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MEAN-VARIANCE HEDGING IN CONTINUOUS TIME

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A hedger is faced with a commitment in one asset and the opportunity to continuously trade futures contracts on another asset whose returns are correlated with those of the committed asset. Optimal futures trading strategies are presented in closed form for several mean-variance and quadratic objectives.

1. Introduction. This paper presents a closed-form solution for the optimal continuous-time futures hedging policy under various mean-variance and quadratic objectives. The results include, as special cases, minimum-variance hedging and policies achieving the minimum variance for a given mean.

Asset prices are assumed to be exponential correlated Brownian motions; that is, each asset price is of the Black–Scholes type, while rates of return between assets are correlated. We allow the coefficients for "return, volatility and correlation" to depend on time. The basic proof of optimality rests on showing that the usual inner product associated with the normal equations for orthogonal projection is defined by an ordinary differential equation in time with an explicit solution. Our solution was conjectured from discrete-time reasoning, and would have been difficult to obtain by standard dynamic programming or variational methods [such as those in He and Pearson (1988)], since the value function is not easy to guess beforehand.

This paper differs from most continuous-time results in that markets are incomplete in an essential way: the hedger is risk averse and has a random endowment that cannot be replicated by security trading, and whose risk cannot, therefore, be eliminated at any cost. Duffie and Jackson (1990) and Svensson and Werner (1990) have other results for somewhat different special cases. Although the related abstract existence results of Shreve and Xu (1988) as well as He and Pearson (1988) do not allow for random endowments, it may be possible to extend their results for this purpose. Svensson and Werner (1990) reviews the earlier literature.

2. Problem statement and solutions. We begin with a loosely stated version of the hedging problem and solutions. The full mathematical definition of the problem is completed in the following section. Asset price processes S and F are determined by the stochastic differential equations

$$(1) \qquad \begin{aligned} dS_t &= \mu_t S_t\,dt + \sigma_t S_t\,dB_t, \\ dF_t &= m_t F_t\,dt + v_t F_t\,d\xi_t, \end{aligned}$$

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where B and ξ are Brownian motions whose increments have correlation $\rho_t \in [0, 1]$ at time t. The functions μ, σ, m, v and ρ are deterministic. (Technical definitions are given at the beginning of the next section.) The hedger is committed to k units of the first asset at some time T in the future. The risk of the corresponding uncertain value kS_T can be hedged by a futures strategy $\theta = \{\theta_t\colon\ 0 \le t \le T\}$, where θ_t is the futures position at time t. A futures position held at some constant level $\bar{\theta}$ between two dates t_1 and t_2 generates a credit to the hedger's margin account of $\bar{\theta}(F(t_2) - F(t_1))$ during that interval. This process of continually crediting profits (or collecting losses) as the futures price changes is called resettlement, or marking to market, a feature of futures contracts that distinguishes them from most other financial securities. We will at first assume that there is a zero interest rate, so this distinction is not important, but later return to consider interest on margin. In the general case, a (stochastic) futures strategy θ generates profits (or losses) of $G(\theta)_t = \int_0^t \theta_s\, dF_s$ by any time t. The total final wealth as a function of the strategy θ is thus $W(\theta) = kS_T + G(\theta)_T$.

The hedger's problem is then

$$\max_{\theta \in \Theta} E(u[W(\theta)]), \tag{2}$$

where $u(w) = w - cw^2$ for some constant c and where Θ is the space of all trading strategies. The futures strategy θ^* solving this problem is given by

$$\theta_t^* = \frac{1}{F_t}\left[\frac{m_t}{v_t^2}[L - G(\theta^*)_t] - \alpha_t S_t\right],$$

where $L = (2c)^{-1}$ and

$$\alpha_t = \frac{m_t + \sigma_t \rho_t v_t}{v_t^2} k \exp\left[-\int_t^T \left(\frac{m_s \sigma_s \rho_s}{v_s} - \mu_s\right) ds\right].$$

Indeed, θ^* also solves

$$\min_{\theta \in \Theta} E\big([W(\theta) - L]^2\big), \tag{3}$$

where L is any given target level for final wealth.

With constant coefficients m, v, μ, σ and ρ, the minimum variance problem

$$\min_{\theta \in \Theta} \operatorname{var}[W(\theta)], \tag{4}$$

where var$[W(\theta)]$ is the variance of $W(\theta)$, is also solved by θ^*, where we choose $L = kS_0 \exp([\mu - (m\sigma\rho/v)]T)$.

Finally, for constant coefficients, we can trace out the mean-variance efficient frontier (provided $m \neq 0$). That is, for any mean wealth level M, the problem

$$\min_{\theta \in \Theta} \operatorname{var}[W(\theta)] \quad \text{subject to} \quad E[W(\theta)] = M \tag{5}$$

is solved by θ^* with

$$L = \frac{M - kS_0 \exp\left(\left[\mu - (m\sigma\rho/v) - (m/v)^2\right]T\right)}{1 - \exp\left(-(m/v)^2 T\right)}. \tag{6}$$

We also solve these problems (2)–(5) with the added consideration of interest rates on futures margin accounts.

There are well known objections to all of the preceding criteria, such as nonmonotonicity outside of certain ranges and increasing absolute risk aversion. In some problem settings, nonmonotonicity can be circumvented by assumptions ensuring that only the monotonic increasing portion of the domain of the utility function is relevant. Since geometric Brownian motion is unbounded, it is generally impossible to obtain this kind of restriction in the present setting, although the probability of entering the decreasing portion of the utility function can be made arbitrarily small. In summary, the preference structure in this model is quite limited from a theoretical viewpoint, despite being relatively standard in practice.

The drawbacks of the quadratic criterion are less severe if one treats the problem as that of a corporation hedging so as to reduce the expected costs of financial distress, or to mitigate incentive costs in managerial decision making. In the latter case, for example, a manager of a corporation may avoid otherwise profitable corporate projects if the unhedged project cash flows impose a risk upon the manager (through compensation or job security) because the manager's actions or abilities are not fully observable to the owners of the firm. Hedging may reduce the costs of giving the manager the appropriate incentives in such situations, and a simple hedging criterion, such as variance minimization, may be quite satisfactory from a practical point of view. [Aside from such incentive effects, or other capital market "imperfections," a standard Modigliani–Miller (1958) argument implies that the shareholders of the firm are indifferent to corporate hedging in financial markets.]

In our problem setting, the optimal policy may generate a final level of wealth that is negative with some probability, however small. Under the additional constraint of almost sure nonnegativity of final wealth, the only feasible policy, in the general case of imperfect correlation between the two Brownian motions B and ξ, is the policy of zero hedging. Again, this is a theoretical limitation of the model that is often ignored in practice.

The next section presents the remaining mathematical details in the problem formulation as well as our solution to problem (3), which is basic to our approach. Section 4 presents results for the other problems and extends the solution to handle interest on margin accounts. The final section displays some numerical examples.

3. The basic problem.

3.1. *Rigorous problem formulation.* First, we complete a rigorous statement of (1)–(5), as follows. Let (B, ε) be a standard Brownian motion in $\mathbb{R}^2$. All probabilistic statements are made (suppressing "almost surely") with respect to a probability space $(\Omega, \mathscr{F}, P)$ on which (B, ε) is defined, and using the filtration $\mathbb{F}$ of σ-algebras which is the augmentation of the filtration generated by (B, ε).

Let μ, σ, m, v and ρ be bounded measurable functions on $[0, T]$ into $\mathbb{R}$ such that v is bounded away from zero and $\rho_t \in [-1, 1]$ for all t. Given this, we can define the Brownian motion ξ of (1) without loss of generality by

$$\xi_t = \int_0^t \rho_s \, dB_s + \int_0^t \sqrt{1 - \rho_s^2} \, d\varepsilon_s, \qquad t \in [0, T].$$

This completes the formalization of (1) for any $S_0 > 0$ and $F_0 > 0$. A trading strategy is an $\mathbb{F}$-predictable process θ such that

$$E\left[\int_0^T \theta_t^2 F_t^2 \, dt\right] < \infty.$$

The set Θ of all trading strategies is a vector space. With the above, the stochastic integral $G(\theta)_t = \int_0^t \theta_s \, dF_s$ is well defined in $L^2(P)$ for all $\theta \in \Theta$ and all t. Thus, problems (2)–(5) are now completely defined.

3.2. *Reduction to an orthogonal projection.* It is well known that $L^2(P)$ is a Hilbert space under the inner product $(\cdot | \cdot)$ defined by $(X|Y) = E(XY)$ and the associated norm $\|\cdot\|$ defined by $\|X\| = [E(X^2)]^{1/2}$. The set

$$M = \{G(\theta)_T : \theta \in \Theta\}$$

is a linear subspace of $L^2(P)$ since G is linear and Θ is a vector space.

Problem (3) is equivalent to

$$\min_{X \in M} \|Y - X\|, \tag{7}$$

where $Y = L - kS_T$. By the Hilbert space projection theorem [which can be found, for example, in Luenberger (1969)], $\hat{X}$ is a solution to (7) if and only if $(Y - \hat{X}|X) = 0$ for all X in M. Equivalently, we have the following characterization of optimal trading strategies.

LEMMA 1. *A trading strategy φ is optimal if and only if, for any trading strategy θ,*

$$(L - kS_T - G(\varphi)_T | G(\theta)_T) = 0. \tag{8}$$

3.3. *Solution of the basic problem.* This subsection presents a solution to problem (3). We define the "tracking process" Z by

$$Z_t = k \exp\left(-\int_t^T \gamma_s \, ds\right) S_t, \qquad t \in [0, T], \tag{9}$$

where

$$\gamma_t = \frac{m_t \sigma_t \rho_t}{v_t} - \mu_t, \qquad t \in [0, T].$$

(Note that $Z_T = kS_T$.) Given a target level L for terminal wealth, we next define G_t^* to be the solution of the stochastic differential equation

$$dG_t^* = \Phi(G_t^*)\,dF_t, \qquad G_0^* = 0, \tag{10}$$

where

$$\Phi(G_t^*) = \frac{1}{F_t}\left[\frac{m_t}{v_t^2}(L - Z_t - G_t^*) - \frac{\sigma_t \rho_t}{v_t} Z_t\right]. \tag{11}$$

A solution G^* to (10) exists by standard arguments, as in Protter (1990), and $G_t^* \in L^2(P)$ for all t. We claim that the strategy φ defined by $\varphi_t = \Phi(G_t^*)$ solves problem (3). Moreover, φ is in a convenient feedback form, since $G_t^* = G(\varphi)_t$ is readily observable as the trading gains to date, while Z_t is a simple function of the observable asset price S_t.

PROPOSITION 1. *The futures strategy* $\Phi(G^*)$ *defined by* (9)–(11) *solves problem* (3).

In order to prove this result, we construct an ordinary differential equation (ODE) for the inner product function H defined by

$$H_t = (L - Z_t - G_t^* | G(\theta)_t), \qquad t \in [0, T]. \tag{12}$$

Then we show that the solution to the ODE is $H_t = 0$ for all t, which completes the proof of Lemma 1. (Recall that $Z_T = kS_T$.)

LEMMA 2. *Let* θ *be an arbitrary trading strategy, let* $\varphi = \Phi(G^*)$ *be defined by* (9)–(11) *and let* H_t *be defined by* (12) *for* $t \in [0,T]$. *Then the derivative* $\dot{H}_t \equiv (d/dt)H_t$ *is well defined and*

$$\dot{H}_t = -\frac{m_t^2}{v_t^2} H_t, \qquad t \in [0, T]. \tag{13}$$

PROOF. The proof is by direct calculation, using Itô's lemma, as follows. First, by Itô's lemma, Z solves the stochastic differential equation

$$dZ_t = (\gamma_t + \mu_t) Z_t + \sigma_t Z_t\, dB_t.$$

We also have, for $G_t = G(\theta)_t$,

$$\begin{aligned} dG_t &= \theta_t m_t F_t\, dt + \theta_t F_t v_t\, d\xi_t \\ &= \theta_t m_t F_t\, dt + \theta_t F_t v_t \left(\rho_t\, dB_t + \sqrt{1 - \rho_t^2}\, d\varepsilon_t\right). \end{aligned}$$

By Itô's lemma and Fubini's theorem, if we let $X_t = Z_t G_t$ and $\overline{X}_t = E(X_t)$, we have for almost every t in $[0, T]$,

$$\frac{d\overline{X}(t)}{dt} = (\gamma_t + \mu_t) E[Z_t G_t] + m_t E[Z_t \theta_t F_t] + \sigma_t v_t \rho_t E[Z_t \theta_t F_t].$$

Likewise, if we let $Y_t = G_t^* G_t$,

$$\frac{d\overline{Y}(t)}{dt} = m_t E(\varphi_t F_t G_t) + m_t E(\theta_t F_t G_t^*) + v_t^2 E(\varphi_t \theta_t F_t^2).$$

Finally, $(d\overline{G}(t))/dt = m_t E(\theta_t F_t)$. This implies that $\dot{H}$ exists and

$$\dot{H}_t = L\frac{d\overline{G}(t)}{dt} - \frac{d\overline{X}(t)}{dt} - \frac{d\overline{Y}(t)}{dt}.$$

After collecting all terms and using the definition of γ_t, we have (13). □

The solution to (13) is

$$H_t = H_0 \exp\left(-\int_0^t \frac{m_s^2}{v_s^2}\, ds\right).$$

Of course, since $G_0 = 0$, we know that $H_0 = 0$, and therefore $H_t = 0$ for all t. In particular, $H_T = 0$. This implies (8) and proves Proposition 1 and the optimality of the policy $\Phi(G^*)$ for problem (3).

4. Related problems.

4.1. *Quadratic utility maximization.* Problem (2) is equivalent to problem (3) with $L = 1/(2c)$. Thus $\Phi(G^*)$ also solves problem (2) if we let $L = 1/(2c)$ in the definition of Φ.

4.2. *Mean-variance efficiency.* The mean-variance efficiency of the futures policy $\varphi = \Phi(G^*)$ for any choice of L is established with the following lemma.

LEMMA 3. *For any L, if φ solves problem* (3), *then φ is mean-variance efficient.*

PROOF. Suppose φ solves problem (3) and $E[W(\varphi)] = M$. The proof is by contradiction. Suppose there is a futures strategy θ such that $E[W(\theta)] = M$ and $\mathrm{var}[W(\theta)] < \mathrm{var}[W(\varphi)]$. Then

$$\begin{aligned}\mathrm{var}[W(\theta)] &= \mathrm{var}[W(\theta) - L] \\ &= \|W(\theta) - L\|^2 - (M - L)^2 \\ &< \|W(\varphi) - L\|^2 - (M - L)^2 = \mathrm{var}[W(\varphi)].\end{aligned}$$

But this implies that $\|W(\theta) - L\|^2 < \|W(\varphi) - L\|^2$, which contradicts the optimality of φ for problem (3). □

4.3. *Minimum variance for a given mean.* In this paragraph, we assume for simplicity that the coefficient functions m, v, μ, σ and ρ are constant and abuse the notation by using the same symbols m, v, μ, σ and ρ for the respective constants. The Appendix includes a calculation of $E(G_t^*)$, the expected gains from the optimal trading policy for a given target level L. We have

$$E(G_T^*) = L\left(1 - \exp\left(-\left(\frac{m}{v}\right)^2 T\right)\right)$$
$$- kS_0\left(\exp(\mu T) - \exp\left(\left[\mu - \left(\frac{\sigma\rho m}{v}\right) - \left(\frac{m}{v}\right)^2\right]T\right)\right).$$

Since $E(kS_T) = kS_0 \exp(\mu T)$, the mean $M(L)$ of the total wealth achieved by the optimal policy given the target level L is

$$M(L) = L\left(1 - \exp\left(-\left(\frac{m}{v}\right)^2 T\right)\right) + kS_0 \exp\left(\left[\mu - \left(\frac{m\rho\sigma}{v}\right) - \left(\frac{m}{v}\right)^2\right]T\right).$$

Unless $m = 0$, this implies that any mean M can be obtained by the unique target level

$$L(M) = \frac{M - kS_0 \exp\left(\left[\mu - (m\rho\sigma/v) - (m/v)^2\right]T\right)}{1 - \exp\left(-(m/v)^2 T\right)}.$$

For example, in order to achieve a mean that is some multiple δ of the unhedged mean $E(kS_T)$, we can choose the target level

$$L_\delta = \frac{kS_0 \exp(\mu T)\left(\delta - \exp\left(-\left[(m\rho\sigma/v) + (m/v)^2\right]T\right)\right)}{1 - \exp\left(-(m/v)^2 T\right)}.$$

4.4. *The minimum variance policy.* This subsection provides the solution to problem (4), minimize the variance of terminal wealth over all futures strategies. For the case of martingale futures prices $m = 0$ this problem is solved in Duffie and Jackson (1987), so we restrict ourselves here to the case $m \neq 0$.

For any given mean M, the previous subsection shows the existence of a target level L and a futures strategy φ_L that solves problem (3) with the property that $E[W(\varphi_L)] = M$. The target level $L = \bar{L} = kS_0 \exp([\mu - (m\rho\sigma/v)]T)$ has the special property that $E[W(\varphi_{\bar{L}})] = \bar{L}$. The following proposition shows that $\varphi_{\bar{L}}$ is the minimum variance futures strategy. The proposition is stated and proved in a manner showing that it does not depend on our particular assumptions (1) about price processes, but only on the assumption that there is a unique target level that achieves a given mean.

LEMMA 4. *Suppose φ solves problem* (3), *where the target level L has the property that $L = E[W(\varphi)]$. Then φ solves problem* (4) (*minimize variance*).

PROOF. If $m = 0$, the result follows from Duffie and Jackson (1987), so we take the case $m \neq 0$. We equip Θ with the norm $\theta \mapsto [E(\int_0^T \theta_t^2 F_t^2\, dt)]^{1/2}$. Since var$(\cdot)$ is continuous on $L^2(P)$ and the total gain $\theta \mapsto G(\theta)_T$ is continuous on

Θ into $L^2(P)$ (by the definition of stochastic integration), the function $\theta \mapsto \mathrm{var}[W(\theta)]$ is continuous on Θ (using the Cauchy–Schwarz inequality). Since Θ is a complete space and variance is bounded below by zero, there is some strategy $\theta^* \in \Theta$ solving the minimum variance problem (4). Let $M = E[W(\theta^*)]$. Then

$$\|W(\theta^*) - M\|^2 \geq \|W(\varphi_M) - M\|^2 \geq \|W(\varphi_M) - E[W(\varphi_M)]\|^2$$
$$\geq \|W(\theta^*) - M\|^2.$$

The first inequality is due to the definition of φ_M as the solution of problem (3) for $L = M$. The second follows from the fact that

$$\|W(\varphi_M) - M\|^2 = \mathrm{var}[W(\varphi_M)] + (E[W(\varphi_M)] - M)^2 \geq \mathrm{var}[W(\varphi_M)].$$

The final inequality follows from the definition of θ^* as the minimum variance policy. Thus

$$\|W(\varphi_M) - M\|^2 = \mathrm{var}[W(\theta^*)].$$

Since there is a unique target level $\bar{L}$ with the property that $E[W(\varphi_{\bar{L}})] = \bar{L}$, we know that $M = \bar{L}$ and the proposition is proved. □

4.5. *Accounting for interest on margin accounts.* So far, we have assumed that futures gains or losses are accumulated without interest until the terminal date T. Suppose, however, that $r = \{r_t\colon 0 \leq t \leq T\}$ is a (bounded measurable deterministic) interest rate process that applies to margin accounts. This implies that the total gain process G^r with interest is defined for any futures strategy θ by the equation

$$dG^r(\theta)_t = \theta_t\, dF_t + r_t G^r(\theta)_t\, dt,$$

the additional term indicating the accumulation of interest. By applying Itô's lemma, we have the solution

$$G^r(\theta)_T = \int_0^T \theta_t \exp\left(\int_t^T r_s\, ds\right) dF_t = G(\theta^{(r)})T,$$

where $\theta^{(r)}$ is the futures policy defined by $\theta_t^{(r)} = \theta_t \exp(\int_t^T r_s\, ds)$. Suppose φ solves problem (4) with $r = 0$, zero interest on margin. If we redefine problem (4) so that $W(\theta) = kS_T + G^r(\theta)$ includes interest on margin, it follows that the *tailed policy* $\varphi^{(-r)}$, defined by

$$\varphi_t^{(-r)} = \varphi_t \exp\left(-\int_t^T r_s\, ds\right),$$

solves problem (4). Likewise, when we consider each of the problems in this paper with the added complication of interest on margin, the solution is given by the tailed version $\varphi^{(-r)}$ of the solution φ without interest. In particular, the location of the mean-variance frontier is not affected by interest on margin. (This would not be the case if the futures contract is replaced with an asset requiring an initial investment.)

We also remark that, with $k = 0$ and $r = 0$ (no hedging motives or interest), the problems faced here are a special case of those solved by Richardson (1989). A related problem is solved by Pliska (1988).

5. Numerical examples. This section compares the continuous-time optimal hedging policy with the fixed optimal hedge. Since continuous-time hedges include discrete-time hedges for arbitrary period length as special cases,

TABLE 1
Parameter cases

Case	μ	σ	m	v	ρ	T*
1	0.20	0.30	0.20	0.30	0.9	1.00
2	0.20	0.30	0.20	0.30	0.0	1.00
3	0.20	0.30	0.20	0.30	0.9	2.00
4	0.20	0.30	0.20	0.30	0.9	0.25
5	−0.20	0.30	0.20	0.30	0.9	1.00
6	0.40	0.30	0.20	0.30	0.9	1.00

*Time measured in years.

TABLE 2
Hedging comparisons

	Unhedged		Fixed hedge		Dynamic hedge	
Cases	Mean	Variance	Mean	Variance	Mean	Variance
1a*	1.2214	0.1405	1.2214	0.1405	1.2214	0.0908
2a	1.2214	0.1405	1.2214	0.1405	1.2214	0.1138
3a	1.4918	0.4389	1.4918	0.4389	1.4918	0.1844
4a	1.0513	0.0251	1.0513	0.0251	1.0513	0.0225
5a	0.8187	0.0631	0.8187	0.0631	0.8187	0.0408
6a	1.4918	0.2096	1.4918	0.2096	1.4918	0.1354
1b†	1.2214	0.1405	1.0230	0.0277	1.0202	0.0184
2b	1.2214	0.1405	1.2214	0.1405	1.2214	0.1138
3b	1.4918	0.4389	1.0533	0.0899	1.0408	0.0424
4b	1.0513	0.0251	1.0052	0.0048	1.0050	0.0043
5b	0.8187	0.0631	0.6858	0.0125	0.6839	0.0083
6b	1.4918	0.2096	1.2496	0.0414	1.2461	0.0275
1c‡	1.2214	0.1405	1.5268	0.7549	1.5268	0.4770
2c	1.2214	0.1405	1.5268	0.4077	1.5268	0.2804
3c	1.4919	0.4389	1.8648	1.2849	1.8648	0.5164
4c	1.0513	0.0251	1.3141	0.9177	1.3141	0.8171
5c	0.8187	0.0631	1.0234	0.3392	1.0234	0.2143
6c	1.4918	0.2096	1.8648	1.1262	1.8648	0.7115

*Cases 1a through 6a: As in Table 1 with minimum variance for unhedged mean.
†Cases 1b through 6b: As in Table 1 with minimum variance.
‡Cases 1c through 6c: As in Table 1 with minimum variance for 1.25 times unhedged mean.

TABLE 3
Hedging comparisons

	Unhedged		Fixed hedge		Dynamic hedge	
Case	Mean	Variance	Mean	Variance	Mean	Variance
1a*	1.2214	0.1405	1.2214	0.1405	1.2214	0.1022
2a	1.2214	0.1405	1.2214	0.1405	1.2214	0.1138
3a	1.4918	0.4389	1.4918	0.4389	1.4918	0.2396
4a	1.0513	0.0251	1.0513	0.0251	1.0513	0.0232
5a	0.8187	0.0631	0.8187	0.0631	0.8187	0.0459
6a	1.4918	0.2096	1.4918	0.2096	1.4918	0.1524
1b†	1.2214	0.1405	1.1132	0.1069	1.1052	0.0780
2b	1.2214	0.1405	1.2214	0.1405	1.2214	0.1138
3b	1.4918	0.4389	1.2570	0.3388	1.2214	0.1885
4b	1.0513	0.0251	1.0258	0.0189	1.0253	0.0174
5b	0.8187	0.0631	0.7462	0.0480	0.7408	0.0351
6b	1.4918	0.2096	1.3597	0.1595	1.3499	0.1164
1c‡	1.2214	0.1405	1.5268	0.5971	1.5268	0.3956
2c	1.2214	0.1405	1.5268	0.4077	1.5268	0.2804
3c	1.4919	0.4389	1.8648	1.0092	1.8648	0.4775
4c	1.0513	0.0251	1.3141	1.0092	1.3141	0.7270
5c	0.8187	0.0631	1.0234	0.2683	1.0234	0.1778
6c	1.4918	0.2096	1.8648	1.8908	1.8648	0.5902

*Cases 1a through 6a: As in Table 2, except $\rho = 0.5$.
†Cases 1b through 6b: As in Table 2, except $\rho = 0.5$.
‡Cases 1c through 6c: As in Table 2, except $\rho = 0.5$.

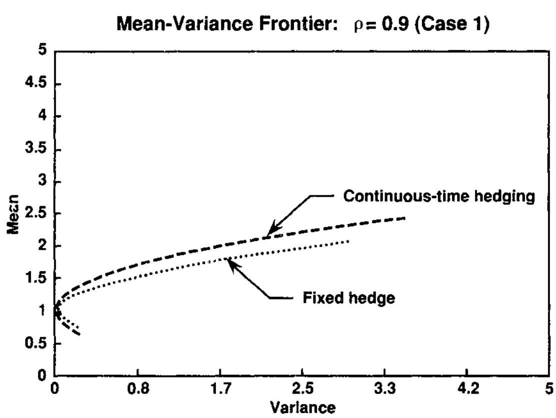

FIG. 1. *Efficient frontier comparison* ($\mu = 0.20, \sigma = 0.30, m = 0.20, v = 0.30, \rho = 0.9, T = 1.00$).

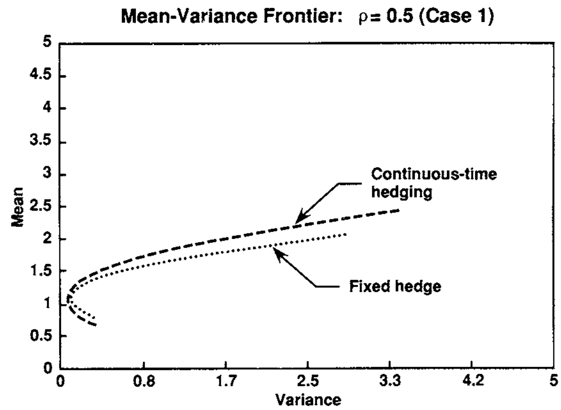

FIG. 2. *Efficient frontier comparison* ($\mu = 0.20, \sigma = 0.30, m = 0.20, v = 0.30, \rho = 0.5, T = 1.00$).

the results of this section bound the results that one can achieve with discretely adjusted hedging strategies.

Table 1 shows some of the numerical examples we explore. Tables 2 and 3 present the mean and variance in each case with that obtained from the optimal fixed hedging position. (The calculations for the fixed hedge are from Appendix B.) Figures 1 and 2 are plots of the mean-variance frontier for the continuous-time and the fixed hedge cases, for two particular parameter cases.

APPENDIX A

Calculation of variance. We wish to calculate the variance of the total wealth $W = G_T^* + Z_T$ of an optimal policy. This variance, $\text{var}(W) = E(W^2) - [E(W)]^2$, can be calculated (tediously) by first calculating

$$E(W^2) = E(G_T^{*2}) + 2E(G_T^* Z_T) + E(Z_T^2),$$

then subtracting

$$[E(W)]^2 = [E(G_T^*)]^2 + 2E(G_T^*)E(Z_T) + [E(Z_T)]^2.$$

Each of these six terms can be calculated explicitly using Itô's lemma and the definitions

$$dZ_t = \frac{m_t \rho_t \sigma_t}{v_t} Z_t\, dt + \sigma_t Z_t\, dB_t,$$

$$dG_t^* = \left[\frac{m_t}{v_t^2}(L - Z_t - G_t^*) - \frac{\sigma_t \rho_t}{v_t} Z_t\right]\left[m_t\, dt + v_t\left(\rho_t\, dB_t + \sqrt{1 - \rho_t^2}\, d\varepsilon_t\right)\right].$$

(a) Let $g_t = E(G_t^*)$. We have

$$g_t = E\left(\int_0^t \left[\frac{m_s^2}{v_s^2}(L - Z_s - G_s^*) - \frac{\sigma_s m_s \rho_s}{v_s} Z_s\right] ds\right)$$

$$= \int_0^t \left[\frac{m_s^2}{v_s^2}(L - \overline{Z}_s - g_s) - \frac{\sigma_s m_s \rho_s}{v_s} \overline{Z}_s\right] ds,$$

where $\overline{Z}_s = E(Z_s)$, using Fubini's theorem. This implies the ordinary differential equation

$$(14) \qquad \dot{g}_t \equiv \frac{dg(t)}{dt} = \frac{m_t^2}{v_t^2}(L - \overline{Z}_t - g_t) - \frac{\sigma_t m_t \rho_t}{v_t}\overline{Z}_t = f_t - \frac{m_t^2}{v_t^2} g_t,$$

where

$$f_t = \frac{m_t^2}{v_t^2} L - \left(\frac{m_t^2}{v_t^2} + \frac{\sigma_t m_t \rho_t}{v_t}\right)\overline{Z}_t.$$

The solution to an ODE like (14) has the standard form

$$g_t = g_0 \exp\left(-\int_0^t \left(\frac{m_s^2}{v_s^2}\right) ds\right) + \int_0^t f_s \exp\left(-\int_s^t \left(\frac{m_u^2}{v_u^2}\right) du\right) ds.$$

Here, $g_0 = 0$ and we can solve the expression in a simple closed form when the coefficient functions m, v, μ, σ and ρ are constant. (For simplicity, we use the same symbols m, v, μ, σ and ρ for the respective constants.) In that case, we have

$$g_t = L\left(1 - \exp\left(-\left(\frac{m}{v}\right)^2 t\right)\right) - Z_0\left(\exp\left(\left(\frac{\sigma m \rho}{v}\right)t\right) - \exp\left(-\left(\frac{m}{v}\right)^2 t\right)\right),$$

where $Z_0 = kS_0 \exp([\mu - (\sigma m \rho / v)]T)$.

(b) Let $R_t = E(Z_t^2)$. Then, following the same procedures outlined for calculating g_t, we have the ODE $\dot{R}_t = (2m\rho\sigma/v + \sigma^2)R_t$, which implies that $R_t = Z_0^2 \exp([(2m\rho\sigma/v) + \sigma^2]t)$.

(c) Let $Q_t = E(Z_t G_t)$. Then

$$\dot{Q}_t = \overline{Z}_t\left[\frac{mL}{v}\left(\frac{m}{v} + \sigma\rho\right)\right] - R_t\left[\left(\frac{m}{v}\right)^2 + 2\frac{\sigma\rho m}{v} + \sigma^2\rho^2\right] - \frac{m^2}{v^2} Q_t,$$

where $\overline{Z}_t = Z_0 \exp(\sigma m \rho / v)t)$. The solution is

$$Q_t = c_1 \exp\left(\left(\frac{\sigma m \rho}{v}\right)t\right) + c_2 \exp\left(\left[\left(\frac{2\sigma m \rho}{v}\right) + \sigma^2\right]t\right) + c_3 \exp\left(-\left(\frac{m}{v}\right)^2 t\right),$$

where

$$c_1 = LZ_0,$$

$$c_2 = -\frac{(m^2 + 2\sigma\rho mv + v^2\sigma^2\rho^2)Z_0^2}{2\sigma m\rho v + \sigma^2 v^2 + m^2},$$

$$c_3 = -(c_1 + c_2).$$

(d) Let $J_t = E[(G_t^*)^2]$. Then

$$\dot{J}_t = \frac{m^2}{v^2}L^2 - \frac{2mL}{v}\left(\frac{m}{v} + \rho\sigma\right)\bar{Z}_t + \left(\frac{m}{v} + \rho\sigma\right)^2 R_t - \frac{m^2}{v^2}J_t.$$

The solution is

$$J_t = L^2 + b_1 \exp\left(-(m/v)^2 t\right) + b_2 \exp((m\rho\sigma/v)t) + b_3 \exp\left(\left[(2m\rho\sigma/v) + \sigma^2\right]t\right),$$

where

$$b_1 = 2LZ_0 - L^2 - \frac{(m + \sigma\rho v)^2 Z_0^2}{\sigma^2 v^2 + 2m\rho\sigma v + m^2},$$

$$b_2 = -2LZ_0,$$

$$b_3 = \frac{(m + \sigma\rho v)^2 Z_0^2}{\sigma^2 v^2 + 2m\rho\sigma v + m^2}.$$

(e) We are ready to calculate $\text{var}(W_T) = \text{var}(G_T^* + Z_T)$. We have

$$\begin{aligned}\text{var}(W) &= E(G_T^{*2}) + 2E(G_T^* Z_T) + E(Z_T^2) - [E(G_T^*)]^2 \\ &\quad - 2E(G_T^*)E(Z_T) - [E(Z_T)]^2 \\ &= J_T + 2Q_T + R_T - g_T^2 - 2g_T\bar{Z}_T - \bar{Z}_T^2,\end{aligned}$$

where J_T, Q_T, R_T, g_T and $\bar{Z}_T$ are stated previously, with $Z_0 = kS_0 \exp([\mu - (m\rho\sigma/v)]T)$. The solution is

$$\text{var}(W) = \frac{Z_0^2\sigma^2(1-\rho^2)v^2}{m^2 + 2\rho\sigma mv + \sigma^2 v^2}\left(\exp\left(\left[\sigma^2 + \left(\frac{2m\rho\sigma}{v}\right)\right]T\right) - \exp\left(-\left(\frac{m}{v}\right)^2 T\right)\right)$$
$$+ (L - Z_0)^2\left(\exp\left(-\left(\frac{m}{v}\right)^2 T\right) - \exp\left(-2\left(\frac{m}{v}\right)^2 T\right)\right).$$

APPENDIX B

The optimal fixed hedge. For comparison purposes, we work out the optimal hedge and its characteristics in the case in which the hedge must be fixed at time zero and not adjusted.

Again, for given L, we start with problem (3), but have the new definition

$$W(\theta) = kS_T + \theta(F_T - F_0), \qquad \theta \in \mathbb{R},$$

for terminal wealth given any fixed futures position θ. It is easy to check from the first order conditions that the solution is

$$\theta^* = \frac{LE(F_T - F_0) - kE[S_T(F_T - F_0)]}{E\left[(F_T - F_0)^2\right]}.$$

Using the properties of the log-normal distribution, this eventually reduces to

$$\theta^* = \frac{L(\exp(mT) - 1) - kS_0[\exp((\mu + m + \rho\sigma v)T) - \exp(\mu T)]}{F_0\left(1 - 2\exp(mT) + \exp((2m + v^2)T)\right)}.$$

In order to achieve a given mean M (provided $m \neq 0$), we can choose the target level L to be

$$L(M) = \frac{\left(M - kS_0\exp(\mu T)(1 - 2\exp(mT) + \exp(2m + v^2)T)\right)}{(\exp(mT) - 1)^2} + \frac{kS_0(\exp((\mu + m + \rho\sigma v)T) - \exp(\mu T))}{\exp(mT) - 1}.$$

In particular, the target level $\bar{L}$ that produces a mean equal to itself (and, therefore, the minimum variance) is

$$\bar{L} = \frac{kS_0\exp(\mu T)}{1 - \exp(v^2 T)} \times \left(\exp(\sigma\rho v T) + \exp(-mT) - \exp((-m + \rho\sigma v)T) - \exp(v^2 T)\right).$$

Finally, the mean and variance of the optimal position are

$$E[kS_t + \theta^*(F_T - F_0)] = kS_0\exp(\mu T) + F_0\theta^*(\exp(mT) - 1),$$

$$\begin{aligned}\operatorname{var}[kS_T + \theta^*(F_T - F_0)] = {} & k^2S_0^2\left(\exp((2\mu + \sigma^2)T) - \exp(2\mu T)\right) \\ & + (\theta^*)^2F_0^2\left(\exp((2m + v^2)T) - \exp(2mT)\right) \\ & + 2\theta^*F_0kS_0\left(\exp((\mu + m + \rho\sigma v)T)\right. \\ & \left. - \exp((\mu + m)T)\right).\end{aligned}$$

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