ELSEVIER

Journal of Mathematical Economics 38 (2002) 191–196

JOURNAL OF Mathematical ECONOMICS

www.elsevier.com/locate/jmateco

# Universal state prices and asymmetric information

Darrell Duffie [a,*], Rui Kan [b]

[a] *Graduate School of Business, Stanford University, Stanford, CA 94305, USA*

[b] *Red Wolf Capital Management, 11 Madison Avenue, New York, NY 10010, USA*

Received 17 November 1999; received in revised form 9 February 2000; accepted 6 July 2000

## 1. Introduction

This note gives a simple, robust, and arbitrage-free example of the absence of a fixed vector of state prices that applies to asymmetrically informed agents. In the same sense, the example is such that there is no universal equivalent martingale measure. The example has a finite number of agents, states, and periods. In this example, moreover, each of the asymmetrically informed agents has complete market. The example is consistent with a full rational expectations general equilibrium with learning from prices.

Duffie and Huang (1986) showed conditions implying the existence of a universal equivalent martingale measure. These conditions include the assumption that one of the agents has more information than each of the others. When there is no agent whose information dominates in this sense, there is no particular reason to believe in the existence of an equivalent martingale measure (EMM) that applies to all. Our example of non-existence depends on a special information structure, as it must since prices are generically fully revealing with a finite number of states. Given this special information structure, the example is robust in that no perturbation of the payoffs of the securities allows for existence of universal state prices.

## 2. The example

We consider an example in which both the states of the world and time periods are finite in number. There are three periods, $t \in \boldsymbol{T} = \{1, 2, 3\}$, and nine states of the world, $\Omega = \{\omega_1, \omega_2, \ldots, \omega_9\}$ in our example. We fix the probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F} = 2^{\Omega}$ (meaning that the state is fully revealed by $\mathcal{F}$.) We assume that $P(\{\omega_i\}) > 0$ for all $i$. There are two agents, $\alpha$ and $\beta$, in the market. How they receive information is formalized by specifying filtrations $\boldsymbol{F}^{\alpha} = \{\mathcal{F}_t^{\alpha} : t \in \boldsymbol{T}\}$ and $\boldsymbol{F}^{\beta} = \{\mathcal{F}_t^{\beta} : t \in \boldsymbol{T}\}$, respectively. Three

* Corresponding author.



PII: S0304-4068(02)00064-2

securities are marketed: a riskless zero-coupon bond maturing at time 3 and two risky securities, A and B. After normalizing by the bond price process, A and B have the price process $S = \{(S_t^{\mathrm{A}}, S_t^{\mathrm{B}}) : t \in \boldsymbol{T}\}$, adapted to both $\boldsymbol{F}^{\alpha}$ and $\boldsymbol{F}^{\beta}$. The filtration generated by the process $S$ is $\boldsymbol{F}^S = \{\mathcal{F}_t^S : t \in \boldsymbol{T}\}$. The join of $\boldsymbol{F}^{\alpha}$ and $\boldsymbol{F}^S$ is denoted $\boldsymbol{G}^{\alpha} = \{\mathcal{G}_t^{\alpha} : t \in \boldsymbol{T}\}$; the join of $\boldsymbol{F}^{\beta}$ and $\boldsymbol{F}^S$ is denoted $\boldsymbol{G}^{\beta} = \{\mathcal{G}_t^{\beta} : t \in \boldsymbol{T}\}$. In the rational expectations framework, it is common to assume that agents also learn from the prices. Therefore, the total information that the agents have is described by $\boldsymbol{G}^{\alpha}$ and $\boldsymbol{G}^{\beta}$, respectively. Interested readers can refer to Duffie and Huang (1986) for more details. As they point out, normalization by the bond price process can "destroy" some of the information revealed by prices, but our example extends easily to cases not requiring normalization.

The well-known results of Harrison and Kreps (1979) apply to the case $\boldsymbol{G}^{\alpha} = \boldsymbol{G}^{\beta}$, that is, the case in which there is no information asymmetry. Duffie and Huang considered the situation in which one agent has superior information, corresponding to $\boldsymbol{G}^{\alpha} \subset \boldsymbol{G}^{\beta}$ or $\boldsymbol{G}^{\beta} \subset \boldsymbol{G}^{\alpha}$ in our example. They showed that, in either of these cases, there is no arbitrage if and only if there is a *universal equivalent martingale measure*, a probability measure $Q$ on $(\Omega, \mathcal{F})$ equivalent to $P$ such that $S$ is a $Q$-martingale for all agents. Our example has neither $\boldsymbol{G}^{\alpha} \subseteq \boldsymbol{G}^{\beta}$ nor $\boldsymbol{G}^{\beta} \subseteq \boldsymbol{G}^{\alpha}$. A necessary condition for this is that $\boldsymbol{F}^S$ is not equal to, and does not include, either $\boldsymbol{F}^{\alpha}$ or $\boldsymbol{F}^{\beta}$.

In our example, $\mathcal{F}_1^{\alpha}$, $\mathcal{F}_1^{\beta}$, $\mathcal{F}_1^S$ are taken to be the trivial $\sigma$-algebra: $\{\phi, \Omega\}$. Let $\mathcal{F}_3^{\alpha}$ and $\mathcal{F}_3^{\beta}$ both be $\mathcal{F}$, the $\sigma$-algebra that fully reveals the state of the world. Let the $\sigma$-algebra $\mathcal{F}_2^{\alpha}$ be generated by the partition $\{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9\}\}$, and let $\mathcal{F}_2^{\beta}$ be generated by the partition $\{\{\omega_1, \omega_3, \omega_5\}, \{\omega_2, \omega_4, \omega_6, \omega_7, \omega_8, \omega_9\}\}$. Finally, let $\mathcal{F}_2^S$ be generated by $\{\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}, \{\omega_7, \omega_8, \omega_9\}\}$, as illustrated in Fig. 1. It is easy to see that $\mathcal{G}_2^{\alpha}$ is generated by $\{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_6\}, \{\omega_7, \omega_8, \omega_9\}\}$ and that $\mathcal{G}_2^{\beta}$ is generated by $\{\{\omega_1, \omega_3, \omega_5\}, \{\omega_2, \omega_4, \omega_6\}, \{\omega_7, \omega_8, \omega_9\}\}$.

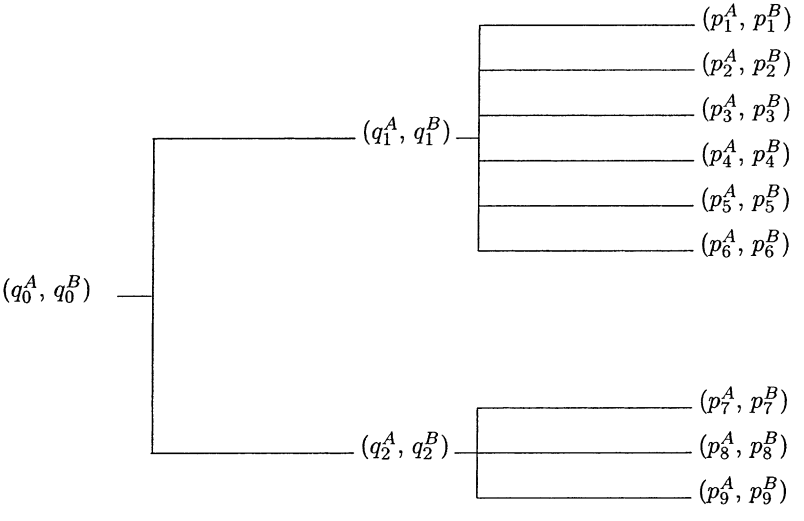


Fig. 1. Illustration of $\boldsymbol{F}^S$.

For notational convenience, we denote the prices, as shown in Fig. 1, in the following way: for $j \in \{A, B\}$, $S_1^j = q_0^j$; $S_2^j(\omega_i) = q_1^j$, $i \in \{1, \dots, 6\}$; $S_2^j(\omega_i) = q_2^j$, $i \in \{7, 8, 9\}$; $S_3^j(\{\omega_i\}) = p_i^j$, $i \in \{1, 2, \dots, 9\}$. The price process is thus uniquely determined by $x \in \boldsymbol{R}^{24}$, where $x = (q_0^{\mathrm{A}}, q_1^{\mathrm{A}}, q_2^{\mathrm{A}}, p_1^{\mathrm{A}}, \dots, p_9^{\mathrm{A}}, q_0^{\mathrm{B}}, q_1^{\mathrm{B}}, q_2^{\mathrm{B}}, p_1^{\mathrm{B}}, \dots, p_9^{\mathrm{B}})$.

We now show a robust example of the absence of a universal equivalent martingale measure, despite the absence of arbitrage for any agent. First, we give the definition of "robust."

**Definition.** Fix the filtrations $\boldsymbol{F}^\alpha$, $\boldsymbol{F}^\beta$, and $\boldsymbol{F}^S$, as above. Let $U$ be the set of all $x \in \boldsymbol{R}^{24}$ such that if the price process is given by $x$, then there is no arbitrage for any agent. Let $V$ be the set of all $x \in U$ such that if the price process is given by $x$, then there is no universal equivalent martingale measure. A *robust* example is a non-empty open subset of $V$ in the relative topology of $U$.

In order to prove our claim that there is a robust example, it suffices to prove that the complement of $V$, $V^c$, is contained by a closed set in the relative topology of $U$, and that $V$ is non-empty.

We first show that $V$ is not empty. A necessary and sufficient condition for no arbitrage for agent $\alpha$ is that there exists an equivalent martingale measure $Q^\alpha$ with respect to the filtration $\boldsymbol{G}^\alpha$, with $Q^\alpha(\{\omega_i\}) = \pi_i^\alpha$ for $i \in \{1, 2, \dots, 9\}$. In order to determine $\pi^\alpha$, we write the associated martingale condition $\mathrm{E}^\alpha(S_{t+1} \mid \mathcal{G}_t^\alpha) = S_t$, $(t = 1, 2)$, where $\mathrm{E}^\alpha$ denotes expectation under $Q^\alpha$, in the explicit form:

$$\sum_{i=1}^{3} \frac{\pi_i^\alpha p_i^j}{\pi_1^\alpha + \pi_2^\alpha + \pi_3^\alpha} = q_1^j, \quad j \in \{\mathrm{A}, \mathrm{B}\}, \tag{1}$$

$$\sum_{i=4}^{6} \frac{\pi_i^\alpha p_i^j}{\pi_4^\alpha + \pi_5^\alpha + \pi_6^\alpha} = q_1^j, \quad j \in \{\mathrm{A}, \mathrm{B}\}, \tag{2}$$

$$\sum_{i=7}^{9} \frac{\pi_i^\alpha p_i^j}{\pi_7^\alpha + \pi_8^\alpha + \pi_9^\alpha} = q_2^j, \quad j \in \{\mathrm{A}, \mathrm{B}\}, \tag{3}$$

$$\sum_{i=1}^{6} \pi_i^\alpha q_1^j + \sum_{i=7}^{9} \pi_i^\alpha q_2^j = q_0^j, \quad j \in \{\mathrm{A}, \mathrm{B}\}, \tag{4}$$

$$\sum_{i=1}^{9} \pi_i^\alpha = 1, \tag{5}$$

$$\pi_i^\alpha > 0, \quad i \in \{1, 2, \dots, 9\}. \tag{6}$$

The associated vector $\pi^\alpha$ of probabilities corresponds to "state prices."

Let $x^* = (8, 4, 12, 3, 14, 2, 1, 8, 5, 8, 10, 20, 10.5, 7, 14, 8, 15, 3, 2, 13, 10, 12, 13, 18)$. It is easy to check that for $q$ and $p$ determined by $x^*$, $\pi^\alpha = (a/2, a/8, 3a/8, 1/4 - a/2, 1/6 - a/3, 1/12 - a/6, 1/6, 1/5, 2/15)$ satisfies all the above equations, where $a$ is any number in $(0, 1/2)$. Therefore, if the price process is given by $x^*$, there is no arbitrage for agent α. Similarly, we can show that, given $x^*$, there is no arbitrage for agent β, and that the equivalent martingale measure $Q^\beta$ for agent β is given by the vector of probabilities $\pi^\beta = (b/5, 1/26 - b/13, b/2, 11/52 - 11b/26, 3b/10, 1/4 - b/2, 1/6, 1/5, 2/15)$, where $b$ is any number in $(0, 1/2)$.

Now, suppose that there is a universal equivalent martingale measure $Q$ determined by the vector $\pi$ of probabilities. Then $\pi$ must satisfy Eqs. (1)–(6) and the corresponding equations for β. For example,

$$\frac{\pi_1 p_1^j + \pi_3 p_3^j + \pi_5 p_5^j}{\pi_1 + \pi_3 + \pi_5} = q_1^j, \quad j \in \{A, B\}. \tag{1'}$$

For $q$ and $p$ given by $x^*$, it is easy to check that Eqs. (1) and (1′) cannot be satisfied at the same time. Therefore, $V$ is not empty.

From (1),

$$\frac{\pi_1}{\pi_3} = \frac{(q_1^{\mathrm{A}} - p_2^{\mathrm{A}})(p_3^{\mathrm{B}} - p_2^{\mathrm{B}}) - (q_1^{\mathrm{B}} - p_2^{\mathrm{B}})(p_3^{\mathrm{A}} - p_2^{\mathrm{A}})}{(q_1^{\mathrm{A}} - p_1^{\mathrm{A}})(p_2^{\mathrm{B}} - p_1^{\mathrm{B}}) - (q_1^{\mathrm{B}} - p_1^{\mathrm{B}})(p_2^{\mathrm{A}} - p_1^{\mathrm{A}})}.$$

From (1′),

$$\frac{\pi_1}{\pi_3} = \frac{(q_1^{\mathrm{A}} - p_3^{\mathrm{A}})(p_5^{\mathrm{B}} - p_3^{\mathrm{B}}) - (q_1^{\mathrm{B}} - p_3^{\mathrm{B}})(p_5^{\mathrm{A}} - p_3^{\mathrm{A}})}{(q_1^{\mathrm{A}} - p_1^{\mathrm{A}})(p_3^{\mathrm{B}} - p_1^{\mathrm{B}}) - (q_1^{\mathrm{B}} - p_1^{\mathrm{B}})(p_3^{\mathrm{A}} - p_1^{\mathrm{A}})}.$$

Since

$$W = \left\{ x \in \boldsymbol{R}^{24} : \frac{(q_1^{\mathrm{A}} - p_2^{\mathrm{A}})(p_3^{\mathrm{B}} - p_2^{\mathrm{B}}) - (q_1^{\mathrm{B}} - p_2^{\mathrm{B}})(p_3^{\mathrm{A}} - p_2^{\mathrm{A}})}{(q_1^{\mathrm{A}} - p_1^{\mathrm{A}})(p_2^{\mathrm{B}} - p_1^{\mathrm{B}}) - (q_1^{\mathrm{B}} - p_1^{\mathrm{B}})(p_2^{\mathrm{A}} - p_1^{\mathrm{A}})} \right.$$
$$\left. = \frac{(q_1^{\mathrm{A}} - p_3^{\mathrm{A}})(p_5^{\mathrm{B}} - p_3^{\mathrm{B}}) - (q_1^{\mathrm{B}} - p_3^{\mathrm{B}})(p_5^{\mathrm{A}} - p_3^{\mathrm{A}})}{(q_1^{\mathrm{A}} - p_1^{\mathrm{A}})(p_3^{\mathrm{B}} - p_1^{\mathrm{B}}) - (q_1^{\mathrm{B}} - p_1^{\mathrm{B}})(p_3^{\mathrm{A}} - p_1^{\mathrm{A}})} \right\}$$

is closed in $\boldsymbol{R}^{24}$, $W \cap U$ is closed in the relative topology of $U$. Since $V^c \subset W \cap U$, we get the desired result.

## 3. Discussion

The existence of a universal equivalent martingale measure (EMM) would imply that all agents have the same marginal rates of transfer of wealth among the various state-date nodes of the tree. These marginal rates of substitution are determined in part by trading opportunities. But the different agents have different trading opportunities because they have different information. Thus, there is nothing from theory that suggests that there should be a universal EMM.

Our example is consistent with a rational expectations equilibrium (REE). A degenerate example is obtained by having linear preferences over terminal consumption of the form $U_i(c) = E^i(c)$, where $E^i$ denotes expectation with respect to $Q^i$. Then any measurable (agent by agent) endowments and no trade is an equilibrium, trivially. For a slightly less trivial example, we could have, say, additive utility of the form $U_i(c) = E[u(c)]$, where the marginal utility map $v = u'(\cdot)$ on $(0, \infty)$ onto $(0, \infty)$ is strictly monotone (as implied by strictly concave utility and Inada conditions). We can then let $\xi_i$ denote the Radon-Nikodym derivative of $Q^i$ with respect to the reference probability measure $P$, and let $c_i = v^{-1}(\xi_i)$ denote the endowment and the consumption level of agent $i$ in a no-trade equilibrium. In this last example, the total endowment need not, in general, by adapted to each agent's filtration, so one would not generally presume that total consumption is observable.

Because at each period $t$, each agent faces a partition of the state space revealing three possible events in the next period, three securities is a necessary and sufficient number (given linear independence) of securities for spanning. In our example, each agent indeed has access to three securities, two risky securities and one risk-free (after normalization) bond. We ensured that their event-contingent payoff matrices are of full rank. As for the prices before normalization, as warned by Duffie and Huang (1986), they may or may not reveal the same information, and they may or may not have the same span. One can in any case treat ours as an example in which normalization is not necessary (as in our first example above, in which the bond paying one unit of consumption has a price of one in the consumption numeraire). More generally, one can, beginning from our example, construct examples in which spanning and non-existence of a universal EMM is present even when the bond price process is random.

Although, in our example, the market for contingent claims that payoff based on future information is complete for each agent, prices do not fully reveal the state of the world before the last period. While one agent can adjust his portfolio to form any state-contingent claim, other agents cannot observe the composition of his or her portfolio.

Despite the completeness of markets for each agent, a derivative security, for example, a call option on one of the original securities, is in general not redundant. Since there is no universal equivalent martigale measure, the price of the call fixed by no arbitrage for one agent will in general introduce an arbitrage for the other agent, unless prices for the underlying assets change in response to the introduction of the call. See also, Kraus and Smith (1990), in this vein. The new equilibrium price process will, in general, generate a finer filtration than the original $\boldsymbol{F}^S$. That is to say, the introduction of a deriviative security will reveal more information to all agents.

A point of connection with the case of homogeneous information, as suggested by a referee, might be made through asymmetric participation constraints, as considered, for example, by Balasko et al. (1990). In their model, the different agents are restricted, asymmetrically, in the portfolios that they are allowed to hold at given state and dates, and the restrictions are linear. (For example, an agent is not allowed to trade certain securities in certain events). Informational asymmetries can likewise be viewed in terms of symmetric information with linear restrictions on portfolio trades. What prevents a direct application of portfolio constraints to treat asymmetric information, in general, is the typical rational-expectations assumption that agents observe prices. (Our example is consistent with that observational assumption.)

As is traditional in the REE setting, we do not assume that portfolio demands by other agents, individually, are observable. (Our example is in any case consistent with learning from demands, because, as mentioned, it is consistent with a no-trade equilibrium.)

## References

Balasko, Y., Cass, D., Siconolfi, 1990. The structure of financial equilibrium with exogenous yields. Journal of Mathematical Economics 19, 195–216.

Duffie, D., Huang, C.-F., 1986. Multiperiod security markets with differential information. Journal of Mathematical Economics 15, 283–303.

Harrison, M.J., Kreps, D., 1979. Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory 20, 381–408.

Kraus, A., Smith, D., 1990. Beliefs about beliefs: the effect on equilibrium of non-payoff-relevant uncertainty. Mimeo, University of British Columbia.