_0}^{s_1} \hat{f}(s)g(s)\,ds = \int_{s_0}^{s_1} f_0(s)g(s)\,ds$ and there exists $s_2 \in (s_0, s_1)$ such that $\hat{f}(s) < f_0(s)$ for $s \in (s_0, s_2)$ and $\hat{f}(s) > f_0(s)$ for $s \in (s_2, s_1)$. We then define

$$\bar{f}(s) = \begin{cases} \hat{f}(s) & if\ s \in [s_0, s_1] \\ f_0(s) & \text{otherwise.} \end{cases}$$

By construction, $\bar{f}$ is feasible (it is continuous, piecewise C^2, and satisfies the constraint that guarantees an unbiased estimator). The difference in the value of the administrator's objective function $\mathcal{P}$ under $\bar{f}$ and f_0 is (using the fact that there are no manipulation in $[s_0, s_1]$ under $\bar{f}$),

$$\int_{s_0}^{s_1} \left(\bar{f}^2(s) - f_0^2(s)\right)(\sigma_Y^2 + v^2(s))g(s)\,ds = \int_{s_0}^{s_1} \left(\bar{f}(s) - f_0(s)\right)\phi(s)g(s)\,ds,$$

where $\phi(s) \equiv \left(\bar{f}(s) + f_0(s)\right)(\sigma_Y^2 + v(s))$ is a strictly decreasing function (we used Assumption 1). By the mean value theorem, there exists $x \in (s_0, s_1)$ such that

$$\int_{s_0}^{s_1} \left(\bar{f}(s) - f_0(s)\right)\phi(s)g(s)\,ds = \phi(s_0)\int_{s_0}^{x} \left(\bar{f}(s) - f_0(s)\right)g(s)\,ds.$$

But $\int_{s_0}^{x} \left(\bar{f}(s) - f_0(s)\right)g(s)\,ds < 0$ because the integrand is (strictly) negative on $[s_0, s_2)$, (strictly) positive on $(s_2, s_1]$ and because $\int_{s_0}^{s_1} \left(\bar{f}(s) - f_0(s)\right)g(s)\,ds = 0$.

Therefore, $\bar{f}$ is feasible and yields a smaller value of the objective function than does f_0.

Proof of Lemma 6

Take a feasible f_0 and suppose it is not concave (if it is, set $\bar{f}(s) = f_0(s)$). This means that we can find an affine function $\varphi(s) = a + bs$ and an interval $[s_0, s_1]$ such that $\varphi(s_0) = f_0(s_0)$, $\varphi(s_1) = f_0(s_1)$ and $\varphi(s) \geq f_0(s)$ for all $s \in (s_0, s_1)$, with a strict inequality for at least some

$\tilde{s} \in (s_0, s_1)$. We first prove that there can be no manipulations[18] in (s_0, s_1). It's enough to show that for generic R and for all $s \in (s_0, s_1)$,

$$Rf_0(s) - \gamma s < \max\{Rf_0(s_0) - \gamma s_0,\, Rf_0(s_1) - \gamma s_1\}.$$

We have

$$\max\{Rf_0(s_0) - \gamma s_0,\, Rf_0(s_1) - \gamma s_1\} = \begin{cases} Ra + (Rb - \gamma)\, s_1 & \text{if } Rb > \gamma \\ Ra + (Rb - \gamma)\, s_0 & \text{if } Rb < \gamma. \end{cases}$$

Take the case $Rb > \gamma$. Then we have, for all $s \in (s_0, s_1)$,

$$Rf_0(s) - \gamma s \leq Ra + (Rb - \gamma)\, s < Ra + (Rb - \gamma)\, s_1.$$

Similarly, for $Rb < \gamma$ and for all $s \in (s_0, s_1)$,

$$Rf_0(s) - \gamma s \leq Ra + (Rb - \gamma)\, s < Ra + (Rb - \gamma)\, s_0.$$

This conclusion depended only on the fact that f_0 lies below the line described by the affine function φ. Thus, if f_0 cannot be improved upon by another feasible function $\bar{f}$, it must be the case that f_0 restricted to the interval $[s_0, s_1]$ arises as a solution to the following optimal control problem:[19]

$$\min_u \int_{s_0}^{s_1} f^2(s)(\sigma_Y^2 + v(s))g(s)\, ds \tag{A.1}$$

subject to

$$\int_{s_0}^{s_1} f(s)g(s)\, ds = \int_{s_0}^{s_1} f_0(s)g(s)\, ds,$$

$$u(s) = f'(s),$$

$$f(s_0) = f_0(s_0),$$

$$f(s_1) = f_0(s_1),$$

$$f(s) \leq \varphi(s).$$

Here, the first derivative plays the role of the control variable, and the value of the function is the state variable. We get rid of the isoperimetric constraint $\int_{s_0}^{s_1} f(s)g(s)\, ds = \int_{s_0}^{s_1} f_0(s)g(s)\, ds$

[18] Strictly speaking, the measure of manipulations is zero.

[19] In the sense that f_0 is the resulting state variable f if u is chosen optimally.

by defining an auxiliary state variable Γ with

$$\Gamma'(s) = f(s)g(s)$$

and

$$\Gamma(s_0) = 0,\, \Gamma(s_1) = \int_{s_0}^{s_1} f_0(s)g(s)\,ds.$$

We can now derive the necessary conditions for a function f to solve the above problem. We formally state the necessary conditions in the form of a Lemma.

Lemma 12. *Let $u^\star(s)$ be an admissible control which solves Problem A.1 above. Let $f(s)$ and $\Gamma(s)$ be the corresponding state variables. Then there exist a constant $\lambda_0 \in \{0, 1\}$, a vector function $\lambda(s) = (\lambda_1(s),\, \lambda_2(s))$ with one-sided limits everywhere, and a non-decreasing function $q(s)$ such that:*

(i) $(\lambda_0,\, \lambda(s),\, q(s_1) - q(s_0)) \neq (0,\, \mathbf{0},\, 0)$ for all s in $[s_0,\, s_1]$.

(ii) $u^\star(s)$ maximizes $\mathcal{H}(f(s),\, u,\, \lambda(s),\, s)$ over all $u \in \mathbb{R}$, and for almost all s, where

$$\mathcal{H}(f(s),\, u,\, \lambda(s),\, s) = -\lambda_0 f^2(s)(\sigma_Y^2 + v(s))g(s) + \lambda_1(s)u + \lambda_2(s)f(s)g(s).$$

(iii) $q(s)$ is constant on any interval on which $\varphi(s) > f(s)$, and is continuous at all $s \in (s_0,\, s_1)$, where $f(s) = \varphi(s)$ and $u^\star(s)$ is discontinuous.

(iv) If we define $\lambda_1^\star(s) = \lambda_1(s) - q(s)$, then $\lambda_1^\star(s)$ and $\lambda_2(s)$ are continuous and have continuous derivatives at all points of continuity of $u^\star(s)$ and $q(s)$, and we have

$$\left(\lambda_1^\star\right)'(s) = \left[2\lambda_0 f(s)(\sigma_Y^2 + v(s)) - \lambda_2(s)\right] g(s) \tag{A.2}$$

$$\lambda_2'(s) = 0. \tag{A.3}$$

Proof. Apply Theorem 2, page 332, of Seierstad and Sydsaeter (1987). □

Suppose that $\lambda_0 = 1$. First, notice that λ_2 must be constant. Indeed, its derivative is equal to 0 almost everywhere, so it is piecewise constant. And since it is continuous by condition (iv), it must be constant. We denote $\eta \equiv \lambda_2(s)$. Second, to satisfy condition (ii), we need $\lambda_1(s) = 0$ almost everywhere. On any interval on which the constraint $f(s) \leq \varphi(s)$ does not bind, we must have, by conditions (iii) and (iv), a constant level of $q(s)$. Hence, at any such s, $\lambda_1'(s) = (\lambda_1^\star)'(s)$, which must be equal to 0 almost everywhere. Thus, on any interval on which the constraint does not bind, an optimal f must coincide with $\frac{\eta}{2(\sigma_Y^2+v(s))}$ for some constant η. By Assumption 1, the function $\kappa(s) \equiv \left((\sigma_Y^2 + v(s)\right)^{-1}$ is concave. This is a contradiction of the fact that f_0 solves the problem A.1. Indeed, since $\frac{\eta}{2(\sigma_Y^2+v(s))}$ is concave, it is impossible that $f_0(s)$ coincides with $\frac{\eta}{2(\sigma_Y^2+v(s))}$ whenever $f_0(s)$ is strictly below $\varphi(s)$ (and this is the case for at least some point, by assumption). At the same time, however, f_0 coincides with the values of an affine function at the ends of the interval $[s_0,\, s_1]$.

Now consider the case $\lambda_0 = 0$. Similar arguments lead to the conclusion that $\lambda_1(s) = 0$ almost everywhere, and $\eta \equiv \lambda_2(s)$. Since the constraint cannot bind everywhere, there must be an interval on which $\lambda_1^\star$ is constant. But because we have $(\lambda_1^\star)'(s) = -\eta g(s)$, this means that $\eta = 0$. Applying Theorem 6 on page 346 from Seierstad and Sydsaeter (1987), which states that $(\lambda_0,\, \lambda(s)) \neq (0,\, \mathbf{0})$ in some open interval contained in $(s_0,\, s_1)$, we get a contradiction.

B Proofs of Propositions 8 and 9

We first prove Proposition 8. We fix some $R < \hat{R}$, which guarantees that the set of functions $f \in \mathcal{F}_c$ that satisfy the constraints of problem $\mathcal{P}(R)$ is non-empty. Given that f must be continuous and piecewise C^1 with an absolutely continuous derivative, we can treat the optimization problem $\mathcal{P}(R)$ as an optimal control problem for which f and f' are state variables, and f'' is the control variable. Moreover, we optimize over a finite set of points $(\tau_j)_{j=1}^k \subset [0,\, \bar{s}]$ at which the first derivative f' may jump down by an amount v_j. Summing up, the problem can be expressed as:

$$\inf_{\substack{-\infty < f'' \leq 0 \\ (\tau_j,\, v_j > 0)_{j=1}^k}} \int_0^{\bar{s}} f^2(s) \left[\left(\sigma_Y^2 + v\right) H(R) dG(s) + \left(\sigma_X^2 + \bar{z}^2\right) dH\left(\frac{\gamma}{f'(s^+)}\right)\right] \tag{B.1}$$

subject to

$$\int_0^{\bar{s}} f(s) \left[H(R) dG(s) + dH\left(\frac{\gamma}{f'(s^+)}\right)\right] = \frac{1}{n}, \tag{B.2}$$

$$f(0) = 0,\ f'(0^+) = \frac{\gamma}{R},$$

$$f'(\tau_j^+) - f'(\tau_j^-) = -v_j,\ \ j = 1,\, 2, \ldots,\, k.$$

We can simplify the objective function by applying integration by parts for the Riemann-

Stieltjes Integral, and, using the fact that f is absolutely continuous, get

$$\int_0^{\bar{s}} f^2(s)\, dH\left(\frac{\gamma}{f'(s^+)}\right) = f^2(\bar{s}) - 2\int_0^{\bar{s}} f(s)f'(s)H\left(\frac{\gamma}{f'(s^+)}\right) ds$$

$$= 2\int_0^{\bar{s}} f(s)f'(s)\left(1 - H\left(\frac{\gamma}{f'(s)}\right)\right) ds.$$

In the last step we replaced the directional derivative by the derivative because f is almost everywhere differentiable (the points at which it is not differentiable do not influence the value of the integral). Therefore, the objective function becomes

$$\int_0^{\bar{s}} \left[f^2(s)\left(\sigma_Y^2 + v(s)\right) H(R)g(s) + 2f(s)f'(s)\left(\sigma_X^2 + \bar{z}^2\right)\left(1 - H\left(\frac{\gamma}{f'(s)}\right)\right)\right] ds.$$

Applying the same method, we get

$$\int_0^{\bar{s}} f(s)\, dH\left(\frac{\gamma}{f'(s^+)}\right) = \int_0^{\bar{s}} f'(s)\left(1 - H\left(\frac{\gamma}{f'(s)}\right)\right) ds,$$

which allows us to express the constraint as

$$\int_0^{\bar{s}} \left[f(s)H(R)g(s) + f'(s)\left(1 - H\left(\frac{\gamma}{f'(s)}\right)\right)\right] ds = \frac{1}{n}.$$

Moreover, we can transform the problem into an unconstrained one by defining an auxiliary state variable Γ by

$$\Gamma(t) = \int_0^{t} \left[f(s)H(R)g(s) + f'(s)\left(1 - H\left(\frac{\gamma}{f'(s)}\right)\right)\right] ds, \;\; t \in [0,\, \bar{s}].$$

This means that

$$\Gamma'(s) = f(s)H(R)g(s) + f'(s)\left(1 - H\left(\frac{\gamma}{f'(s)}\right)\right)$$

with $\Gamma(0) = 0$ and $\Gamma(\bar{s}) = 1/n$.

Finally, to simplify notation, we denote the state variables f, f' and Γ by x_1, x_2, and x_3, respectively, and the control variable, f'', by u. Then, the full statement of the optimal control problem, suppressing dependence on s in the notation, is

$$\sup_{\substack{-\infty < u \leq 0 \\ (\tau_j,\, v_j \geq 0)_{j=1}^k}} \; -\int_0^{\bar{s}} \left[x_1^2\left(\sigma_Y^2 + v\right) H(R)g + 2x_1x_2\left(\sigma_X^2 + \bar{z}^2\right)\left(1 - H\left(\frac{\gamma}{x_2}\right)\right)\right] ds \tag{B.3}$$

subject to

$$x_1' = x_2, \quad x_1(0) = 0, \quad x_1(\bar{s}) \ \text{– free}, \quad x_1(\tau_j^+) - x_1(\tau_j^-) = 0,$$

$$x_2' = u, \quad x_2(0) = \frac{\gamma}{R}, \quad x_2(\bar{s}) \ \text{– free}, \quad x_2(\tau_j^+) - x_2(\tau_j^-) = -v_j,$$

$$x_3' = x_1 H(R) g + x_2 \left(1 - H\left(\frac{\gamma}{x_2}\right)\right), \quad x_3(0) = 0, \quad x_3(\bar{s}) = \frac{1}{n}, \quad x_3(\tau_j^+) - x_3(\tau_j^-) = 0.$$

We now apply the Maximum Principle for an optimal control problem with jumps in the state variable to identify necessary conditions for the optimal control $u^\star$. These necessary conditions are introduced in the Lemma below.

Lemma 13. *Let $u^\star(s)$, $(\tau_j^\star, v_j^\star)_{j=1}^k$ be an admissible control that solves the problem above. Let $x(s) = (x_1(s), x_2(s), x_3(s))$ be the corresponding vector of state variables. Then there exist a constant $\lambda_0 \in \{0, 1\}$ and a continuous function $\lambda(t) = (\lambda_1(t), \lambda_2(t), \lambda_3(t))$ such that:*

1. $(\lambda_0, \lambda(\bar{s}^+)) \neq (0, \mathbf{0})$.
2. If the Hamiltonian $\mathcal{H}$ is defined by

$$\mathcal{H}(x, u, \lambda) = -\lambda_0 x_1^2 \left(\sigma_Y^2 + v\right) H(R) g - 2\lambda_0 x_1 x_2 \left(\sigma_X^2 + \bar{z}^2\right) \left[1 - H\left(\frac{\gamma}{x_2}\right)\right]$$

$$+\lambda_1 x_2 + \lambda_2 u + \lambda_3 \left[x_1 H(R) g + x_2 \left(1 - H\left(\frac{\gamma}{x_2}\right)\right)\right],$$

then for all $s \in [0, \bar{s}]$ such that $s \neq \tau_j^\star$ for $j = 1, \ldots, k$, we have

$$\mathcal{H}(x^\star(s), u, \lambda(s)) \leq \mathcal{H}(x^\star(s), u^\star(s), \lambda(s)), \quad u \in (-\infty, 0].$$

3. For all $s \in [0, \bar{s}]$ such that $s \neq \tau_j^\star$ for $j = 1, \ldots, k$, and except for points of discontinuity of $u^\star$, λ is continuously differentiable and

$$\lambda_i'(s) = -\frac{\partial \mathcal{H}\left(x^\star(s), u^\star(s), \lambda(s)\right)}{\partial x_i}, \quad i \in \{1, 2, 3\}.$$

4. The transversality conditions $\lambda_1(\bar{s}^+) = 0$ and $\lambda_2(\bar{s}^+) = 0$ are satisfied.
5. At the jump points $\tau_j^\star$, for $j = 1, \ldots, k$, $\lambda_2(\tau_j^\star) = 0$,
6. For all $s \neq \tau_j^\star$, for $j = 1, \ldots, k$, $\lambda_2(s) \geq 0$.

Proof. Apply Theorem 7, p. 196, of Seierstad and Sydsaeter (1987).

□

Before we proceed, we state a simple lemma that will be used throughout the remainder.

Lemma 14. *Suppose X is a nonnegative random variable with a finite variance and a continuously differentiable decreasing density h on $(0, \infty)$. Then* $\lim_{x\to\infty} h(x)x^2 = 0$ *and* $\lim_{x\to\infty} h'(x)x^3 = 0$.

Proof. The first claim follows directly from the definition of variance, and the second can be obtained by applying integration by parts.

□

We now simplify the necessary conditions. First, it is easy to rule out the degenarate case $\lambda_0 = 0$, and thus $\lambda_0 = 1$.[20] Second, except possibly at jump points, we have

$$\lambda_1' = \left[2x_1\left(\sigma_Y^2 + v\right) - \lambda_3\right] H(R)g + 2x_2\left(\sigma_X^2 + \bar{z}^2\right)\left[1 - H\left(\frac{\gamma}{x_2}\right)\right], \tag{B.4}$$

$$\lambda_2' = \left[2x_1\left(\sigma_X^2 + \bar{z}^2\right) - \lambda_3\right]\left[1 - H\left(\frac{\gamma}{x_2}\right) + h\left(\frac{\gamma}{x_2}\right)\frac{\gamma}{x_2}\right] - \lambda_1, \tag{B.5}$$

$$\lambda_3' = 0. \tag{B.6}$$

Third, each λ_k function is continuous and continuously differentiable between jump points, and $\lambda_2 = 0$ at jump points. Moreover, λ_3 is a constant, and we denote $\eta \equiv \lambda_3$. Since $\lambda_2 \geq 0$, u must be zero if $\lambda_2 > 0$, and can be an arbitrary negative number if $\lambda_2 = 0$.

If $\lambda_2 = 0$ on an interval (which is required for $u < 0$), then we must have $\lambda_2'' = 0$ almost everywhere. That is,

$$2\gamma\left(\sigma_X^2 + \bar{z}^2\right) h\left(\frac{\gamma}{x_2}\right) + \left[\eta - 2x_1\left(\sigma_X^2 + \bar{z}^2\right)\right] h'\left(\frac{\gamma}{x_2}\right)\frac{\gamma^2 u}{x_2^3} + \left[\eta - 2x_1\left(\sigma_Y^2 + v\right)\right] H(R) = 0.$$

If $x_2 = 0$, then (using Lemma 14) this equality boils down to $\left[\eta - 2x_1\left(\sigma_Y^2 + v\right)\right] H(R) = 0$, so we must have $\eta = 2x_1\left(\sigma_Y^2 + v\right)$. Otherwise, we may rewrite the equality as

$$u = \frac{\left[2x_1\left(\sigma_Y^2 + v\right) - \eta\right] H(R)g - 2\gamma\left(\sigma_X^2 + \bar{z}^2\right) h\left(\frac{\gamma}{x_2}\right)}{\left[2x_1\left(\sigma_X^2 + \bar{z}^2\right) - \eta\right]\left(-h'\left(\frac{\gamma}{x_2}\right)\right)\frac{\gamma^2}{x_2^3}}. \tag{B.7}$$

[20] This claim does require a proof but we skip it because it is not difficult and uses similar arguments to the ones below.

If we treat $0/0$ as 0, then equation (B.7) gives an "if and only if" condition for $\lambda_2 = 0$. Notice that if $2x_1\left(\sigma_X^2 + \bar{z}^2\right) < \eta$, then

$$2x_1\left(\sigma_Y^2 + v\right) - \eta < 2x_1\left(\sigma_X^2 + \bar{z}^2\right) - \eta < 0,$$

so that the numerator in (B.7) becomes negative and $u > 0$. This contradicts feasibility. Thus we need $2x_1\left(\sigma_X^2 + \bar{z}^2\right) \geq \eta$ in any interval $[s_0,\, s_1]$ in which $\lambda_2 = 0$. In fact this must be a strict inequality if $x_2(s_0) > 0$. Since x_1 is strictly increasing when $x_2 > 0$, it is obvious that $2x_1\left(\sigma_X^2 + \bar{z}^2\right) > \eta$ for all $s > s_0$. And if $2x_1(s_0)\left(\sigma_X^2 + \bar{z}^2\right) = \eta$, then it can be shown that there does not exist a finite, concave and nondecreasing solution to the ODE (B.7). For future use, notice that in the interval with $\lambda_2 = 0$, λ_1 is uniquely pinned down, in that

$$\lambda_1 = \left[2x_1\left(\sigma_X^2 + \bar{z}^2\right) - \eta\right]\left[1 - H\left(\frac{\gamma}{x_2}\right) + h\left(\frac{\gamma}{x_2}\right)\frac{\gamma}{x_2}\right]. \tag{B.8}$$

On the other hand, if $\lambda_2 > 0$, we must have $u = 0$. Therefore, the optimal u is either 0 or determined by equation (B.7).

Lemma 15. *For every optimal solution, $x_1(s) = \gamma s/R$ on an interval of the form $[0,\, s_0]$, for some $s_0 > 0$.*

Proof. The claim follows immediately from the observation that if $\eta > 0$ (which is indeed the case and will be demonstrated later), then the solution to the ODE (B.7) cannot be concave for s close to 0. Therefore $\lambda_2 > 0$, and we must have $u = 0$ initially. Since $x_1(0) = 0$ and $x_2(0) = \frac{\gamma}{R}$, we obtain $x_1(s) = \frac{\gamma s}{R}$ in some interval $[0,\, s_0]$.

□

From now on, we let $s_0 = \max\left\{s:\ f(s) = \frac{\gamma s}{R}\right\}$. Notice that this is well defined (because f is continuous), strictly larger than zero (by the Lemma above), and strictly smaller than $\bar{s}$ (because we assumed $R < \hat{R}$), and also that $f(s) = \frac{\gamma s}{R}$ for all $s \leq s_0$.

Lemma 16. *There is at most one jump in x_2 in the optimal solution. If there is a jump, then it must be at s_0. Further, x_1 must be linear (with strictly positive slope) in a right neighborhood of s_0.*

Proof. First, notice that λ_2 is continuous and equal to zero at every jump point. We apply Note 7 on p. 197 of Seierstad and Sydsaeter (1987), which states that the Hamiltonian is continuous at the jump points (for all jump points in $(0,\, \bar{s})$). In particular,

$$\phi(x_2) = \left[\eta - 2x_1\left(\sigma_X^2 + \bar{z}^2\right)\right] x_2 \left[1 - H\left(\frac{\gamma}{x_2}\right)\right] + \lambda_1 x_2$$

must be continuous at any such jump points. Suppose that $\lambda_2 = 0$ a.e. in a subinterval contained in the neighborhood of some jump point τ (on either side). Then we have $\phi(x_2) = \left[\eta - 2x_1\left(\sigma_X^2 + \bar{z}^2\right)\right] \gamma h(\frac{\gamma}{x_2})$ (using equation (B.8), and continuity of λ_1). Therefore, if there is a jump in x_2 in this case, then continuity of ϕ at τ requires that $\eta = 2x_1(\tau)\left(\sigma_X^2 + \bar{z}^2\right)$. However, by the remark made above, $2x_1\left(\sigma_X^2 + \bar{z}^2\right) > \eta$ on every closed interval in which $\lambda_2 = 0$ almost everywhere. Since x_1 is continuous, we get a contradiction.

So we are left to explore the possibility of a jump when $\lambda_2 > 0$ on both sides of τ. (This means that x_1 is linear on both sides of τ.) Since $\lambda_2(\tau) = 0$, and $\lambda_2 > 0$ for all $s \neq \tau$ in the neighborhood of τ, we must have $\lambda_2'(\tau^-) \leq 0$ and $\lambda_2'(\tau^+) \geq 0$. This means that

$$\begin{aligned}\left[2x_1(\tau)\left(\sigma_X^2 + \bar{z}^2\right) - \eta\right]\left[1 - H\left(\frac{\gamma}{x_2(\tau^-)}\right) + h\left(\frac{\gamma}{x_2(\tau^-)}\right)\frac{\gamma}{x_2(\tau^-)}\right] &\leq \lambda_1(\tau) \\ \leq \left[2x_1(\tau)\left(\sigma_X^2 + \bar{z}^2\right) - \eta\right]\left[1 - H\left(\frac{\gamma}{x_2(\tau^+)}\right) + h\left(\frac{\gamma}{x_2(\tau^+)}\right)\frac{\gamma}{x_2(\tau^+)}\right]&.\end{aligned}$$

Because x_2 can only jump down, and because

$$1 - H\left(\frac{\gamma}{x_2}\right) + h\left(\frac{\gamma}{x_2}\right)\frac{\gamma}{x_2}$$

is an increasing function of x_2, we must have $2x_1(\tau)\left(\sigma_X^2 + \bar{z}^2\right) \leq \eta$, and, moreover, $\lambda_1(\tau) \leq 0$. As an immediate conclusion, we cannot have jumps in the interval $[\tilde{s},\, \bar{s}]$, where $\tilde{s}$ is a point such that $2x_1(\tilde{s})\left(\sigma_X^2 + \bar{z}^2\right) > \eta$. In particular, if $\lambda_2 = 0$ a.e. in some interval, then there are no jumps in x_2 to the right of this interval.

In the maximal interval $(\tau,\, s_1]$ within which $u = 0$ a.e., we have

$$\lambda_1' = \left[2x_1\left(\sigma_Y^2 + v\right) - \eta\right] H(R) g + 2x_2\left(\sigma_X^2 + \bar{z}^2\right)\left[1 - H\left(\frac{\gamma}{x_2}\right)\right] \tag{B.9}$$

$$\lambda_2' = \left[2x_1\left(\sigma_X^2 + \bar{z}^2\right) - \eta\right]\left[1 - H\left(\frac{\gamma}{x_2}\right) + h\left(\frac{\gamma}{x_2}\right)\frac{\gamma}{x_2}\right] - \lambda_1 \tag{B.10}$$

$$\lambda_2'' = 2\gamma\left(\sigma_X^2 + \bar{z}^2\right) h\left(\frac{\gamma}{x_2}\right) - \left[2x_1\left(\sigma_Y^2 + v\right) - \eta\right] H(R) g. \tag{B.11}$$

Since $\lambda_2(\tau) = \lambda_2(s_1) = 0$ (by Lemma 13), $\lambda_2 > 0$ for almost all $s \in (\tau,\, s_1)$, and λ_2' is differentiable in $(\tau,\, s_1)$, we must have $\lambda_2'' < 0$ at least in some subinterval. But this means that $2x_1\left(\sigma_Y^2 + v\right) > \eta$ at least at one point $\tilde{s} \in (\tau,\, s_1)$. Since x_1 is non-decreasing, it follows that $2x_1(\tilde{s})\left(\sigma_X^2+\bar{z}^2\right)>\eta$. From the remark above, there can be no more jumps in x_2 after the (possible) jump at τ, that is, there is at most one jump in x_2. Summarizing what we have shown so far, and using Lemma 15, we conclude that $\tau = s_0$.

So if there is a jump, then we have the following: $\lambda_2(s_1)=\lambda_2(s_0)=0$ and $\lambda_1(s_0)\le 0\le \lambda_1(s_1)$, where $(s_0,\, s_1]$ is the maximal interval in which $u=0$ a.s. Notice that in this interval x_2 is constant.

We can now rule out the case $x_2=0$ in $[s_0,\, s_1]$. We would have $s_1=\bar{s}$, x_1 constant (i.e. $x_1(s)=\frac{\gamma s_0}{R}$ for $s\in[s_0,\,\bar{s}]$), and

$$\lambda_1' = \left[2\frac{\gamma s_0}{R}\left(\sigma_Y^2+v\right)-\eta\right]H(R)g.$$

Since we have

$$2\frac{\gamma s_0}{R}\left(\sigma_Y^2+v\right)-\eta < 2\frac{\gamma s_0}{R}\left(\sigma_X^2+\bar{z}^2\right)-\eta \le 0,$$

we get $\lambda_1'<0$ which in conjunction with $\lambda_1(s_0)\le 0$ gives a contradiction with the transversality condition $\lambda_1(\bar{s})=0$ from Lemma 13. This concludes the proof of the Lemma.

□

We now establish that in any optimal solution, x_1 follows (B.7) in some interval with $\bar{s}$ as the right endpoint. In particular, this rules out solutions that are piecewise linear. Moreover, $x_2(\bar{s})=0$, that is, the optimal function x_1 becomes flat as s gets closer to $\bar{s}$.

Lemma 17. *In any optimal solution, there exists $s_1<\bar{s}$ such that in the interval $[s_1,\bar{s}]$, x_1 follows* (B.7)*, and $x_2(\bar{s})=0$.*

Proof. There are two possible cases: either u follows (B.7) or $u=0$, in some interval containing $\bar{s}$ as the right endpoint. In the first case, to satisfy the transversality condition for λ_1, we need

$$\left[2x_1(\bar{s})\left(\sigma_X^2+\bar{z}^2\right)-\eta\right]\left[1-H\left(\frac{\gamma}{x_2(\bar{s})}\right)+h\left(\frac{\gamma}{x_2(\bar{s})}\right)\frac{\gamma}{x_2(\bar{s})}\right]=0.$$

Because $2x_1(\bar{s})\left(\sigma_X^2+\bar{z}^2\right)>\eta$, this means that we must have $x_2(\bar{s})=0$ (applying Lemma 14). In the second case, suppose that $x_2(\bar{s})>0$. Because λ_1 is continuous, and $\lambda_1(\bar{s})=0$, $\lambda_1(s)$ is arbitrarily close to 0 for s close to $\bar{s}$. But this means that $\lambda_2'(s)>0$ for s close to $\bar{s}$. Because $\lambda_2(s)\ge 0$, we get a contradiction with the transversality condition $\lambda_2(\bar{s})=0$. This establishes that $x_2(\bar{s})=0$.

Because we have excluded the possibility that x_1 is flat on $[s_0,\bar{s}]$ in Lemma 16, it follows that x_1 must satisfy (B.7) in some subinterval contained in $[s_0,\bar{s}]$. So to finish the proof of the Lemma, we need to show that it is impossible that the optimal x_1 has $x_2=0$ with $\lambda_2>0$

in some $(s_1, \bar{s})$ when the solution to the left of s_1 satisfies (B.7). In such a case we would have $\lambda_1(s_1) = \lambda_2(s_1) = \lambda_2(\bar{s}) = \lambda_1(\bar{s}) = 0$, and in the interval $(s_1, \bar{s})$,

$$\lambda_1' = \left[2x_1(\bar{s})\left(\sigma_Y^2 + v\right) - \eta\right] H(R)g,$$

$$\lambda_2' = -\lambda_1,$$

which leads to an immediate contradiction.[21]

□

Finally, having restricted the set of candidate solutions, we prove that at least one of these candidates is indeed a solution, thereby concluding the proof of Proposition 8.

Lemma 18. *There exists a solution to Problem* (B.3).

Proof. Since $\lambda_2 \geq 0$ in the optimal solution, we can view the control variable u as coming from a bounded interval $[-M, 0]$, and this modification doesn't change anything in the analysis above (taking M large enough that the solution to (B.7) does not violate the constraint $u \geq -M$). Similarly, we can impose a constraint that there is at most $K < \infty$ jumps in x_2, and, by Lemma 16, this also doesn't change the set of candidate solutions. Define formally

$$\mathcal{F}_c^{M,K} = \{f \in \mathcal{F}_c : |f''(s)| \leq M \text{ a.e. and } |\{s \in [0, s_0] : f'(s^-) \neq f'(s^+)\}| \leq K\}. \tag{B.12}$$

Functions from the space $\mathcal{F}_c^{M,K}$ are uniformly bounded in the $\|\cdot\|_{C^2}$ norm, and thus, by Arzela-Ascoli Theorem, relatively compact as a subset of the space $\mathcal{F}_c$ endowed with the $\|\cdot\|_{C^1}$ norm. Because Lipschitz continuity is preserved in the limit, $\mathcal{F}_c^{M,K}$ is closed, and thus compact. Moreover, the functional we are maximizing (and the constraint) is continuous (by Lebesgue Dominated Convergence Theorem and Lemma 14) on $\mathcal{F}_c$ with this norm. By Weierstrass' Theorem, the maximum is attained when we restrict attention to the space $\mathcal{F}_c^{M,K}$.

Thus, we have existence of solution for a sequence of problems indexed by N obtained by taking the original problem and replacing $u \in (-\infty, 0]$ with $u \in [-N, 0]$ (and restricting the number of allowed jumps in x_2 to at most N). Let $f^{(N)}$ be the corresponding sequence of solutions. By the remark above, for large enough N, $f^{(N)} = f^\star$, for some $f^\star$. But the space $\mathcal{F}_c^{N,N}$ approximates the space $\mathcal{F}_c$ as $N \to \infty$ in the $\|\cdot\|_{C^1}$ norm, thus by continuity of the functional that we are maximizing, $f^\star$ must also be a solution to the original problem.

□

[21]Except for the special case when v is constant and $2x_1(\bar{s})\left(\sigma_Y^2 + v\right) = \eta$. But then x_1 is the solution to (B.7) in $[s_1, \bar{s}]$ (using Lemma 14), and $\lambda_2 = 0$.

Proof of Proposition 9.

In this subsection we examine the case in which either the observation variance $v(s)$ is constant, or the distribution of unmanipulated transaction sizes is uniform, that is, $g(s)$ is constant. We start with another simple lemma.

Lemma 19. *Whenever $u = 0$ in some interval $[\underline{t}, \bar{t}]$, then the function $2x_1(\sigma_Y^2 + v)$ is quasi-convex in this interval.*

Proof. When $u = 0$, x_1 is an affine function, in particular it's convex. Recall that $\kappa(s) = (\sigma_Y^2 + v(s))^{-1}$ is a concave function by Assumption 1. Thus $2x_1(\sigma_Y^2 + v)$ can be seen as a positive convex function divided by a positive concave function, and is thus quasi-convex. □

Lemma 20. *When either (i) $v(s)$ is constant, or (ii) $g(s)$ is constant, we cannot have $u \equiv 0$ in any interval $(t_1, t_2) \subset (s_0, \bar{s})$.*

Proof. Suppose to the contrary that there exists such interval (t_1, t_2), and take it to be maximal (i.e. such that it cannot be enlarged to a bigger interval in which $u \equiv 0$). By what was shown above, we know that x_2 is constant and strictly positive in (t_1, t_2), and $\lambda_2(t_1) = \lambda_2(t_2) = 0$. (Note that t_1 can be equal to s_0 and there can be a jump at s_0). By the same argument as in the proof of Lemma 16 , if $u = 0$ and $\lambda_2 > 0$ in (t_1, t_2), then there must be a point $\tilde{s} \in (t_1, t_2)$ such that $\lambda_2''(\tilde{s}) < 0$, in particular $2x_1(\tilde{s})(\sigma_Y^2 + v(\tilde{s})) > \eta$. From now on, we consider the cases (i) and (ii) seprately.

In case (i), we know that $2x_1(s)(\sigma_Y^2 + v(s))$ is nondecreasing in s, and thus $2x_1(s)(\sigma_Y^2 + v(s)) > \eta$ for all $s \geq \tilde{s}$. By Lemma 17, we know that x_1 follows equation (B.7) in some terminal interval $[s_1, \bar{s}]$, and that $x_2(s) \to 0$ as $s \nearrow \bar{s}$. But then we would have

$$\left[2x_1(s)\left(\sigma_Y^2 + v\right) - \eta\right] H(R)g(s) > 2\gamma\left(\sigma_X^2 + \bar{z}^2\right) h(\frac{\gamma}{x_2(s)})$$

for points s close to $\bar{s}$ (since g is bounded away from zero and $2x_1\left(\sigma_Y^2 + v\right) > \eta$ for all $s > \tilde{s}$). This means that $u > 0$, contradicting concavity of x_1.

In case (ii), we make use of equation (B.11). We know that λ_2'' is negative at some point $\tilde{s} \in (t_1, t_2)$. It is also easy to observe that $\lambda_2''(t_1) \geq 0$. By Lemma 19, the function $2x_1(\sigma_Y^2 + v)$ is quasi-convex. Thus, using the assumption that g is constant, λ_2'' is quasi-concave in (t_1, t_2). This means that it is increasing on $(t_1, \tilde{t})$ and decreasing on $(\tilde{t}, t_2)$, for some $\tilde{t} \in [t_1, t_2]$. Therefore, λ_2'' must cross zero from above, and once it does, it stays below zero until t_2. This produces a contradiction. Indeed, since $\lambda_2'(t_1) \geq 0$ and $\lim_{s \to t_2} \lambda_2'(s) = 0$, it would mean that $\lambda_2'(s) > 0$ for all $s \in (t_1, t_2)$ which is impossible given that $\lambda_2(t_1) = \lambda_2(t_2) = 0$.

□

Combining Lemma 16 with Lemma 20, we get the following corollary.

Corollary 21. *When either (i) $v(s)$ is constant, or (ii) $g(s)$ is constant, there can be no jumps in x_2.*

It follows that the optimal x_1 is the solution to equation (B.7) on $(s_0, \bar{s})$. Because we have established that there are no jumps in x_2, it must be the case that $x_2(s_0) = \frac{\gamma}{R}$ which gives the second initial condition (the first one is $x_1(s_0) = \frac{\gamma s_0}{R}$). Moreover, we have the condition $x_2(\bar{s}) = 0$ by Lemma 17 which uniquely pins down the constant η for a given s_0. Finally, s_0 must be chosen so that the constraint is satisfied. Recalling that $f(s) = x_1(s)$, $f'(s) = x_2(s)$, and $f''(s) = u(s)$ concludes the proof of Proposition 9.