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INFORMATION PERCOLATION WITH EQUILIBRIUM SEARCH DYNAMICS

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We solve for the equilibrium dynamics of information sharing in a large population. Each agent is endowed with signals regarding the likely outcome of a random variable of common concern. Individuals choose the effort with which they search for others from whom they can gather additional information. When two agents meet, they share their information. The information gathered is further shared at subsequent meetings, and so on. Equilibria exist in which agents search maximally until they acquire sufficient information precision and then search minimally. A tax whose proceeds are used to subsidize the costs of search improves information sharing and can, in some cases, increase welfare. On the other hand, endowing agents with public signals reduces information sharing and can, in some cases, decrease welfare.

KEYWORDS: Information percolation, search, learning, equilibrium.

1. INTRODUCTION

WE CHARACTERIZE THE EQUILIBRIUM DYNAMICS of information sharing in a large population. An agent's optimal current effort to search for information sharing opportunities depends on that agent's current level of information and on the cross-sectional distribution of information quality and search efforts of other agents. Under stated conditions, in equilibrium, agents search maximally until their information quality reaches a trigger level and then search minimally. In general, it is not the case that raising the search-effort policies of all agents causes an improvement in information sharing. This monotonicity property does, however, apply to trigger strategies and enables a fixed-point algorithm for equilibria.

In our model, each member of the population is endowed with signals regarding the likely outcome of a Gaussian random variable Y of common concern. The ultimate utility of each agent is increasing in the agent's conditional precision of Y . Individuals therefore seek out others from whom they can gather additional information about Y . When agents meet, they share their information. The information gathered is then further shared at subsequent meetings and so on. Agents meet according to a technology for search and random matching, versions of which are common in the economics literatures covering labor markets, monetary theory, and financial asset markets. A distinction is that the search intensities in our model vary cross sectionally in a manner that depends on the endogenously chosen efforts of agents.

¹We are grateful for a conversation with Alain-Sol Sznitman, and for helpful remarks and suggestions from three referees and a co-editor. Duffie acknowledges support from the Swiss Finance Institute while visiting The University of Lausanne. Malamud gratefully acknowledges financial support from the National Centre of Competence in Research program "Financial Valuation and Risk Management" (NCCR FINRISK).

Going beyond prior work in this setting, we capture implications of the incentive to search more intensively whenever there is greater expected utility to be gained from the associated improvement in the information arrival process. Of course, the amount of information that can be gained from others depends on the efforts that the others themselves have made to search in the past. Moreover, the current expected rate at which a given agent meets others depends not only on the search efforts of that agent, but also on the current search efforts of the others. We assume complementarity in search efforts. Specifically, we suppose that the intensity of arrival of matches by a given agent increases in proportion to the current search effort of that agent, given the search efforts of the other agents. Each agent is modeled as fully rational, in a subgame-perfect Bayes–Nash equilibrium.

The existence and characterization of an equilibrium involve incentive consistency conditions on the jointly determined search efforts of all members of the population simultaneously. Each agent's lifetime search intensity process is the solution of a stochastic control problem, whose rewards depend on the search intensity processes chosen by other agents. We state conditions for a stationary equilibrium in which each agent's search effort at a given time depends only on that agent's current level of precision regarding the random variable Y of common concern.

We show that if the cost of search is increasing and convex in effort, then, taking as given the cross-sectional distribution of other agents' information quality and search efforts, the optimal search effort of any given agent is declining in the current information precision of that agent. This property holds, even out of equilibrium, because the marginal valuation of additional information for each agent declines as that agent gathers additional information. With proportional search costs, this property leads to equilibria with trigger policies that reduce search efforts to a minimum once a sufficient amount of information is obtained. Our proof of existence relies on a monotonicity result: Raising the assumed trigger level at which all agents reduce their search efforts leads to a first-order-dominant cross-sectional distribution of information arrivals.

We show by counterexample, however, that for general forms of search-effort policies, it is not generally true that the adoption of more intensive search policies leads to an improvement in population-wide information sharing. Just the opposite can occur. More intensive search at given levels of information can, in some cases, advance the onset of a reduction of the search efforts of agents who may be a rich source of information to others. This can lower access to richly informed agents in such a manner that, in some cases, information sharing is actually poorer.

We also analyze the welfare effects of some policy interventions. First, we analyze welfare gains that can be achieved with a lump-sum tax whose proceeds are used to subsidize the costs of search efforts. Under stated conditions, we show that this promotes positive search externalities that would not otherwise arise in equilibrium. Finally, we show that, with proportional search costs, additional public information leads in equilibrium to an unambiguous reduction

in the sharing of private information to the extent that there is, in some cases, a net negative welfare effect.

2. RELATED LITERATURE

Previous research in economics has investigated the issue of information aggregation. A large literature has focused on the aggregation of information through prices. For example, Grossman (1981) proposed the concept of rational-expectations equilibrium to capture the idea that prices aggregate information that is initially dispersed across investors. Wilson (1977), Milgrom (1981), Pesendorfer and Swinkels (1997), and Reny and Perry (2006) provided strategic foundations for the rational-expectations equilibrium concept in centralized markets.

In many situations, however, information aggregation occurs through local interactions rather than through common observation of market prices. For example, in decentralized markets, such as those for real estate and over-the-counter securities, agents learn from the bids of other agents in private auctions or bargaining sessions. Wolinsky (1990) and Blouin and Serrano (2001) studied information percolation in decentralized markets. In the literature on social learning, agents communicate with each other and choose actions based on information received from others. Banerjee and Fudenberg (2004), for example, studied information aggregation in a social-learning context.

Previous literature has shown that some forms of information externalities may slow down or prevent information aggregation. For example, Vives (1993) showed that information aggregation may be slowed when agents base their actions on public signals (price) rather than on private signals, making inference noisier. Bikhchandani, Hirshleifer, and Welch (1992) and Banerjee (1992) showed that agents may rely on publicly observed actions, ignoring their private signals, giving rise to informational cascades that prevent social learning.

Burguet and Vives (2000) and Amador and Weill (2008) posed related questions. Burguet and Vives (2000) studied a model with endogenous private information acquisition and public revelation of a noisy statistic of agents' predictions. Improving public information reduces agents' incentives to collect private information that could potentially slow down learning and reduce social welfare. Amador and Weill (2008) studied a model in which agents learn from public information as well as from the private observation of other agents' actions. Improving public information allows agents to rely less heavily on their private signals when choosing their actions, thus slowing down the diffusion of information and potentially reducing social welfare.

Our paper studies information aggregation in a social learning context. In contrast to previous studies, we analyze the equilibria of a game in which agents seek out other agents from whom they can gather information. This introduces a new source of information externality. If an agent chooses a high search intensity, he produces an indirect benefit to other agents by increasing both the

mean arrival rate at which the other agents will be matched and receive additional information, as well as the amount of information that the given agent is able to share when matched. We show that because agents do not take this externality into account when choosing their search intensities, social learning may be relatively inefficient or even collapse. We also show that endowing agents with public signals reduces their effort in searching for other agents from whom they can gather information. This reduces information sharing and can, in some cases, reduce social welfare.

In addition to the information externality problem, our paper shows that coordination problems may be important in information aggregation problems. In our model, there are multiple equilibria that are Pareto-ranked in terms of the search intensities of the agents. If agents believe that other agents are searching with lower intensity, they will also search with lower intensity, producing an equilibrium with slower social learning. Pareto-dominant equilibria, in which all agents search with higher intensity, may be possible, but it is not clear how agents coordinate to achieve such equilibria.

Our technology of search and matching is similar to that used in search-theoretic models that have provided foundations for competitive general equilibrium and for models of equilibrium in markets for labor, money, and financial assets.² Unlike these prior studies, we allow for information asymmetry about a common-value component, with learning from matching and with endogenously chosen search efforts.

Our model is related to those of Duffie and Manso (2007) and Duffie, Giroux, and Manso (2009), who provided an explicit solution for the evolution of posterior beliefs when agents are randomly matched in groups over time, exchanging their information with each other when matched. In contrast to these prior studies, however, we model the endogenous choice of search intensities. Moreover, we deal with Gaussian uncertainty, as opposed to the case of binary uncertainty that is the focus of these prior two papers. Further, we allow for the entry and exit of agents, and analyze the resulting stationary equilibria.

3. MODEL PRIMITIVES

A probability space (Ω, \mathcal{F}, P) and a nonatomic measure space (A, \mathcal{A}, α) of agents are fixed. We rely throughout on applications of the exact law of large numbers (LLN) for a continuum of random variables. A suitably precise version can be found in Sun (2006), based on technical conditions on the measurable subsets of $\Omega \times A$. As in the related literature, we also rely formally on

²Examples of theoretical work using random matching to provide foundations for competitive equilibrium include that of Rubinstein and Wolinsky (1985) and Gale (1987). Examples in labor economics include Pissarides (1985) and Mortensen (1986); examples in monetary theory include Kiyotaki and Wright (1993) and Trejos and Wright (1995); examples in finance include Duffie, Gârleanu, and Pedersen (2005), Lagos and Rocheteau (2009), and Weill (2008).

a continuous-time LLN for random search and matching that has only been rigorously justified in discrete-time settings.³ An alternative, which we avoid for simplicity, would be to describe limiting results for a sequence of models with discrete-time periods or finitely many agents as the lengths of time periods shrink or as the number of agents gets large.

All agents benefit, in a manner to be explained, from information about a particular random variable Y . Agents are endowed with signals from a space \mathcal{S} . The signals are jointly Gaussian with Y . Conditional on Y , the signals are pairwise independent. We assume that Y and all of the signals in \mathcal{S} have zero mean and unit variance, which is without loss of generality because they play purely informational roles.

Agent i enters the market with a random number N_{i0} of signals that is independent of Y and \mathcal{S} . The probability distribution π of N_{i0} does not depend on i . For almost every pair (i, j) of agents, N_{i0} and N_{j0} are independent, and their signal sets are disjoint.

When present in the market, agents meet other agents according to endogenous search and random matching dynamics to be described. Under these dynamics, for almost every pair (i, j) of agents, conditional on meeting at a given time t , there is zero probability that they meet at any other time and zero probability that the set of agents that i has met before t overlaps with the set of agents that j has met before t .

Whenever two agents meet, they share with each other enough information to reveal their respective current conditional distributions of Y . Although we do not model any strict incentive for matched agents to share their information, they have no reason not to do so. We could add to the model a joint production decision that would provide a strict incentive for agents to reveal their information when matched, but we have avoided this for simplicity.

By the joint Gaussian assumption and by induction in the number of prior meetings of each of a pair of currently matched agents, it is enough when sharing information relevant to Y that each of the two agents tells the other his or her immediately prior conditional mean and variance of Y . The conditional variance of Y given any n signals is

$$v(n) = \frac{1 - \rho^2}{1 + \rho^2(n - 1)},$$

where ρ is the correlation between Y and any signal. Thus, it is equivalent for the purpose of updating the agents' conditional distributions of Y that agent i tells his counterparty at any meeting at time t his or her current conditional mean X_{it} of Y and the total number N_{it} of signals that played a role in calculating the agent's current conditional distribution of Y . This number of signals is initially the endowed number N_{i0} , and is then incremented at each meeting

³See Duffie and Sun (2007).

by the number N_{jt} of signals that similarly influenced the information about Y that had been gathered by his counterparty j by time t .

Because the precision $1/v(N_{it})$ of the conditional distribution of Y given the information set \mathcal{F}_{it} of agent i at time t is strictly monotone in N_{it} , we speak of “precision” and N_{it} interchangeably.

Agents remain in the market for exponentially distributed times that are independent (pairwise) across agents, with parameter η' . If exiting at time t , agent i chooses an action A , measurable with respect to his current information \mathcal{F}_{it} , with cost $(Y - A)^2$. Thus, to minimize the expectation of this cost, agent i optimally chooses the action $A = E(Y|\mathcal{F}_{it})$ and incurs an optimal expected exit cost equal to the \mathcal{F}_{it} -conditional variance σ_{it}^2 of Y . Thus, while in the market, the agent has an incentive to gather information about Y in order to reduce the expected exit cost. We will shortly explain how search for other agents according to a costly effort process ϕ influences the current mean rate of arrival of matches and, thus, the information filtration $\{\mathcal{F}_{it} : t \geq 0\}$. Given a discount rate r , the agent’s lifetime utility (measuring time from the point of that agent’s market entrance) is

$$U(\phi) = E\left(-e^{-r\tau}\sigma_{i\tau}^2 - \int_0^\tau e^{-rt}K(\phi_t)dt \mid \mathcal{F}_{i0}\right),$$

where τ is the exit time and $K(c)$ is the cost rate for search effort level c , which is chosen at each time from some interval $[c_L, c_H] \subset \mathbb{R}_+$. We take the cost function K to be bounded and measurable, so $U(\phi)$ is bounded above and finite.

As we will show, essentially any exit utility formulation that is concave and increasing in $\sigma_{i\tau}^{-2}$ would result in precisely the same characterization of equilibrium that we shall provide.

The agent is randomly matched at a stochastic intensity that is proportional to the current effort of the agent, given the efforts of other agents. The particular pairings of counterparties are randomly chosen, in the sense of the law of large numbers for pairwise random matching described by Duffie and Sun (2007). The proportionality of matching intensities to effort levels means that an agent who exerts search effort c at time t has a current intensity (conditional mean arrival rate) of cbq_b of being matched to some agent from the set of agents currently using effort level b at time t , where q_b is the current fraction of the population using effort b . More generally, if the current cross-sectional distribution of effort by other agents is given by a measure ϵ , then the intensity of a match with agents whose current effort levels are in a set B is $c \int_B b d\epsilon(b)$. Our equilibrium analysis rests significantly on the complementarity of the search and matching technology, meaning that the more effort that an agent makes to be found, the more effective are the efforts of his counterparties to find him.

One can relax the parameterization of the search technology by rescaling the “effort” variable and making a corresponding adjustment of the effort cost

function $K(\cdot)$. Search incentives ultimately depend only on the mapping from the search costs of two types of agents to the expected contact rate between these types per unit mass of each.⁴

Agents enter the market at a rate proportional to the current mass q_t of agents in the market for some proportional “birth rate” $\eta > 0$. Because agents exit the market pairwise independently at intensity η' , the law of large numbers implies that the total quantity q_t of agents in the market at time t is $q_t = q_0 e^{(\eta - \eta')t}$ almost surely.

An advantage of the Gaussian informational setting is that the cross-sectional distribution of agents’ current conditional distributions of Y can be described in terms of the joint cross-sectional distribution of conditional precisions and conditional means. We now turn to a characterization of the dynamics of the cross-sectional distribution of posteriors as the solution of a particular deterministic differential equation in time.

The cross-sectional distribution μ_t of information precision at time t is defined, at any set B of positive integers, as the fraction $\mu_t(B) = \alpha(\{i : N_{it} \in B\})/q_t$ of agents whose precisions are currently in the set B . We sometimes abuse notation by writing $\mu_t(n)$ for the fraction of agents with precision n .

In the equilibria that we shall demonstrate, each agent chooses an effort level at time t that depends only on that agent’s current precision, according to a policy $C : \mathbb{N} \rightarrow [c_L, c_H]$ used by all agents. Assuming that such a search effort policy C is used by all agents, the cross-sectional precision distribution satisfies (almost surely) the differential equation

$$(1) \quad \frac{d}{dt} \mu_t = \eta(\pi - \mu_t) + \mu_t^C * \mu_t^C - \mu_t^C \mu_t^C(\mathbb{N}),$$

where $\mu_t^C(n) = C_n \mu_t(n)$ is the effort-weighted measure, $\mu * \nu$ denotes the convolution of two measures μ and ν , and

$$\mu_t^C(\mathbb{N}) = \sum_{n=1}^{\infty} C_n \mu_t(n)$$

is the cross-sectional average search effort. The mean exit rate η' plays no role in (1) because exit removes agents with a cross-sectional distribution that is the same as the current population cross-sectional distribution. The first term on the right-hand side of (1) represents the replacement of agents with newly entering agents. The convolution term $\mu_t^C * \mu_t^C$ represents the gross rate at which new agents of a given precision are created through matching and information

⁴For example, some of our results allow the cost function $K(\cdot)$ to be increasing and convex, so for these results we can allow mean contact rates to have decreasing returns to scale in effort costs. For our characterization of equilibrium trigger strategies, however, we rely on the assumption that expected matching rates are linear with respect to the cost of each agent’s effort.

sharing. For example, agents of a given posterior precision n can be created by pairing agents of prior respective precisions k and $n - k$, for any $k < n$, so the total gross rate of increase of agents with precision n from this source is

$$(\mu_t^C * \mu_t^C)(n) = \sum_{k=1}^{n-1} \mu_t(k)C(k)C(n - k)\mu_t(n - k).$$

The final term of (1) captures the rate $\mu_t^C(n)\mu_t^C(\mathbb{N})$ of replacement of agents with prior precision n with those of some new posterior precision that is obtained through matching and information sharing.

We anticipate that, in each state of the world ω and at each time t , the joint cross-sectional population distribution of precisions and posterior means of Y has a density f_t on $\mathbb{N} \times \mathbb{R}$, with evaluation $f_t(n, x)$ at precision n and posterior mean x of Y . This means that the fraction of agents whose conditional precision–mean pair (n, x) is in a given measurable set $B \subset \mathbb{N} \times \mathbb{R}$ is $\sum_n \int_{-\infty}^{+\infty} f_t(n, x)1_{\{(n,x) \in B\}} dx$. When it is important to clarify the dependence of this density on the state of world $\omega \in \Omega$, we write $f_t(n, x, \omega)$.

PROPOSITION 3.1: *For any search-effort policy function C , the cross-sectional distribution f_t of precisions and posterior means of the agents is almost surely given by*

$$(2) \quad f_t(n, x, \omega) = \mu_t(n)p_n(x|Y(\omega)),$$

where μ_t is the unique solution of the differential equation (1) and $p_n(\cdot|Y)$ is the Y -conditional Gaussian density of $E(Y|X_1, \dots, X_n)$ for any n signals X_1, \dots, X_n . This density has conditional mean

$$\frac{n\rho^2 Y}{1 + \rho^2(n - 1)}$$

and conditional variance

$$(3) \quad \sigma_n^2 = \frac{n\rho^2(1 - \rho^2)}{(1 + \rho^2(n - 1))^2}.$$

Appendix A provides a proof based on a formal application of the law of large numbers, and an independent proof by direct solution of the differential equation for f_t that arises from matching and information sharing. As n goes to infinity, the measure with density $p_n(\cdot|Y)$ converges, ω by ω (almost surely), to a Dirac measure at $Y(\omega)$. In other words, those agents who have collected a large number of signals have posterior means that cluster (cross sectionally) close to Y .

4. STATIONARY MEASURE

In our eventual equilibrium, all agents adopt an optimal search-effort policy function C , taking as given the presumption that all other agents adopt the same policy C and taking as given a stationary cross-sectional distribution of posterior conditional distributions of Y . Proposition 3.1 implies that this cross-sectional distribution is determined by the cross-sectional precision distribution μ_t . In a stationary setting, from (1), this precision distribution μ solves

$$(4) \quad 0 = \eta(\pi - \mu) + \mu^C * \mu^C - \mu^C \mu^C(\mathbb{N}),$$

which can be viewed as a form of algebraic Riccati equation. We consider only solutions that have the correct total mass $\mu(\mathbb{N})$ of 1. For brevity, we use the notation $\mu_i = \mu(i)$ and $C_i = C(i)$.

LEMMA 4.1: *Given a policy C , there is a unique measure μ satisfying the stationary-measure equation (4). This measure μ is characterized as follows. For any $\bar{C} \in [c_L, c_H]$, construct a measure $\bar{\mu}(\bar{C})$ by the algorithm*

$$\bar{\mu}_1(\bar{C}) = \frac{\eta \pi_1}{\eta + C_1 \bar{C}}$$

and then, inductively,

$$\bar{\mu}_k(\bar{C}) = \frac{\eta \pi_k + \sum_{l=1}^{k-1} C_l C_{k-l} \bar{\mu}_l(\bar{C}) \bar{\mu}_{k-l}(\bar{C})}{\eta + C_k \bar{C}}.$$

There is a unique solution \bar{C} to the equation $\bar{C} = \sum_{n=1}^{\infty} \bar{\mu}_n(\bar{C}) \bar{C}$. Given such a \bar{C} , we have $\mu = \bar{\mu}(\bar{C})$.

An important question is stability. That is, if the initial condition μ_0 is not sufficiently near the stationary measure, will the solution path $\{\mu_t : t \geq 0\}$ converge to the stationary measure? The dynamic equation (1) is an infinite-dimensional nonlinear dynamical system that could, in principle, have potentially complicated oscillatory behavior. In fact, a technical condition on the tail behavior of the effort policy function $C(\cdot)$ implies that the stationary distribution is globally attractive: From any initial condition, μ_t converges to the unique stationary distribution.

PROPOSITION 4.2: *Suppose that there is some integer N such that $C_n = C_N$ for $n \geq N$, and suppose that $\eta \geq c_H C_N$. Then the unique solution μ_t of (1) converges pointwise to the unique stationary measure μ .*

The proof, given in Appendix B, is complicated by the factor $\mu_t^c(\mathbb{N})$, which is nonlocal and involves $\mu_t(n)$ for each n . The proof takes the approach of representing the solution as a series $\{\mu_t(1), \mu_t(2), \dots\}$, each term of which solves an equation similar to (1), but without the factor $\mu_t^c(\mathbb{N})$. Convergence is proved for each term of the series. A tail estimate completes the proof. The convergence of μ_t does not guarantee that the limit measure is, in fact, the unique stationary measure μ . Appendix B includes a demonstration of this, based on Proposition B.13. As we later show in Proposition 5.3, the assumption that $C_n = C_N$ for all n larger than some integer N is implied merely by individual agent optimality under a mild condition on search costs.

Our eventual equilibrium will, in fact, be in the form of a trigger policy C^N , which for some integer $N \geq 1$ is defined by

$$C_n^N = \begin{cases} c_H, & n < N, \\ c_L, & n \geq N. \end{cases}$$

In other words, a trigger policy exerts maximal search effort until sufficient information precision is reached and then exerts minimal search effort thereafter. A trigger policy automatically satisfies the “flat-tail” condition of Proposition 4.2.

A key issue is whether search policies that exert more effort at each precision level actually generate more information sharing. This is an interesting question in its own right and also plays a role in obtaining a fixed-point proof of the existence of equilibria. For a given agent, access to information from others is entirely determined by the weighted measure μ^c , because if the given agent searches at some rate c , then the arrival rate of agents that offer n units of additional precision is $cC_n\mu_n = c\mu_n^c$. Thus, a first-order stochastically dominant shift in the measure μ^c is an unambiguous improvement in the opportunity of any agent to gather information. (A measure ν has first-order stochastic dominance (FOSD) relative to a measure θ if, for any nonnegative bounded increasing sequence f , we have $\sum_n f_n \nu_n \geq \sum_n f_n \theta_n$.)

The next result states that, at least when comparing trigger policies, a more intensive search policy results in an improvement in information sharing opportunities.

PROPOSITION 4.3: *Let μ^M and ν^N be the unique stationary measures corresponding to trigger policies C^M and C^N , respectively. Let $\mu_n^{c,N} = \mu_n^N C_n^N$ denote the associated search-effort-weighted measure. If $N > M$, then $\mu^{c,N}$ has the first-order dominance property over $\mu^{c,M}$.*

Superficially, this result may seem obvious. It says merely that if all agents extend their high-intensity search to a higher level of precision, then there will be an unambiguous upward shift in the cross-sectional distribution of information transmission rates. Our proof, shown in Appendix B, is not simple. Indeed,

we provide a counterexample below to the similarly “obvious” conjecture that any increase in the common search-effort policy function leads to a first-order-dominant improvement in information sharing.

Whether or not raising search efforts at each given level of precision improves information sharing involves two competing forces. The direct effect is that higher search efforts at a given precision increase the speed of information sharing, holding constant the precision distribution μ . The opposing effect is that if agents search harder, then they may earlier reach a level of precision at which they reduce their search efforts, which could, in principle, cause a downward shift in the cross-sectional average rate of information arrival. To make precise these competing effects, we return to the construction of the measure $\bar{\mu}(\bar{C})$ in Lemma 4.1 and write $\bar{\mu}(C, \bar{C})$ to show the dependence of this candidate measure on the conjectured average search effort \bar{C} as well as the given policy C . We emphasize that μ is the stationary measure for C provided $\mu = \bar{\mu}(C, \bar{C})$ and $\bar{C} = \sum_n C_n \mu_n$. From the algorithm stated in Lemma 4.1, $\bar{\mu}_k(C, \bar{C})$ is increasing in C and *decreasing* in \bar{C} for all k . (A proof, by induction in k , is given in Appendix B.) Now the relevant question is, “What effect does increasing C have on the stationary average search effort, \bar{C} , solving the equation $\bar{C} = \sum_n \bar{\mu}_n(C, \bar{C}) C_n$?” The following proposition shows that increasing C has a positive effect on \bar{C} and, thus, through this channel has a negative effect on $\bar{\mu}_k(C, \bar{C})$.

PROPOSITION 4.4: *Let μ and ν be the stationary measures associated with policies C and D . If $D \geq C$, then $\sum_n D_n \nu_n \geq \sum_n C_n \mu_n$. That is, any increase in search policy increases the equilibrium average search effort.*

For trigger policies, the direct effect of increasing the search-effort policy C dominates the “feedback” effect on the cross-sectional average rate of effort. For other types of policies, this need not be the case, as shown by the following counterexample, whose proof is given in Appendix B.

EXAMPLE 4.5: Suppose that $\pi_2 > 2\pi_1$, that is, the probability of being endowed with two signals is more than double the probability of being endowed with only one signal. Consider a policy C with $C_n = 0$ for $n \geq 3$. Fix $C_2 > 0$ and consider a variation of C_1 . For C_1 sufficiently close to C_2 , we show in Appendix B that

$$\sum_{k=2}^{\infty} C_k \mu_k = C_2 \mu_2$$

is monotone decreasing in C_1 . Thus, if we consider the increasing sequence

$$\begin{aligned} f_1 &= 0, \\ f_n &= 1 \end{aligned} \qquad (n \geq 2),$$

we have $f \cdot \mu^C = C_2 \mu_2$ strictly decreasing in C_1 for C_1 in a neighborhood of C_2 , so that we do not have FOSD of μ^C with increasing C . In fact, more search effort by those agents with precision 1 can actually lead to poorer information sharing. To see this, consider the policies $D = (1, 1, 0, 0, \dots)$ and $C = (1 - \epsilon, 1, 0, 0, \dots)$. The measure μ^C has FOSD over the measure μ^D for any⁵ sufficiently small $\epsilon > 0$.

5. OPTIMALITY

In this section, we study the optimal policy of a given agent who presumes that precision is distributed in the population according to some fixed measure μ and further presumes that other agents search according to a conjectured policy function C . We let $\bar{C} = \sum_n C_n \mu_n$ denote the average search effort.

Given the conjectured market properties (μ, C) , each agent i chooses some search-effort process $\phi : \Omega \times [0, \infty) \rightarrow [c_L, c_H]$ that is progressively measurable with respect to that agent's information filtration $\{\mathcal{F}_{it} : t \geq 0\}$, meaning that ϕ_t is based only on current information. The posterior distribution of Y given \mathcal{F}_{it} has conditional variance $v(N_t)$, where N is the agent's precision process and $v(n)$ is the variance of Y given any n signals.

For a discount rate r on future expected benefits and given the conjectured market properties (μ, C) , an agent solves the problem

$$(5) \quad U(\phi) = \sup_{\phi} E \left(-e^{-r\tau} v(N_{\tau}^{\phi}) - \int_0^{\tau} e^{-st} K(\phi_t) dt \mid \mathcal{F}_{i0} \right),$$

where τ is the time of exit, exponentially distributed with parameter η' , and where the agent's precision process N^{ϕ} is the pure-jump process with a given initial condition N_0 , with jump-arrival intensity $\phi_t \bar{C}$, and with jump-size probability distribution μ^C / \bar{C} , that is, with probability $C(j)\mu(j) / \bar{C}$ of jump size j . We have abused notation by measuring calendar time for the agent from the time of that agent's market entry.

For generality, we relax from this point on the assumption that the exit disutility is the conditional variance $v(N_{\tau}^{\phi})$ and we allow the exit utility to be of the more general form $u(N_{\tau}^{\phi})$ for any bounded increasing concave⁶ function $u(\cdot)$ on the positive integers. It can be checked that $u(n) = -v(n)$ is indeed a special case.

We say that ϕ^* is an optimal search-effort process given (μ, C) if ϕ^* attains the supremum (5). We further say that a policy function $\Gamma : \mathbb{N} \rightarrow [c_L, c_H]$ is optimal given (μ, C) if the search-effort process $\{\Gamma(N_t) : t \geq 0\}$ is optimal, where the precision process N uniquely satisfies the stochastic differential equation

⁵For this, we can without loss of generality take $f_1 = 1$ and calculate that $h(\epsilon) = f \cdot \mu^C$ is decreasing in ϵ for sufficiently small $\epsilon > 0$.

⁶We say that a real-valued function F on the integers is concave if $F(j+2) + F(j) \leq 2F(j+1)$.

with jump-arrival intensity $\Gamma(N_t)\bar{C}$ and with jump-size distribution μ^C/\bar{C} . (Because $\Gamma(n)$ is bounded by c_H , there is a unique solution N to this stochastic differential equation; see Protter (2005).)

We characterize agent optimality given (μ, C) using the principle of dynamic programming, showing that the indirect utility, or “value,” V_n for precision n satisfies the Hamilton–Jacobi–Bellman (HJB) equation for optimal search effort given by

$$(6) \quad 0 = -(r + \eta')V_n + \eta'u_n + \sup_{c \in [c_L, c_H]} \left\{ -K(c) + c \sum_{m=1}^{\infty} (V_{n+m} - V_n)\mu_m^C \right\}.$$

A standard martingale-based verification proof of the following result is found in Appendix C.

LEMMA 5.1: *If V is a bounded solution of the Hamilton–Jacobi–Bellman equation (6) and Γ is a policy with the property that, for each n , the supremum in (6) is attained at Γ_n , then Γ is an optimal policy function given (μ, C) , and V_{N_0} is the value of this policy.*

We begin to lay out some of the properties of optimal policies, based on conditions on the search-cost function $K(\cdot)$.

PROPOSITION 5.2: *Suppose that K is increasing, convex, and differentiable. Then, given (μ, C) , there is a policy Γ that is optimal for all agents and the optimal search effort Γ_n is monotone decreasing in the current precision n .*

To calculate a precision threshold \bar{N} , independent of the conjectured population properties (μ, C) , above which it is optimal to search minimally, we let $\bar{u} = \lim_n u(n)$, which exists because u_n is increasing in n and bounded, and we let

$$\bar{N} = \sup \{ n : c_H \eta'(r + \eta')(\bar{u} - u(n)) \geq K'(c_L) \},$$

which is finite if $K'(c_L) > 0$. A proof of the following result is found in Appendix C.

PROPOSITION 5.3: *Suppose that $K(\cdot)$ is increasing, differentiable, and convex, with $K'(c_L) > 0$. Then, for any optimal search-effort policy Γ ,*

$$\Gamma_n = c_L \quad (n \geq \bar{N}).$$

In the special case of proportional and nontrivial search costs, it is, in fact, optimal for all agents to adopt a trigger policy that searches at maximal effort until a trigger level of precision is reached and searches at minimal effort

thereafter. This result, stated next, is a consequence of our prior results that an optimal policy is decreasing and eventually reaches c_L , and of the fact that with linear search costs, an optimal policy is “bang-bang,” therefore taking the maximal effort level c_H at first and then eventually switching to the minimal effort c_L at a sufficiently high precision.

PROPOSITION 5.4: *Suppose that $K(c) = \kappa c$ for some scalar $\kappa > 0$. Then, given (μ, C) , there is a trigger policy that is optimal for all agents.*

6. EQUILIBRIUM

An equilibrium is a search-effort policy function C that satisfies the statements: (i) there is a unique stationary cross-sectional precision measure μ satisfying the associated equation (4) and (ii) taking as given the market properties (μ, C) , the search-effort policy function C is indeed optimal for each agent. Our main result is that with proportional search costs, there exists an equilibrium in the form of a trigger policy.

THEOREM 6.1: *Suppose that $K(c) = \kappa c$ for some scalar $\kappa > 0$. Then there exists a trigger policy that is an equilibrium.*

The theorem is proved using the following Proposition 6.2 and Corollary 6.3. We let C^N be the trigger policy with trigger at precision level N and we let μ^N denote the associated stationary measure. We let $\mathcal{N}(N) \subset \mathbb{N}$ be the set of trigger levels that are optimal given the conjectured market properties (μ^N, C^N) associated with a trigger level N . We can look for an equilibrium in the form of a fixed point of the optimal trigger-level correspondence $\mathcal{N}(\cdot)$, that is, some N such that $N \in \mathcal{N}(N)$. Theorem 6.1 does not rely on the stability result that from any initial condition, μ_t converges to μ . This stability applies, by Proposition 4.2, provided that $\eta \geq c_H c_L$.

PROPOSITION 6.2: *Suppose that $K(c) = \kappa c$ for some scalar $\kappa > 0$. Then $\mathcal{N}(N)$ is increasing in N in the sense that if $N' \geq N$ and if $k \in \mathcal{N}(N)$, then there exists some $k' \geq k$ in $\mathcal{N}(N')$. Further, there exists a uniform upper bound on $\mathcal{N}(N)$, independent of N , given by*

$$\bar{N} = \max\{j : c_H \eta'(r + \eta)(\bar{u} - u(j)) \geq \kappa\}.$$

Theorem 6.1 then follows from the corollary:

COROLLARY 6.3: *The correspondence \mathcal{N} has a fixed point N . An equilibrium is given by the associated trigger policy C^N .*

Our proof, found in Appendix D, leads to the following algorithm for computing symmetric pure strategy equilibria of the game. The algorithm finds all such equilibria in trigger strategies.

ALGORITHM: Start by letting $N = \bar{N}$.

Step 1. Compute $\mathcal{N}(N)$. If $N \in \mathcal{N}(N)$, then output C^N (an equilibrium of the game). Go to the next step.

Step 2. If $N > 0$, go back to Step 1 with $N = N - 1$; otherwise quit.

There may exist multiple equilibria of the game. The following proposition shows that the equilibria are Pareto-ranked according to their associated trigger levels and that there is never “too much” search in equilibrium.

PROPOSITION 6.4: Suppose that $K(c) = \kappa c$ for some scalar $\kappa > 0$. If C^N is an equilibrium of the game, then it Pareto dominates a setting in which all agents employ a policy $C^{N'}$ for a trigger level $N' < N$. In particular, an equilibrium associated with a trigger level N Pareto dominates an equilibrium with a trigger level lower than N .

6.1. Equilibria With Minimal Search

We now consider conditions under which there are equilibria with minimal search, corresponding to the trigger precision $N = 0$. The idea is that such equilibria can arise because a presumption that other agents make minimal search efforts can lead to a conjecture of such poor information sharing opportunities that any given agent may not find it worthwhile to exert more than minimal search effort. We give an explicit sufficient condition for such equilibria, a special case of which is $c_L = 0$. Clearly, with $c_L = 0$, it is pointless for any agent to expend any search effort if he or she assumes that all other agents make no effort to be found.

Let μ^0 denote the stationary precision distribution associated with minimal search, so that $\bar{C} = c_L$ is the average search effort. The value function V of any agent solves

$$(7) \quad (r + \eta' + c_L^2)V_n = \eta' u_n - K(c_L) + c_L^2 \sum_{m=1}^{\infty} V_{n+m} \mu_m^0.$$

Consider the bounded increasing sequence f given by

$$f_n = (r + \eta' + c_L^2)^{-1}(\eta' u_n - K(c_L)).$$

Define the operator A on the space of bounded sequences by

$$(A(g))_n = \frac{c_L^2}{r + \eta' + c_L^2} \sum_{m=1}^{\infty} g_{n+m} \mu_m^0.$$

LEMMA 6.5: *The unique, bounded solution V to (7) is given by*

$$V = (I - A)^{-1}(f) = \sum_{j=0}^{\infty} A^j(f),$$

which is concave and monotone increasing.

To provide simple conditions for minimal-search equilibria, let

$$(8) \quad B = c_L \sum_{m=1}^{\infty} (V_{1+m} - V_1) \mu_m^0 \geq 0.$$

THEOREM 6.6: *Suppose that $K(\cdot)$ is convex, increasing, and differentiable. Then the minimal-search policy C , that with $C(n) = c_L$ for all n , is an equilibrium if and only if $K'(c_L) \geq B$. In particular, if $c_L = 0$, then $B = 0$ and minimal search is always an equilibrium.*

Intuitively, when the cost of search is small, there should exist equilibria with active search.

PROPOSITION 6.7: *Suppose that $K(c) = \kappa c$ and $c_L = 0$. If $\pi_1 > 0$ and*

$$(9) \quad \kappa - \frac{\eta'(u(2) - u(1))c_H\mu_1^1}{r + \eta'} < 0,$$

then there exists an equilibrium trigger policy C^N with $N \geq 1$. This equilibrium strictly Pareto dominates the no-search equilibrium.

7. POLICY INTERVENTIONS

In this section, we discuss the potential welfare implications of policy interventions. First, we analyze the potential to improve welfare by a tax whose proceeds are used to subsidize the costs of search efforts. This has the potential benefit of positive search externalities that may not otherwise arise in equilibrium because each agent does not search unless others are searching, even though there are feasible search efforts that would make all agents better off.

Then, we study the potentially adverse implications of providing all entrants with some additional common information. Although there is some direct benefit of the additional information, we show that with proportional search costs, additional public information leads to an unambiguous reduction in the sharing of private information, to the extent that there is, in some cases, a net negative welfare effect.

In both cases, welfare implications are judged in terms of the utilities of agents as they enter the market. In this sense, the welfare effect of an intervention is said to be positive if it improves the utility of every agent at the point in time that the agent enters and to be negative if it causes a reduction in the utilities of all entering agents.

7.1. *Subsidizing Search*

The adverse welfare implications of low information sharing may be attenuated by a tax whose proceeds are used to subsidize search costs. An example could be research subsidies aimed at the development of technologies that reduce communication costs. Another example is a subsidy that defrays some of the cost of using existing communication technologies.

We assume in this subsection that each agent pays a lump-sum tax τ at entry. Search costs are assumed to be proportional, at rate κc for some $\kappa > 0$. Each agent is also offered a proportional reduction δ in search costs, so that the after-subsidy search-cost function of each agent is $K_\delta(c) = (\kappa - \delta)c$. The lump-sum tax has no effect on equilibrium search behavior, so we can solve for an equilibrium policy C , as before, based on an after-subsidy proportional search cost of $\kappa - \delta$. Because of the law of large numbers, the total per capita rate $\tau\eta$ of tax proceeds can then be equated to the total per capita rate of subsidy by setting

$$\tau = \frac{1}{\eta} \delta \sum_n \mu_n C_n.$$

The search subsidy can potentially improve welfare by addressing the failure, in a low-search equilibrium, to exploit positive search externalities. As Proposition 6.4 shows, there is never too much search in equilibrium. The following Lemma 7.1 and Proposition 7.2 show that, indeed, equilibrium search effort is increasing in the search subsidy rate δ .

LEMMA 7.1: *Suppose that $K(c) = \kappa c$ for some $\kappa > 0$. For given market conditions (μ, C) , the trigger precision level N of an optimal search-effort policy is increasing in the search subsidy rate δ . That is, if N is an optimal trigger level given a subsidy δ , then for any search subsidy $\delta' \geq \delta$, there exists a higher optimal trigger $N' \geq N$.*

Coupled with Proposition 4.3, Lemma 7.1 implies that an increase in the subsidy allows an increase (in the sense of first-order dominance) in information sharing. A direct consequence of this lemma is the following proposition.

PROPOSITION 7.2: *Suppose that $K(c) = \kappa c$ for some $\kappa > 0$. If C^N is an equilibrium with proportional search subsidy δ , then for any $\delta' \geq \delta$, there exists some $N' \geq N$ such that $C^{N'}$ is an equilibrium with proportional search subsidy δ' .*

EXAMPLE: Suppose, for some integer $N > 1$, that $\pi_0 = 1/2$, $\pi_N = 1/2$, and $c_L = 0$. This setting is equivalent to that of Proposition 6.7, after noting that every information transfer is in blocks of N signals each, resulting in a model isomorphic to one in which each agent is endowed with one private signal of a particular higher correlation. Recalling that inequality (9) determines whether zero search is optimal, we can exploit continuity of the left-hand side of this inequality to choose parameters so that, given market conditions (μ^N, C^N) , agents have a strictly positive but arbitrarily small increase in utility when choosing search policy C^0 over policy C^N . With this, C^0 is the unique equilibrium. This is before considering a search subsidy. We now consider a model that is identical with the exception that each agent is taxed at entry and given search subsidies at the proportional rate δ . We can choose δ so that all agents strictly prefer C^N to C^0 (the nonzero search condition (9) is satisfied) and C^N is an equilibrium. For sufficiently large N , all agents have strictly higher indirect utility in the equilibrium with the search subsidy than they do in the equilibrium with the same private-signal endowments and no subsidy.

7.2. Educating Agents at Birth

A policy that might superficially appear to mitigate the adverse welfare implications of low information sharing is to “educate” all agents by giving all agents additional public signals at entry. We assume for simplicity that the $M \geq 1$ additional public signals are drawn from the same signal set \mathcal{S} . When two agents meet and share information, they take into account that the information reported by the other agent contains the effect of the additional public signals. (The implications of the reported conditional mean and variance for the conditional mean and variance associated with a counterparty’s nonpublic information can be inferred from the public signals by using Lemma A.1.) Because of this, our prior analysis of information sharing dynamics can be applied without alteration merely by treating the precision level of a given agent as the total precision less the public precision and by treating the exit utility of each agent for n nonpublic signals as $\hat{u}(n) = u(n + M)$.

The public signals influence optimal search efforts. Given the market conditions (μ, C) , the indirect utility V_n for nonpublic precision n satisfies the Hamilton–Jacobi–Bellman equation for optimal search effort given by

$$(10) \quad 0 = -(r + \eta')V_n + \eta' u_{M+n} + \sup_{c \in [c_L, c_H]} \left\{ -K(c) + c \sum_{m=1}^{\infty} (V_{n+m} - V_n) \mu_m^C \right\}.$$

Educating agents at entry with public signals has two effects. On one hand, when agents enter the market they are better informed than if they had not received the extra signals. On the other hand, this extra information may reduce agents’ incentives to search for more information, slowing down information

percolation. Below, we show an example in which the net effect is a strict welfare loss. First, however, we establish that adding public information causes an unambiguous reduction in the sharing of private information.

LEMMA 7.3: *Suppose that $K(c) = \kappa c$ for some $\kappa > 0$. For given market conditions (μ, C) , the trigger level N in nonpublic precision of an optimal policy C^N is decreasing in the precision M of the public signals. (That is, if N is an optimal trigger level of precision given public-signal precision M , then for any higher public precision $M' \geq M$, there exists a lower optimal trigger $N' \leq N$.)*

Coupled with Proposition 4.3, Lemma 7.3 implies that adding public information leads to a reduction (in the sense of first-order dominance) in information sharing. A direct consequence of Lemma 7.3 is the following result.

PROPOSITION 7.4: *Suppose that $K(c) = \kappa c$ for some $\kappa > 0$. If C^N is an equilibrium with M public signals, then for any $M' \leq M$, there exists some $N' \geq N$ such that $C^{N'}$ is an equilibrium with M' public signals.*

In particular, by removing all public signals, as in the following example, we can get strictly superior information sharing and, in some cases, a strict welfare improvement.

EXAMPLE: As in the previous example, suppose, for some integer $N > 1$, that $\pi_0 = 1/2$, $\pi_N = 1/2$, and $c_L = 0$. This setting is equivalent to that of Proposition 6.7, after noting that every information transfer is in blocks of N signals each, resulting in a model isomorphic to one in which each agent is endowed with one private signal of a particular higher correlation. Analogously with the previous example, we can exploit continuity in the model parameters of the left-hand side of inequality (9), determining whether zero search is optimal, to choose the parameters so that, given market conditions (μ^N, C^N) , agents have a strict but arbitrarily small preference of policy C^N over C^0 . We now consider a model that is identical with the exception that each agent is given $M = 1$ public signal at entry. With this public signal, again using continuity, we can choose parameters so that all agents strictly prefer C^0 to C^N (the nonzero search condition (9) fails) and C^0 is the only equilibrium. For sufficiently large N or, equivalently, for any $N \geq 2$ and sufficiently small signal correlation ρ , all agents have strictly lower indirect utility in the equilibrium with the public signal at entry than they do in the equilibrium with the same private-signal endowments and no public signal.

8. COMPARATIVE STATICS

We conclude with a brief selection of comparative statics. We say that the set of equilibrium trigger levels is increasing in a parameter α if for any $\alpha_1 \geq \alpha$

and any equilibrium trigger level N for α , there exists an equilibrium trigger level $N_1 \geq N$ corresponding to α_1 . For simplicity, we take the specific exit disutility given by conditional variance, rather than allowing an arbitrary bounded concave increasing exit utility.

PROPOSITION 8.1: *Suppose that the exit disutility is conditional variance; that is, $-u_n = v_n$. Then the set of equilibrium trigger levels*

- *Is increasing in the exit intensity η' .*
- *Is decreasing in the discount rate r .*
- *Is decreasing in the signal “quality” ρ^2 provided that $\rho^2 \geq \sqrt{2} - 1 \approx 0.414$.*

A proof is given in the final Appendix F. The first two results, regarding η' and r , would apply for an arbitrary bounded concave increasing exit utility.

Roughly speaking, scaling up the number of primitively endowed signals by a given integer multiple has the same effect as increasing the signal quality ρ^2 by a particular amount, so one can provide a corresponding comparative static concerning the distribution π of the number of endowed signals. For sufficiently small $\rho^2 < \sqrt{2} - 1$, we suspect that nothing general can be said about the monotonicity of equilibrium search efforts with respect to signal quality.

APPENDIX A: PROOFS FOR SECTION 3: INFORMATION SHARING MODEL

LEMMA A.1: *Suppose that $Y, X_1, \dots, X_n, Z_1, \dots, Z_m$ are joint Gaussian, and that X_1, \dots, X_n and Z_1, \dots, Z_m all have correlation ρ with Y and are Y -conditionally independent and identically distributed (i.i.d.). Then*

$$\begin{aligned}
 E(Y|X_1, \dots, X_n, Z_1, \dots, Z_m) \\
 &= \frac{\gamma_n}{\gamma_{n+m}} E(Y|X_1, \dots, X_n) + \frac{\gamma_m}{\gamma_{m+n}} E(Y|Z_1, \dots, Z_m),
 \end{aligned}$$

where $\gamma_k = 1 + \rho^2(k - 1)$.

PROOF: The proof is by calculation. If (Y, W) are joint mean-zero Gaussian and W has an invertible covariance matrix, then by a well known result,

$$E(Y|W) = W^\top \text{cov}(W)^{-1} \text{cov}(Y, W).$$

It follows by calculation that

$$(A.1) \quad E(Y|X_1, \dots, X_n) = \beta_n(X_1 + \dots + X_n),$$

where

$$\beta_n = \frac{\rho}{1 + \rho^2(n - 1)}.$$

Likewise,

$$\begin{aligned} & E(Y|X_1, \dots, X_n, Z_1, \dots, Z_m) \\ &= \beta_{n+m}(X_1 + \dots + X_n + Z_1 + \dots + Z_m) \\ &= \beta_{n+m} \left(\frac{E(Y|X_1, \dots, X_n)}{\beta_n} + \frac{E(Y|Z_1, \dots, Z_m)}{\beta_m} \right). \end{aligned}$$

The result follows from the fact that $\beta_{n+m}/\beta_n = \gamma_n/\gamma_{n+m}$. Q.E.D.

COROLLARY A.2: *The conditional probability density $p_n(\cdot|Y)$ of $E(Y|X_1, \dots, X_n)$ given Y is almost surely Gaussian with conditional mean*

$$\frac{n\rho^2 Y}{1 + \rho^2(n - 1)}$$

and with conditional variance

$$(A.2) \quad \sigma_n^2 = \frac{n\rho^2(1 - \rho^2)}{(1 + \rho^2(n - 1))^2}.$$

PROOF OF PROPOSITION 3.1: We use the conditional law of large numbers (LLN) to calculate the cross-sectional population density f_t . Later, we independently calculate f_t , given the appropriate boundary condition f_0 , by a direct solution of the particular dynamic equation that arises from updating beliefs at matching times.

Taking the first, more abstract, approach, we fix a time t and state of the world ω , and let $W_n(\omega)$ denote the set of all agents whose current precision is n . We note that $W_n(\omega)$ depends nontrivially on ω . This set $W_n(\omega)$ has an infinite number of agents whenever $\mu_i(n)$ is nonzero, because the space of agents is nonatomic. In particular, the restriction of the measure on agents to $W_n(\omega)$ is nonatomic. Agent i from this set $W_n(\omega)$ has a current conditional mean of Y that is denoted $U_i(\omega)$. Now consider the cross-sectional distribution $q_n(\omega)$ —a measure on the real line—of $\{U_i(\omega) : i \in W_n(\omega)\}$. Note that the random variables U_i and U_j are Y -conditionally independent for almost every distinct pair (i, j) , by the random matching model, which implies by induction in the number of their finitely many prior meetings that they have conditioned on distinct subsets of signals, and that the only source of correlation in U_i and U_j is the fact that each of these posteriors is a linear combination of Y and of other pairwise-independent variables that are also jointly independent of Y .

Conditional on the event $\{N_{it} = n\}$ that agent i is in the set $W_n(\omega)$ and conditional on Y , U_i has the Gaussian conditional density $p_n(\cdot|Y)$ recorded in Corollary A.2. This conditional density function does not depend on i . Thus, by a formal application of the law of large numbers, in almost every state of

the world ω , $q_n(\omega)$ has the same distribution as the (W_n, Y) -conditional distribution of U_i for any i . Thus, for almost every ω , the cross-sectional distribution $q_n(\omega)$ of posteriors over the subset $W_n(\omega)$ of agents has the density $p_n(\cdot|Y(\omega))$. In summary, for almost every state of the world, the fraction $\mu_t(n)$ of the population that has received n signals has a cross-sectional density $p_n(\cdot|Y(\omega))$ over their posteriors for Y .

We found it instructive to consider a more concrete proof based on a computation of the solution of the appropriate differential equation for f_t , using the LLN to set the initial condition f_0 . Lemma A.1 implies that when an agent with joint type (n, x) exchanges all information with an agent whose type is (m, y) , both agents achieve posterior type

$$\left(m + n, \frac{\gamma_n}{\gamma_{m+n}}x + \frac{\gamma_m}{\gamma_{m+n}}y\right).$$

We therefore have the dynamic equation

$$\begin{aligned} \text{(A.3)} \quad \frac{d}{dt}f_t(n, x) &= \eta(\Pi(n, x) - f_t(n, x)) + (f_t \circ f_t)(n, x) \\ &\quad - C_n f_t(n, x) \sum_{m=1}^{\infty} C_m \int_{\mathbb{R}} f_t(m, x) dx, \end{aligned}$$

where $\Pi(n, x) = \pi(n) p_n(x|Y(\omega))$ and

$$\begin{aligned} (f_t \circ f_t)(n, x) &= \sum_{m=1}^{n-1} \frac{\gamma_n}{\gamma_{n-m}} C_{n-m} C_m \\ &\quad \times \int_{-\infty}^{+\infty} f_t\left(n - m, \frac{\gamma_n x - \gamma_m y}{\gamma_{n-m}}\right) f_t(m, y) dy. \end{aligned}$$

It remains to solve this ordinary differential equation (ODE) for f_t . We will use the following calculation.

LEMMA A.3: *Let $q_1(x)$ and $q_2(x)$ be the Gaussian densities with respective means M_1, M_2 and variances σ_1^2, σ_2^2 . Then,*

$$\frac{\gamma_n}{\gamma_{n-m}} \int_{-\infty}^{+\infty} q_1\left(\frac{\gamma_n x - \gamma_m y}{\gamma_{n-m}}\right) q_2(y) dy = q(x),$$

where $q(x)$ is the density of a Gaussian with mean

$$M = \frac{\gamma_{n-m}}{\gamma_n} \mu_1 + \frac{\gamma_m}{\gamma_n} \mu_2$$

and variance

$$\sigma^2 = \frac{\gamma_{n-m}^2}{\gamma_n^2} \sigma_1^2 + \frac{\gamma_m^2}{\gamma_n^2} \sigma_2^2.$$

PROOF: Let X be a random variable with density $q_1(x)$ and let Y be an independent variable with density $q_2(x)$. Then

$$Z = \gamma_n^{-1}(\gamma_{n-m}X + \gamma_m Y)$$

is also normal with mean M and variance σ^2 . On the other hand, $\gamma_n^{-1}\gamma_{n-m}X$ and $\gamma_n^{-1}\gamma_m Y$ are independent with densities

$$\frac{\gamma_n}{\gamma_{n-m}} q_1\left(\frac{\gamma_n}{\gamma_{n-m}} x\right)$$

and

$$\frac{\gamma_n}{\gamma_m} q_2\left(\frac{\gamma_n}{\gamma_m} x\right),$$

respectively. Consequently, the density of Z is the convolution

$$\begin{aligned} \text{(A.4)} \quad & \frac{\gamma_n^2}{\gamma_{n-m}\gamma_m} \int_{\mathbb{R}} q_1\left(\frac{\gamma_n}{\gamma_{n-m}}(x-y)\right) q_2\left(\frac{\gamma_n}{\gamma_m}y\right) dy \\ & = \frac{\gamma_n}{\gamma_{n-m}} \int_{-\infty}^{+\infty} q_1\left(\frac{\gamma_n x - \gamma_m y}{\gamma_{n-m}}, \sigma_{n-m}\right) q_2(z) dz, \end{aligned}$$

where we have made the transformation $z = \gamma_n \gamma_m^{-1} y$.

Q.E.D.

LEMMA A.4: *The density*

$$f_t(n, x, \omega) = \mu_t(n) p_n(x|Y(\omega))$$

solves the evolution equation (A.3) if and only if the distribution μ_t of precisions solves the evolution equation (1).

PROOF: By Lemma A.3 and Corollary A.2,

$$\frac{\gamma_n}{\gamma_{n-m}} \int_{-\infty}^{+\infty} p_{n-m}\left(\frac{\gamma_n x - \gamma_m y}{\gamma_{n-m}} \middle| Y(\omega)\right) p_m(y|Y(\omega)) dy$$

is conditionally Gaussian with mean

$$\frac{\gamma_{n-m}}{\gamma_n} \frac{(n-m)\rho^2 Y}{1 + \rho^2(n-m-1)} + \frac{\gamma_m}{\gamma_n} \frac{m\rho^2 Y}{1 + \rho^2(m-1)} = \frac{n\rho^2 Y}{1 + \rho^2(n-1)}$$

and conditional variance

$$\begin{aligned} \sigma^2 &= \frac{\gamma_{n-m}^2}{\gamma_n^2} \frac{(n-m)\rho^2(1-\rho^2)}{(1+\rho^2(n-m-1))^2} + \frac{\gamma_m^2}{\gamma_n^2} \sigma_2^2 \frac{m\rho^2(1-\rho^2)}{(1+\rho^2(m-1))^2} \\ &= \frac{n\rho^2(1-\rho^2)}{(1+\rho^2(n-1))^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} (f_t \circ f_t)(n, x) &= \sum_{m=1}^{n-1} \frac{\gamma_n}{\gamma_{n-m}} C_{n-m} C_m \int_{-\infty}^{+\infty} f_t\left(n-m, \frac{\gamma_n x - \gamma_m y}{\gamma_{n-m}}\right) f_t(m, y) dy \\ &= \sum_{m=1}^{n-1} C_{n-m} C_m \mu_t(n-m) \mu_t(m) \frac{\gamma_n}{\gamma_{n-m}} \\ &\quad \times \int_{-\infty}^{+\infty} p_{n-m}\left(\frac{\gamma_n x - \gamma_m y}{\gamma_{n-m}} \middle| Y(\omega)\right) p_m(y | Y(\omega)) dy \\ &= \sum_{m=1}^{n-1} C_{n-m} C_m \mu_t(n-m) \mu_t(m) p_n(x | Y(\omega)). \end{aligned}$$

Substituting the last identity into (A.3), we get the required result. *Q.E.D.*

APPENDIX B: PROOFS FOR SECTION 4: STATIONARY DISTRIBUTIONS

This appendix provides proofs of the results on the existence, stability, and monotonicity properties of the stationary cross-sectional precision measure μ .

B.1. Existence of the Stationary Measure

PROOF OF LEMMA 4.1: If a positive, summable sequence $\{\mu_n\}$ indeed solves (4), then, adding up the equations over n , we get that $\mu(\mathbb{N}) = 1$, that is, μ is indeed a probability measure. Thus, it remains to show that the equation

$$\bar{C} = \sum_{n=1}^{\infty} \bar{\mu}_n(\bar{C}) C_n$$

has a unique solution. By construction, the function $\bar{\mu}_k(\bar{C})$ is monotone decreasing in \bar{C} and

$$\eta \bar{\mu}_k(\bar{C}) = \eta \pi_k + \sum_{l=1}^{k-1} C_l C_{k-l} \bar{\mu}_l(\bar{C}) \bar{\mu}_{k-l}(\bar{C}) - C_k \bar{\mu}_k(\bar{C}) \bar{C}.$$

Clearly, $\bar{\mu}_1(\bar{C}) < \pi_1 \leq 1$. Suppose that $\bar{C} \geq c_H$. Then, adding up the above identities, we get

$$\eta \sum_{k=1}^n \mu_k(\bar{C}) \leq \eta + c_H \sum_{l=1}^{k-1} C_l \bar{\mu}_l - \bar{C} \sum_{l=1}^{k-1} C_l \bar{\mu}_l \leq \eta.$$

Hence, for $\bar{C} \geq c_H$ we have that

$$\sum_{k=1}^{\infty} \mu_k(\bar{C}) \leq 1.$$

Consequently, the function

$$f(\bar{C}) = \sum_{k=1}^{\infty} C_k \bar{\mu}_k(\bar{C})$$

is strictly monotone decreasing in \bar{C} and satisfies

$$f(\bar{C}) \leq \bar{C}, \quad \bar{C} \geq c_H.$$

It may happen that $f(x) = +\infty$ for some $C_{\min} \in (0, \bar{C})$; otherwise, we set $C_{\min} = 0$. The function

$$g(\bar{C}) = \bar{C} - f(\bar{C})$$

is continuous (by the monotone convergence theorem for infinite series (see, e.g., Yeh (2006, p. 168)) and strictly monotone increasing, and satisfies $g(C_{\min}) \leq 0$ and $g(c_H) \geq 0$. Hence, it has a unique zero. *Q.E.D.*

B.2. Stability of the Stationary Measure

PROOF OF PROPOSITION 4.2: The ordinary differential equation for $\mu_k(t)$ can be written as

$$(B.1) \quad \mu'_k = \eta \pi_k - \eta \mu_k - C_k \mu_k \sum_{i=1}^{\infty} C_i \mu_i + \sum_{l=1}^{k-1} C_l \mu_l C_{k-l} \mu_{k-l}.$$

We will need to establish the right to interchange infinite summation and differentiation. We will use the following known lemma:

LEMMA B.1: *Let $g_k(t)$ be C^1 functions such that*

$$\sum_k g'_k(t) \quad \text{and} \quad \sum_k g_k(0)$$

converge for all t and

$$\sum_k |g'_k(t)|$$

is locally bounded (in t). Then $\sum_k g_k(t)$ is differentiable and

$$\left(\sum_k g_k(t) \right)' = \sum_k g'_k(t).$$

We will also need the next lemma:

LEMMA B.2: *Suppose that f solves*

$$f' = -a(t)f + b(t),$$

where $a(t) \geq \varepsilon > 0$ and

$$\lim_{t \rightarrow \infty} \frac{b(t)}{a(t)} = c.$$

Then

$$f(t) = e^{-\int_0^t a(s) ds} \int_0^t e^{\int_0^s a(u) du} b(s) ds + f_0 e^{-\int_0^t a(s) ds}$$

and $\lim_{t \rightarrow \infty} f(t) = c$.

PROOF: The formula for the solution is well known. By l'Hôpital's rule,

$$\lim_{t \rightarrow \infty} \frac{\int_0^t e^{\int_0^s a(u) du} b(s) ds}{e^{\int_0^t a(s) ds}} = \lim_{t \rightarrow \infty} \frac{e^{\int_0^t a(s) ds} b(t)}{a(t) e^{\int_0^t a(s) ds}} = c. \quad \text{Q.E.D.}$$

The following proposition shows existence and uniqueness of the solution.

PROPOSITION B.3: *There exists a unique solution $\{\mu_k(t)\}$ to (B.1) and this solution satisfies*

$$(B.2) \quad \sum_{k=1}^{\infty} \mu_k(t) = 1$$

for all $t \geq 0$.

PROOF: Let $l_1(\mathbb{N})$ be the space of absolutely summable sequences $\{\mu_k\}$ with

$$\|\{\mu_k\}\|_{l_1(\mathbb{N})} = \sum_{k=1}^{\infty} |\mu_k|.$$

Consider the mapping $F : l_1(\mathbb{N}) \rightarrow l_1(\mathbb{N})$ defined by

$$(F(\{\mu_i\}))_k = \eta \pi_k - \eta \mu_k - C_k \mu_k \sum_{i=1}^{\infty} C_i \mu_i + \sum_{l=1}^{k-1} C_l \mu_l C_{k-l} \mu_{k-l}.$$

Then (B.1) takes the form $(\{\mu_k\})' = F(\{\mu_k\})$. A direct calculation shows that

$$(B.3) \quad \sum_{k=1}^{\infty} \left| \sum_{l=1}^{k-1} (C_l a_l C_{k-l} a_{k-l} - C_l b_l C_{k-l} b_{k-l}) \right| \\ \leq c_H^2 \sum_{k=1}^{\infty} \sum_{l=1}^{k-1} (|a_l - b_l| |a_{k-l}| + |a_{k-l} - b_{k-l}| |b_l|) \\ = c_H^2 (\|\{a_k\}\|_{l_1(\mathbb{N})} + \|\{b_k\}\|_{l_1(\mathbb{N})}) \|\{a_k - b_k\}\|_{l_1(\mathbb{N})}.$$

Thus,

$$\|F(\{a_k\}) - F(\{b_k\})\|_{l_1(\mathbb{N})} \\ \leq (\eta + 2c_H^2 (\|\{a_k\}\|_{l_1(\mathbb{N})} + \|\{b_k\}\|_{l_1(\mathbb{N})})) \|\{a_k - b_k\}\|_{l_1(\mathbb{N})},$$

so F is locally Lipschitz continuous. By a standard existence result (Dieudonné (1960, Theorem 10.4.5)), there exists a unique solution to (B.1) for $t \in [0, T_0)$ for some $T_0 > 0$ and this solution is locally bounded. Furthermore, $[0, T_0)$ can be chosen to be the maximal existence interval, such that the solution $\{\mu_k\}$ cannot be continued further. It remains to show that $T_0 = +\infty$. Because, for any $t \in [0, T_0)$,

$$\|(\{\mu_k\})'\|_{l_1(\mathbb{N})} = \|F(\{\mu_k\})\|_{l_1(\mathbb{N})} \leq \eta + \eta \|\{\mu_k\}\|_{l_1(\mathbb{N})} + 2c_H^2 \|\{\mu_k\}\|_{l_1(\mathbb{N})}^2$$

is locally bounded, Lemma B.1 implies that

$$\begin{aligned} \left(\sum_{k=1}^{\infty} \mu_k\right)' &= \sum_{k=1}^{\infty} \left(\eta \pi_k - \eta \mu_k - C_k \mu_k \sum_{i=1}^{\infty} C_i \mu_i + \sum_{l=1}^{k-1} C_l \mu_l C_{k-l} \mu_{k-l} \right) \\ &= 0, \end{aligned}$$

and hence (B.2) holds. We will now show that $\mu_k(t) \geq 0$ for all $t \in [0, T_0]$. For $k = 1$, we have

$$\mu_1' = \eta \pi_1 - \eta \mu_1 - C_1 \mu_1 \sum_{i=1}^{\infty} C_i \mu_i.$$

Denote $a_1(t) = -\eta - C_1 \sum_{i=1}^{\infty} C_i \mu_i$. Then we have

$$\mu_1' = \eta \pi_1 + a_1(t) \mu_1.$$

Lemma B.2 implies that $\mu_1 \geq 0$ for all $t \in [0, T_0)$. Suppose we know that $\mu_l \geq 0$ for $l \leq k - 1$. Then

$$\mu_k' = z_k(t) + a_k(t) \mu_k(t),$$

where

$$\begin{aligned} a_k(t) &= -\eta - C_k \sum_{i=1}^{\infty} C_i \mu_i, \\ z_k(t) &= \eta \pi_k + \sum_{l=1}^{k-1} C_l \mu_l C_{k-l} \mu_{k-l}. \end{aligned}$$

By the induction hypothesis, $z_k(t) \geq 0$ and Lemma B.2 implies that $\mu_k \geq 0$. Thus,

$$\|\{\mu_k\}\|_{l_1(\mathbb{N})} = \sum_{k=1}^{\infty} \mu_k = 1,$$

so the solution to (B.1) is uniformly bounded on $[0, T_0)$, and can, therefore, be continued beyond T_0 (Dieudonné (1960, Theorem 10.5.6)). Since $[0, T_0)$ is, by assumption, the maximal existence interval, we have $T_0 = +\infty$. *Q.E.D.*

We now expand the solution μ in a special manner. Namely, denote

$$c_H - C_i = f_i \geq 0.$$

We can then rewrite the equation as

$$(B.4) \quad \mu'_k = \eta \pi_k - (\eta + c_H^2) \mu_k + c_H f_k \mu_k + C_k \mu_k \sum_{i=1}^{\infty} f_i \mu_i + \sum_{l=1}^{k-1} C_l \mu_l C_{k-l} \mu_{k-l}.$$

Now, we will (formally) expand

$$\mu_k = \sum_{j=0}^{\infty} \mu_{kj}(t),$$

where μ_{k0} does not depend on (the whole sequence) (f_i) , μ_{k1} is linear in (the whole sequence) (f_i) , μ_{k2} is quadratic in (f_i) , and so on. The main idea is to expand so that all terms of the expansion are nonnegative.

Later, we will prove that the expansion indeed converges and coincides with the unique solution to the evolution equation.

Substituting the expansion into the equation, we get

$$(B.5) \quad \mu'_{k0} = \eta \pi_k - (\eta + c_H^2) \mu_{k0} + \sum_{l=1}^{k-1} C_l \mu_{l0} C_{k-l} \mu_{k-l,0}$$

with the given initial conditions $\mu_{k0}(0) = \mu_k(0)$ for all k . Furthermore,

$$(B.6) \quad \begin{aligned} \mu'_{k1} = & -(\eta + c_H^2) \mu_{k1} + c_H f_k \mu_{k0} + C_k \mu_{k0} \sum_{i=1}^{\infty} f_i \mu_{i0} \\ & + 2 \sum_{l=1}^{k-1} C_l \mu_{l0} C_{k-l} \mu_{k-l,1} \end{aligned}$$

and then

$$(B.7) \quad \begin{aligned} \mu'_{kj} = & -(\eta + c_H^2) \mu_{kj} + c_H f_k \mu_{kj-1} \\ & + 2 C_k \sum_{m=0}^{j-1} \mu_{km} \sum_{i=1}^{\infty} f_i \mu_{i,j-1-m} + 2 \sum_{l=1}^{k-1} \sum_{m=0}^{j-1} C_l \mu_{lm} C_{k-l} \mu_{k-l,j-m} \end{aligned}$$

with initial conditions $\mu_{kj}(0) = 0$. Equations (B.6) and (B.7) are only well defined if μ_{i0} exists for all t and the infinite series

$$\sum_{i=1}^{\infty} \mu_{ij}(t)$$

converges for all t and all j .

Thus, we can solve these linear ODEs with the help of Lemma B.2. This is done through a recursive procedure. Namely, the equation for μ_{10} is linear and we have

$$\begin{aligned} \mu'_{10} &= \eta \pi_1 - (\eta + c_H^2) \mu_{1,0} \\ &\leq \eta \pi_k - (\eta + c_H^2) \mu_{10} + c_H f_1 \mu_1 + C_1 \mu_1 \sum_{i=1}^{\infty} f_i \mu_i. \end{aligned}$$

A comparison theorem for ODEs (Hartman (1982, Theorem 4.1, p. 26)) immediately implies that $\mu_{10} \leq \mu_1$ for all t . By definition, μ_{k0} solves

$$\mu'_{k0} = \eta \pi_k - (\eta + c_H^2) \mu_{k0} + z_{k0}$$

with

$$z_{k0} = \sum_{l=1}^{k-1} C_l \mu_{l0} C_{k-l} \mu_{k-l,0}$$

depending on only those μ_{l0} with $l < k$. Since μ_{10} is nonnegative, it follows by induction that all equations for μ_{k0} have nonnegative inhomogeneities and hence μ_{k0} is nonnegative for each k . Suppose now that $\mu_{l0} \leq \mu_l$ for all $l \leq k - 1$. Then

$$z_{k0} = \sum_{l=1}^{k-1} C_l \mu_{l0} C_{k-l} \mu_{k-l,0} \leq \sum_{l=1}^{k-1} C_l \mu_l C_{k-l} \mu_{k-l},$$

and (B.4) and the same comparison theorem imply $\mu_{k0} \leq \mu_k$. Thus, $\mu_{k0} \leq \mu_k$ for all k . It follows that the series

$$\sum_k \mu_{k0} \leq 1$$

converges and, therefore, equations (B.6) are well defined. Let now

$$\mu_k^{(N)} = \sum_{j=0}^N \mu_{kj}.$$

Suppose that we have shown that $\mu_{kj} \geq 0$ for all k , and all $j \leq N - 1$, and that

$$(B.8) \quad \mu_k^{(N-1)} \leq \mu_k$$

for all k . Equations (B.6) and (B.7) are again linear inhomogeneous and can be solved using Lemma B.2 and the nonnegativity of μ_{kN} follows. By adding

(B.5), (B.6), and (B.7), and using the induction hypothesis (B.8), we get

$$(B.9) \quad (\mu_k^{(N)})' \leq \eta \pi_k - (\eta + c_H^2) \mu_k^{(N)} + c_H f_k \mu_k^{(N)} + C_k \mu_k^{(N)} \sum_{i=1}^{\infty} f_i \mu_i \sum_{l=1}^{k-1} C_l \mu_l C_{k-l} \mu_{k-l}.$$

The comparison theorem applied to (B.9) and (B.4) implies that $\mu_k^{(N)} \leq \mu_k$. Thus, we have shown by induction that $\mu_{kj} \geq 0$ and

$$\sum_{j=0}^N \mu_{kj} \leq \mu_k$$

for any $N \geq 0$. The infinite series $\sum_{j=0}^{\infty} \mu_{kj}(t)$ consists of nonnegative terms and is uniformly bounded from above. Therefore, it converges to a function $\tilde{\mu}_k(t)$. Using Lemma B.1 and adding up (B.5), (B.6), and (B.7), we get that the sequence $\{\tilde{\mu}_k(t)\}$ is continuously differentiable and satisfies (B.4) and $\tilde{\mu}_k(0) = \mu_k(0)$. Since, by Proposition B.3, the solution to (B.4) is unique, we get $\tilde{\mu}_k = \mu_k$ for all k . Thus, we have proved the following statement:

THEOREM B.4: *We have*

$$\mu_k = \sum_{j=0}^{\infty} \mu_{kj}.$$

It remains to prove that $\lim_{t \rightarrow \infty} \mu_k(t)$ exists. The strategy for this consists of two steps:

Step 1. Prove that $\lim_{t \rightarrow \infty} \mu_{kj}(t) = \mu_{kj}(\infty)$ exists.

Step 2. Prove that

$$\lim_{t \rightarrow \infty} \sum_{j=0}^{\infty} \mu_{kj} = \sum_{j=0}^{\infty} \lim_{t \rightarrow \infty} \mu_{kj}.$$

Equation (D.1) and Lemma B.2 directly imply the convergence of μ_{k0} . But, the next step is tricky because of the appearance of the infinite sums of the form

$$\sum_{i=0}^{\infty} f_i \mu_{ij}(t)$$

in equations (B.6) and (B.7). If we prove convergence of these infinite sums a subsequent application of Lemma B.2 to (B.6) and (B.7) will imply convergence of $\mu_{i,j+1}$. Unfortunately, convergence of $\mu_{ij}(t)$ for each i, j is not enough for convergence of these infinite sums.

Recall that, by assumption, there exists an N such that $C_i = C_N$ for all $i \geq N$. Thus, we need only show that

$$M_j(t) = \sum_{i=N}^{\infty} \mu_{ij}(t)$$

converges for each j .

We will start with the case $j = 0$. Then, adding up (B.5), (B.6), and (B.7) and using Lemma B.1, we get

$$M'_0(t) + \sum_{i=1}^{N-1} \mu'_{i0} = \eta - (\eta + c_H^2) \left(M_0(t) + \sum_{i=1}^{N-1} \mu_{i0} \right) + \left(\sum_{i=1}^{N-1} C_i \mu_{i0} + C_N M_0(t) \right)^2.$$

Opening the brackets, we can rewrite this equation as a Riccati equation for M_0 :

$$M'_0(t) = a_0(t) + b_0(t)M_0(t) + C_N^2 M_0(t)^2.$$

A priori, we know that M_0 stays bounded and, by (B.5) and Lemma B.2, the coefficients a_0 and b_0 converge to finite limits.

LEMMA B.5: $M_0(t)$ converges to a finite limit as $t \rightarrow \infty$.

To prove Lemma B.5, we will need an auxiliary lemma:

LEMMA B.6: Let $N(t)$ be the solution, for $c > 0$, to

$$N'(t) = a + bN(t) + cN^2(t), \quad N(0) = N_0.$$

If the characteristic polynomial $q(\lambda) = a + b\lambda + c\lambda^2$ has real zeros $\lambda_1 \geq \lambda_2$, then two situations can occur:

- (i) If $N_0 < \lambda_1$, then $\lim_{t \rightarrow \infty} N(t) = \lambda_2$.
- (ii) If $N_0 > \lambda_1$, then $\lim_{t \rightarrow \infty} N(t) = +\infty$.

If $q(\lambda)$ does not have real zeros, then $\lim_{t \rightarrow \infty} N(t) = +\infty$ for any N_0 .

PROOF: The stationary solutions are $N = \lambda_{1,2}$. If $N_0 < \lambda_2$, then $N(t) < \lambda_2$ for all t by uniqueness. Hence, $N'(t) = a + bN(t) + cN^2(t) > 0$, and $N(t)$ increases and converges to a limit $N(\infty) \leq \lambda_2$. This limit should be a stationary solution, that is, $N(\infty) = \lambda_2$. If $N_0 \in (\lambda_2, \lambda_1)$, then $N(t) \in (\lambda_2, \lambda_1)$ for all t by

uniqueness, and, therefore, $N'(t) < 0$ and $N(t)$ decreases to $N(\infty) \geq \lambda_2$, and we again should have $N(\infty) = \lambda_2$. If $N_0 > \lambda_1$, $N' > 0$ and hence

$$N'(t) = a + bN(t) + cN^2(t) > a + bN_0 + cN_0^2 > 0$$

for all t and the claim follows. If $q(\lambda)$ has no real zeros, its minimum $\min_{\lambda \in \mathbb{R}} q(\lambda) = \delta$ is strictly positive. Hence, $N'(t) > \delta > 0$ and the claim follows. *Q.E.D.*

PROOF OF LEMMA B.5: Consider the quadratic polynomial

$$q_\infty(\lambda) = a_0(\infty) + b_0(\infty)\lambda + C_N^2\lambda^2 = 0.$$

We will consider three cases:

CASE 1: $q_\infty(\lambda)$ does not have real zeros, that is, $\min_{\lambda \in \mathbb{R}} q_\infty(\lambda) = \delta > 0$. Then, for all sufficiently large t ,

$$\begin{aligned} \text{(B.10)} \quad M'_0(t) &= a_0(t) + b_0(t)M_0(t) + C_N^2M_0(t)^2 \\ &\geq \delta/2 > 0, \end{aligned}$$

so $M_0(t)$ will converge to $+\infty$, which is impossible.

CASE 2: $q_\infty(\lambda)$ has a double zero λ_∞ . Then we claim that $\lim_{t \rightarrow \infty} M(t) = \lambda_\infty$. Indeed, suppose it is not true. Then there exists an $\varepsilon > 0$ such that either $\sup_{t > T} (M(t) - \lambda_\infty) > \varepsilon$ for any $T > 0$ or $\sup_{t > T} (\lambda_\infty - M(t)) > \varepsilon$ for any $T > 0$. Suppose that the first case takes place. Pick a $\delta > 0$ and choose $T > 0$ so large that

$$a_0(t) \geq a_0(\infty) - \delta, \quad b_0(t) \geq b_0(\infty) - \delta$$

for all $t \geq T$. The quadratic polynomial

$$a_0(\infty) - \delta + (b_0(\infty) - \delta)\lambda + C_N^2\lambda^2$$

has two real zeros $\lambda_1(\delta) > \lambda_2(\delta)$ and, for sufficiently small δ , $|\lambda_{1,2}(\delta) - \lambda_\infty| < \varepsilon/2$. Let $T_0 > T$ be such that $M_0(T_0) > \lambda_\infty + \varepsilon$. Consider the solution $N(t)$ to

$$\begin{aligned} N'(t) &= a_0(\infty) - \delta + (b_0(\infty) - \delta)N(t) + C_N^2N(t)^2, \\ N(T_0) &= M(T_0) > \lambda_1(\delta). \end{aligned}$$

By the comparison theorem for ODEs, $M_0(t) \geq N(t)$ for all $t \geq T_0$ and Lemma B.6 implies that $N(t) \rightarrow \infty$, which is impossible. Suppose now $\sup_{t > T} (\lambda_\infty - M(t)) > \varepsilon$ for any $T > 0$. Consider the same $N(t)$ as above and choose δ so small that $M(T_0) < \lambda_2(\delta)$. Then $M(t) \geq N(t)$ and, by Lemma B.6,

$N(t) \rightarrow \lambda_2(\delta)$. For sufficiently small δ , $\lambda_2(\delta)$ can be made arbitrarily close to δ_∞ and hence $\sup_{t>T}(\lambda_\infty - M(t)) > \varepsilon$ cannot hold for sufficiently large T , which is a contradiction.

CASE 3: $q(\lambda)$ has two distinct real zeros $\lambda_1(\infty) > \lambda_2(\infty)$. Then we claim that either $\lim_{t \rightarrow \infty} M(t) = \lambda_1(\infty)$ or $\lim_{t \rightarrow \infty} M(t) = \lambda_2(\infty)$. Suppose the contrary. Then there exists an $\varepsilon > 0$ such that $\sup_{t>T} |M(t) - \lambda_1(\infty)| > \varepsilon$ and $\sup_{t>T} |M(t) - \lambda_2(\infty)| > \varepsilon$. Now, an argument completely analogous to that of Case 2 applies. *Q.E.D.*

With this result, we go directly to (B.6) and get the convergence of μ_{j1} from Lemma B.2. Then adding equations (B.6), we get a linear equation for $M_1(t)$ and again get convergence from Lemma B.2. Note that we are in an even simpler situation of linear equations. Proceeding inductively, we arrive at the following statement:

PROPOSITION B.7: *The limit*

$$\lim_{t \rightarrow \infty} \mu_{kj}(t) \leq 1$$

exists for any k, j .

The next important observation is that if we subtract an ε from c in the first quadratic term, then the measure remains bounded. This is a nontrivial issue, since the derivative could become large.

LEMMA B.8: *If ε is sufficiently small, then there exists a constant $K > 0$ such that the system*

$$\mu'_k = \eta \pi_k - \eta \mu_k - (C_k - \varepsilon) \mu_k \sum_{i=1}^{\infty} (C_i - \varepsilon) \mu_i + \sum_{l=1}^{k-1} C_l \mu_l C_{k-l} \mu_{k-l}$$

has a unique solution $\{\mu_k(t)\} \in l_1(\mathbb{N})$ and this solution satisfies

$$\sum_{k=1}^{\infty} \mu_k(t) \leq K$$

for all $t > 0$.

PROOF: An argument completely analogous to that in the proof of Proposition B.3 implies that the solution exists on an interval $[0, T_0)$, and is unique and nonnegative. Letting

$$M = \sum_{k=1}^{\infty} \mu_k(t)$$

and adding up the equations, we get

$$M' \leq \eta - \eta M + 2\varepsilon c_H M^2.$$

Consider now the solution N to the equation

$$(B.11) \quad N' = \eta - \eta N + 2\varepsilon c_H N^2.$$

By a standard comparison theorem for ODEs (Hartman (1982, Theorem 4.1, p. 26)), if $N(0) = M(0)$, then $M(t) \leq N(t)$ for all t . Thus, we only need to show that $N(t)$ stays bounded. The stationary points of (B.11) are

$$d_{1,2} = \frac{\eta \pm \sqrt{\eta^2 - 8\varepsilon c_H \eta}}{4\varepsilon c_H},$$

so, for sufficiently small ε , the larger stationary point d_1 is arbitrarily large. Therefore, by uniqueness, if $N(0) < d_1$, then $N(t) < d_1$ for all t and we are done. Now, the same argument as in the proof of Proposition B.3 implies that $T_0 = +\infty$ and the proof is complete. *Q.E.D.*

Since (f_i) is bounded, for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$f_i + \varepsilon \geq (1 + \delta)f_i$$

for all i . Consider now the solution $(\mu_k^{(\varepsilon)})$ to the equation of Lemma B.8. Then, using the same expansion

$$\mu_k^{(\varepsilon)} = \sum_{j=0}^{\infty} \mu_{kj}^{(\varepsilon)},$$

we immediately get (by direct calculation) that

$$(1 + \delta)^j \mu_{kj} \leq \mu_{kj}^{(\varepsilon)}.$$

By Lemma B.8,

$$\mu_{kj}^{(\varepsilon)} \leq \sum_{k,j} \mu_{kj}^{(\varepsilon)} = \sum_k \mu_k^{(\varepsilon)} < K,$$

and, therefore,

$$\mu_{kj}(t) \leq \frac{K}{(1 + \delta)^j},$$

for some (possibly very large) constant K , independent of t . Now we are ready to prove the next theorem:

THEOREM B.9: *We have*

$$\lim_{t \rightarrow \infty} \mu_k(t) = \sum_{j=0}^{\infty} \mu_{kj}(\infty).$$

PROOF: Take N so large that

$$\sum_{j=N}^{\infty} \mu_{kj}(t) \leq \sum_{j=N}^{\infty} \frac{K}{(1+\delta)^j} < \frac{\varepsilon}{2}$$

for all t . Then choose T so large that

$$\sum_{j=0}^{N-1} |\mu_{kj}(t) - \mu_{kj}(\infty)| < \varepsilon/2$$

for all $t > T$. Then

$$\left| \sum_{j=0}^{\infty} (\mu_{kj}(t) - \mu_{kj}(\infty)) \right| < \varepsilon.$$

Since ε is arbitrary, we are done.

Q.E.D.

LEMMA B.10: *Consider the limit*

$$\bar{C} = \lim_{t \rightarrow \infty} \sum_n C_n \mu_n(t).$$

In general,

$$\bar{C} \geq \sum_n C_n \lim_{t \rightarrow \infty} \mu_n(t) = \tilde{C}$$

with equality if and only if

$$\sum_n \lim_{t \rightarrow \infty} \mu_n(t) = 1.$$

Furthermore,

$$\bar{C} - \tilde{C} = C_N \left(1 - \sum_n \lim_{t \rightarrow \infty} \mu_n(t) \right).$$

Based on Lemma B.10, we have another lemma:

LEMMA B.11: *The limit $\mu_\infty(n) = \lim_{t \rightarrow \infty} \mu_t(n)$ satisfies the equation*

$$0 = \eta(\pi - \mu_\infty) + \mu_\infty^c * \mu_\infty^c - \mu_\infty^c \bar{C},$$

where, in general,

$$\bar{C} \geq \mu_\infty^c(\mathbb{N})$$

and

$$\mu_\infty(\mathbb{N}) = 1 - \eta^{-1} \mu_\infty^c(\bar{C} - \mu_\infty^c) \leq 1.$$

An immediate consequence is the next lemma.

LEMMA B.12: *The limit distribution μ_∞ is a probability measure and coincides with the unique solution to (4) if and only if $\bar{C} = \tilde{C}$.*

PROPOSITION B.13: *Under the same tail condition $C_n = C_N$ for $n \geq N$ and the condition that*

$$(B.12) \quad \eta \geq C_N c_H,$$

we have

$$\bar{C} = \tilde{C}$$

and, therefore, μ_∞ is a probability measure that coincides with the unique solution to (4).

PROOF: Recall that the equation for the limit measure is

$$(B.13) \quad -\eta \mu_\infty(k) + \eta \pi_k - \bar{C} C_k \mu_\infty(k) + \sum_l C_l \mu_\infty(l) C_{k-l} \mu_\infty(k-l) = 0,$$

where

$$\bar{C} = \lim_{t \rightarrow \infty} \sum_n C_n \mu_n(t) \geq \sum_n C_n \mu_\infty(n) = \tilde{C}.$$

The difference

$$\bar{C} - \tilde{C} = (1 - M) C_N$$

with

$$M = \sum_k \mu_\infty(k)$$

is nonzero if and only if there is a loss of mass, that is, $M < 1$.

Adding (B.13) up over k , we get

$$\begin{aligned}
 \text{(B.14)} \quad & -\eta M + \eta - \bar{C}\tilde{C} + (\tilde{C})^2 \\
 & = 0 \\
 & = -\eta M + \eta - \bar{C}(\bar{C} - (1 - M)C_N) + (\bar{C} - (1 - M)C_N)^2.
 \end{aligned}$$

If $M \neq 1$, we get, dividing this equation by $(1 - M)$, that

$$M = 1 + \frac{\eta - C_N \bar{C}}{C_N^2}.$$

Since $M \leq 1$, we immediately get that if

$$\eta \geq C_N c_H,$$

then there is no loss of mass, proving the result.

Q.E.D.

B.3. Monotonicity Properties of the Stationary Measure

We recall that the equation for the stationary measure $\mu = (\mu_k, k \geq 1)$ is

$$-\eta \mu_k + \eta \pi_k - C_k \mu_k \sum_{i=1}^{\infty} C_i \mu_i + \sum_{l=1}^{k-1} C_l C_{k-l} \mu_l \mu_{k-l} = 0.$$

We write $\{\bar{\mu}_k(\bar{C}, C_1, \dots, C_k) : k \geq 1\}$ for the measure constructed recursively in the statement of Lemma 4.1 from a given \bar{C} and a given policy C .

LEMMA B.14: *For each k , the function that maps C to*

$$C_k \bar{\mu}_k = C_k \bar{\mu}_k(\bar{C}, C_1, \dots, C_k)$$

is monotone increasing in $C_i, i = 1, \dots, k$, and monotone decreasing in \bar{C} .

PROOF: The proof is by induction. For $k = 1$, there is nothing to prove. For $k > 1$,

$$C_k \bar{\mu}_k = \frac{C_k}{\eta + \bar{C}C_k} \left(\eta \pi_k + \sum_{l=1}^{k-1} (C_l \bar{\mu}_l)(C_{k-l} \bar{\mu}_{k-l}) \right)$$

is monotone by the induction hypothesis.

Q.E.D.

Steps Toward a Proof of Proposition 4.3

Our proof of Proposition 4.3 is based on the next series of results, Lemmas B.15–B.26, for which we use the notation

$$\nu_k = \nu_k(C_1, \dots, C_k, \bar{C}) = C_k \bar{\mu}_k(\bar{C}, C_1, \dots, C_k)$$

and

$$Z_k = \frac{C_k}{\eta + C_k \bar{C}}.$$

Note that multiplying $\sqrt{\eta}$ and C_k by the same number λ does not change the equation and, hence, does not change the stationary measure. Thus, without loss of generality, we can normalize for simplicity to the case $\eta = 1$.

LEMMA B.15: *The sequence ν_k satisfies*

$$\begin{aligned} \nu_1 &= Z_1 \pi_1, \\ \nu_k &= Z_k \left(\pi_k + \sum_{l=1}^{k-1} \nu_l \nu_{k-l} \right), \end{aligned}$$

and

$$(B.15) \quad \sum_{k=1}^{\infty} \nu_k(C_1, \dots, C_k, \bar{C}) = \bar{C}.$$

LEMMA B.16: *Differentiating (B.15) with respect to C_k , we get*

$$\frac{\partial \bar{C}}{\partial C_k} = \frac{\sum_{i=k}^{\infty} \frac{\partial \nu_i}{\partial C_k}}{1 - \sum_{i=1}^{\infty} \frac{\partial \nu_i}{\partial \bar{C}}}.$$

We now let $\bar{C}(C)$ denote the unique solution of the equation (B.15), and for any policy C we define

$$\xi_k(C) = \nu_k(C_1, \dots, C_k, \bar{C}(C)).$$

LEMMA B.17: *Let $N - 1 < k$. We have*

$$\frac{\partial}{\partial C_{N-1}} \sum_{i=k}^{\infty} \xi_i \geq 0$$

if and only if

$$(B.16) \quad \left(1 - \sum_{i=1}^{k-1} \frac{\partial v_i}{\partial \bar{C}}\right) \sum_{i=k}^{\infty} C_{N-1}^2 \frac{\partial v_i}{\partial C_{N-1}} \geq \left(\sum_{i=1}^{k-1} C_{N-1}^2 \frac{\partial v_i}{\partial C_{N-1}}\right) \sum_{i=k}^{\infty} \left(-\frac{\partial v_i}{\partial \bar{C}}\right).$$

PROOF: We have

$$\begin{aligned} & \frac{\partial}{\partial C_{N-1}} \sum_{i=k}^{\infty} \xi_i \\ &= \sum_{i=k}^{\infty} \left(\frac{\partial v_i}{\partial C_{N-1}} + \frac{\partial v_i}{\partial \bar{C}} \frac{\partial \bar{C}}{\partial C_{N-1}} \right) \\ &= \frac{\left(1 - \sum_{i=1}^{\infty} \frac{\partial v_i}{\partial \bar{C}}\right) \sum_{i=k}^{\infty} \frac{\partial v_i}{\partial C_{N-1}} + \sum_{i=k}^{\infty} \frac{\partial v_i}{\partial \bar{C}} \left(\sum_{i=N-1}^{k-1} + \sum_{i=k}^{\infty}\right) \frac{\partial v_i}{\partial C_{N-1}}}{1 - \sum_{i=1}^{\infty} \frac{\partial v_i}{\partial \bar{C}}} \\ &= \frac{\left(1 - \sum_{i=1}^{k-1} \frac{\partial v_i}{\partial \bar{C}}\right) \sum_{i=k}^{\infty} \frac{\partial v_i}{\partial C_{N-1}} + \sum_{i=k}^{\infty} \frac{\partial v_i}{\partial \bar{C}} \sum_{i=N-1}^{k-1} \frac{\partial v_i}{\partial C_{N-1}}}{1 - \sum_{i=1}^{\infty} \frac{\partial v_i}{\partial \bar{C}}}, \end{aligned}$$

and the claim follows.

Q.E.D.

Suppose now that we have the “flat-tail” condition that for some N , $C_n = C_N$ for all $n \geq N$. We define the moment-generating function $m(\cdot)$ of ν by

$$(B.17) \quad m(x) = \sum_{i=1}^{\infty} \nu_i x^i.$$

By definition, the first $N - 1$ coefficients ν_i of the power-series expansion of $m(x)$ satisfy

$$\nu_1 = \pi_1 Z_1,$$

and then

$$\nu_i = \left(\pi_i + \sum_{l=1}^{i-1} \nu_l \nu_{i-l} \right) Z_i$$

for $i \leq N - 1$ and

$$v_i = Z_N \left(\pi_i + \sum_{l=1}^{i-1} v_l v_{i-l} \right)$$

for $i \geq N$. For $N \geq 2$, let

$$m^b(x, N) = \sum_{i=N}^{\infty} \pi_i x^i.$$

Using Lemma B.15 and comparing the coefficients in the power-series expansions, we get the following statement:

LEMMA B.18: *If $C_n = C_N$ for all $n \geq N$, then*

$$m(x) - \sum_{i=1}^{N-1} v_i x^i = Z_N \left(m^b(x, N) + m^2(x) - \sum_{i=2}^{N-1} x^i \sum_{l=1}^{i-1} v_l v_{i-l} \right).$$

Thus,

$$\begin{aligned} 0 &= Z_N \left(m^b(x, N) + m^2(x) - \sum_{i=2}^{N-1} x^i \sum_{l=1}^{i-1} v_l v_{i-l} \right) - m(x) + \sum_{i=1}^{N-1} v_i x^i \\ &= Z_N \left(m^b(x, N) + m^2(x) - \sum_{i=2}^{N-1} x^i \sum_{l=1}^{i-1} v_l v_{i-l} \right) \\ &\quad - m(x) + \pi_1 Z_1 x + \sum_{i=2}^{N-1} Z_i \left(\pi_i + \sum_{l=1}^{i-1} v_l v_{i-l} \right) x^i \\ &= Z_N m^2(x) - m(x) \\ &\quad + Z_N \left(m^b(x, N) + \sum_{i=2}^{N-1} \pi_i x^i - \sum_{i=2}^{N-1} \left(\pi_i + \sum_{l=1}^{i-1} v_l v_{i-l} \right) x^i \right) \\ &\quad + \pi_1 Z_1 x + \sum_{i=2}^{N-1} Z_i \left(\pi_i + \sum_{l=1}^{i-1} v_l v_{i-l} \right) x^i \\ &= Z_N m^2(x) - m(x) + Z_N m^b(x, 2) + \pi_1 Z_1 x \\ &\quad + \sum_{i=2}^{N-1} x^i (Z_i - Z_N) \left(\pi_i + \sum_{l=1}^{i-1} v_l v_{i-l} \right). \end{aligned}$$

Solving this quadratic equation for $m(x)$ and picking the branch that satisfies $m(0) = 0$, we arrive at the next lemma.

LEMMA B.19: *The moment-generating function $m(\cdot)$ of ν is given by*

$$(B.18) \quad m(x) = \frac{1 - \sqrt{1 - 4Z_N M(x)}}{2Z_N},$$

where

$$(B.19) \quad M(x) = Z_N m^b(x, 2) + \pi_1 Z_1 x + \sum_{i=2}^{N-1} x^i (Z_i - Z_N) \left(\pi_i + \sum_{l=1}^{i-1} \nu_l \nu_{i-l} \right).$$

The derivatives of the functions Z_k satisfy interesting algebraic identities, summarized in the following lemma.

LEMMA B.20: *We have $Z_k < 1$ for all k . Moreover,*

$$C_k^2 \frac{\partial Z_k}{\partial C_k} = - \frac{\partial Z_k}{\partial \bar{C}} = Z_k^2.$$

These identities allow us to calculate derivatives in an elegant form. Let

$$\gamma(x) = (1 - 4Z_N M(x))^{-1/2}.$$

Differentiating identity (B.18) with respect to \bar{C} and C_{N-1} , and using Lemma B.20, we arrive at the following statements:

LEMMA B.21: *We have*

$$(B.20) \quad - \frac{\partial m(x)}{\partial \bar{C}} = - \frac{1}{2} + \gamma(x) \left(\frac{1}{2} + \sum_{i=1}^{N-1} x^i Z_i (Z_i - Z_N) \left(\pi_i + \sum_{l=1}^{i-1} \nu_l \nu_{i-l} \right) + \sum_{i=2}^{N-1} x^i (Z_i - Z_N) \frac{-\partial}{\partial \bar{C}} \sum_{l=1}^{i-1} \nu_l \nu_{i-l} \right)$$

and

$$(B.21) \quad C_{N-1}^2 \frac{\partial m(x)}{\partial C_{N-1}} = \gamma(x) x^{N-1} Z_{N-1}^2 \left(\pi_{N-1} + \sum_{l=1}^{N-2} \nu_l \nu_{N-1-l} \right).$$

Let now

$$(B.22) \quad \gamma(x) = \sum_{j=0}^{\infty} \gamma_j x^j$$

and let $\gamma_j = 0$ for $j < 0$.

LEMMA B.22: *We have $\gamma_j \geq 0$ for all j .*

PROOF: By (B.19), the function $M(\cdot)$ has nonnegative Taylor coefficients. Thus, it suffices to show that the function that maps x to $(1 - x)^{-1/2}$ has nonnegative Taylor coefficients. We have

$$(1 - x)^{-1/2} = 1 + \sum_{k=1}^{\infty} x^k \beta_k,$$

with

$$\begin{aligned} \beta_k &= (-1)^k \frac{(-0.5)(-0.5 - 1)(-0.5 - 2) \cdots (-0.5 - k + 1)}{k!} \\ &= \frac{(0.5)(0.5 + 1)(0.5 + 2) \cdots (0.5 + k - 1)}{k!} > 0. \end{aligned}$$

Therefore,

$$\gamma(x) = 1 + \sum_{k=1}^{\infty} \beta_k (4Z_N M(x))^k$$

also has nonnegative Taylor coefficients.

Q.E.D.

Let

$$Q_{N-1} = \sum_{l=1}^{N-2} \nu_l \nu_{N-1-l} + \pi_{N-1}.$$

Define also

$$R_0 = \frac{1}{2}, \quad R_1 = Z_1(Z_1 - Z_N)\pi_1,$$

and

$$R_i = (Z_i - Z_N) \left(\nu_i + \frac{-\partial}{\partial C} \sum_{l=1}^{i-1} \nu_l \nu_{i-l} \right).$$

Recall that $Z_i > Z_N$ if and only if $C_i > C_N$ and, therefore, $R_i \geq 0$ for all i as soon as $C_i \geq C_N$ for all $i \leq N - 1$.

LEMMA B.23: *We have*

$$C_{N-1}^2 \frac{\partial v_j}{\partial C_{N-1}} = Z_{N-1}^2 Q_{N-1} \gamma_{j-N+1}$$

and

$$-\frac{\partial v_j}{\partial \bar{C}} = \sum_{i=0}^{N-1} R_i \gamma_{j-i} = Z_{N-1}^{-2} Q_{N-1}^{-1} \sum_{i=0}^{N-1} R_i \frac{\partial v_{j-i+N-1}}{\partial C_{N-1}}.$$

PROOF: Identity (B.20) implies that

$$\frac{-\partial m(x)}{\partial \bar{C}} = \sum_{j=1}^{\infty} x^j \sum_{i=0}^{N-1} R_i \gamma_{j-i}.$$

On the other hand, by (B.17),

$$\frac{-\partial m(x)}{\partial \bar{C}} = \sum_{j=1}^{\infty} x^j \frac{-\partial v_j}{\partial \bar{C}}.$$

Comparing Taylor coefficients in the above identities, we get the required result. The case of the derivative $\partial/\partial C_{N-1}$ is analogous. *Q.E.D.*

LEMMA B.24: *We have*

$$(R_0 + \dots + R_{N-1}) Z_{N-1}^{-2} Q_{N-1}^{-1} \sum_{j=k}^{\infty} \frac{\partial v_j}{\partial C_{N-1}} \geq \sum_{j=k}^{\infty} \frac{-\partial v_j}{\partial \bar{C}}$$

and

$$R_0 \gamma_0 + \sum_{j=1}^{k-1} \frac{-\partial v_j}{\partial \bar{C}} \geq (R_0 + \dots + R_{N-1}) Z_{N-1}^{-2} Q_{N-1}^{-1} \sum_{j=N-1}^{k-1} \frac{\partial v_j}{\partial C_{N-1}}.$$

PROOF: By Lemma B.23,

$$\begin{aligned} \sum_{j=k}^{\infty} \frac{-\partial v_j}{\partial \bar{C}} &= \sum_{j=k}^{\infty} \sum_{i=0}^{N-1} R_i \gamma_{j-i} = \sum_{i=0}^{N-1} R_i \sum_{j=k-i}^{\infty} \gamma_j \leq \sum_{i=0}^{N-1} R_i \sum_{j=k-N+1}^{\infty} \gamma_j \\ &= (R_0 + \dots + R_{N-1}) Z_{N-1}^{-2} Q_{N-1}^{-1} \sum_{j=k}^{\infty} \frac{\partial v_j}{\partial C_{N-1}} \end{aligned}$$

and

$$\begin{aligned}
 R_0\gamma_0 + \sum_{j=1}^{k-1} \frac{-\partial v_j}{\partial \bar{C}} &= R_0\gamma_0 + \sum_{j=1}^{k-1} \sum_{i=0}^{N-1} R_i\gamma_{j-i} \\
 &= R_0 \sum_{i=0}^{k-1} \gamma_j + \sum_{i=1}^{N-1} R_i \sum_{j=0}^{k-1-i} \gamma_i \\
 &\geq (R_0 + \dots + R_{N-1}) \sum_{j=0}^{k-N+1} \gamma_j \\
 &= (R_0 + \dots + R_{N-1}) Z_{N-1}^{-2} Q_{N-1}^{-1} \sum_{j=N-1}^{k-1} \frac{\partial v_j}{\partial C_{N-1}}.
 \end{aligned}$$

Q.E.D.

By definition, $R_0 = 1/2$ and $\gamma_0 = 1$. Hence, we get the following lemma:

LEMMA B.25: *Suppose that $C_i \geq C_N$ for all $i \leq N$ and $C_i = C_N$ for all $i \geq N$. Then, for all $k \geq N$,*

$$\left(1 - \sum_{j=1}^{k-1} \frac{\partial v_j}{\partial \bar{C}}\right) \sum_{j=k}^{\infty} \frac{\partial v_j}{\partial C_{N-1}} \geq \sum_{j=N-1}^{k-1} \frac{\partial v_j}{\partial C_{N-1}} \sum_{j=k}^{\infty} \frac{-\partial v_j}{\partial \bar{C}}.$$

LEMMA B.26: *The function λ_k defined by*

$$\lambda_k((C_i)) = \sum_{i=k}^{\infty} \xi_i((C_i))$$

is monotone increasing in C_{N-1} for all $k \leq N - 1$.

PROOF: By Proposition 4.4,

$$\lambda_1 = \bar{C} = \sum_{j=1}^{\infty} \xi_j$$

is monotone increasing in C_{N-1} for each $N - 1$. By Lemma 4.1 and Proposition 4.4, $\xi_j = \bar{\mu}_j(C_1, \dots, C_{j-1}, \bar{C})$ is monotone decreasing in C_{N-1} for all $N - 1 \geq j$. Thus,

$$\lambda_k = \bar{C} - \sum_{j=1}^{k-1} \xi_j$$

is monotone increasing in C_{N-1} .

Q.E.D.

We are now ready to prove Proposition 4.3.

PROOF OF PROPOSITION 4.3: It suffices to prove the claim for the case $N = M + 1$. It is known that $\mu^{C,N}$ dominates $\mu^{C,M}$ in the sense of first-order stochastic dominance if and only if

$$(B.23) \quad \sum_{j=k}^{\infty} \mu_j^{C,N} \geq \sum_{j=k}^{\infty} \mu_j^{C,M}$$

for any $k \geq 1$. The only difference between policies C^M and C^{M+1} is that

$$C_M^{M+1} = c_H > c_L = C_M^M.$$

By Lemma B.26, (B.23) holds for any $k \leq M$. By Lemmas B.17 and B.25, this is also true for $k > M$. The proof of Proposition 4.3 is complete. *Q.E.D.*

PROOF OF PROPOSITION 4.4: By Lemma B.14, the function f that maps (\bar{C}, C) to

$$f(\bar{C}, (C_i, i \geq 1)) = \sum_{k=1}^{\infty} C_k \bar{\mu}_k(\bar{C}, C_1, \dots, C_k)$$

is monotone increasing in C_i and decreasing in \bar{C} . Therefore, given C , the unique solution \bar{C} to the equation

$$(B.24) \quad \bar{C} - f(\bar{C}, (C_i, i \geq 1)) = 0$$

is monotone increasing in C_i for all i .

Q.E.D.

PROOF OF EXAMPLE 4.5 OF NONMONOTONICITY: Let $C_n = 0$ for $n \geq 3$, as stipulated. We will check the condition

$$\frac{\partial v_2}{\partial C_1} > 0.$$

By the above, we need to check that

$$\left(1 - \frac{\partial v_1}{\partial \bar{C}}\right) \frac{\partial v_2}{\partial C_1} > \frac{\partial v_1}{\partial C_1} \frac{-\partial v_j}{\partial \bar{C}}.$$

We have

$$v_1 = \pi_1 Z_1$$

and

$$\nu_2 = (\pi_2 + (\pi_1 Z_1)^2) Z_2.$$

Since $\nu_i = 0$ for $i \geq 2$, using the properties of the function Z_k , the inequality takes the form

$$\begin{aligned} & (1 + \pi_1 Z_1^2) C_1^{-2} \pi_1 \pi_1 2 Z_1^3 Z_2 \\ & > C_1^{-2} \pi_1 Z_1^2 (\pi_2 Z_2^2 + \pi_1 \pi_1 2 Z_1^3 Z_2 + (\pi_1 Z_1)^2 Z_2^2). \end{aligned}$$

Opening the brackets,

$$\pi_1 \pi_1 2 Z_1^3 Z_2 > \pi_1 Z_1^2 (\pi_2 Z_2^2 + (\pi_1 Z_1)^2 Z_2^2).$$

Consider the case $C_1 = C_2$ in which $Z_1 = Z_2$. Then, the inequality takes the form

$$2\pi_1 > \pi_2 + (\pi_1 Z_1)^2.$$

If $\pi_2 > 2\pi_1$, this cannot hold.

Q.E.D.

APPENDIX C: PROOFS FOR SECTION 5: OPTIMALITY

PROOF OF LEMMA 5.1: Let ϕ be any search-effort process, and let

$$\begin{aligned} \theta_t^\phi &= - \int_0^t e^{-rs} K(\phi_s) ds + e^{-rt} V(N_t^\phi) & (t < \tau) \\ &= - \int_0^\tau e^{-rs} K(\phi_s) ds + e^{-r\tau} u(N_\tau^\phi) & (t \geq \tau). \end{aligned}$$

By Itô's formula and the fact that V solves the HJB equation (6), θ^ϕ is a supermartingale, so

$$(C.1) \quad V(N_{i0}) = \theta_0^\phi \geq U(\phi).$$

For the special case of $\phi_t^* = \Gamma(N_t)$, where N satisfies the stochastic differential equation associated with the specified search-effort policy function Γ , the HJB equation implies that $\theta^* = \theta^{\phi^*}$ is actually a martingale. Thus,

$$V(N_{i0}) = \theta_0^* = E(\theta_\tau^*) = U(\phi^*).$$

It follows from (C.1) that $U(\phi^*) \geq U(\phi)$ for any control ϕ .

Q.E.D.

LEMMA C.1: *The operator $M : \ell_\infty \rightarrow \ell_\infty$, defined by*

$$(C.2) \quad (MV)_n = \max_c \left(\frac{\eta' u(n) - K(c)}{c\bar{C} + r + \eta'} + \frac{c}{c\bar{C} + r + \eta'} \sum_{m=1}^\infty V_{n+m} \mu_m^c \right),$$

is a contraction, satisfying, for candidate value functions V_1 and V_2 ,

$$(C.3) \quad \|MV_1 - MV_2\|_\infty \leq \frac{c_H \bar{C}}{c_H \bar{C} + r + \eta'} \|V_1 - V_2\|_\infty.$$

In addition, M is monotonicity preserving.

PROOF: The fact that M preserves monotonicity follows because the pointwise maximum of two monotone increasing functions is also monotone. To prove that M is a contraction that satisfies (C.3), we verify Blackwell’s sufficient conditions (see Stockey and Lucas (1989, Theorem 3.3, p. 54)).⁷ Clearly, $V_1 \leq V_2$ implies $MV_1 \leq MV_2$, so M is monotone. Furthermore, for any $a \geq 0$,

$$(C.4) \quad \begin{aligned} M(V + a) &= \max_c \left(\frac{\eta' u(n) - K(c)}{c\bar{C} + r + \eta'} + \frac{c}{c\bar{C} + r + \eta'} \sum_{m=1}^\infty (V_{n+m} + a) \mu_m^c \right) \\ &\leq MV + a \frac{c_H \bar{C}}{c_H \bar{C} + r + \eta'} \end{aligned}$$

and so the discounting condition also holds.

Q.E.D.

The contraction mapping theorem implies the following corollary.

COROLLARY C.2: *The value function V is the unique fixed point of M and is a monotone increasing function.*

LEMMA C.3: *Let*

$$\mathcal{L} = \{V \in \ell_\infty : V_n - (r + \eta')^{-1} \eta' u_n \text{ is monotone decreasing}\}.$$

If $u(\cdot)$ is concave, then the operator M maps \mathcal{L} into itself. Consequently, the unique fixed point of M also belongs to \mathcal{L} .

PROOF: Suppose that $V \in \mathcal{L}$. Using the identity

$$\frac{u(n)}{c\bar{C} + r + \eta'} - \frac{u(n)}{r + \eta'} = - \frac{u(n)}{r + \eta'} \frac{c\bar{C}}{c\bar{C} + r + \eta'},$$

⁷We thank a referee for suggesting a simplification of the proof by invoking Blackwell’s theorem.

we get

$$\begin{aligned}
 M_c V_n - \frac{\eta' u_n}{r + \eta'} &= \frac{\eta' u_n - K(c)}{c\bar{C} + r + \eta'} - \frac{\eta' u_n}{r + \eta'} + \frac{c}{c\bar{C} + r + \eta'} \sum_{m=1}^{\infty} V_{n+m} \mu_m^C \\
 &= \frac{-K(c)}{c\bar{C} + r + \eta'} + \frac{c}{c\bar{C} + r + \eta'} \\
 &\quad \times \sum_{m=1}^{\infty} \left(V_{n+m} - \frac{\eta' u_{n+m}}{r + \eta'} + \frac{\eta' u_{n+m} - \eta' u_n}{r + \eta'} \right) \mu_m^C.
 \end{aligned}$$

Since $V \in \mathcal{L}$ and u is concave, the sequence

$$(C.5) \quad B_n = \sum_{m=1}^{\infty} \left(V_{n+m} - \frac{\eta' u_{n+m}}{r + \eta'} + \frac{\eta' u_{n+m} - \eta' u_n}{r + \eta'} \right) \mu_m^C$$

$$(C.6) \quad = \sum_{m=1}^{\infty} V_{n+m} \mu_m^C - \frac{\bar{C} \eta' u_n}{r + \eta'}$$

is monotone decreasing. Since the maximum of decreasing sequences is again decreasing, the sequence

$$(MV)_n - \frac{\eta' u_n}{r + \eta'} = \max_c (M_c V)_n - \frac{\eta' u_n}{r + \eta'}$$

is decreasing, proving the result. Q.E.D.

We will need the following auxiliary lemma.

LEMMA C.4: *If $K(c)$ is a convex differentiable function, then $c \mapsto K(c) - K'(c)c$ is monotone decreasing for $c > 0$.*

PROOF: For any c and b in $[c_L, c_H]$ with $c \geq b$, using first the convexity property that $K(c) - K(b) \leq K'(c)(c - b)$ and then the fact that the derivative of a convex function is an increasing function, we have

$$\begin{aligned}
 (K(c) - K'(c)c) - (K(b) - K'(b)b) &\leq K'(c)(c - b) - K'(c)c + K'(b)b \\
 &= b(K'(b) - K'(c)) \leq 0,
 \end{aligned}$$

the desired result. Q.E.D.

PROPOSITION C.5: *Suppose that the search-cost function K is convex, differentiable, and increasing. Given (μ, C) , any optimal search-effort policy function $\Gamma(\cdot)$ is monotone decreasing. If $K(c) = \kappa c$ for some $\kappa > 0$, then there is an optimal policy of the trigger form.*

PROOF: The optimal V solves the alternative Bellman equation

$$(C.7) \quad V_n = \max_c \left(\frac{\eta' u_n - K(c)}{c\bar{C} + r + \eta'} + \frac{c}{c\bar{C} + r + \eta'} \sum_{m=1}^{\infty} V_{n+m} \mu_m^c \right).$$

We want to solve

$$\max_c f(c) = \max_c \frac{\eta' u(n) + cY_n - K(c)}{c\bar{C} + r + \eta'}$$

with

$$Y_n = \sum_{m=1}^{\infty} V_{n+m} \mu_m^c.$$

Then

$$f'(c) = [(Y_n(r + \eta') - \eta' u(n)\bar{C}) + \bar{C}(K(c) - K'(c)c) - (r + \eta')K'(c)] / (c\bar{C} + r + \eta')^2.$$

By Lemma C.4, the function $K(c) - K'(c)c$ is monotone decreasing and the function $-(r + \eta')K'(c)$ is decreasing because $K(\cdot)$ is convex. Therefore, the function

$$\bar{C}(K(c) - K'(c)c) - (r + \eta')K'(c)$$

is also monotone decreasing. There are three possibilities. If the unique solution z_n to

$$\bar{C}(K(z_n) - K'(z_n)z_n) - (r + \eta')K'(z_n) + (Y_n(r + \eta') - \eta' u(n)\bar{C}) = 0$$

belongs to the interval $[c_L, c_H]$, then $f'(c)$ is positive for $c < z_n$ and is negative for $c > z_n$. Therefore, $f(c)$ attains its global maximum at z_n and the optimum is $c_n = z_n$. If

$$\bar{C}(K(c) - K'(c)c) - (r + \eta')K'(c) + (Y_n(r + \eta') - \eta' u(n)\bar{C}) > 0$$

for all $c \in [c_L, c_H]$, then $f'(c) > 0$, so f is increasing and the optimum is $c = c_H$. Finally, if

$$\bar{C}(K(c) - K'(c)c) - (r + \eta')K'(c) + (Y_n(r + \eta') - \eta' u(n)\bar{C}) < 0$$

for all $c \in [c_L, c_H]$, then $f'(c) < 0$, so f is decreasing and the optimum is $c = c_L$. By (C.5), the sequence

$$Y_n(r + \eta') - \bar{C}\eta'u(n) = (r + \eta')B_n$$

is monotone decreasing. The above analysis directly implies that the optimal policy is then also decreasing.

If K is linear, it follows from the above discussion that the optimum is c_H if $Y_n(r + \eta') - \bar{C}\eta'u(n) > 0$ and c_L if $Y_n(r + \eta') - \bar{C}\eta'u(n) < 0$. Thus, we have a trigger policy. *Q.E.D.*

PROOF OF PROPOSITION 5.3: Using the fact that

$$(r + \eta')V_n = \sup_{c \in [c_L, c_H]} \eta'u_n - K(c) + c \sum_{m=1}^{\infty} (V_{n+m} - V_n)C_m\mu_m,$$

which is a concave maximization problem over a convex set, the supremum is achieved at c_L if and only if some element of the supergradient of the objective function at c_L includes zero or a negative number. (See Rockafellar (1970).) This is the case provided that

$$\sum_{m=1}^{\infty} (V_{n+m} - V_n)C_m\mu_m \leq K'(c_L),$$

where $K'(c_L)$. By Lemma C.3,

$$\begin{aligned} \sum_{m=1}^{\infty} (V_{n+m} - V_n)C_m\mu_m &\leq \eta'(r + \eta') \sum_{m=1}^{\infty} (u_{n+m} - u_n)C_m\mu_m \\ &\leq \eta'(r + \eta') \sum_{m=1}^{\infty} (\bar{u} - u_n)C_m\mu_m \\ &< K'(c_L) \quad \text{for } n > \bar{N}, \end{aligned}$$

completing the proof. *Q.E.D.*

APPENDIX D: PROOFS FOR SECTION 6: EQUILIBRIUM

This appendix contains proofs of the results in Section 6.

D.1. *Monotonicity of the Value Function in Other Agents' Trigger Level*

From the results of Section 5, we can restrict attention to equilibria in the form of a trigger policy C^N , with trigger precision level N . For any constant c

in $[c_L, c_H]$, we define the operator L_c^N , at any bounded increasing sequence g , by

$$(L_c^N g)_n = \frac{1}{c_H^2 + \eta' + r} \times \left\{ \eta' u_n - K(c) + (c_H^2 - c\bar{C}^N)g_n + c \sum_{m=1}^{\infty} g_{n+m} C_m^N \mu_m^N \right\},$$

where

$$\bar{C}^N = \sum_{i=1}^{\infty} C_i^N \mu_i^N.$$

LEMMA D.1: *Given (μ^N, C^N) , the value function V^N of any given agent solves*

$$V_n^N = \max_{c \in [c_L, c_H]} \{L_c^N V_n^N\}.$$

PROOF: By Corollary C.2,

$$(D.1) \quad V_n^N \geq \frac{\eta' u_n - K(c)}{c\bar{C}^N + r + \eta'} + \frac{c}{c\bar{C}^N + r + \eta'} \sum_{m=1}^{\infty} V_{n+m}^N \mu_m^N$$

for all $c \in [c_L, c_H]$ and the equality is attained for some $c \in [c_L, c_H]$. Multiplying (D.1) by $(c\bar{C}^N + r + \eta')$, adding $(c_H^2 - c\bar{C}^N)V_n^N$ to both sides of (D.1), and then dividing (D.1) by $c_H^2 + \eta' + r$, we get

$$V_n^N \geq \frac{1}{c_H^2 + \eta' + r} \times \left\{ \eta' u_n - K(c) + (c_H^2 - c\bar{C}^N)V_n + c \sum_{m=1}^{\infty} V_{n+m} C_m^N \mu_m^N \right\}$$

for all $c \in [c_L, c_H]$ and the equality is attained for some $c \in [c_L, c_H]$. The proof is complete. *Q.E.D.*

LEMMA D.2: *The operator L_c^N is monotone increasing in N . That is, for any increasing sequence g ,*

$$L_c^{N+1} g \geq L_c^N g.$$

PROOF: It is enough to show that if $f(n)$ and $g(n)$ are increasing, bounded functions with $f(n) \geq g(n) \geq 0$, then $L_c^{N+1} f(n) \geq L_c^N g(n)$. For that it suffices

to show that

$$(D.2) \quad (c_H^2 - c\bar{C}^{N+1})f(n) + c \sum_{m=1}^{\infty} f(n+m)C_m^{N+1}\mu_m^{N+1} \\ \geq (c_H^2 - c\bar{C}^N)g(n) + c \sum_{m=1}^{\infty} g(n+m)C_m^N\mu_m^N.$$

Because f and g are increasing and $f \geq g$, inequality (D.2) holds because

$$\sum_{m=k}^{\infty} C_m^{N+1}\mu_m^{N+1} \geq \sum_{m=k}^{\infty} C_m^N\mu_m^N$$

for all $k \geq 1$, based on Proposition 4.3.

Q.E.D.

PROPOSITION D.3: *If $N' \geq N$, then $V^{N'}(n) \geq V^N(n)$ for all n .*

PROOF: Let

$$L^N V = \sup_{c \in [c_L, c_H]} L_c^N V.$$

By Lemmas D.1 and D.2,

$$V^{N'} = L^{N'} V^{N'} \geq L^N V^{N'}.$$

Thus, for any $c \in [c_L, c_H]$,

$$V^{N'}(n) \geq L_c^N V^{N'}(n) \\ = \frac{1}{c_H^2 + \eta' + r} \left[\eta' u(n) - K(c) + (c_H^2 - c\bar{C}^N)V^{N'}(n) \right. \\ \left. + c \sum_{m=1}^{\infty} V^{N'}(n+m)c_m^N\mu_m^N \right].$$

Multiplying this inequality by $c_H^2 + \eta + r$, adding $(\bar{C}^N c - c_H^2)V^{N'}(n)$, dividing by $c\bar{C}^N + r + \eta$, and maximizing over c , we get

$$V^{N'} \geq M V^{N'},$$

where the M operator is defined in Lemma C.1, corresponding to c^N . Since M is monotone, we get

$$V^{N'} \geq M^k V^{N'}$$

for any $k \in \mathbb{N}$. By Lemma C.1 and Corollary C.2,

$$\lim_{k \rightarrow \infty} M^k V^{N'} = V^N$$

and the proof is complete. Q.E.D.

D.2. Nash Equilibria

We define the operator $Q: \ell_\infty \rightarrow \ell_\infty$ by

$$\begin{aligned} (Q\tilde{C})(n) = \arg \max_c & \left(\frac{\eta' u(n) - K(c)}{\bar{C}c + r + \eta'} \right. \\ & \left. + \frac{c}{\bar{C}c + r + \eta'} \sum_{m=1}^{\infty} V^{\tilde{C}}(n+m) \mu^{\tilde{C}}(m) \right), \end{aligned}$$

where

$$\begin{aligned} V^{\tilde{C}}(n) = \max_c & \left(\frac{\eta' u(n) - K(c)}{\bar{C}c + r + \eta'} \right. \\ & \left. + \frac{c}{\bar{C}c + r + \eta'} \sum_{m=1}^{\infty} V^{\tilde{C}}(n+m) \mu^{\tilde{C}}(m) \right). \end{aligned}$$

We then define the $\mathcal{N}(N) \subset \mathbb{N}$ as

$$\mathcal{N}(N) = \{n \in \mathbb{N}; C^n \in Q(C^N)\}.$$

The following proposition is a direct consequence of Proposition 5.4 and the definition of the correspondence \mathcal{N} .

PROPOSITION D.4: *Symmetric pure strategy Nash equilibria of the game are given by trigger policies with trigger precision levels that are the fixed points of the correspondence \mathcal{N} .*

LEMMA D.5: *The correspondence $\mathcal{N}(N)$ is monotone increasing in N .*

PROOF: From the Hamilton–Jacobi–Bellman equation (6),

$$(r + \eta')V_n^N = \max_{c \in [c_L, c_H]} \eta' u_n + c \left(-\kappa + \sum_{m=1}^{\infty} (V_{n+m}^N - V_n^N) C_m^N \mu_m^N \right).$$

Let first $c_L > 0$. Then it is optimal for an agent to choose $c = c_H$ if

$$(D.3) \quad -\kappa + \sum_{m=1}^{\infty} (V_{n+m}^N - V_n^N) C_m^N \mu_m^N > 0 \Leftrightarrow (r + \eta') V_n^N - \eta' u_n > 0,$$

the agent is indifferent between choosing c_H or c_L if

$$(D.4) \quad -\kappa + \sum_{m=1}^{\infty} (V_{n+m}^N - V_n^N) C_m^N \mu_m^N = 0 \Leftrightarrow (r + \eta') V_n^N - \eta' u_n = 0,$$

and the agent will choose c_L if the less than inequality holds in (D.3). By Lemma C.3, the set of n for which $(r + \eta') V_n^N - \eta' u_n = 0$ is either empty or is an interval $N_1 \leq n \leq N_2$. Proposition D.3 implies the required monotonicity.

Let now $c_L = 0$. Then, by the same reasoning, it is optimal for an agent to choose $c = c_H$ if and only if $(r + \eta') V_n^N - \eta' u_n > 0$. Alternatively, $(r + \eta') V_n^N - \eta' u_n = 0$. By Lemma C.3, the set of n for which $(r + \eta') V_n^N - \eta' u_n > 0$ is an interval $n < N_1$ and, hence, $(r + \eta') V_n^N - \eta' u_n = 0$ for all $n \geq N_1$. Consequently, since u_n is monotone increasing and concave, the sequence

$$\begin{aligned} Z_n &:= -\kappa + \sum_{m=1}^{\infty} (V_{n+m}^N - V_n^N) C_m^N \mu_m^N \\ &= -\kappa + \frac{\eta'}{r + \eta'} \sum_{m=1}^{\infty} (u_{n+m} - u_n) C_m^N \mu_m^N \end{aligned}$$

is decreasing for $n \geq N_1$. Therefore, the set of n for which $Z_n = 0$ is either empty or is an interval $N_1 \leq n \leq N_2$. Proposition 4.3 implies the required monotonicity. *Q.E.D.*

PROPOSITION D.6: *The correspondence \mathcal{N} has at least one fixed point. Any fixed point is less than or equal to \bar{N} .*

PROOF: If $0 \in \mathcal{N}(0)$, we are done; otherwise, $\inf\{\mathcal{N}(0)\} \geq 1$. By monotonicity, $\inf\{\mathcal{N}(1)\} \geq 1$. Again, if $1 \in \mathcal{N}(1)$, we are done; otherwise, continue inductively. Since there is only a finite number \bar{N} of possible outcomes, we must arrive at some n in $\mathcal{N}(n)$. *Q.E.D.*

PROOF OF PROPOSITION 6.4: The result follows directly from Proposition D.3 and the definition of equilibrium. *Q.E.D.*

D.3. *Equilibria With Minimal Search Intensity*

Lemma 6.5 is an immediate consequence of the following.

LEMMA D.7: *The operator A is a contraction on ℓ_∞ with*

$$\|A\|_{\ell_\infty \rightarrow \ell_\infty} \leq \frac{c_L^2}{r + \eta' + c_L^2} < 1.$$

Furthermore, A preserves positivity, monotonicity, and concavity.

PROOF: We have, for any bounded sequence g ,

$$(D.5) \quad |A(g)_n| \leq \frac{c_L^2}{r + \eta' + c_L^2} \sum_{m=1}^\infty |g_{n+m}| \mu_m \leq \frac{c_L^2}{r + \eta' + c_L^2} \sup_n g_n,$$

which establishes the first claim. Furthermore, if g is increasing, we have $g_{n_1+m} \geq g_{n_2+m}$ for $n_1 \geq n_2$ and for any m . Summing up these inequalities, we get the required monotonicity. Preservation of concavity is proved similarly. *Q.E.D.*

PROOF OF THEOREM 6.6: It suffices to show that, taking the minimal-search (μ^0, C^0) behavior as given, the minimal-effort search effort c_L achieves the supremum defined by the Hamilton–Jacobi–Bellman equation at each precision n if and only if $K'(c_L) \geq B$, where B is defined by (8).

By the previous lemma, $V(n)$ is concave in n . Thus, the sequence

$$\sum_{m=1}^\infty (V_{n+m} - V_n) \mu_m^0$$

is monotone decreasing with n . Therefore,

$$\sum_{m=1}^\infty (V_{1+m} - V_1) \mu_m^0 = \max_n \sum_{m=1}^\infty (V_{n+m} - V_n) \mu_m^0.$$

We need to show that the objective function, mapping c to

$$-K(c) + cc_L \sum_{m=1}^\infty (V_{n+m} - V_n) \mu_m^0,$$

achieves its maximum at $c = c_L$. Because K is convex, this objective function is decreasing on $[c_L, c_H]$ and the claim follows. *Q.E.D.*

The following lemma follows directly from the proof of Proposition D.6.

LEMMA D.8: *If $C^N \in \mathcal{Q}(C^1)$ for some $N \geq 1$, then the correspondence \mathcal{N} has a fixed point $n \geq 1$.*

PROOF OF PROPOSITION 6.7: By Lemma D.8, it suffices to show that $\mathcal{Q}(C^1) \neq C^0$. Suppose on the contrary that $\mathcal{Q}(C^1) = C^0$. Then the value function is simply

$$V_n^1 = \frac{\eta' u(n)}{r + \eta'}$$

It follows from (D.3) that $C_1 = 0$ is optimal if and only if

$$\frac{\eta'(u(2) - u(1))c_H \mu_1^1}{r + \eta'} = \sum_{m=1}^{\infty} (V_{n+m}^1 - V_n^1) C_m^1 \mu_m^1 < \kappa.$$

By (D.3), we will have the inequality $V_n^1 \geq V_n^0$ for all $n \geq 2$ and a strict inequality $V_1^1 > V_1^0$, which is a contradiction. The proof is complete. Q.E.D.

APPENDIX E: PROOFS OF RESULTS IN SECTION 7: POLICY INTERVENTIONS

PROOF OF LEMMA 7.1: By construction, the value function V^δ associated with proportional search subsidy rate δ is monotone increasing in δ . By (D.3), the optimal trigger N is that n at which the sequence

$$(r + \eta')V_n^\delta - \eta' u(n)$$

crosses zero. Hence, the optimal trigger is also monotone increasing in δ . Q.E.D.

PROOF OF LEMMA 7.3: Letting $V_{n,M}$ denote the value associated with n private signals and M public signals, we define

$$Z_{n,M} = V_{n,M} - \frac{\eta' u_{n+M}}{r + \eta'}$$

We can rewrite the HJB equation for the agent, educated with M signals at birth, in the form

$$\begin{aligned} (r + \eta')Z_{n,M} &= \sup_{c \in [c_L, c_H]} c \left(-\kappa + \sum_{m=1}^{\infty} (V_{n+m,M} - V_{n,M}) \mu_m^C \right) \\ &= \sup_{c \in [c_L, c_H]} c \left(W_{n,M} - \kappa + \sum_{m=1}^{\infty} (Z_{n+m,M} - Z_{n,M}) \mu_m^C \right), \end{aligned}$$

where

$$W_{n,M} = \frac{\eta'}{r + \eta'} \sum_{m=1}^{\infty} (u_{n+M+m} - u_{n+M}) \mu_m^C.$$

The quantity $W_{n,M}$ is monotone decreasing with M and n by the concavity of $u(\cdot)$. Equivalently,

$$(E.1) \quad Z_{n,M} = \sup_{c \in [c_L, c_H]} \frac{c}{\bar{C}c + r + \eta'} \left(W_{n,M} - \kappa + \sum_{m=1}^{\infty} Z_{n+m,M} \mu_m^C \right).$$

Treating the right-hand side as the image of an operator at Z , this operator is a contraction and, by Lemma C.2, $Z_{n,M}$ is its unique fixed point. Since $W_{n,M}$ decreases with M , so does $Z_{n,M}$.

By Lemma C.3, $Z_{n,M}$ is also monotone decreasing with n . Hence, an optimal trigger policy, attaining the supremum in (E.1), is an n at which the sequence

$$W_{n,M} - \kappa + \sum_{m=1}^{\infty} Z_{n+m,M} \mu_m^C$$

crosses zero. Because both $W_{n,M}$ and $Z_{n,M}$ are decreasing with M , the trigger is also decreasing with M . *Q.E.D.*

PROOF OF PROPOSITION 7.4: Suppose that C^N is an equilibrium with M public signals, which we express as

$$C^N \in \mathcal{Q}_M(C^N).$$

By Lemma 7.3, we have

$$C^{N_1} \in \mathcal{Q}_{M-1}(C^N)$$

for some $N_1 \geq N$. It follows from the algorithm at the end of Section 6 that there exists some $N' \geq N$ with $N' \in \mathcal{Q}_{M-1}(C^{N'})$. The proof is completed by induction in M . *Q.E.D.*

APPENDIX F: PROOFS OF COMPARATIVE STATICS

This appendix provides proofs of the comparative statics of Proposition 8.1.

PROOF OF PROPOSITION 8.1: We first study the effect of a shift in the exit intensity η' . Given the market conditions (C, μ) , let $V(\eta')$ be the value function corresponding to intensity η' and define

$$\tilde{Z}_n(\eta') = (r + \eta') \left(V_n(\eta') - \frac{\eta' u_n}{r + \eta'} \right).$$

Then the argument used in the proof of Lemma 7.3 implies that

$$(F.1) \quad \tilde{Z}_n(\eta') = \sup_{c \in [c_L, c_H]} c \left(W_n(\eta') - \kappa + \frac{1}{r + \eta'} \sum_{m=1}^{\infty} (\tilde{Z}_{n+m} - \tilde{Z}_n) \mu_m^C \right),$$

where

$$W_n(\eta') = \frac{\eta'}{r + \eta'} \sum_{m=1}^{\infty} (u_{n+m} - u_n) \mu_m^c.$$

Since u_n is monotone increasing, W_n is increasing in η' . By Lemma C.3, \tilde{Z}_n is monotone decreasing in n , so $\tilde{Z}_{n+m} - \tilde{Z}_n$ is negative. Let $\eta'_1 < \eta'$. Then, for any $c \in [c_L, c_H]$,

$$\begin{aligned} \text{(E.2)} \quad \tilde{Z}_n(\eta') &= \sup_{c \in [c_L, c_H]} c \left(W_n(\eta') - \kappa + \frac{1}{r + \eta'} \sum_{m=1}^{\infty} (\tilde{Z}_{n+m}(\eta') - \tilde{Z}_n(\eta')) \mu_m^c \right) \\ &\geq c \left(W_n(\eta'_1) - \kappa + \frac{1}{r + \eta'_1} \sum_{m=1}^{\infty} (\tilde{Z}_{n+m}(\eta') - \tilde{Z}_n(\eta')) \mu_m^c \right). \end{aligned}$$

Define the operator $\tilde{M}_{\eta'_1}$ by

$$[\tilde{M}_{\eta'_1} g](n) = \sup_c \frac{c}{r + \eta'_1 + c\bar{C}} \left((W_n(\eta'_1) - \kappa)(r + \eta'_1) + \sum_{m=1}^{\infty} g_{n+m} \mu_m^c \right).$$

Then, multiplying (E.2) by $r + \eta'_1$, adding $c\bar{C}\tilde{Z}_n$ to both sides, dividing by $r + \eta'_1 + c\bar{C}$, and taking the supremum over c , we get that

$$\tilde{Z}(\eta') \geq \tilde{M}_{\eta'_1}(\tilde{Z}(\eta')).$$

Since $\tilde{M}_{\eta'_1}$ is clearly monotone increasing, by iterating we get, for any $k \geq 0$,

$$\tilde{Z}(\eta') \geq (\tilde{M}_{\eta'_1})^k(\tilde{Z}(\eta')).$$

By the argument used in the proof of Lemma C.1, $\tilde{M}_{\eta'_1}$ is a contraction and $Z(\eta'_1)$ is its unique fixed point. Therefore, $Z(\eta') \geq \lim_{k \rightarrow \infty} (\tilde{M}_{\eta'_1})^k(\tilde{Z}(\eta')) = Z(\eta'_1)$. That is, Z is monotone increasing in η' .

Let $\mathcal{N}^{\eta'}$ be the optimal-trigger correspondence associated with exit rate η' . Let also $C = C^N$ and $\mu = \mu^N$. It follows from the proof of Lemma D.5 that $\mathcal{N}^{\eta'}(N)$ is an interval $[n_1, n_2]$ with the property that $Z_n(\eta')$ is positive for $n < n_1$ and is negative for $n > n_2$. Since $Z(\eta')$ is monotone increasing in η' , so are both n_1 and n_2 . Monotonicity of the correspondence \mathcal{N} combined with the same argument as in the proof of Proposition D.6 imply the required comparative static in η' .

As for the effect of shifting the discount rate r , we rewrite the Bellman equation (6) in the form

$$\tilde{V}_n(r) = \max_{c \in [c_L, c_H]} \eta' u_n + c \left(-\kappa + \frac{1}{r + \eta'} \sum_{m=1}^{\infty} (\tilde{V}_{n+m}(r) - \tilde{V}_n(r)) C_m \mu_m \right),$$

where $\tilde{V}_n(r) = (r + \eta') V_n(r)$. Let $r_1 > r$. Then, since \tilde{V}_n is increasing in n , we get

$$\begin{aligned} \text{(F.3)} \quad \tilde{V}_n(r) &= \max_{c \in [c_L, c_H]} \eta' u_n + c \left(-\kappa + \frac{1}{r + \eta'} \sum_{m=1}^{\infty} (\tilde{V}_{n+m}(r) - \tilde{V}_n(r)) C_m \mu_m \right) \\ &\geq \max_{c \in [c_L, c_H]} \eta' u_n + c \left(-\kappa + \frac{1}{r_1 + \eta'} \sum_{m=1}^{\infty} (\tilde{V}_{n+m}(r) - \tilde{V}_n(r)) C_m \mu_m \right). \end{aligned}$$

An argument analogous to that used in the proof of the comparative static for η' implies that $\tilde{V}(r) \geq \tilde{V}(r_1)$. That is, $\tilde{V}(r)$ is decreasing in r . Therefore, $\tilde{Z}_n = \tilde{V}_n - \eta' u_n$ also decreases in r and the claim follows in just the same way as it did for η' .

Finally, to study the impact of shifting the information-quality parameter ρ^2 , we note the impact of ρ^2 on the exit utility

$$u_n = -\frac{1 - \rho^2}{1 + \rho^2(n - 1)}.$$

Using (F.1) and following the same argument as for the parameter η' , we see that it suffices to show that $W_n(\rho)$ is monotone increasing in ρ^2 . To this end, it suffices to show that $u_{n+1}(\rho) - u_n(\rho)$ is increasing in ρ^2 . A direct calculation shows that the function

$$k_n(x) = -\frac{1 - x}{1 + xn} + \frac{1 - x}{1 + x(n - 1)}$$

is monotone decreasing for

$$x > b_n = \frac{1 - \sqrt{n/(n + 1)}}{n\sqrt{n/(n + 1)} - n + 1}.$$

It is not difficult to see that $b_n < b_1 = \sqrt{2} - 1$, and the claim follows. *Q.E.D.*

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Manuscript received September, 2008; final revision received April, 2009.