# Supplement to "Continuous Time Random Matching"

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The supplement is organized as follows. We first prove Theorem A.1 in Appendix E. Since continuous-time independent random matching *without* enduring partnerships can be viewed as a special case of the model considered in Appendix A with all the enduring probabilities being zero, most results in Section 2 are covered by the corresponding results in Appendix A. Some remaining properties are then checked in Appendix F.

## E Proof of Theorem A.1

This section is organized as follows. Subsection E.1 presents a static random partial matching model with finitely many agents as well as some estimations on the relevant matching probabilities. Such a static model will be used in the construction of a finite-period dynamic random matching model with finitely many agents in Subsection E.2. To make the proof of Theorem A.1 more accessible, we first state in Subsection E.3 some properties of the finite-agent dynamic matching model (Lemmas E.2 – E.10) that are needed for proving Theorem A.1 in Subsection E.4. The proofs of the technical results in Lemmas E.1 through E.10 are postponed to Subsection E.5. In particular, Lemma E.1 is proved in Subsection E.5.1. In order to prove Lemmas E.2 – E.10, some additional technical results are presented as Lemmas E.11 through E.21 in Subsection E.5.2. Then, Lemmas E.2 through E.10 are shown in Subsections E.5.3 through E.5.11, respectively.

### E.1 Finite-agent static random partial matching with general matching probabilities

Let $I = \{1, \ldots, \hat{M}\}$ be a finite set with $\hat{M}$ an even integer in the set $\mathbb{N}$ of positive integers, $\mathcal{I}_0$ the power set on $I$, and $\lambda_0$ the counting probability measure on $\mathcal{I}_0$ with $\lambda_0(A) = |A|/|I|$ for any $A \in \mathcal{I}_0$, where $|A|$ is the cardinality of $|A|$. A partial matching $\psi$ on $I$ is an involution from $I$ to $I$ in the sense that $\psi(\psi(i)) = i$ for any $i \in I$. When $\psi(i) \neq i$ ($\psi(i) = i$), agent $i$ is matched

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with agent $\psi(i)$ (agent $i$ is not matched). When $\psi(i) \neq i$ for each $i \in I$, $\psi$ is said to be a full matching on $I$. For a given probability space $(\Omega, \mathcal{F}_0, P_0)$, a random (partial) matching $\pi$ on $I$ is a mapping from $I \times \Omega$ to $I$ such that $\pi_\omega = \pi(\,\cdot\,, \omega)$ is a partial matching on $I$ for each $\omega \in \Omega$.

The following result is essential to the construction of finite matching model with multiple periods (for the matching steps) in Subsection E.2.

**Lemma E.1.** *Let $(I, \mathcal{I}_0, \lambda_0)$ be the finite counting probability space as above. Then, there exists a finite set $\Omega$ with its power set $\mathcal{F}_0$ such that for any type function $\alpha^0$ from $I$ to $S$ and partial matching $\pi^0$ on $I$ with*

$$g^0(i) = \begin{cases} \alpha^0(\pi^0(i)) & \text{if } \pi^0(i) \neq i \\ J & \text{if } \pi^0(i) = i, \end{cases}$$

*and for any function $q$ from $S \times S$ to $\mathbb{R}_+$ with $\sum_{r \in S} q_{kr} \leq 1$ and $\hat{\rho}_{kJ} q_{kl} = \hat{\rho}_{lJ} q_{lk}$ for any $k, l \in S$, where $\hat{\rho} = \lambda_0 \left(\alpha^0, g^0\right)^{-1}$ is the extended type distribution induced by $\left(\alpha^0, g^0\right)$ on $\hat{S}$,*[1] *there exists a random matching $\pi$ from $I \times \Omega$ to $I$ and a probability measure $P_0$ on $(\Omega, \mathcal{F}_0)$ with the following properties.*

*(i) Let $H = \{i \in I : \pi^0(i) \neq i\}$. Then $P_0\left(\{\omega \in \Omega : \pi_\omega(i) = \pi^0(i) \text{ for any } i \in H\}\right) = 1$.*

*(ii) Let $g$ be the mapping from $I \times \Omega$ to $S \cup \{J\}$, defined by*

$$g(i, \omega) = \begin{cases} \alpha^0(\pi(i, \omega)) & \text{if } \pi(i, \omega) \neq i \\ J & \text{if } \pi(i, \omega) = i \end{cases}$$

*for any $(i, \omega) \in I \times \Omega$.*

*Fix any $i, j \in I$ with $i \neq j$, $\pi^0(i) = i$ and $\pi^0(j) = j$; denote $\alpha^0(i)$ and $\alpha^0(j)$ by $k_1$ and $k_2$ respectively. For any $l_1, l_2 \in S$, the random matching $\pi$ and the associated type process $g$ satisfy the following inequalities:*

$$P_0\left(\pi_i = j\right) \leq \frac{2}{\hat{M} \hat{\rho}_{k_1 J}},$$

$$q_{k_1 l_1} - \frac{2}{\hat{M}^{\frac{2}{3}}} \leq P_0(g_i = l_1) \leq q_{k_1 l_1} \text{ if } \hat{\rho}_{k_1 J} \geq \frac{1}{\hat{M}^{\frac{1}{3}}},$$

$$q_{k_1 l_1} q_{k_2 l_2} - \frac{5}{\hat{M}^{\frac{2}{3}}} \leq P_0(g_i = l_1, g_j = l_2) \leq q_{k_1 l_1} q_{k_2 l_2} + \frac{1}{\hat{M}^{\frac{2}{3}}} \text{ if } \hat{\rho}_{k_1 J} \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \text{ and } \hat{\rho}_{k_2 J} \geq \frac{1}{\hat{M}^{\frac{1}{3}}}.$$

*(iii) For any $k, l \in S$ and any $\omega \in \Omega$,*

$$\left|\lambda_0\left(\{i \in I : \alpha^0(i) = k, g^0(i) = J, g(i, \omega) = l\}\right) - \hat{\rho}_{kJ} q_{kl}\right| \leq \frac{2}{\hat{M}}.$$

[1] That is, for any subset $C$ of $\hat{S}$, $\hat{\rho}(C) = \lambda_0\left(\left(\alpha^0, g^0\right)^{-1}(C)\right)$. In particular, $\hat{\rho}_{kJ} = \lambda_0\left(\left(\alpha^0, g^0\right) = (k, J)\right)$. Note that $g^0$ represents the partners' types for the initially matched agents.

*To reflect their dependence on* $(\alpha^0, \pi^0, q)$*,* $\pi$ *and* $P_0$ *will also be denoted by* $\pi_{(\alpha^0,\pi^0,q)}$ *and* $P_{(\alpha^0,\pi^0,q)}$ *respectively.*[2]

Part (i) means that initially matched agents are not rematched. The three inequalities in Part (ii) provide respectively (1) an upper bound on the probability for two single agents to be matched, (2) an estimation on the distribution of the partner's type of a newly matched agent, (3) the approximate pairwise independence of the random types of the partners for the newly matched agents. Part (iii) provides an estimation of the cross-sectional extended type distributions for the newly matched agents.

### E.2 Finite-agent dynamic matching model

What we need to do is to construct a sequence of transition probabilities and a sequence of extended type functions. Since we need to consider random mutation, random matching, random type changing and break-up at each time period, three finite spaces with transition probabilities will be constructed at each time period.

Before the formal construction, we briefly describe the timeline. In each period, there are three steps. The first step is the mutation step, agents (single or matched) change their types independently. The second step is the matching step, only single agents take part in a static random matching described in Lemma E.1. The third step is the type changing with break-up step, at which agents who were just matched in the last step either enter into a long-term partnership or do not, and then experience a change in their types according to the specified type-changing probabilities. At this step, agents who have been matched for more than one step may break up with some probability, and change their types according to the specified type-changing probabilities if they indeed break up.

Denote

$$\begin{aligned}
\bar{\eta} &= \max\{\eta_{kl} : k, l \in S, k \neq l\}, \\
\bar{q} &= \max\{\theta_{kl}(\hat{p}) : k, l \in S, \hat{p} \in \hat{\Delta}\}, \\
\bar{\vartheta} &= \max\{\vartheta_{kl} : k, l \in S\}, \\
\bar{a} &= \max\{\bar{\eta}, \bar{q}, \bar{\vartheta}\} + 1.
\end{aligned}$$

Let $M$ be an integer in $\mathbb{N}$ with $M \geq \max\{K\bar{a}, 3\}$. Let $\hat{M}$ be an even integer in $\mathbb{N}$ and sufficiently larger than $M$ (an explicit expression for $\hat{M}$ will be given after Lemma E.2). As in Subsection E.1, let $I = \{1, 2, \ldots, \hat{M}\}$, $\mathcal{I}_0$ the power set on $I$, and $\lambda_0$ the counting probability measure on

[2]The above equation shows that $g_i, i \in I$ are approximately pairwise independent. In fact, we can use similar techniques to prove that $g_i, i \in I$ are approximately mutually independent. For simplicity, we only demonstrate the case for approximate pairwise independence.

$\mathcal{I}_0$. Let $\mathbb{T}_0$ be the finite set $\{n\}_{n=0}^{M^2}$. The corresponding time line is $\{n/M\}_{n=0}^{M^2}$ so that the time length for each period is $1/M$.

We define the parameters for the dynamical system as follows. For any $k, k', l, l' \in S$, and $\hat{p} \in \hat{\Delta}$, let

$$\begin{aligned}
&\hat{\eta}_{kl} = \begin{cases} \frac{1}{M}\eta_{kl} + \frac{1}{M^2} & \text{if } k \neq l \\ 1 - \sum_{r \in (S \setminus \{k\})} \hat{\eta}_{kr} & \text{if } k = l, \end{cases} \\
&\hat{q}_{kl}(\hat{p}) = \frac{1}{M}\theta_{kl}(\hat{p}) \text{ and } \hat{q}_k(\hat{p}) = 1 - \sum_{l \in S} \hat{q}_{kl}(\hat{p}), \\
&\hat{\xi}_{kl} = \min\{\xi_{kl}, 1 - \frac{1}{M^2}\}, \\
&\hat{\sigma}_{kl} = \sigma_{kl}, \\
&\hat{\varsigma}_{kl} = \varsigma_{kl}, \\
&\hat{\vartheta}_{kl} = \frac{1}{M}\vartheta_{kl} + \frac{1}{M^2}.
\end{aligned}$$

Note that $M \geq K\bar{a}$ and $\bar{a} = \max\{\bar{\eta}, \bar{q}, \bar{\vartheta}\} + 1$. Then, we can obtain that

$$\begin{aligned}
&\hat{\eta}_{kl} \leq \frac{\bar{a}-1}{K\bar{a}} + \frac{1}{K^2\bar{a}^2} \leq \frac{\bar{a}-1}{K\bar{a}} + \frac{1}{K\bar{a}} = \frac{1}{K} \quad \text{if } k \neq l, \\
&\hat{\eta}_{kl} \geq 1 - K\frac{1}{K} = 0 \quad \text{if } k = l, \\
&\hat{q}_{kl}(\hat{p}) \leq \frac{\bar{a}}{K\bar{a}} = \frac{1}{K}, \\
&\hat{q}_k(\hat{p}) \geq 1 - K\frac{1}{K} = 0.
\end{aligned}$$

In this way, we have defined $\hat{\eta}_{kl}$, $\hat{\xi}_{kl}$ and $\hat{\vartheta}_{kl}$ so that $\hat{\eta}_{kl}$ and $\hat{\vartheta}_{kl}$ have lower bound $\frac{1}{M^2}$, and $\hat{\xi}_{kl}$ has upper bound $1 - \frac{1}{M^2}$. Such bounds will be used in the proof of Lemma E.15.

For the initial stage at period 0, let $\hat{\alpha}^0$ be the initial type function from $I$ to $S$, and $\hat{\pi}^0$ the initial partial matching from $I$ to $I$. Let $\hat{g}^0$ be the mapping from $I$ to $S \cup \{J\}$ defined by

$$\hat{g}^0(i) = \begin{cases} \hat{\alpha}^0(\hat{\pi}^0(i)) & \text{if } \hat{\pi}^0(i) \neq i \\ J & \text{if } \hat{\pi}^0(i) = i, \end{cases}$$

for any $i \in I$. Let $\hat{\rho}^0 = \lambda_0 \left(\hat{\alpha}^0, \hat{g}^0\right)^{-1}$ be the initial cross-sectional extended type distribution on $\hat{S}$. We require that $\hat{\rho}^0_{kJ} \geq \frac{1}{M^2}$ for any $k \in S$. Since the initial stage is deterministic, we can let $(\Omega_0, \mathcal{E}_0, Q_0)$ be the trivial probability space over the single set $\{0\}$. A function on $I$ can be trivially viewed as a function on $I \times \Omega_0$, and vice versa.

Suppose that the construction for the dynamical system $\mathbb{D}$ has been done up to time period $n-1$ for $n \geq 1$. Thus, $\{(\Omega_m, \mathcal{E}_m, Q_m)\}_{m=0}^{3n-3}$ and $\{\hat{\alpha}^m, \hat{\pi}^m, \hat{g}^m\}_{m=0}^{3n-3}$ have been constructed, where each $\Omega_m$ is a finite set with its power set $\mathcal{E}_m$, $Q_m$ a transition probability from $\Omega^{m-1}$ to $(\Omega_m, \mathcal{E}_m)$, $\hat{\alpha}^m$ a type function from $I \times \Omega^{m-1}$ to the type space $S$, and $\hat{\pi}^m$ a random partial matching from $I \times \Omega^{m-1}$ to $I$. Here, $\Omega^m = \prod_{j=0}^{m} \Omega_j$, and $\{\omega_j\}_{j=1}^{m}$ will also be denoted by $\omega^m$ when there is no confusion. Denote the product transition probability $Q_0 \otimes Q_1 \otimes \cdots \otimes Q_m$ by $Q^m$, and $\otimes_{j=0}^{m} \mathcal{E}_j$ by $\mathcal{E}^m$ (which is simply the power set on $\Omega^m$). Then, $Q^m$ is the product of the transition probability $Q_m$ with the probability measure $Q^{m-1}$.

We shall now consider the constructions for period $n$. We first work with the random mutation step. Let $\Omega_{3n-2} = S^I$ (the space of all functions from $I$ to $S$) with its power set $\mathcal{E}_{3n-2}$. For each $\omega^{3n-3} \in \Omega^{3n-3}$ and $i \in I$, if $\hat{\alpha}^{3n-3}(i, \omega^{3n-3}) = k$, define a probability measure $\gamma_i^{\omega^{3n-3}}$ on $S$ by letting $\gamma_i^{\omega^{3n-3}}(l) = \hat{\eta}_{kl}$ for each $l \in S$. Define a probability measure $Q_{3n-2}^{\omega^{3n-3}}$ on $(S^I, \mathcal{E}_{3n-2})$ to be the product measure $\prod_{i \in I} \gamma_i^{\omega^{3n-3}}$. Let $\hat{\alpha}^{3n-2} : \left(I \times \prod_{m=0}^{3n-2} \Omega_m\right) \to S$ be such that $\hat{\alpha}^{3n-2}\left(i, \omega^{3n-2}\right) = \omega_{3n-2}(i)$. Let $\hat{\pi}^{3n-2} : \left(I \times \prod_{m=0}^{3n-2} \Omega_m\right) \to I$ be such that $\hat{\pi}^{3n-2}\left(i, \omega^{3n-2}\right) = \hat{\pi}^{3n-3}\left(i, \omega^{3n-3}\right)$. Let $\hat{g}^{3n-2} : \left(I \times \prod_{m=0}^{3n-2} \Omega_m\right) \to S \cup \{J\}$ be such that

$$\hat{g}^{3n-2}\left(i, \omega^{3n-2}\right) = \begin{cases} \hat{\alpha}^{3n-2}(\hat{\pi}^{3n-2}(i, \omega^{3n-2}), \omega^{3n-2}) & \text{if } \hat{\pi}^{3n-2}(i, \omega^{3n-2}) \neq i \\ J & \text{if } \hat{\pi}^{3n-2}(i, \omega^{3n-2}) = i. \end{cases}$$

Let $\hat{\rho}_{\omega^{3n-2}}^{3n-2} = \lambda_0 \left(\hat{\alpha}_{\omega^{3n-2}}^{3n-2}, \hat{g}_{\omega^{3n-2}}^{3n-2}\right)^{-1}$ be the cross-sectional extended type distribution after the random mutation step.

Next, we consider the step of random matching. Let $\Omega_{3n-1}$ be the finite space constructed in Lemma E.1 with the power set $\mathcal{E}_{3n-1}$. For any given $\omega^{3n-2} \in \Omega^{3n-2}$, the type function is $\hat{\alpha}_{\omega^{3n-2}}^{3n-2}(\cdot)$, while the partial matching function is $\hat{\pi}_{\omega^{3n-3}}^{3n-3}(\cdot)$. We can construct a probability measure $Q_{3n-1}^{\omega^{3n-2}} = P_{\hat{\alpha}_{\omega^{3n-2}}^{3n-2}, \hat{\pi}_{\omega^{3n-3}}^{3n-3}, \hat{q}(\hat{\rho}_{\omega^{3n-2}}^{3n-2})}$ and a random matching $\pi_{\hat{\alpha}_{\omega^{3n-2}}^{3n-2}, \hat{\pi}_{\omega^{3n-3}}^{3n-3}, \hat{q}(\hat{\rho}_{\omega^{3n-2}}^{3n-2})}$ by Lemma E.1. Let $\hat{\alpha}^{3n-1} : \left(I \times \prod_{m=0}^{3n-1} \Omega_m\right) \to S$, $\hat{\pi}^{3n-1} : \left(I \times \prod_{m=0}^{3n-1} \Omega_m\right) \to I$ and $\hat{g}^{3n-1} : \left(I \times \prod_{m=0}^{3n-1} \Omega_m\right) \to S \cup \{J\}$ be such that

$$\begin{aligned} \hat{\alpha}^{3n-1}\left(i, \omega^{3n-1}\right) &= \hat{\alpha}^{3n-2}\left(i, \omega^{3n-2}\right), \\ \hat{\pi}^{3n-1}\left(i, \omega^{3n-1}\right) &= \pi_{\hat{\alpha}_{\omega^{3n-2}}^{3n-2}, \hat{\pi}_{\omega^{3n-3}}^{3n-3}, \hat{q}(\hat{\rho}_{\omega^{3n-2}}^{3n-2})}\left(i, \omega_{3n-1}\right), \\ \hat{g}^{3n-1}\left(i, \omega^{3n-1}\right) &= \begin{cases} \hat{\alpha}^{3n-2}(\hat{\pi}^{3n-1}(i, \omega^{3n-1}), \omega^{3n-2}) & \text{if } \hat{\pi}^{3n-1}(i, \omega^{3n-1}) \neq i \\ J & \text{if } \hat{\pi}^{3n-1}(i, \omega^{3n-1}) = i. \end{cases} \end{aligned}$$

Let $\hat{\rho}_{\omega^{3n-1}}^{3n-1} = \lambda_0 \left(\hat{\alpha}_{\omega^{3n-1}}^{3n-1}, \hat{g}_{\omega^{3n-1}}^{3n-1}\right)^{-1}$ be the cross-sectional extended type distribution after the random matching step.

Now, we consider the final step of random type changing with break-up for matched agents. Let $\Omega_{3n} = (S \times \{0,1\})^I$ with its power set $\mathcal{E}_{3n}$, where 0 represents "unmatched" and 1 represents "paired". Each point $\omega_{3n} = (\omega_{3n}^1, \omega_{3n}^2) \in \Omega_{3n}$ is a function from $I$ to $S \times \{0,1\}$. Define a new type function $\hat{\alpha}^{3n} : (I \times \Omega^{3n}) \to S$ by letting $\hat{\alpha}^{3n}(i, \omega^{3n}) = \omega_{3n}^1(i)$. Fix $\omega^{3n-1} \in \Omega^{3n-1}$. For each $i \in I$,

1. if $\hat{\pi}^{3n-1}(i,\omega^{3n-1}) = i$ ($i$ is not paired after the matching step at period $n$), let $\tau_i^{\omega^{3n-1}}$ be the probability measure on the space $S \times \{0,1\}$ that gives probability one to $\left(\hat{\alpha}^{3n-2}(i,\omega^{3n-2}), 0\right)$ and zero for the rest.

2. if $\hat{\pi}^{3n-1}(i,\omega^{3n-1}) \neq i$ and $\hat{\pi}^{3n-3}(i,\omega^{3n-3}) = i$ ($i$ is newly paired after the matching step at period $n$), $\hat{\alpha}^{3n-2}(i,\omega^{3n-2}) = k$, $\hat{\pi}^{3n-1}(i,\omega^{3n-1}) = j$ and $\hat{\alpha}^{3n-2}(j,\omega^{3n-2}) = l$, define a probability measure $\tau_{ij}^{\omega^{3n-1}}$ on $(S \times \{0,1\}) \times (S \times \{0,1\})$ such that

$$\tau_{ij}^{\omega^{3n-1}}\left((k',1),(l',1)\right) = \hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')$$

and

$$\tau_{ij}^{\omega^{3n-1}}\left((k',0),(l',0)\right) = \left(1-\hat{\xi}_{kl}\right)\hat{\varsigma}_{kl}(k')\hat{\varsigma}_{lk}(l')$$

for $k', l' \in S$, and zero for the rest.

3. if $\hat{\pi}^{3n-1}(i,\omega^{3n-1}) \neq i$ and $\hat{\pi}^{3n-3}(i,\omega^{3n-3}) \neq i$ ($i$ is already paired at time $n-1$), $\hat{\alpha}^{3n-2}(i,\omega^{3n-2}) = k$, $\hat{\pi}^{3n-1}(i,\omega^{3n-1}) = j$ and $\hat{\alpha}^{3n-2}(j,\omega^{3n-2}) = l$, define a probability measure $\tau_{ij}^{\omega^{3n-1}}$ on $(S \times \{0,1\}) \times (S \times \{0,1\})$ such that

$$\tau_{ij}^{\omega^{3n-1}}\left((k',1),(l',1)\right) = \left(1-\hat{\vartheta}_{kl}\right)\delta_k(k')\delta_l(l')$$

and

$$\tau_{ij}^{\omega^{3n-1}}\left((k',0),(l',0)\right) = \hat{\vartheta}_{kl}\hat{\varsigma}_{kl}(k')\hat{\varsigma}_{lk}(l')$$

for $k', l' \in S$, and zero for the rest, where $\delta_k(k')$ is one for $k = k'$ and zero for $k \neq k'$.

Let

$$\begin{aligned} A^n_{\omega^{3n-1}} &= \{(i,j) \in I \times I : i < j,\ \hat{\pi}^{3n-1}(i,\omega^{3n-1}) = j\} \\ B^n_{\omega^{3n-1}} &= \{i \in I : \hat{\pi}^{3n-1}(i,\omega^{3n-1}) = i\}. \end{aligned}$$

Define a probability measure $Q_{3n}^{\omega^{3n-1}}$ on $(S \times \{0,1\})^I$ to be the product measure

$$\prod_{i \in B^n_{\omega^{3n-1}}} \tau_i^{\omega^{3n-1}} \otimes \prod_{(i,j) \in A^n_{\omega^{3n-1}}} \tau_{ij}^{\omega^{3n-1}}.$$

We define $\hat{\pi}^{3n}$ and $\hat{g}^{3n}$ such that for any $(i,\omega^{3n}) \in I \times \Omega^{3n}$,

$$\begin{aligned} \hat{\pi}^{3n}(i,\omega^{3n}) &= \begin{cases} i & \text{if } \hat{\pi}^{3n-1}(i,\omega^{3n-1}) = i \text{ or } \omega^2_{3n}(i) = 0 \text{ or } \omega^2_{3n}(\hat{\pi}^{3n-1}(i,\omega^{3n-1})) = 0 \\ \hat{\pi}^{3n-1}(i,\omega^{3n-1}) & \text{otherwise,} \end{cases} \\ \hat{g}^{3n}\left(i,\omega^{3n}\right) &= \begin{cases} \hat{\alpha}^{3n}(\hat{\pi}^{3n}(i,\omega^{3n}),\omega^{3n}) & \text{if } \hat{\pi}^{3n}(i,\omega^{3n}) \neq i \\ J & \text{if } \hat{\pi}^{3n}(i,\omega^{3n}) = i. \end{cases} \end{aligned}$$

It is to check that for each $\omega^{3n} \in \Omega^{3n}$, $\hat{\pi}^{3n}_{\omega^{3n}}(\,\cdot\,)$ is indeed a partial matching on $I$.[3] Let $\hat{\rho}^{3n}_{\omega^{3n}} = \lambda_0 \left(\hat{\alpha}^{3n}_{\omega^{3n}}, \hat{g}^{3n}_{\omega^{3n}}\right)^{-1}$ be the cross-sectional extended type distribution after the step of random type changing with break-up for matched agents.

Repeating this construction, we can construct a sequence of transition probabilities $\{(\Omega_m, \mathcal{E}_m, Q_m)\}_{m=0}^{3M^2}$ and a sequence of functions $\{(\hat{\alpha}^m, \hat{\pi}^m, \hat{g})\}_{m=0}^{3M^2}$.

Let $(I \times \Omega^{3M^2}, \mathcal{I}_0 \otimes \mathcal{E}^{3M^2}, \lambda_0 \otimes Q^{3M^2})$ be the product probability space of $(I, \mathcal{I}_0, \lambda_0)$ and $(\Omega^{3M^2}, \mathcal{E}^{3M^2}, Q^{3M^2})$. For simplicity, we denote $\Omega^{3M^2}$ by $\Omega$ and $Q^{3M^2}$ by $P_0$. For a natural number $N$, any function $f$ from $(\Omega^{m+1}, \mathcal{E}^{m+1}, Q^{m+1})$ to $\mathbb{R}^N$ and $\omega^m \in \Omega^m$, $\mathbb{E}^{\omega^m}(f)$ and $\mathrm{Var}^{\omega^m}(f)$ are defined to be $\int_{\Omega_{m+1}} f(\omega^{m+1})\, dQ^{\omega^m}_{m+1}$ and $\int_{\Omega_{m+1}} ||f(\omega^{m+1}) - \mathbb{E}^{\omega^m} f||^2_\infty \, dQ^{\omega^m}_{m+1}$, respectively.

In the following, we will often work with functions or sets that are measurable in $(\Omega^m, \mathcal{E}^m, Q^m)$ for some $m \le 3M^2$, which may be viewed as functions or sets based on $(\Omega^{3M^2}, \mathcal{E}^{3M^2}, Q^{3M^2})$ by allowing for dummy components for the tail part.

### E.3 Properties of the finite-agent dynamic matching model

In this subsection, we first introduce a process $\tilde{\beta}^m$ to capture the types of the agents and their partners, and whether the agents are newly matched. For $1 \le m \le 3M^2$ and $i \in I$, let $\tilde{\beta}^m_i = (\hat{\alpha}^m_i, \hat{g}^m_i, \hat{h}^m_i)$, where

$$\hat{h}^m_i = \begin{cases} 0 & \text{if } \hat{g}^m_i \neq J \text{ and } \hat{g}^{m-1}_i \neq J \\ 1 & \text{otherwise.} \end{cases}$$

It is clear that $\hat{h}^m_i = 0$ if and only if agent $i$ has been matched with another agent for at least two steps. Note that in the third step of each time period, agents who have been matched for at least two steps break up with some probability; agents who have just been matched in the previous step (the matching step) form a long-term partnership with some probability. That is why we need $\hat{h}$ to identify agents who have been matched for at least two steps. By the construction of the model, if an agent has a partner at the end of the mutation step, he or she must have the same partner in the previous step. It is easy to verify that for any $n \in \{1, \dots, M\}$,

$$\hat{h}^{3n-2}_i = \begin{cases} 0 & \text{if } \hat{g}^{3n-2}_i \neq J \\ 1 & \text{if } \hat{g}^{3n-2}_i = J. \end{cases} \tag{E.1}$$

[3] For any given $\omega^{3n} \in \Omega^{3n}$, let $C^n_{\omega^{3n}} = \{i \in I : \hat{\pi}^{3n-1}(i, \omega^{3n-1}) = i \text{ or } \omega^2_{3n}(i) \cdot \omega^2_{3n}\left(\hat{\pi}^{3n-1}(i, \omega^{3n-1})\right) = 0\}$. Then, for any $i \in C^n_{\omega^{3n}}$, we have $\hat{\pi}^{3n}_{\omega^{3n}}(i) = i$ by the definition of $\hat{\pi}^{3n}$. For any $i \notin C^n_{\omega^{3n}}$, we know that $\hat{\pi}^{3n-1}(i, \omega^{3n-1}) = j \neq i$, and $\omega^2_{3n}(i) \cdot \omega^2_{3n}(j) = 1$. The definition of $\hat{\pi}^{3n}$ indicates that $\hat{\pi}^{3n}_{\omega^{3n}}(i) = \hat{\pi}^{3n-1}(i, \omega^{3n-1}) = j$. Since $\hat{\pi}^{3n-1}_{\omega^{3n-1}}(\cdot)$ is a matching, we know that $\hat{\pi}^{3n-1}(j, \omega^{3n-1}) = i \neq j$. It is also clear that $\omega^2_{3n}(j) \cdot \omega^2_{3n}(i) = 1$, which implies that $j \notin C^n_{\omega^{3n}}$. It follows from the definition of $\hat{\pi}^{3n}$ that $\hat{\pi}^{3n}_{\omega^{3n}}(j) = \hat{\pi}^{3n-1}(j, \omega^{3n-1}) = i$. Therefore, $\hat{\pi}^{3n}_{\omega^{3n}}(\cdot)$ is a partial matching on $I$.

Similarly, for the type changing with break-up step,

$$\hat{h}_i^{3n} = \begin{cases} 0 & \text{if } \hat{g}_i^{3n} \neq J \\ 1 & \text{if } \hat{g}_i^{3n} = J. \end{cases} \tag{E.2}$$

Let $\tilde{S} = S \times (S \cup \{J\}) \times \{0, 1\}$. Any $(k, l, r) \in \tilde{S}$ is called an expanded type. Let $\tilde{\Delta}$ be the space whose elements consist of any probability measure $\tilde{p}$ on $\tilde{S} = S \times (S \cup \{J\}) \times \{0, 1\}$ satisfying $\tilde{p}_{klr} = \tilde{p}_{lkr}$ and $\tilde{p}_{kJ0} = 0$ (which means that $\tilde{p}_{kJ1} = \hat{p}_{kJ}$) for any $k, l \in S$ and $r \in \{0, 1\}$, which can be viewed as a compact and convex subset of the simplex in a Euclidean space. We will work with the sup norm $|| \cdot ||_\infty$ on the relevant Euclidean space. For each $k, l \in S$, we use the same notation $\hat{q}_{kl}$ to denote the matching probability from $\tilde{\Delta} \to \mathbb{R}$ that is defined by letting $\hat{q}_{kl}(\tilde{\rho}) = \hat{q}_{kl}(\hat{\rho})$, where $\hat{\rho}_{kl} = \tilde{\rho}_{kl0} + \tilde{\rho}_{kl1}$.

Let $\tilde{\rho}^m$ be the cross-sectional expanded type distribution $\lambda_0 \left(\tilde{\beta}^m\right)^{-1}$. For $k, l \in S$, $\tilde{\rho}_{kl0}^m$ is the fraction of agents who are of type $k$, matched with type-$l$ agents at the $m$-th step and paired at the $(m-1)$-th step as well, while $\tilde{\rho}_{kl1}^m$ is the fraction of agents who are of type $k$, matched with type-$l$ agents at the $m$-th step and single at the $(m-1)$-th step. Note that $\hat{\rho}_{kl}^m$ is the proportion of type-$k$ agents matched with type-$l$ agents at the $m$-th step, which implies $\hat{\rho}_{kl}^m = \tilde{\rho}_{kl0}^m + \tilde{\rho}_{kl1}^m$. It is clear that $\tilde{\rho}^m$ belongs to $\tilde{\Delta}$.

Next, we define three mappings $T_1$, $T_2$, $T_3$ on $\tilde{\Delta}$ to represent the transformation of the expanded type distribution after each step of random mutation, random matching, and random type changing and break-up.[4] For any $\rho \in \tilde{\Delta}$, let

$$[T_1(\tilde{\rho})]_{kl0} = \begin{cases} \sum_{k',l' \in S} \tilde{\rho}_{k'l'0} \, \hat{\eta}_{k'k} \, \hat{\eta}_{l'l} & \text{if } l \neq J \\ 0 & \text{if } l = J, \end{cases}$$

$$[T_1(\tilde{\rho})]_{kl1} = \begin{cases} 0 & \text{if } l \neq J \\ \sum_{k' \in S} \tilde{\rho}_{k'J1} \, \hat{\eta}_{k'k} & \text{if } l = J, \end{cases}$$

$$[T_2(\tilde{\rho})]_{kl0} = \begin{cases} \tilde{\rho}_{kl0} & \text{if } l \neq J \\ 0 & \text{if } l = J, \end{cases}$$

$$[T_2(\tilde{\rho})]_{kl1} = \begin{cases} \tilde{\rho}_{kJ1} \hat{q}_{kl}(\tilde{\rho}) & \text{if } l \neq J \\ \tilde{\rho}_{kJ1} \hat{q}_k(\tilde{\rho}) & \text{if } l = J, \end{cases}$$

$$[T_3(\tilde{\rho})]_{kl0} = \begin{cases} \tilde{\rho}_{kl0} \left(1 - \hat{\vartheta}_{kl}\right) + \sum_{k',l' \in S} \tilde{\rho}_{k'l'1} \, \hat{\xi}_{k'l'} \, \hat{\sigma}_{k'l'}(k, l) & \text{if } l \neq J \\ 0 & \text{if } l = J, \end{cases}$$

$$[T_3(\tilde{\rho})]_{kl1} = \begin{cases} 0 & \text{if } l \neq J \\ \sum_{k',l' \in S} \tilde{\rho}_{k'l'1} \left(1 - \hat{\xi}_{k'l'}\right) \hat{\varsigma}_{k'l'}(k) + \sum_{k',l' \in S} \tilde{\rho}_{k'l'0} \, \hat{\vartheta}_{k'l'} \, \hat{\varsigma}_{k'l'}(k) + \tilde{\rho}_{kJ1} & \text{if } l = J. \end{cases}$$

[4] If the expanded type distribution at the beginning of step $3n-2$ is $\tilde{\rho}$, Lemma E.11 indicates that the expected expanded type distribution at the end of step $3n-2$ is $T_1(\tilde{\rho})$. Similarly, Lemma E.13 says that the expected expanded type distribution at the end of step $3n$ is $T_3(\tilde{\rho})$ if the expanded type distribution at the beginning of step $3n$ is $\tilde{\rho}$. However, $T_2(\tilde{\rho})$ is not the expected expanded type distribution at the end of step $3n-1$ if the type distribution at the beginning of step $3n-1$ is $\tilde{\rho}$. Nevertheless, by Lemma E.12, $T_2(\tilde{\rho})$ is a good approximation of the expected expanded type distribution at the end of step $3n-1$.

The following lemma shows the equicontinuity of $T_1$, $T_2$, $T_3$ and $\hat{q}$.

**Lemma E.2.** *There exists a sequence of positive numbers* $\{\xi_m\}_{m=-1}^{3M^2+1}$ *with* $\xi_{-1} = \frac{1}{M^{M^M}}$ *and* $3M^2\xi_m \leq \xi_0 \leq \xi_{-1}$ *for any* $m \in \{1, \ldots, 3M^2+1\}$ *such that for any* $m \in \{-1, 0, \ldots, 3M^2\}$, $r \in \{1, 2, 3\}$, $\tilde{\rho}, \tilde{\rho}' \in \tilde{\Delta}$, *if* $\|\tilde{\rho} - \tilde{\rho}'\|_\infty \leq \xi_{m+1}$, *then*

$$\|T_r(\tilde{\rho}) - T_r(\tilde{\rho}')\|_\infty \leq \xi_m,$$

$$\|\hat{q}(\tilde{\rho}) - \hat{q}(\tilde{\rho}')\|_\infty \leq \xi_m.$$

In the rest of this paper, we shall take $\hat{M}$ to be the smallest even integer greater than $\left(\frac{1}{\xi_{3M^2+1}}\right)^3$.

Let $e(m) = [\frac{m+2}{3}]$ and $f(m) = m - 3e(m) + 3$. Then for any $m \in \{1, \ldots, 3M^2\}$, the $m$-th step in the finite dynamical system is also the $f(m)$-th step in the $e(m)$-th period. For integers $1 \leq m_1 \leq m_2 \leq 3M^2$, we use $U_{m_1}^{m_2}$ to represent $T_{f(m_2)} \circ T_{f(m_2-1)} \circ \cdots \circ T_{f(m_1)}$. For convenience, when $1 \leq m_2 < m_1 \leq 3M^2$, $U_{m_1}^{m_2}$ is defined to be the identity mapping on $\tilde{\Delta}$.

The following lemma provides an upper bound on the difference between the expected expanded type distribution at the $m$-th step $\mathbb{E}\tilde{\rho}^m$ and the repeated applications of the transformations $T_1$, $T_2$, $T_3$.

**Lemma E.3.** *There exists a sequence* $\{B_1(n)\}_{n=1}^\infty$ *of positive real numbers with* $\lim_{n\to\infty} B_1(n) = 0$ *such that for any* $m \in \{1, 2, \ldots, 3M^2\}$, *we have* $\|\mathbb{E}(\tilde{\rho}^m) - U_1^m(\tilde{\rho}^0)\|_\infty \leq B_1(M)$.

Let $\mathcal{F}^m = \{F \in \mathcal{E}^{3M^2} : F = F^m \times \Pi_{m'=m+1}^{3M^2}\Omega_{m'}$ and $F^m \in \mathcal{E}^m\}$. Any set $F$ in $\mathcal{F}^m$ represents an event that "happens" by step $m$. For example, we use $\left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}$ to represent some event that happens by step $3n-2$ in which $\tilde{\beta}_i^{3n-2} = (k, J, 1)$. The following two lemmas consider conditional probabilities[5] of the form of $P_0\left(\tilde{\beta}_i^{m+1} = b \,\middle|\, \left(\tilde{\beta}_i^m = a\right) \cap F^m\right)$ for $F^m \in \mathcal{F}^m$, which will be used in Subsection E.4 below.

The following lemma provides an upper bound on the difference between $\hat{q}_{kl}\left(U_1^{3n-2}(\mathbb{E}(\tilde{\rho}^0))\right)$ and $P_0\left(\tilde{\beta}_i^{3n-1} = (k, l, 1) \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right)$.

**Lemma E.4.** *For any* $i \in I$, $n \in \{1, 2, \ldots, M^2\}$, $k, l \in S$, *and* $F^{3n-2} \in \mathcal{F}^{3n-2}$ *with* $P_0\left(\left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right) > 0$, *we have*

$$\left|P_0\left(\tilde{\beta}_i^{3n-1} = (k, l, 1) \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right) - \hat{q}_{kl}\left(U_1^{3n-2}(\tilde{\rho}^0)\right)\right|$$
$$\leq \quad \frac{1}{M^3 P_0\left(\left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right)} + \frac{1}{M^2}.$$

[5]For given events $A$ and $B$ with $P_0(A) = 0$, we can define the value of the conditional probability $P_0(B|A)$ to be any number in $[0, 1]$ that suits a particular context.

The next lemma shows the relationship between $P_0\left(\tilde{\beta}_i^{m+1}=b \,\middle|\, \tilde{\beta}_i^m=a\right)$ and $P_0\left(\tilde{\beta}_i^{m+1}=b \,\middle|\, \left(\tilde{\beta}_i^m=a\right)\cap F^m\right)$.

**Lemma E.5.** *Fix any $i\in I$, $a,b\in\tilde{S}$, and $n\in\{1,2,\ldots,M^2\}$.*

*(i) For any $F^{3n-3}\in\mathcal{F}^{3n-3}$ with $P_0\left(\left(\tilde{\beta}_i^{3n-3}=a\right)\cap F^{3n-3}\right)>0$, the following identity holds:*

$$P_0\left(\tilde{\beta}_i^{3n-2}=b \,\middle|\, \left(\tilde{\beta}_i^{3n-3}=a\right)\cap F^{3n-3}\right)=P_0\left(\tilde{\beta}_i^{3n-2}=b \,\middle|\, \tilde{\beta}_i^{3n-3}=a\right).$$

*(ii) For any $F^{3n-2}\in\mathcal{F}^{3n-2}$ with $P_0\left(\left(\tilde{\beta}_i^{3n-2}=a\right)\cap F^{3n-2}\right)>0$, we have the following inequality*

$$\begin{aligned}&\left|P_0\left(\tilde{\beta}_i^{3n-1}=b \,\middle|\, \left(\tilde{\beta}_i^{3n-2}=a\right)\cap F^{3n-2}\right)-P_0\left(\tilde{\beta}_i^{3n-1}=b \,\middle|\, \tilde{\beta}_i^{3n-2}=a\right)\right|\\ \leq\;& \frac{1}{M^3P_0\left(\left(\tilde{\beta}_i^{3n-2}=a\right)\cap F^{3n-2}\right)}+\frac{1}{M^2}.\end{aligned}$$

*(iii) For any $F^{3n-1}\in\mathcal{F}^{3n-1}$ with $P_0\left(\left(\tilde{\beta}_i^{3n-1}=a\right)\cap F^{3n-1}\right)>0$, we have*

$$P_0\left(\tilde{\beta}_i^{3n}=b \,\middle|\, \left(\tilde{\beta}_i^{3n-1}=a\right)\cap F^{3n-1}\right)=P_0\left(\tilde{\beta}_i^{3n}=b \,\middle|\, \tilde{\beta}_i^{3n-1}=a\right).$$

For any $i\in I$ and $m\in\{0,1,\ldots,3M^2\}$, let $\mathcal{F}_i^m$ be the algebra generated by $\{\tilde{\beta}_i^{m'}\}_{m'=0}^m$. Any set in $\mathcal{F}_i^m$ represents an event for agent $i$ that happens by step $m$.

An approximate Markov property for the expanded type process is presented below.

**Lemma E.6.** *There exists a sequence $\{B_2(n)\}_{n=1}^\infty$ of positive real numbers with $\lim_{n\to\infty}B_2(n)=0$. Fox any $i\in I$, $\tilde{\beta}_i$ satisfies the approximate Markov property in the sense that for any $m,m'\in\{0,1,\ldots,3M^2\}$ with $m>m'$, $a,a'\in\tilde{S}$, and $F_i^{m'-1}\in\mathcal{F}_i^{m'-1}$,*

$$\begin{aligned}&P_0\left(\left(\tilde{\beta}_i^m=a,\tilde{\beta}_i^{m'}=a'\right)\cap F_i^{m'-1}\right)P_0\left(\tilde{\beta}_i^{m'}=a'\right)-P_0\left(\tilde{\beta}_i^m=a,\tilde{\beta}_i^{m'}=a'\right)P_0\left(\left(\tilde{\beta}_i^{m'}=a'\right)\cap F_i^{m'-1}\right)\\ \leq\;& B_2(M).\end{aligned}$$

The following lemma shows that the expanded type process satisfies an approximate pairwise independence condition.

**Lemma E.7.** *There exists a sequence $\{B_3(n)\}_{n=1}^\infty$ of positive real numbers with $\lim_{n\to\infty}B_3(n)=0$ such that for any $i,j\in I$ with $i\neq j$ and $\tilde{\pi}_i^0\neq j$, $m\in\{0,1,\ldots,3M^2\}$, $F_i^m\in\mathcal{F}_i^m$, and $F_j^m\in\mathcal{F}_j^m$ we have*

$$\left|P_0\left(F_i^m\cap F_j^m\right)-P_0\left(F_i^m\right)P\left(F_j^m\right)\right|\leq B_3(M).$$

In the rest of this subsection, we consider an estimation of the number of mutations, matchings and break-ups that can happen within any time interval, and consider the expected cross-sectional expanded type distribution. For any $\omega \in \Omega$, let

$$\hat{H}_i^m(\omega) = \left|\{n \in \mathbb{T}_0 : \hat{\alpha}_i^{3n-2}(\omega) \neq \hat{\alpha}_i^{3n-3}(\omega) \text{ or } \hat{g}_i^{3n-2}(\omega) \neq \hat{g}_i^{3n-3}(\omega), 3n-2 \leq m\}\right|,$$
$$\hat{N}_i^m(\omega) = \left|\{n \in \mathbb{T}_0 : \hat{g}_i^{3n-1}(\omega) \neq \hat{g}_i^{3n-2}(\omega), 3n-1 \leq m\}\right|,$$
$$\hat{R}_i^m(\omega) = \left|\{n \in \mathbb{T}_0 : \hat{g}_i^{3n}(\omega) = J \text{ and } \hat{h}_i^{3n-1}(\omega) = 0, 3n \leq m\}\right|.$$

Here, $\hat{H}_i^m$ is the number of mutations of agent $i$ and of the partner of agent $i$, by the $m$-th step, while $\hat{N}_i^m$ and $\hat{R}_i^m$ are the numbers of matchings and breakups of agent $i$ by the $m$-th step. Let $\hat{X}_i^m = \hat{H}_i^m + \hat{N}_i^m + \hat{R}_i^m$.

The following lemma provides a lower bound for the probability that there is no jump for the counting process $\hat{X}_i$ between two different steps.

**Lemma E.8.** *For any $m, \Delta m \in \{0, \ldots, 3M^2\}$ and $F^m \in \mathcal{F}^m$ such that $m + \Delta m \leq 3M^2$ and $P_0(F^m) > 0$, we have*

$$P_0(\hat{X}_i^{m+\Delta m} = \hat{X}_i^m | F^m) \geq \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m}.$$

An estimation on the probability of changing type twice in a given time interval is presented below.

**Lemma E.9.** *For any $m, \Delta m \in \{0, \ldots, 3M^2\}$ such that $m + \Delta m \leq 3M^2$ and $P_0(F^m) > 0$, we have*

$$P_0\left(\hat{X}_i^{m+\Delta m} - \hat{X}_i^m \geq 2 \,\middle|\, F^m\right) \leq \left(1 - \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m}\right)^2.$$

An upper bound is provided below for $\|\mathbb{E}\left(\tilde{\rho}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}^m\right)\|_\infty$.

**Lemma E.10.** *For any $m, \Delta m \in \{0, \ldots, 3M^2\}$ such that $m + \Delta m \leq 3M^2$, we have*

$$\|\mathbb{E}\left(\tilde{\rho}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}^m\right)\|_\infty \leq 1 - \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m}.$$

### E.4 Existence of continuous-time random matching

The proof in this subsection makes extensive use of some basic results in nonstandard analysis, of which a comprehensive introduction is provided in the first three chapters of the book Loeb and Wolff (2015).

As noted in Loeb and Wolff (2015), hyperfinite sets are important objects in nonstandard analysis, which can be viewed as equivalence classes of sequences of finite sets. The transfer principle indicates that any results about finite sets can be restated on hyperfinite sets. In particular, the dynamic matching model and its properties as developed in Subsections E.2 and E.3 can be recast in the setting with a hyperfinite number of agents and time periods with the same notations.

We recall some notations in the hyperfinite setting as follows. First, we take $M$, as used in the finite dynamic matching model, to be an unlimited hyperfinite integer in ${}^*\mathbb{N}_\infty$, and $\hat{M}$ the smallest even hyperinteger in ${}^*\mathbb{N}_\infty$ which is greater than $\left(\frac{1}{\xi_{3M^2+1}}\right)^3$ (as described in the paragraph below Lemma E.2).[6] Then, let $I$ be the hyperfinite set $\{1, 2, \ldots, \hat{M}\}$ with its internal power set $\mathcal{I}_0$ and the internal counting probability measure $\lambda_0$ on $\mathcal{I}_0$, and $\mathbb{T}_0$ the hyperfinite set $\{n\}_{n=0}^{M^2}$ with the corresponding time line $\{n/M\}_{n=0}^{M^2}$ (i.e., the time length for each period is the infinitesimal $1/M$). The parameters for the dynamical system, $\hat{\eta}_{kl}$, $\hat{q}_{kl}$, $\hat{\xi}_{kl}$, $\hat{\sigma}_{kl}$, $\hat{\varsigma}_{kl}$, $\hat{\vartheta}_{kl}$ remain the same. As in the finite dynamic matching model, we denote $\Omega^{3M^2}$, $\mathcal{E}^{3M^2}$ (the internal power set on $\Omega^{3M^2}$) and $Q^{3M^2}$ by $\Omega$, $\mathcal{F}_0$ and $P_0$ respectively. Let $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes P_0)$ be the internal product probability space of $(I, \mathcal{I}_0, \lambda_0)$ and $(\Omega, \mathcal{F}_0, P_0)$. Note that $\mathcal{I}_0 \otimes \mathcal{F}_0$ is also the internal power set on $I \times \Omega$. By the Transfer Principle, we know that Lemmas E.1 to E.10 still hold in the hyperfinite setting. We will not distinguish the statements of Lemmas E.1 to E.10 in the finite and hyperfinite settings when there is no confusion.

Let $(I, \mathcal{I}, \lambda)$, $(\Omega, \mathcal{F}, P)$ and $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be the standard probability spaces that are obtained from the internal probability spaces $(I, \mathcal{I}_0, \lambda_0)$, $(\Omega, \mathcal{F}_0, P_0)$ and $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes P_0)$ respectively by taking their corresponding Loeb probability spaces. Note that $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a Fubini extension of the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$. We need to prove that there exist $\alpha : I \times \Omega \times \mathbb{R}_+ \to S$, $\pi : I \times \Omega \times \mathbb{R}_+ \to I$, and $g : I \times \Omega \times \mathbb{R}_+ \to S \cup \{J\}$ satisfying all the properties described in Appendix A.1. Towards this end, we divide the proof into six parts. In Part 1, we define these processes, and discuss their basic properties. In Part 2, we prove that $(\alpha, g)$ is Markovian and independent. We then check that the transition-intensity matrix of the relevant Markov chains at time $t$ is $Q(\check{p}(t))$. In particular, we consider Cases 1, 2, 3 and 4 of Table 1 in Parts 3, 4, 5, and 6 respectively.

**Part 1:** Recall that $\hat{p}^0$ is the initial extended type distribution. Let $\{A_{kl}\}_{(k,l) \in \hat{S}}$ be an internal partition of $I$ such that $\frac{|A_{kl}|}{\hat{M}} \simeq \hat{p}^0_{kl}$ for any $k \in S$ and $l \in S \cup \{J\}$, $\frac{|A_{kJ}|}{\hat{M}} \geq \frac{1}{M^2}$ for any $k \in S$, and $|A_{kl}| = |A_{lk}|$ for any $k, l \in S$, and $|A_{kk}|$ is even for any $k \in S$. Let $\hat{\alpha}^0$ be an

[6] A positive hyperreal number is said to be infinite or unlimited if it is greater than every standard natural number. As usual, ${}^*\mathbb{N}_\infty$ denotes the set of unlimited hyperfinite integers. A hyperreal number is said to be finite or limited if its absolute value is less than some standard natural number. Two hyperreal numbers $a$ and $b$ are said to be infinitely close to each other if $a - b$ is an infinitesimal, which is denoted by $a \simeq b$. We also use monad($a$) to denote the set of all the hyperreal numbers infinitely close to $a$. When $a$ is limited, it is infinitely close to a standard real number $b$, which is called the standard part of $a$, denoted by ${}^\circ a$ or st($a$).

internal function from $(I, \mathcal{I}_0, \lambda_0)$ to $S$ such that $\hat{\alpha}^0(i) = k$ if $i \in \bigcup_{l \in S \cup \{J\}} A_{kl}$. Let $\hat{\pi}^0$ be an internal partial matching on $I$ such that $\hat{\pi}^0(i) = i$ on $\bigcup_{k \in S} A_{kJ}$, and the restriction $\hat{\pi}^0|_{A_{kl}}$ is an internal bijection from $A_{kl}$ to $A_{lk}$ for any $k, l \in S$. Let $\hat{g}^0(i) = \hat{\alpha}^0(\hat{\pi}^0(i))$. It is clear that $\lambda_0(\{i : \hat{\alpha}^0(i) = k, \hat{g}^0(i) = l\}) \simeq \hat{p}^0_{kl}$ for any $k \in S$ and $l \in S \cup \{J\}$.

Fix any $t \in \mathbb{R}_+$, and denote the hyperinteger part of $tM$ by $\bar{n}$. Based on the hyperfinite dynamic system transferred from Appendix E.2, let $\alpha'(t) = \hat{\alpha}^{3\bar{n}}$, $\pi(t) = \hat{\pi}^{3\bar{n}}$, $g'(t) = \hat{g}^{3\bar{n}}$. Since $\hat{\alpha}^{3\bar{n}}$, $\hat{\pi}^{3\bar{n}}$, and $\hat{g}^{3\bar{n}}$ are all internal, it is clear that $\alpha'(t)$, $\pi(t)$, and $g'(t)$ are measurable on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

Fix any $i \in I$. The stochastic processes $\alpha'_i$ and $g'_i$ may not be right-continuous with left limits (RCLL). We will show that up to any finite time, any agent can only change their types finitely many times with probability one. Recall that $\hat{X}^m_i(\omega)$ is defined in the paragraph below Lemma E.7. Let

$$A_i = \{\omega \in \Omega : \hat{X}^m_i(\omega) \text{ is finite for any } m \in {}^*\mathbb{N} \text{ such that } \frac{m}{M} \text{ is finite}\}.$$

For any $N$ in the set $\mathbb{N}$ of (standard) positive integers, let $A^N_i = \{\omega \in \Omega : \hat{X}^{NM}_i(\omega) \text{ is finite}\}$. It is clear that $A_i = \cap_{N=1}^{\infty} A^N_i$, and

$$A^N_i = \bigcup_{k=1}^{\infty} \{\omega \in \Omega : \hat{X}^{NM}_i(\omega) \le k\}.$$

Since the set $\{\omega \in \Omega : \hat{X}^{NM}_i(\omega) \le k\}$ is internal, $A^N_i$ is measurable in $\mathcal{F}$ Fix any $N \in \mathbb{N}$. For any $n \in \mathbb{N}$ and $j \in \{0, 1, \dots, n\}$, let $m_j$ be the hyperinteger part of $\frac{jNM}{n}$. Then $m_0 = 0$, $m_n = NM$ and $m_j - m_{j-1} < \frac{2NM}{n}$ for any $j \in \{1, \dots, n\}$. Fix any $n \in \mathbb{N}$. For any $\omega \notin A^N_i$, $\hat{X}^{NM}_i(\omega)$ is infinite, which implies that there exists $j \in \{1, \dots, n\}$ such that $\hat{X}^{m_j}_i(\omega) - \hat{X}^{m_{j-1}}_i(\omega) \ge 2$. Therefore, we know that

$$\Omega \setminus A^N_i \subseteq \bigcup_{j=1}^{n} \{\omega \in \Omega : \hat{X}^{m_j}_i(\omega) - \hat{X}^{m_{j-1}}_i(\omega) \ge 2\},$$

which implies that

$$P\left(\Omega \setminus A^N_i\right) \le \sum_{j=1}^{n} P\left(\hat{X}^{m_j}_i - \hat{X}^{m_{j-1}}_i \ge 2\right). \tag{E.3}$$

It follows from Lemma E.9 that

$$P_0\left(\hat{X}^{m_j}_i - \hat{X}^{m_{j-1}}_i \ge 2\right) \le \left(1 - \left(1 - \frac{K\bar{a}}{M}\right)^{2(m_j - m_{j-1})}\right)^2. \tag{E.4}$$

Note that $M$ is unlimited. It is clear that for any $\Delta m$ such that $\frac{\Delta m}{M}$ is finite,

$$\left(1 - \frac{K\bar{a}}{M}\right)^{\Delta m} = \left(1 - \frac{K\bar{a}}{M}\right)^{\frac{M}{K\bar{a}} \frac{K\bar{a}\Delta m}{M}} \simeq e^{-\frac{K\bar{a}\Delta m}{M}} \simeq e^{-{}^\circ\left(\frac{\Delta m}{M}\right) K\bar{a}}. \tag{E.5}$$

Since the standard part of $\frac{m_j - m_{j-1}}{M}$ is $\frac{N}{n}$, it follows from Equations (E.4) and (E.5) that

$$P\left(\hat{X}_i^{m_j} - \hat{X}_i^{m_{j-1}} \geq 2\right) \leq \left(1 - e^{-\frac{2KN\bar{a}}{n}}\right)^2. \tag{E.6}$$

By combining Equations (E.3) and (E.6), we obtain that

$$P\left(\Omega \setminus A_i^N\right) \leq n\left(1 - e^{-\frac{2KN\bar{a}}{n}}\right)^2.$$

Note that $n\left(1 - e^{-\frac{2KN\bar{a}}{n}}\right)^2 \to 0$ as $n \to \infty$. Then $P\left(\Omega \setminus A_i^N\right) = 0$, which implies that $P\left(A_i^N\right) = 1$. Therefore, we have $P\left(A_i\right) = P\left(\cap_{N=1}^{\infty} A_i^N\right) = 1$.

Let $A = \{(i, \omega) \in I \times \Omega : \hat{X}_i^m(\omega)$ is finite for any $m \in {}^*\mathbb{N}$ such that $\frac{m}{M}$ is finite$\}$. Then, it is clear that

$$A = \bigcap_{N=1}^{\infty} \bigcup_{k=1}^{\infty} \{(i, \omega) \in I \times \Omega : \hat{X}_i^{NM}(\omega) \leq k\},$$

which also implies that $A$ is measurable in $\mathcal{I} \boxtimes \mathcal{F}$. Since $A = \{(i, \omega) \in I \times \Omega : \omega \in A_i\}$ and $P(A_i) = 1$ for any $i \in I$, we have $\lambda \boxtimes P(A) = 1$ by the Fubini property.

We define

$$\alpha_i(\omega, t) = \begin{cases} \lim_{t' \to t^+} \alpha'_i(\omega, t') & \text{if } (i, \omega) \in A \\ \hat{\alpha}_i^{3\bar{n}}(\omega) & \text{otherwise,} \end{cases}$$

$$g_i(\omega, t) = \begin{cases} \lim_{t' \to t^+} g'_i(\omega, t') & \text{if } (i, \omega) \in A \\ \hat{g}_i^{3\bar{n}}(\omega) & \text{otherwise.} \end{cases}$$

Now we prove that $\alpha$ and $g$ are well defined and measurable on $(I \times \Omega \times \mathbb{R}_+, (\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}(\mathbb{R}_+))$. For any $(i, \omega) \in A$, $\alpha'_i(\omega, t')$ can only change finitely many times in the time interval $[0, t+1]$. Then there exists $\epsilon > 0$ such that $\alpha'_i(\omega, t')$ are constant on $(t, t+\epsilon)$. Then, for any $(i, \omega) \in A$, $\lim_{t' \to t^+} \alpha'_i(\omega, t')$ is well defined, and the sample path $\alpha_i(\omega, t)$ is RCLL in $t \in \mathbb{R}_+$. For any $i \in I$, since $P(A_i) = 1$, the stochastic process $\alpha_i$ is RCLL. We can prove that $g$ is well defined with the RCLL property in the same way. By the definition of $\alpha$ and $g$, and the fact that $A$ is measurable, it is clear that for any $t \in \mathbb{R}_+$, $\alpha(i, \omega, t)$ and $g(i, \omega, t)$ are measurable on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. By Proposition 1.13 in Karatzas and Shreve (1991), $\alpha$ and $g$ are measurable on $(I \times \Omega \times \mathbb{R}_+, (\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}(\mathbb{R}_+))$.

For each $n \in \mathbb{T}_0 = \{n\}_{n=0}^{M^2}$ and $\omega \in \Omega$, since $\hat{\pi}_\omega^{3n}$ is an internal involution on $I$ and $\lambda_0$ is the hyperfinite counting probability measure on $\mathcal{I}_0$, it is obvious that the particular case $\hat{\pi}_\omega^{3\bar{n}}$ is measure-preserving from the Loeb space $(I, \mathcal{I}, \lambda)$ to itself. Hence, for any $t \in \mathbb{R}_+$ and $\omega \in \Omega$, $\pi_{\omega t}(\cdot)$ is an internal involution on $I$ and is measure-preserving.

**Part 2:** Fix any $i \in I$ and $t \in \mathbb{R}_+$. Letting $E_t = \{n \in {}^*\mathbb{N} : \frac{n}{M} \in \text{monad}(t)\}$, it is obvious that $\bar{n} \in E_t$. Define the following $\mathcal{F}$ -measurable set

$$B_i(t) = \{\omega \in \Omega : \hat{X}_i^{3n}(\omega) = \hat{X}_i^{3\bar{n}}(\omega) \text{ for any } n \in E_t\}. \tag{E.7}$$

For any $n_1, n_2 \in \mathbb{T}_0$ such that $\mathrm{st}(\frac{n_1}{M}) < t < \mathrm{st}(\frac{n_2}{M})$ for $t > 0$, $n_1 = 0$ and $\mathrm{st}(\frac{n_2}{M}) > 0$ for $t = 0$, Lemma E.8 implies that

$$P\left(\Omega \setminus B_i(t)\right) \leq P\left(\hat{X}_i^{3n_1} \neq \hat{X}_i^{3n_2}\right) \leq 1 - \mathrm{st}\left(e^{-\frac{6K\bar{a}(n_2-n_1)}{M}}\right).$$

If $\mathrm{st}\left(\frac{n_2-n_1}{M}\right) \to 0$, then $\mathrm{st}\left(e^{-\frac{6K\bar{a}(n_2-n_1)}{M}}\right) \to 1$. Hence, we have $P\left(\Omega \setminus B_i(t)\right) = 0$, which implies that $P\left(B_i(t)\right) = 1$.

Fix any $\omega \in A_i$. If $\hat{\alpha}_i^{3n}(\omega) \equiv C$ for any $n \in E_t$, then the Spillover Principle implies that there exists $n_1, n_2 \in \mathbb{T}_0$ such that $\mathrm{st}(\frac{n_1}{M}) < t < \mathrm{st}(\frac{n_2}{M})$ for $t > 0$, $n_1 = 0$ and $\mathrm{st}(\frac{n_2}{M}) > 0$ for $t = 0$, and $\hat{\alpha}_i^{3n}(\omega) \equiv C$ for any $n \in \{n_1, n_1 + 1, \ldots, n_2\}$. Hence for any $t'$ in the time interval $\left(\mathrm{st}(\frac{n_1}{M}), \mathrm{st}(\frac{n_2}{M})\right)$, $\alpha_i'(t') = C$. Therefore, for any $n \in E_t$, we have

$$\alpha_i(\omega, t) = \lim_{t' \to t^+} \alpha_i'(\omega, t') = C = \hat{\alpha}_i^{3n}(\omega).$$

Fix any $n_0 \in E_t$. For any $\omega \in A_i$, if $\hat{\alpha}_i^{3n_0}(\omega) \neq \alpha_i(\omega, t)$, then $\hat{\alpha}_i^{3n}(\omega)$ can not be constant for $n \in E_t$ by the argument above. In this case, there is a mutation, matching, or break up at some period in $E_t$, which implies that the event $\{\omega \in A_i : \hat{\alpha}_i^{3n_0}(\omega) \neq \alpha_i(\omega, t)\}$ is a subset of $\Omega \setminus B_i(t)$. Since $P(A_i) = 1$, we have

$$P\left(\hat{\alpha}_i^{3n_0} \neq \alpha_{it}\right) \leq P\left(\Omega \setminus B_i(t)\right) = 0,$$

which implies that $P\left(\hat{\alpha}_i^{3n_0} = \alpha_{it}\right) = 1$. Similarly, we can prove that $P\left(\hat{g}_i^{3n_0} = g_{it}\right) = 1$. Denote $\beta_i(t) = (\alpha_i(t), g_i(t))$ and $\hat{\beta}_i^m = (\hat{\alpha}_i^m, \hat{g}_i^m)$ for any $0 \leq m \leq 3M^2$. Then we have

$$P\left(\omega \in \Omega : \hat{\beta}_i^{3n_0}(\omega) = \beta_i(\omega, t)\right) = 1. \tag{E.8}$$

Fix any $\omega \in B_i(t)$. The Spillover Principle implies that there exists $n_1, n_2 \in \mathbb{T}_0$ such that $\mathrm{st}(\frac{n_1}{M}) < t < \mathrm{st}(\frac{n_2}{M})$ for $t > 0$, $n_1 = 0$ and $\mathrm{st}(\frac{n_2}{M}) > 0$ for $t = 0$, and $\hat{X}_i^{3n_1}(\omega) = \hat{X}_i^{3n_2}(\omega)$. Then, we know that for any $n \in \{n_1, n_1 + 1, \ldots, n_2\}$, $\hat{\pi}_i^{3n}(\omega) = \hat{\pi}_i^{3\bar{n}}(\omega)$ and $\hat{\alpha}^{3n}\left(\hat{\pi}_i^{3n}(\omega), \omega\right) = \hat{\alpha}^{3\bar{n}}\left(\hat{\pi}_i^{3\bar{n}}(\omega), \omega\right)$. Hence, for any $t'$ in the time interval $\left(\mathrm{st}(\frac{n_1}{M}), \mathrm{st}(\frac{n_2}{M})\right)$,

$$\alpha'\left(\pi(i, \omega, t), \omega, t'\right) = \alpha'\left(\pi(i, \omega, t'), \omega, t'\right) = \hat{\alpha}^{3\bar{n}}\left(\hat{\pi}_i^{3\bar{n}}(\omega), \omega\right).$$

Therefore, we can obtain that

$$\lim_{t' \to t^+} \alpha'\left(\pi(i, \omega, t), \omega, t'\right) = \hat{\alpha}^{3\bar{n}}\left(\hat{\pi}_i^{3\bar{n}}(\omega), \omega\right).$$

We consider two cases as in the definition of $\alpha$. If $\omega \in A_{\pi(i,\omega,t)}$, then

$$\alpha\left(\pi(i, \omega, t), \omega, t\right) = \lim_{t' \to t^+} \alpha'\left(\pi(i, \omega, t), \omega, t'\right) = \hat{\alpha}^{3\bar{n}}\left(\hat{\pi}_i^{3\bar{n}}(\omega), \omega\right).$$

If $\omega \notin A_{\pi(i,\omega,t)}$, then $\alpha\left(\pi(i,\omega,t),\omega,t\right) = \hat{\alpha}^{3\bar{n}}\left(\hat{\pi}_i^{3\bar{n}}(\omega),\omega\right)$. Since $P(B_i(t)) = 1$, we can claim that for $P$-almost all $\omega \in \Omega$, $\alpha\left(\pi(i,\omega,t),\omega,t\right) = \hat{\alpha}^{3\bar{n}}\left(\hat{\pi}_i^{3\bar{n}}(\omega),\omega\right)$. Hence, we can derive that for $P$-almost all $\omega \in \Omega$,

$$g_i(\omega,t) = \hat{g}_i^{3\bar{n}}(\omega) = \begin{cases} \hat{\alpha}^{3\bar{n}}\left(\hat{\pi}_i^{3\bar{n}}(\omega),\omega\right) & \text{if } \hat{\pi}_i^{3\bar{n}}(\omega) \neq i \\ J & \text{if } \hat{\pi}_i^{3\bar{n}}(\omega) = i \end{cases} = \begin{cases} \alpha\left(\pi(i,\omega,t),\omega,t\right) & \text{if } \pi(i,\omega,t) \neq i \\ J & \text{if } \pi(i,\omega,t) = i. \end{cases}$$

Therefore, the above equation together with Part 1 imply that Property 1 of the independent dynamical system $\hat{\mathbb{D}}$ is verified.

By the hyperfinite analog of Lemma E.6, we know that $B_2(M) \simeq 0$ (since $M$ is unlimited), and for any $r \in \mathbb{N}$, $m_1 = 3n_1, m_2 = 3n_2, \ldots, m_r = 3n_r$ with $n_1 > n_2 > \cdots > n_r$ in $\mathbb{T}_0$, and any expanded types $a_1, a_2, \ldots, a_r$ in $\tilde{S}$,

$$\begin{aligned} & P\left(\tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{m_2} = a_2, \ldots, \tilde{\beta}_i^{m_r} = a_r\right) P\left(\tilde{\beta}_i^{m_2} = a_2\right) \\ = \;& P\left(\tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{m_2} = a_2\right) P\left(\tilde{\beta}_i^{m_2} = a_2, \ldots, \tilde{\beta}_i^{m_r} = a_r\right). \end{aligned} \tag{E.9}$$

For any $n \in \mathbb{T}_0$, Equation (E.2) indicates that

$$\hat{h}_i^{3n} = \begin{cases} 0 & \text{if } \hat{g}_i^{3n} \neq J \\ 1 & \text{if } \hat{g}_i^{3n} = J \end{cases} = \mathbf{1}_{\{J\}}\left(\hat{g}_i^{3n}\right),$$

which means that the value of $\hat{h}_i^{3n}$ is completely determined by $\hat{g}_i^{3n}$. Hence, Equation (E.9) implies that for any $n_1 > n_2 > \cdots > n_r$ in $\mathbb{T}_0$, and any extended types $b_1, b_2, \ldots, b_r$ in $\hat{S}$,

$$\begin{aligned} & P\left(\hat{\beta}_i^{3n_1} = b_1, \hat{\beta}_i^{3n_2} = b_2, \ldots, \hat{\beta}_i^{3n_r} = b_r\right) P\left(\hat{\beta}_i^{3n_2} = b_2\right) \\ = \;& P\left(\hat{\beta}_i^{3n_1} = b_1, \hat{\beta}_i^{3n_2} = b_2\right) P\left(\hat{\beta}_i^{3n_2} = b_2, \ldots, \hat{\beta}_i^{3n_r} = b_r\right). \end{aligned} \tag{E.10}$$

For any $r \in \mathbb{N}$, and real time sequence $t_1 > t_2 > \cdots > t_r$ in $\mathbb{R}_+$, choose $n_k \in \mathbb{T}_0$ such that $\frac{n_k}{M} \simeq t_k$ for $1 \leq k \leq r$. Then, it follows from Equations (E.8) and (E.10) that for any extended types $b_1, b_2, \ldots, b_r$ in $\hat{S}$

$$\begin{aligned} & P\left(\beta_i(t_1) = b_1, \beta_i(t_2) = b_2, \ldots, \beta_i(t_r) = b_r\right) P\left(\beta_i(t_2) = b_2\right) \\ = \;& P\left(\beta_i(t_1) = b_1, \beta_i(t_2) = b_2\right) P\left(\beta_i(t_2) = b_2, \ldots, \beta_i(t_r) = b_r\right), \end{aligned}$$

which implies that the stochastic process $\beta_i = (\alpha_i, g_i)$ has the Markov property.

Fix any $j \in I$ with $j \neq i$ and $\hat{\pi}_i^0 \neq j$. By the hyperfinite analog of Lemma E.7, we know that $B_3(M) \simeq 0$ (since $M$ is unlimited), and for any $n_1 > n_2 > \cdots > n_r$ in $\mathbb{T}_0$, and any extended types $b_1, c_1, b_2, c_2, \ldots, b_r, c_r$ in $\hat{S}$,

$$\begin{aligned} & P\left(\hat{\beta}_i^{3n_1} = b_1, \hat{\beta}_j^{3n_1} = c_1, \ldots, \hat{\beta}_i^{3n_r} = b_r, \hat{\beta}_j^{3n_r} = c_r\right) \\ = \;& P\left(\hat{\beta}_i^{3n_1} = b_1, \ldots, \hat{\beta}_i^{3n_r} = b_r\right) P\left(\hat{\beta}_j^{3n_1} = c_1, \ldots, \hat{\beta}_j^{3n_r} = c_r\right). \end{aligned} \tag{E.11}$$

For any $r \in \mathbb{N}$, and real time sequence $t_1 > t_2 > \cdots > t_r$ in $\mathbb{R}_+$, choose $n_k \in \mathbb{T}_0$ such that $\frac{n_k}{M} \simeq t_k$ for $1 \le k \le r$. We can obtain from Equations (E.8) and (E.11) that for any extended types $b_1, c_1, b_2, c_2, \ldots, b_r, c_r$ in $\hat{S}$,

$$\begin{aligned} & P\left(\beta_i(t_1) = b_1, \beta_j(t_1) = c_1, \ldots, \beta_i(t_r) = b_r, \beta_j(t_r) = c_r\right) \\ = \; & P\left(\beta_i(t_1) = b_1, \ldots, \beta_i(t_r) = b_r\right) P\left(\beta_j(t_1) = c_1, \ldots, \beta_j(t_r) = c_r\right), \end{aligned} \tag{E.12}$$

which implies that the stochastic processes $(\alpha_i, g_i)$ and $(\alpha_j, g_j)$ are independent.

**Part 3:** Fix any $i \in I$, $t \in \mathbb{R}_+$, $k, l, k', l' \in S$ with $(k, l) \neq (k', l')$ and $P\left(\beta_i(t) = (k, l)\right) > 0$. The purpose of this part is to verify that the transition intensity for agent $i$ from expanded type $(k, l)$ to expanded type $(k', l')$ at time $t$ is given in Case 1 of Table 1.

For any $\Delta t \in \mathbb{R}_{++}$ (the set of positive real numbers), let $n, \Delta n \in {}^*\mathbb{N}$ such that $\frac{n}{M} \in$ monad$(t)$ and $\frac{\Delta n}{M} \in$ monad$(\Delta t)$. By Equation (E.8),

$$P\left(\beta_i(t + \Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k, l)\right) \simeq P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', l') \,\middle|\, \hat{\beta}_i^{3n} = (k, l)\right).$$

Lemma E.9 indicates that

$$P_0\left(\hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} \ge 2 \,\middle|\, \hat{\beta}_i^{3n} = (k, l)\right) \le \left(1 - \left(1 - \frac{K\bar{a}}{M}\right)^{6\Delta n}\right)^2 \simeq \left(1 - e^{-\frac{6K\bar{a}\Delta n}{M}}\right)^2 \simeq \left(1 - e^{-6K\bar{a}\Delta t}\right)^2,$$

which implies that the probability for agent $i$ to change her extended type twice in the time interval $[t, t + \Delta t]$ is of order $\Delta t^2$. Hence,

$$\begin{aligned} & P\left(\beta_i(t + \Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k, l)\right) \\ & \quad = P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', l'), \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, l)\right) + O(\Delta t^2). \end{aligned} \tag{E.13}$$

For any $k_1, l_1 \in S$ and $m, m' \in \{3n, 3n + 1, \ldots, 3M^2\}$ with $m > m'$, let

$$B^m_{k_1 l_1} = \{\omega \in \Omega : \hat{\beta}_i^m(\omega) = (k_1, l_1), \hat{X}_i^{3n}(\omega) = \hat{X}_i^{m-1}(\omega), \hat{\beta}_i^{3n}(\omega) = (k, l)\},$$

which is the event that $\hat{\beta}_i^m = (k_1, l_1)$, $\hat{\beta}_i^{3n} = (k, l)$, and there is neither mutation, nor matching, nor break-up for agent $i$ between $3n$-th step and $(m-1)$-th step. Further,

$$C^m_{m'} = \{\omega \in \Omega : \hat{X}_i^{m'}(\omega) = \hat{X}_i^m(\omega)\}$$

is the event that there is neither mutation, nor matching, nor break-up for agent $i$ between $m'$-th step and $m$-th step. In particular, when the event $C^m_{m'}$ happens, agent $i$ does not change her extended type between $m'$-th step and $m$-th step.

If the events $\left(\hat{\beta}_i^{3n} = (k, l)\right)$ and $\left(\hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1\right)$ happen, then mutation is the only way for agent $i$ to change her extended type to $(k', l')$ by the end of step $3n + 3\Delta n$

(since the other two steps must involve single agents). Based on the definition of conditional probabilities, Equation (E.13) can be expanded as follows:

$$\begin{aligned}
&P\left(\beta_i(t+\Delta t)=(k',l') \,\middle|\, \beta_i(t)=(k,l)\right)\\
&\quad= P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',l'), \hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1 \,\middle|\, \hat{\beta}_i^{3n}=(k,l)\right)+O(\Delta t^2)\\
&\quad= \sum_{r=n}^{n+\Delta n-1} P_0\left(B_{k'l'}^{3r+1}\cap C_{3r+1}^{3n+3\Delta n} \,\middle|\, \hat{\beta}_i^{3n}=(k,l)\right)+O(\Delta t^2)\\
&\quad= \sum_{r=n}^{n+\Delta n-1} \left[P_0\left(B_{k'l'}^{3r+1} \,\middle|\, \hat{\beta}_i^{3n}=(k,l)\right) P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'l'}^{3r+1}\right)\right]+O(\Delta t^2)\\
&\quad= \sum_{r=n}^{n+\Delta n-1} \left[P_0\left(\hat{\beta}_i^{3r+1}=(k',l') \,\middle|\, C_{3n}^{3r}\cap\left(\hat{\beta}_i^{3n}=(k,l)\right)\right) P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,l)\right)\right.\\
&\qquad\qquad \left. P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'l'}^{3r+1}\right)\right]+O(\Delta t^2).
\end{aligned}$$

By Equation (E.1) and Lemma E.5, we obtain that

$$\begin{aligned}
&P_0\left(\hat{\beta}_i^{3r+1}=(k',l') \,\middle|\, C_{3n}^{3r}\cap\left(\hat{\beta}_i^{3n}=(k,l)\right)\right)\\
&\quad= P_0\left(\hat{\beta}_i^{3r+1}=(k',l') \,\middle|\, C_{3n}^{3r}\cap\left(\hat{\beta}_i^{3r}=(k,l)\right)\right)\\
&\quad= P_0\left(\tilde{\beta}_i^{3r+1}=(k',l',0) \,\middle|\, C_{3n}^{3r}\cap\left(\tilde{\beta}_i^{3r}=(k,l,0)\right)\right)\\
&\quad= P_0\left(\tilde{\beta}_i^{3r+1}=(k',l',0) \,\middle|\, \tilde{\beta}_i^{3r}=(k,l,0)\right)\\
&\quad= P_0\left(\hat{\beta}_i^{3r+1}=(k',l') \,\middle|\, \hat{\beta}_i^{3r}=(k,l)\right)\\
&\quad= \hat{\eta}_{kk'}\hat{\eta}_{ll'},
\end{aligned}$$

where the last identity follows from the step of random mutation for matched agents in the construction of the dynamic matching model. Then, the above identities imply that

$$\begin{aligned}
&P\left(\beta_i(t+\Delta t)=(k',l') \,\middle|\, \beta_i(t)=(k,l)\right)\\
&\quad= \sum_{r=n}^{n+\Delta n-1} \left[P_0\left(\hat{\beta}_i^{3r+1}=(k',l') \,\middle|\, \hat{\beta}_i^{3n}=(k,l)\right) P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,l)\right)\right.\\
&\qquad\qquad \left. P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'l'}^{3r+1}\right)\right]+O(\Delta t^2). \qquad \text{(E.14)}
\end{aligned}$$

When $k\neq k'$ and $l=l'$, $P_0\left(\hat{\beta}_i^{3r+1}=(k',l) \,\middle|\, \hat{\beta}_i^{3r}=(k,l)\right)=\hat{\eta}_{kk'}\hat{\eta}_{ll}$, which implies that

$$\begin{aligned}
&\left|P_0\left(\hat{\beta}_i^{3r+1}=(k',l) \,\middle|\, \hat{\beta}_i^{3r}=(k,l)\right)-\hat{\eta}_{kk'}\right|\\
&\quad= |\hat{\eta}_{kk'}\hat{\eta}_{ll}-\hat{\eta}_{kk'}| = \hat{\eta}_{kk'}(1-\hat{\eta}_{ll}) = \hat{\eta}_{kk'}\sum_{l''\in S\setminus\{l\}}\hat{\eta}_{ll''} \leq K\left(\frac{\bar{a}}{M}\right)^2.
\end{aligned}$$

Now, we estimate the difference

$$\left|P\left(\beta_i(t+\Delta t)=(k',l)\,\middle|\,\beta_i(t)=(k,l)\right)-\sum_{r=n}^{n+\Delta n-1}\hat{\eta}_{kk'}\right|$$

$$\begin{aligned}
&\le \sum_{r=n}^{n+\Delta n-1}\left|\left(P_0\left(\hat{\beta}_i^{3r+1}=(k',l)\,\middle|\,\hat{\beta}_i^{3n}=(k,l)\right)-\hat{\eta}_{kk'}\right)P_0\left(C_{3n}^{3r}\,\middle|\,\hat{\beta}_i^{3n}=(k,l)\right)\right.\\
&\qquad\qquad \left.P_0\left(C_{3r+1}^{3n+3\Delta n}\,\middle|\,B_{k'l'}^{3r+1}\right)\right|\\
&\qquad + \sum_{r=n}^{n+\Delta n-1}\hat{\eta}_{kk'}\left|P_0\left(C_{3n}^{3r}\,\middle|\,\hat{\beta}_i^{3n}=(k,l)\right)P_0\left(C_{3r+1}^{3n+3\Delta n}\,\middle|\,B_{k'l'}^{3r+1}\right)-1\right|+O(\Delta t^2)\\
&\le \sum_{r=n}^{n+\Delta n-1}K\left(\frac{\bar{a}}{M}\right)^2\\
&\qquad + \sum_{r=n}^{n+\Delta n-1}\hat{\eta}_{kk'}\left|P_0\left(C_{3n}^{3r}\,\middle|\,\hat{\beta}_i^{3n}=(k,l)\right)P_0\left(C_{3r+1}^{3n+3\Delta n}\,\middle|\,B_{k'l'}^{3r+1}\right)-1\right|+O(\Delta t^2).
\end{aligned}$$

Since $\bar{a}$ is finite, we know that $\sum_{r=n}^{n+\Delta n-1}K\left(\frac{\bar{a}}{M}\right)^2=\frac{\bar{a}^2}{M}\frac{\Delta n}{M}$ is infinitesimal and can be absorbed into $O(\Delta t^2)$. By Lemma E.8, we have

$$\begin{aligned}
&P_0\left(C_{3n}^{3r}\,\middle|\,\hat{\beta}_i^{3n}=(k,l)\right)P_0\left(C_{3r+1}^{3n+3\Delta n}\,\middle|\,B_{k'l'}^{3r+1}\right)\\
\ge\;& \left(1-\frac{K\bar{a}}{M}\right)^{6(r-n)}\left(1-\frac{K\bar{a}}{M}\right)^{6(n+\Delta n-r)}\simeq e^{-\frac{6K(r-n)\bar{a}}{M}}e^{-\frac{6K(n+\Delta n-r)\bar{a}}{M}}.
\end{aligned}$$

Then, it follows from the above inequalities that

$$\begin{aligned}
&\left|P\left(\beta_i(t+\Delta t)=(k',l)\,\middle|\,\beta_i(t)=(k,l)\right)-\sum_{r=n}^{n+\Delta n-1}\hat{\eta}_{kk'}\right|\\
&\quad\le \sum_{r=n}^{n+\Delta n-1}\frac{\bar{a}}{M}\left(1-e^{-\frac{6K(r-n)\bar{a}}{M}}e^{-\frac{6K(n+\Delta n-r)\bar{a}}{M}}\right)+O(\Delta t^2)\\
&\quad= \bar{a}\Delta t\left(1-e^{-6K\bar{a}\Delta t}\right)+O(\Delta t^2)\\
&\quad= O(\Delta t^2).
\end{aligned}$$

Therefore, we obtain the estimation

$$\begin{aligned}
P\left(\beta_i(t+\Delta t)=(k',l)\,\middle|\,\beta_i(t)=(k,l)\right) &= \sum_{r=n}^{n+\Delta n-1}\hat{\eta}_{kk'}+O(\Delta t^2)\\
&= \eta_{kk'}\Delta t+O(\Delta t^2). \qquad \text{(E.15)}
\end{aligned}$$

When $k=k'$ and $l\ne l'$, we can also prove in the same way as above that

$$P\left(\beta_i(t+\Delta t)=(k,l')\,\middle|\,\beta_i(t)=(k,l)\right)=\eta_{ll'}\Delta t+O(\Delta t^2). \qquad \text{(E.16)}$$

It remains to consider the case that $k \neq k'$ and $l \neq l'$. It is clear that

$$P_0\left(\hat{\beta}_i^{3r+1} = (k', l') \,\middle|\, \hat{\beta}_i^{3r} = (k, l)\right) = \hat{\eta}_{kk'}\hat{\eta}_{ll'} \leq \left(\frac{\bar{a}}{M}\right)^2.$$

Therefore, Equation (E.14) implies that

$$\begin{aligned}
&P\left(\beta_i(t+\Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k, l)\right) \\
&= \sum_{r=n}^{n+\Delta n-1} \Big[P_0\left(\hat{\beta}_i^{3r+1} = (k', l') \,\middle|\, \hat{\beta}_i^{3n} = (k, l)\right) P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n} = (k, l)\right) \\
&\qquad P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n} = (k, l)\right)\right)\Big] + O(\Delta t^2) \\
&\leq \sum_{r=n}^{n+\Delta n-1} \left(\frac{\bar{a}}{M}\right)^2 + O(\Delta t^2) \\
&= O(\Delta t^2). \qquad \text{(E.17)}
\end{aligned}$$

By combining Equations (E.15), (E.16), (E.17), we obtain that

$$P\left(\beta_i(t+\Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k, l)\right) = \left(\eta_{kk'}\delta_l(l') + \eta_{ll'}\delta_k(k')\right)\Delta t + O(\Delta t^2).$$

Hence, agent $i$'s transition intensity for her expanded types from $(k, l)$ to $(k', l')$ at time $t$ is indeed $Q_{(k,l)(k',l')}(\check{p}(t))$, as given in Case 1 of Table 1.

**Part 4:** Fix any $i \in I$, $t \in \mathbb{R}_+$, $k, l, k' \in S$ with $P(\beta_i(t) = (k, l)) > 0$. The purpose of this part is to verify that agent $i$'s transition intensity for her expanded types from $(k, l)$ to $(k', J)$ at time $t$ is given in Case 2 of Table 1.

For any $\Delta t \in \mathbb{R}_{++}$, let $n, \Delta n \in {}^*\mathbb{N}$ such that $\frac{n}{M} \in \text{monad}(t)$ and $\frac{\Delta n}{M} \in \text{monad}(\Delta t)$. By Equation (E.8), we have

$$P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k, l)\right) \simeq P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J) \,\middle|\, \hat{\beta}_i^{3n} = (k, l)\right).$$

By Lemma E.9, the probability for agent $i$ to change her extended type twice in the time interval $[t, t+\Delta t]$ is at level of $\Delta t^2$. Hence, we have

$$\begin{aligned}
&P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k, l)\right) \\
&= P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, l)\right) + O(\Delta t^2). \qquad \text{(E.18)}
\end{aligned}$$

For any $k_1 \in S$ and $m, m' \in \{3n, 3n+1, \ldots, 3M^2\}$ with $m > m'$, let

$$B_{k_1 J}^m = \{\omega \in \Omega : \hat{\beta}_i^m(\omega) = (k_1, J), \hat{X}_i^{3n}(\omega) = \hat{X}_i^{m-1}(\omega), \hat{\beta}_i^{3n}(\omega) = (k, l)\}$$

and $C_{m'}^m = \{\omega \in \Omega : \hat{X}_i^{m'}(\omega) = \hat{X}_i^m(\omega)\}$. Then, $B_{k_1 l_1}^m$ is the event that $\hat{\beta}_i^m = (k_1, J)$, $\hat{\beta}_i^{3n} = (k, l)$, and there is neither mutation, nor matching, nor break-up for agent $i$ between

$3n$-th step and $(m-1)$-th step; $C^m_{m'}$ is the event that there is neither mutation, nor matching, nor break-up for agent $i$ between $m'$-th step and $m$-th step. In particular, when the event $C^m_{m'}$ happens, agent $i$ does not change her extended type between $m'$-th step and $m$-th step.

If the events $\left(\hat{\beta}_i^{3n} = (k,l)\right)$ and $\left(\hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1\right)$ happen, break-up is the only way for agent $i$ to change her extended type to $(k', J)$ by the end of step $3n + 3\Delta n$ (since, in the other two steps, paired agents must stay paired). Based on the definition of conditional probabilities, Equation (E.18) can be expanded as follows:

$$
\begin{aligned}
& P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k,l)\right) \\
= \;& P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k,l)\right) + O(\Delta t^2) \\
= \;& \sum_{r=n}^{n+\Delta n -1} P_0\left(B^{3r+3}_{k'J} \cap C^{3n+3\Delta n}_{3r+3} \,\middle|\, \hat{\beta}_i^{3n} = (k,l)\right) + O(\Delta t^2) \\
= \;& \sum_{r=n}^{n+\Delta n -1} \left[P_0\left(B^{3r+3}_{k'J} \,\middle|\, \hat{\beta}_i^{3n} = (k,l)\right) P_0\left(C^{3n+3\Delta n}_{3r+3} \,\middle|\, B^{3r+3}_{k'J}\right)\right] + O(\Delta t^2) \\
= \;& \sum_{r=n}^{n+\Delta n -1} \left[P_0\left(\hat{\beta}_i^{3r+3} = (k', J) \,\middle|\, C^{3r+2}_{3n} \cap \left(\hat{\beta}_i^{3n} = (k,l)\right)\right) P_0\left(C^{3r+2}_{3n} \,\middle|\, \hat{\beta}_i^{3n} = (k,l)\right)\right. \\
& \qquad \left. P_0\left(C^{3n+3\Delta n}_{3r+3} \,\middle|\, B^{3r+3}_{k'J}\right)\right] + O(\Delta t^2).
\end{aligned}
$$

It follows from Equation (E.2) and Lemma E.5 that

$$
\begin{aligned}
& P_0\left(\hat{\beta}_i^{3r+3} = (k', J) \,\middle|\, C^{3r+2}_{3n} \cap \left(\hat{\beta}_i^{3n} = (k,l)\right)\right) \\
= \;& P_0\left(\hat{\beta}_i^{3r+3} = (k', J) \,\middle|\, C^{3r+2}_{3n} \cap \left(\hat{\beta}_i^{3r+2} = (k,l)\right)\right) \\
= \;& P_0\left(\tilde{\beta}_i^{3r+3} = (k', J, 1) \,\middle|\, C^{3r+2}_{3n} \cap \left(\tilde{\beta}_i^{3r+2} = (k,l,0)\right)\right) \\
= \;& P_0\left(\tilde{\beta}_i^{3r+3} = (k', J, 1) \,\middle|\, \tilde{\beta}_i^{3r+2} = (k,l,0)\right) \\
= \;& \hat{\vartheta}_{kl} \hat{\varsigma}_{kl}(k'),
\end{aligned}
$$

where the last identity follows from the step of random type changing with break-up for agents (who are not newly matched, but break up the partnership) in the construction of the dynamic matching model. Then, the above identities imply that

$$
\begin{aligned}
& P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k,l)\right) \\
= \;& \sum_{r=n}^{n+\Delta n -1} \hat{\vartheta}_{kl} \hat{\varsigma}_{kl}(k') P_0\left(C^{3r+2}_{3n} \,\middle|\, \hat{\beta}_i^{3n} = (k,l)\right) P_0\left(C^{3n+3\Delta n}_{3r+3} \,\middle|\, B^{3r+3}_{k'J}\right) + O(\Delta t^2). \quad \text{(E.19)}
\end{aligned}
$$

Next, we estimate the difference

$$\begin{aligned}
& \left| P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k,l)\right) - \sum_{r=n}^{n+\Delta n-1} \hat{\vartheta}_{kl} \hat{\varsigma}_{kl}(k') \right| \\
\leq & \sum_{r=n}^{n+\Delta n-1} \hat{\vartheta}_{kl} \hat{\varsigma}_{kl}(k') \left| P_0\left(C_{3n}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n} = (k,l)\right) P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, B_{k'J}^{3r+3}\right) - 1 \right| \\
+ & \; O(\Delta t^2).
\end{aligned}$$

By Lemma E.8, we obtain that

$$\begin{aligned}
& P_0\left(C_{3n}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n} = (k,l)\right) P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, B_{k'J}^{3r+3}\right) \\
\geq & \left(1 - \frac{K\bar{a}}{M}\right)^{2(3r-3n+2)} \left(1 - \frac{K\bar{a}}{M}\right)^{2(3n+3\Delta n-3r-3)} \\
\simeq & \; e^{-\frac{2K\bar{a}(3r-3n+2)}{M}} e^{-\frac{2K\bar{a}(3n+3\Delta n-3r-3)}{M}} \\
\simeq & \; e^{-6K\bar{a}\Delta t}.
\end{aligned}$$

Then, it follows from the above inequalities that

$$\begin{aligned}
& \left| P\left(\beta_i(t+\Delta t) = (k', l) \,\middle|\, \beta_i(t) = (k,l)\right) - \sum_{r=n}^{n+\Delta n-1} \hat{\vartheta}_{kl} \hat{\varsigma}_{kl}(k') \right| \\
\leq & \sum_{r=n}^{n+\Delta n-1} \frac{\bar{a}}{M}\left(1 - e^{-6K\bar{a}\Delta t}\right) + O(\Delta t^2) \\
= & \; \bar{a}\Delta t\left(1 - e^{-6K\bar{a}\Delta t}\right) + O(\Delta t^2) \\
= & \; O(\Delta t^2).
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k,l)\right) &= \sum_{r=n}^{n+\Delta n-1} \hat{\vartheta}_{kl} \hat{\varsigma}_{kl}(k') + O(\Delta t^2) \\
&= \vartheta_{kl} \varsigma_{kl}(k') \Delta t + O(\Delta t^2), \qquad \text{(E.20)}
\end{aligned}$$

which implies that agent $i$'s transition intensity for her expanded types from $(k,l)$ to $(k',J)$ at time $t$ is $Q_{(k,l)(k',J)}(\check{p}(t))$, as given in Case 2 of Table 1.

**Part 5:** Fix any $i \in I$, $t \in \mathbb{R}_+$, $k, k', l' \in S$ with $P(\beta_i(t) = (k,J)) > 0$. The purpose of this part is to verify that agent $i$'s transition intensity for her expanded types from $(k,J)$ to $(k',l')$ at time $t$ is given in Case 3 of Table 1.

For any $\Delta t \in \mathbb{R}_{++}$, let $n, \Delta n \in {}^*\mathbb{N}$ such that $\frac{n}{M} \in \text{monad}(t)$ and $\frac{\Delta n}{M} \in \text{monad}(\Delta t)$. By Equation (E.8), we have

$$P\left(\beta_i(t+\Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k,J)\right) \simeq P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', l') \,\middle|\, \hat{\beta}_i^{3n} = (k,J)\right).$$

Lemma E.9 says that the probability for agent $i$ to change her extended type twice in the time interval $[t, t+\Delta t]$ is at level of $\Delta t^2$. Hence, we have

$$\begin{aligned}
& P\left(\beta_i(t+\Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k, J)\right) \\
= \;& P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', l'), \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) + O(\Delta t^2).
\end{aligned} \tag{E.21}$$

For any $k_1, l_1 \in S$ and $m, m' \in \{3n, 3n+1, \ldots, 3M^2\}$ with $m > m'$, let

$$B_{k_1 l_1}^m = \{\omega \in \Omega : \hat{\beta}_i^m(\omega) = (k_1, l_1), \hat{X}_i^{3n}(\omega) = \hat{X}_i^{m-1}(\omega), \hat{\beta}_i^{3n}(\omega) = (k, J)\}$$

and $C_{m'}^m = \{\omega \in \Omega : \hat{X}_i^{m'}(\omega) = \hat{X}_i^m(\omega)\}$. Then $B_{k_1 l_1}^m$ is the event that $\hat{\beta}_i^m = (k_1, l_1)$, $\hat{\beta}_i^{3n} = (k, J)$ and there is neither mutation, nor matching, nor break-up for agent $i$ between $3n$-th step and $(m-1)$-th step; $C_{m'}^m$ is the event that there is neither mutation, nor matching, nor break-up for agent $i$ between $m'$-th step and $m$-th step. In particular, when the event $C_{m'}^m$ happens, agent $i$ does not change her extended type between $m'$-th step and $m$-th step. It is clear that

$$B_{k_1 l_1}^m = \{\omega \in \Omega : \hat{\beta}_i^m(\omega) = (k_1, l_1), \hat{\beta}_i^{3n}(\omega) = (k, J)\} \cap C_{3n}^{m-1}. \tag{E.22}$$

If the events $\left(\hat{\beta}_i^{3n} = (k, J)\right)$ and $\left(\hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1\right)$ happen, then matching is the only way for agent $i$ to change her extended type to $(k', l')$ by the end of step $3n + 3\Delta n$ (since, in the other two steps, single agents must stay single). Equation (E.21) can be expanded as follows:

$$\begin{aligned}
& P\left(\beta_i(t+\Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k, J)\right) \\
= \;& P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', l'), \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) + O(\Delta t^2) \\
= \;& \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} P_0\left(\left(\hat{\beta}_i^{3r+3} = (k', l')\right) \cap B_{kl}^{3r+2} \cap C_{3r+3}^{3n+3\Delta n} \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) + O(\Delta t^2) \\
= \;& \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} \Big[ P_0\left(\left(\hat{\beta}_i^{3r+3} = (k', l')\right) \cap B_{kl}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\
& \qquad P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left(\hat{\beta}_i^{3r+3} = (k', l')\right) \cap B_{kl}^{3r+2}\right) \Big] \\
& + O(\Delta t^2) \\
= \;& \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} \Big[ P_0\left(\hat{\beta}_i^{3r+3} = (k', l') \,\middle|\, B_{kl}^{3r+2}\right) P_0\left(B_{kl}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\
& \qquad P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left(\hat{\beta}_i^{3r+3} = (k', l')\right) \cap B_{kl}^{3r+2}\right) \Big] \\
& + O(\Delta t^2).
\end{aligned}$$

By Equations (E.2), (E.22) and Lemma E.5,

$$
\begin{aligned}
& P_0\left(\hat{\beta}_i^{3r+3}=(k',l') \,\middle|\, B_{kl}^{3r+2}\right) \\
= \;& P_0\left(\hat{\beta}_i^{3r+3}=(k',l') \,\middle|\, \left(\hat{\beta}_i^{3r+2}=(k,l)\right) \cap C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n}=(k,J)\right)\right) \\
= \;& P_0\left(\tilde{\beta}_i^{3r+3}=(k',l',0) \,\middle|\, \left(\tilde{\beta}_i^{3r+2}=(k,l,1)\right) \cap C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n}=(k,J)\right)\right) \\
= \;& P_0\left(\tilde{\beta}_i^{3r+3}=(k',l',0) \,\middle|\, \tilde{\beta}_i^{3r+2}=(k,l,1)\right) \\
= \;& \hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l'),
\end{aligned}
$$

where the last identity follows from the step of random type changing with break-up for agents (who are newly matched with an enduring relationship) in the construction of the dynamic matching model. Then, the above identities and Equation (E.22) imply that

$$
\begin{aligned}
& P\left(\beta_i(t+\Delta t)=(k',l') \,\middle|\, \beta_i(t)=(k,J)\right) \\
= \;& \sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')P_0\left(B_{kl}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right) \\
& \qquad P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left(\hat{\beta}_i^{3r+3}=(k',l')\right)\cap B_{kl}^{3r+2}\right) \\
& +O(\Delta t^2) \\
= \;& \sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\Big[\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')P_0\left(\hat{\beta}^{3r+2}=(k,l) \,\middle|\, C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right) \\
& \qquad P_0\left(C_{3n}^{3r+1} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left(\hat{\beta}_i^{3r+3}=(k',l')\right)\cap B_{kl}^{3r+2}\right)\Big] \\
& +O(\Delta t^2). \qquad\qquad (E.23)
\end{aligned}
$$

Fix any sample realization $\omega^{3r+1}\in\Omega^{3r+1}$ such that $\hat{\beta}_i^{3r+1}\left(\omega^{3r+1}\right)=(k,J)$. By the definition of $\hat{h}_i^m$, we know that $\hat{h}_i^{3r+1}\left(\omega^{3r+1}\right)=1$, and $\hat{h}_i^{3r+2}\left(\omega^{3r+1},\omega_{3r+2}\right)=1$ for any $\omega_{3r+2}\in\Omega_{3r+2}$. Hence, these facts together with Lemma E.4 imply that

$$
\begin{aligned}
& \left|\hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right)-P_0\left(\hat{\beta}^{3r+2}=(k,l) \,\middle|\, C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\right| \\
= \;& \left|\hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right)-P_0\left(\hat{\beta}^{3r+2}=(k,l) \,\middle|\, C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3r+1}=(k,J)\right)\right)\right| \\
= \;& \left|\hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right)-P_0\left(\tilde{\beta}^{3r+2}=(k,l,1) \,\middle|\, C_{3n}^{3r+1}\cap\left(\tilde{\beta}_i^{3r+1}=(k,J,1)\right)\right)\right| \\
\leq \;& \frac{1}{M^3P_0\left(C_{3n}^{3r+1}\cap\left(\tilde{\beta}_i^{3r+1}=(k,J,1)\right)\right)}+\frac{1}{M^2} \\
= \;& \frac{1}{M^3P_0\left(C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)}+\frac{1}{M^2}.
\end{aligned}
$$

By Lemma E.8, $P\left(C_{3n}^{3r+1} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)>0$, which implies that $P\left(C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)>$

0. Then $P_0\left(C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)$ is not infinitesimal. It is then clear that

$$\frac{1}{M^3P_0\left(C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)}<\frac{1}{M^2}.$$

Therefore, we obtain the following estimation

$$\left|\hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right)-P_0\left(\hat{\beta}^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\right|\leq\frac{2}{M^2}, \tag{E.24}$$

which implies that

$$\begin{aligned}
& P_0\left(\hat{\beta}^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\\
\leq\ & \hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right)+\frac{2}{M^2}\\
\leq\ & \frac{\bar{a}}{M}+\frac{2}{M^2}.
\end{aligned}$$

It follows from the above inequality that

$$\begin{aligned}
& \Bigg|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\Big[\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')P_0\left(\hat{\beta}^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\\
& \qquad P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',l')\right)\cap B_{kl}^{3r+2}\right)\Big]\\
& \quad-\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')P_0\left(\hat{\beta}^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\Bigg|\\
=\ & \sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\Big[\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')P_0\left(\hat{\beta}^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\\
& \left(1-P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',l')\right)\cap B_{kl}^{3r+2}\right)\right)\Big]\\
\leq\ & \sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\left(\frac{\bar{a}}{M}+\frac{2}{M^2}\right)\\
& \left(1-P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',l')\right)\cap B_{kl}^{3r+2}\right)\right).
\end{aligned} \tag{E.25}$$

By Lemma E.8, we obtain that

$$\begin{aligned}
& P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',l')\right)\cap B_{kl}^{3r+2}\right)\\
\geq\ & \left(1-\frac{K\bar{a}}{M}\right)^{2(3r-3n+1)}\left(1-\frac{K\bar{a}}{M}\right)^{2(3n+3\Delta n-3r-3)}\\
\simeq\ & e^{-\frac{2K\bar{a}(3r-3n+1)}{M}}e^{-\frac{2K\bar{a}(3n+3\Delta n-3r-3)}{M}}\\
\simeq\ & e^{-6K\bar{a}\Delta t}.
\end{aligned} \tag{E.26}$$

For $x, z \in {}^*\mathbb{R}$, we use $x \lesssim z$ ($x \gtrsim z$ ) to denote that there exists $y \in {}^*\mathbb{R}$ with $y \simeq x$ such that $y \le z$ ($y \ge z$). Then, Equations (E.25) and (E.26) imply that

$$\begin{aligned}
& \Bigg| \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} \Big[ \hat{\xi}_{kl} \hat{\sigma}_{kl}(k', l') P_0 \left( \hat{\beta}^{3r+2} = (k,l) \,\middle|\, C_{3n}^{3r+1} \cap \left( \hat{\beta}_i^{3n} = (k, J) \right) \right) \\
& \qquad P_0 \left( C_{3n}^{3r+1} \,\middle|\, \hat{\beta}_i^{3n} = (k, J) \right) P_0 \left( C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left( \hat{\beta}_i^{3r+3} = (k', l') \right) \cap B_{kl}^{3r+2} \right) \Big] \\
- & \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} \hat{\xi}_{kl} \hat{\sigma}_{kl}(k', l') P_0 \left( \hat{\beta}^{3r+2} = (k,l) \,\middle|\, C_{3n}^{3r+1} \cap \left( \hat{\beta}_i^{3n} = (k, J) \right) \right) \Bigg| \\
\lesssim & \sum_{r=n}^{n+\Delta n-1} K \left( \frac{\bar{a}}{M} + \frac{2}{M^2} \right) \left( 1 - e^{-6K\bar{a}\Delta t} \right) \\
\lesssim & \; K\bar{a} \left( 1 - e^{-6K\bar{a}\Delta t} \right) \Delta t \\
= & \; O(\Delta t^2).
\end{aligned} \tag{E.27}$$

Therefore, Equations (E.23) and (E.27) lead to the following estimation

$$\begin{aligned}
& P \left( \beta_i(t + \Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k, J) \right) \\
= & \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} \hat{\xi}_{kl} \hat{\sigma}_{kl}(k', l') P_0 \left( \hat{\beta}^{3r+2} = (k,l) \,\middle|\, C_{3n}^{3r+1} \cap \left( \hat{\beta}_i^{3n} = (k, J) \right) \right) \\
& + O(\Delta t^2).
\end{aligned} \tag{E.28}$$

We can use Equation (E.24) to deduce that

$$\begin{aligned}
& \Bigg| \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} \hat{\xi}_{kl} \hat{\sigma}_{kl}(k', l') \hat{q}_{kl} \left( U_1^{3r+1} \left( \tilde{\rho}^0 \right) \right) \\
& \quad - \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} \hat{\xi}_{kl} \hat{\sigma}_{kl}(k', l') P_0 \left( \hat{\beta}^{3r+2} = (k,l) \,\middle|\, C_{3n}^{3r+1} \cap \left( \hat{\beta}_i^{3n} = (k, J) \right) \right) \Bigg| \\
\le & \left| \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} \hat{\xi}_{kl} \hat{\sigma}_{kl}(k', l') \frac{2}{M^2} \right| \\
\le & \; \Delta n K \frac{2}{M^2},
\end{aligned} \tag{E.29}$$

which is an infinitesimal and can be absorbed into the term $O(\Delta t^2)$. Therefore, Equations (E.28) and (E.29) imply that

$$\begin{aligned}
& P \left( \beta_i(t + \Delta t) = (k', l') \,\middle|\, \beta_i(t) = (k, J) \right) \\
= & \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} \hat{\xi}_{kl} \hat{\sigma}_{kl}(k', l') \hat{q}_{kl} \left( U_1^{3r+1} \left( \tilde{\rho}^0 \right) \right) + O(\Delta t^2).
\end{aligned} \tag{E.30}$$

Equation (E.8) implies that $\check{p}_t = \mathbb{E}(\hat{p}_t) \simeq \mathbb{E}(\hat{\rho}^{3n})$. By Lemma E.3, $U_1^{3r+1}(\tilde{\rho}^0) \simeq \mathbb{E}(\tilde{\rho}^{3r+1})$. By the continuity of $\theta_{kl}$, we obtain the following estimation

$$\begin{aligned}
&\frac{1}{\Delta t}\left|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right)-\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\theta_{kl}\left(\check{p}_t\right)\Delta t\right| \\
\lesssim\ &\frac{1}{\Delta t}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\left|\sum_{r=n}^{n+\Delta n-1}\frac{1}{M}{}^*\theta_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right)-{}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3n}\right)\right)\frac{\Delta n}{M}\right| \\
\lesssim\ &\frac{1}{M\Delta t}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\sum_{r=n}^{n+\Delta n-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3r+1}\right)\right)-{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3n}\right)\right)\right| \\
\lesssim\ &\frac{K}{\Delta n}\sum_{r=n}^{n+\Delta n-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3r+1}\right)\right)-{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3n}\right)\right)\right|.
\end{aligned}$$

Fix any $\Delta n' \in \mathbb{T}_0$ such that $\frac{\Delta n'}{M}$ is infinitesimal. Lemma E.10 implies that $\|\mathbb{E}(\tilde{\rho}^{3r+1}) - \mathbb{E}(\tilde{\rho}^{3n})\|_\infty$ is infinitesimal for any $r$ between $n$ and $n+\Delta n'$. By the continuity of $\theta_{kl}$, $\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3r+1}\right)\right)-{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3n}\right)\right)\right|$ is also infinitesimal. Then, we obtain that

$$\frac{K}{\Delta n'}\sum_{r=n}^{n+\Delta n'-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3r+1}\right)\right)-{}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3n}\right)\right)\right| \simeq 0.$$

By the Spillover Principle, it is easy to show that for any $\epsilon \in \mathbb{R}_{++}$, there exists $\delta \in \mathbb{R}_{++}$ such that for any $\Delta n \in \mathbb{T}_0$ with $\mathrm{st}\left(\frac{\Delta n}{M}\right) < \delta$, the standard part of

$$\frac{K}{\Delta n}\sum_{r=n}^{n+\Delta n-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3r+1}\right)\right)-{}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3n}\right)\right)\right|$$

is less than $\epsilon$. We can then claim that

$$\left|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right)-\sum_{l\in S}\hat{\xi}_{kl}\hat{\sigma}_{kl}(k',l')\theta_{kl}\left(\check{p}_t\right)\Delta t\right| = o(\Delta t).$$

Hence, Equation (E.30) implies that

$$P\left(\beta_i(t+\Delta t) = (k',l') \,\middle|\, \beta_i(t) = (k,J)\right) = \sum_{l\in S}\xi_{kl}\sigma_{kl}(k',l')\theta_{kl}\left(\check{p}_t\right)\Delta t + o(\Delta t),$$

which implies agent $i$'s transition intensity for her expanded types from $(k,J)$ to $(k',l')$ at time $t$ to be $Q_{(k,J)(k',l')}(\check{p}(t))$ as in Case 3 of Table 1.

**Part 6:** Fix any $i \in I$, $t \in \mathbb{R}_+$, $k, k' \in S$ with $k \neq k'$ and $P(\beta_i(t) = (k,J)) > 0$. The purpose of this part is to verify that agent $i$'s transition intensity for her expanded types from $(k,J)$ to $(k',J)$ at time $t$ is given in Case 4 of Table 1.

For any $\Delta t \in \mathbb{R}_{++}$, let $n, \Delta n \in {}^*\mathbb{N}$ such that $\frac{n}{M} \in \text{monad}(t)$ and $\frac{\Delta n}{M} \in \text{monad}(\Delta t)$. By Equation (E.8), we have

$$P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k, J)\right) \simeq P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J) \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right).$$

Lemma E.9 says that the probability for agent $i$ to change her extended type twice in the time interval $[t, t+\Delta t]$ is at level of $\Delta t^2$. Hence, we have

$$\begin{aligned} & P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k, J)\right) \\ = \;& P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) + O(\Delta t^2). \end{aligned} \tag{E.31}$$

For any $k_1 \in S$ and $m, m' \in \{3n, 3n+1, \ldots, 3M^2\}$ with $m > m'$, let

$$B_{k_1 J}^m = \{\omega \in \Omega : \hat{\beta}_i^m(\omega) = (k_1, J), \hat{X}_i^{3n}(\omega) = \hat{X}_i^{m-1}(\omega), \hat{\beta}_i^{3n}(\omega) = (k, J)\}$$

and $C_{m'}^m = \{\omega \in \Omega : \hat{X}_i^{m'}(\omega) = \hat{X}_i^m(\omega)\}$. Then $B_{k_1 J}^m$ is the event that $\hat{\beta}_i^m = (k_1, J)$, $\hat{\beta}_i^{3n} = (k, J)$ and there is neither mutation, nor matching, nor break-up for agent $i$ between $3n$-th step and $(m-1)$-th step; $C_{m'}^m$ is the event that there is neither mutation, nor matching, nor break-up for agent $i$ between $m'$-th step and $m$-th step. In particular, when the event $C_{m'}^m$ happens, agent $i$ does not change her extended type between $m'$-th step and $m$-th step. It is clear that

$$B_{k_1 J}^m = \{\omega \in \Omega : \hat{\beta}_i^m(\omega) = (k_1, J), \hat{\beta}_i^{3n}(\omega) = (k, J)\} \cap C_{3n}^{m-1}. \tag{E.32}$$

If the events $\left(\hat{\beta}_i^{3n} = (k, J)\right)$ and $\left(\hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1\right)$ happen, agent $i$ can become an agent with extended type $(k', J)$ via mutation, or matching (without entering an enduring partnership) by the end of step $3n + 3\Delta n$ (since a single agent does not involve in the break-up of a long-term relationship). Equation (E.31) can be expanded as follows:

$$\begin{aligned} & P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k, J)\right) \\ = \;& P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{H}_i^{3n+3\Delta n} - \hat{H}_i^{3n} = 1, \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\ + \;& P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{N}_i^{3n+3\Delta n} - \hat{N}_i^{3n} = 1, \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\ + \;& O(\Delta t^2). \end{aligned} \tag{E.33}$$

The first term in the right hand side can be expanded as follows:

$$\begin{aligned}
& P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J),\hat{H}_i^{3n+3\Delta n}-\hat{H}_i^{3n}=1,\hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1 \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)\\
= & \sum_{r=n}^{n+\Delta n-1} P_0\left(B_{k'J}^{3r+1}\cap C_{3r+1}^{3n+3\Delta n} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)\\
= & \sum_{r=n}^{n+\Delta n-1}\left[P_0\left(B_{k'J}^{3r+1} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'J}^{3r+1}\right)\right]\\
= & \sum_{r=n}^{n+\Delta n-1}\left[P_0\left(\hat{\beta}_i^{3r+1}=(k',J) \,\middle|\, C_{3n}^{3r}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)\right.\\
& \left.P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'J}^{3r+1}\right)\right].
\end{aligned}$$

By Equation (E.1) and Lemma E.5 (i), we obtain that

$$\begin{aligned}
& P_0\left(\hat{\beta}_i^{3r+1}=(k',J) \,\middle|\, C_{3n}^{3r}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\\
= & P_0\left(\hat{\beta}_i^{3r+1}=(k',J) \,\middle|\, C_{3n}^{3r}\cap\left(\hat{\beta}_i^{3r}=(k,J)\right)\right)\\
= & P_0\left(\tilde{\beta}_i^{3r+1}=(k',J,1) \,\middle|\, C_{3n}^{3r}\cap\left(\tilde{\beta}_i^{3r}=(k,J,1)\right)\right)\\
= & P_0\left(\tilde{\beta}_i^{3r+1}=(k',J,1) \,\middle|\, \tilde{\beta}_i^{3r}=(k,J,1)\right)\\
= & P_0\left(\hat{\beta}_i^{3r+1}=(k',J) \,\middle|\, \hat{\beta}_i^{3r}=(k,J)\right)\\
= & \hat{\eta}_{kk'},
\end{aligned}$$

where the last identity follows from the step of random mutation for matched agents in the construction of the dynamic matching model. Then, the above identities imply that

$$\begin{aligned}
& P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J),\hat{H}_i^{3n+3\Delta n}-\hat{H}_i^{3n}=1,\hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1 \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)\\
= & \sum_{r=n}^{n+\Delta n-1}\left[\hat{\eta}_{kk'}P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'J}^{3r+1}\right)\right]. \qquad \text{(E.34)}
\end{aligned}$$

Now, we estimate the difference

$$\begin{aligned}
& \left|P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J),\hat{H}_i^{3n+3\Delta n}-\hat{H}_i^{3n}=1,\hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1 \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)-\sum_{r=n}^{n+\Delta n-1}\hat{\eta}_{kk'}\right|\\
\leq & \sum_{r=n}^{n+\Delta n-1}\hat{\eta}_{kk'}\left|P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'J}^{3r+1}\right)-1\right|.
\end{aligned}$$

We can obtain from Lemma E.8 that

$$\begin{aligned}
& P_0\left(C_{3n}^{3r} \,\middle|\, \hat{\beta}_i^{3n}=(k,l)\right)P_0\left(C_{3r+1}^{3n+3\Delta n} \,\middle|\, B_{k'l'}^{3r+1}\right)\\
\geq & \left(1-\frac{K\bar{a}}{M}\right)^{6(r-n)}\left(1-\frac{K\bar{a}}{M}\right)^{2(3n+3\Delta n-3r-1)} \simeq e^{-\frac{6K(r-n)\bar{a}}{M}}e^{-\frac{2K(3n+3\Delta n-3r-1)\bar{a}}{M}}.
\end{aligned}$$

Then, it follows from the above inequalities that

$$
\begin{aligned}
& \left| P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{H}_i^{3n+3\Delta n} - \hat{H}_i^{3n} = 1, \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) - \sum_{r=n}^{n+\Delta n-1} \hat{\eta}_{kk'} \right| \\
\leq & \sum_{r=n}^{n+\Delta n-1} \frac{\bar{a}}{M}\left(1 - e^{-\frac{6K(r-n)\bar{a}}{M}} e^{-\frac{2K(3n+3\Delta n-3r-1)\bar{a}}{M}}\right) \\
\simeq & \bar{a}\Delta t\left(1 - e^{-6K\bar{a}\Delta t}\right) \\
= & O(\Delta t^2).
\end{aligned}
$$

Therefore, we obtain the following estimation

$$
\begin{aligned}
& P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{H}_i^{3n+3\Delta n} - \hat{H}_i^{3n} = 1, \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\
= & \sum_{r=n}^{n+\Delta n-1} \hat{\eta}_{kk'} + O(\Delta t^2) \\
= & \eta_{kk'}\Delta t + O(\Delta t^2). \qquad \text{(E.35)}
\end{aligned}
$$

Next, we need to estimate the second term on the right hand side of Equation (E.33). The proof for such an estimation is very close to the proof in Part 5. For the sake of completeness and readability, we present the detailed proof below.

The second term on the right hand side of Equation (E.33) can be expanded as follows:

$$
\begin{aligned}
& P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{N}_i^{3n+3\Delta n} - \hat{N}_i^{3n} = 1, \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\
= & \sum_{r=n}^{n+\Delta n-1} \sum_{l\in S} P_0\left(\left(\hat{\beta}_i^{3r+3} = (k', J)\right) \cap B_{kl}^{3r+2} \cap C_{3r+3}^{3n+3\Delta n} \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\
= & \sum_{r=n}^{n+\Delta n-1} \sum_{l\in S} \Big[P_0\left(\left(\hat{\beta}_i^{3r+3} = (k', J)\right) \cap B_{kl}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\
& \qquad P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left(\hat{\beta}_i^{3r+3} = (k', J)\right) \cap B_{kl}^{3r+2}\right)\Big] \\
= & \sum_{r=n}^{n+\Delta n-1} \sum_{l\in S} \Big[P_0\left(\hat{\beta}_i^{3r+3} = (k', J) \,\middle|\, B_{kl}^{3r+2}\right) P_0\left(B_{kl}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\
& \qquad P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left(\hat{\beta}_i^{3r+3} = (k', J)\right) \cap B_{kl}^{3r+2}\right)\Big].
\end{aligned}
$$

It follows from Equations (E.2) and (E.32), and Lemma E.5 that

$$
\begin{aligned}
& P_0\left(\hat{\beta}_i^{3r+3} = (k', J) \,\middle|\, B_{kl}^{3r+2}\right) \\
= & P_0\left(\hat{\beta}_i^{3r+3} = (k', J) \,\middle|\, \left(\hat{\beta}_i^{3r+2} = (k, l)\right) \cap C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n} = (k, J)\right)\right) \\
= & P_0\left(\tilde{\beta}_i^{3r+3} = (k', J, 1) \,\middle|\, \left(\tilde{\beta}_i^{3r+2} = (k, l, 1)\right) \cap C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n} = (k, J)\right)\right) \\
= & P_0\left(\tilde{\beta}_i^{3r+3} = (k', J, 1) \,\middle|\, \tilde{\beta}_i^{3r+2} = (k, l, 1)\right) \\
= & (1 - \hat{\xi}_{kl})\hat{\varsigma}_{kl}(k'),
\end{aligned}
$$

where the last identity follows from the step of random type changing with break-up for matched agents (without entering an enduing partnership) in the construction of the dynamic matching model. Then, the above identities and Equation (E.32) imply that

$$\begin{aligned}
& P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{N}_i^{3n+3\Delta n} - \hat{N}_i^{3n} = 1, \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\
= & \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} (1 - \hat{\xi}_{kl}) \hat{\varsigma}_{kl}(k') P_0\left(B_{kl}^{3r+2} \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\
& \qquad P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left(\hat{\beta}_i^{3r+3} = (k', J)\right) \cap B_{kl}^{3r+2}\right) \\
= & \sum_{r=n}^{n+\Delta n-1} \sum_{l \in S} \left[(1 - \hat{\xi}_{kl}) \hat{\varsigma}_{kl}(k') P_0\left(\hat{\beta}^{3r+2} = (k, l) \,\middle|\, C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n} = (k, J)\right)\right)\right. \\
& \qquad \left. P_0\left(C_{3n}^{3r+1} \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) P_0\left(C_{3r+3}^{3n+3\Delta n} \,\middle|\, \left(\hat{\beta}_i^{3r+3} = (k', J)\right) \cap B_{kl}^{3r+2}\right)\right]
\end{aligned} \tag{E.36}$$

Fix any sample realization $\omega^{3r+1} \in \Omega^{3r+1}$ such that $\hat{\beta}_i^{3r+1}\left(\omega^{3r+1}\right) = (k, J)$. By the definition of $\hat{h}_i^m$, we know that $\hat{h}_i^{3r+1}\left(\omega^{3r+1}\right) = 1$, and $\hat{h}_i^{3r+2}\left(\omega^{3r+1}, \omega_{3r+2}\right) = 1$ for any $\omega_{3r+2} \in \Omega_{3r+2}$. Hence, these facts together with Lemma E.4 imply that

$$\begin{aligned}
& \left|\hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right) - P_0\left(\hat{\beta}^{3r+2} = (k, l) \,\middle|\, C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n} = (k, J)\right)\right)\right| \\
= & \left|\hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right) - P_0\left(\hat{\beta}^{3r+2} = (k, l) \,\middle|\, C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3r+1} = (k, J)\right)\right)\right| \\
= & \left|\hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right) - P_0\left(\tilde{\beta}^{3r+2} = (k, l, 1) \,\middle|\, C_{3n}^{3r+1} \cap \left(\tilde{\beta}_i^{3r+1} = (k, J, 1)\right)\right)\right| \\
\leq & \frac{1}{M^3 P_0\left(C_{3n}^{3r+1} \cap \left(\tilde{\beta}_i^{3r+1} = (k, J, 1)\right)\right)} + \frac{1}{M^2} \\
= & \frac{1}{M^3 P_0\left(C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n} = (k, J)\right)\right)} + \frac{1}{M^2}.
\end{aligned}$$

By Lemma E.8, $P\left(C_{3n}^{3r+1} \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) > 0$. Then we have $P\left(C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n} = (k, J)\right)\right) > 0$. It is clear that $P_0\left(C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n} = (k, J)\right)\right)$ is not infinitesimal, which implies that

$$\frac{1}{M^3 P_0\left(C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n} = (k, J)\right)\right)} < \frac{1}{M^2}.$$

Therefore, we obtain the following estimation

$$\left|\hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right) - P_0\left(\hat{\beta}^{3r+2} = (k, l) \,\middle|\, C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n} = (k, J)\right)\right)\right| \leq \frac{2}{M^2}, \tag{E.37}$$

which implies

$$\begin{aligned}
& P_0\left(\hat{\beta}^{3r+2} = (k, l) \,\middle|\, C_{3n}^{3r+1} \cap \left(\hat{\beta}_i^{3n} = (k, J)\right)\right) \\
\leq & \hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right) + \frac{2}{M^2} \\
\leq & \frac{\bar{a}}{M} + \frac{2}{M^2}.
\end{aligned}$$

It follows from the above inequality that

$$\begin{aligned}
&\Bigg|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\Big[(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')P_0\left(\hat{\beta}^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\\
&\qquad P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',J)\right)\cap B_{kl}^{3r+2}\right)\Big]\\
&\quad-\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')P_0\left(\hat{\beta}^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\Bigg|\\
=&\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\Big[(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')P_0\left(\hat{\beta}^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\\
&\left(1-P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',J)\right)\cap B_{kl}^{3r+2}\right)\right)\Big]\\
\leq&\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\left(\frac{\bar{a}}{M}+\frac{2}{M^2}\right)\\
&\left(1-P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',J)\right)\cap B_{kl}^{3r+2}\right)\right).
\end{aligned}\tag{E.38}$$

It follows from Lemma E.8 that

$$\begin{aligned}
&P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',J)\right)\cap B_{kl}^{3r+2}\right)\\
\geq&\left(1-\frac{K\bar{a}}{M}\right)^{2(3r-3n+1)}\left(1-\frac{K\bar{a}}{M}\right)^{2(3n+3\Delta n-3r-3)}\\
\simeq&\ e^{-\frac{2K\bar{a}(3r-3n+1)}{M}}e^{-\frac{2K\bar{a}(3n+3\Delta n-3r-3)}{M}}\\
\simeq&\ e^{-6K\bar{a}\Delta t}.
\end{aligned}\tag{E.39}$$

Then, Equations (E.38) and (E.39) imply that

$$\begin{aligned}
&\Bigg|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}\Big[(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')P_0\left(\hat{\beta}^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\\
&\qquad P_0\left(C_{3n}^{3r+1}\,\middle|\,\hat{\beta}_i^{3n}=(k,J)\right)P_0\left(C_{3r+3}^{3n+3\Delta n}\,\middle|\,\left(\hat{\beta}_i^{3r+3}=(k',J)\right)\cap B_{kl}^{3r+2}\right)\Big]\\
-&\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')P_0\left(\hat{\beta}^{3r+2}=(k,l)\,\middle|\,C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\Bigg|\\
\lesssim&\sum_{r=n}^{n+\Delta n-1}K\left(\frac{\bar{a}}{M}+\frac{2}{M^2}\right)\left(1-e^{-6K\bar{a}\Delta t}\right)\\
\lesssim&\ K\bar{a}\left(1-e^{-6K\bar{a}\Delta t}\right)\Delta t\\
=&\ O(\Delta t^2).
\end{aligned}\tag{E.40}$$

By Equations (E.36) and (E.40), we have the following estimation

$$\begin{aligned}
& P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J), \hat{N}_i^{3n+3\Delta n}-\hat{N}_i^{3n}=1, \hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1 \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right) \\
= & \sum_{r=n}^{n+\Delta n-1} \sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k') P_0\left(\hat{\beta}^{3r+2}=(k,l) \,\middle|\, C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right) \\
& +O(\Delta t^2). \qquad (E.41)
\end{aligned}$$

It follows from Equation (E.37) that

$$\begin{aligned}
& \Bigg|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right) \\
& \quad -\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k') P_0\left(\hat{\beta}^{3r+2}=(k,l) \,\middle|\, C_{3n}^{3r+1}\cap\left(\hat{\beta}_i^{3n}=(k,J)\right)\right)\Bigg| \\
\leq & \left|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\frac{2}{M^2}\right| \\
\leq & \Delta n K\frac{2}{M^2}, \qquad (E.42)
\end{aligned}$$

which is an infinitesimal and can be absorbed into the term $O(\Delta t^2)$. Therefore, Equations (E.41) and (E.42) imply that

$$\begin{aligned}
& P_0\left(\hat{\beta}_i^{3n+3\Delta n}=(k',J), \hat{N}_i^{3n+3\Delta n}-\hat{N}_i^{3n}=1, \hat{X}_i^{3n+3\Delta n}-\hat{X}_i^{3n}=1 \,\middle|\, \hat{\beta}_i^{3n}=(k,J)\right) \\
= & \sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right)+O(\Delta t^2). \qquad (E.43)
\end{aligned}$$

Equation (E.8) implies that $\check{p}_t = \mathbb{E}\left(\hat{p}_t\right) \simeq \mathbb{E}\left(\hat{\rho}^{3n}\right)$. By Lemma E.3, $U_1^{3r+1}(\tilde{\rho}^0) \simeq \mathbb{E}\left(\tilde{\rho}^{3r+1}\right)$. By the continuity of $\theta_{kl}$, we obtain the following estimation

$$\begin{aligned}
& \frac{1}{\Delta t}\left|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right)-\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\theta_{kl}\left(\check{p}_t\right)\Delta t\right| \\
\lesssim & \frac{1}{\Delta t}\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\left|\sum_{r=n}^{n+\Delta n-1}\frac{1}{M}{}^*\theta_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right)-{}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3n}\right)\right)\frac{\Delta n}{M}\right| \\
\lesssim & \frac{1}{M\Delta t}\sum_{l\in S}(1-\hat{\xi}_{kl})\hat{\varsigma}_{kl}(k')\sum_{r=n}^{n+\Delta n-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3r+1}\right)\right)-{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3n}\right)\right)\right| \\
\lesssim & \frac{K}{\Delta n}\sum_{r=n}^{n+\Delta n-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3r+1}\right)\right)-{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3n}\right)\right)\right|.
\end{aligned}$$

Fix any $\Delta n' \in \mathbb{T}_0$ such that $\frac{\Delta n'}{M}$ is infinitesimal. Lemma E.10 implies that $\|\mathbb{E}\left(\tilde{\rho}^{3r+1}\right) - \mathbb{E}\left(\tilde{\rho}^{3n}\right)\|_\infty$ is infinitesimal for any $r$ between $n$ and $n+\Delta n'$. By the continuity of $\theta_{kl}$,

$\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3r+1}\right)\right) - {}^*\theta_{kl}\left(\mathbb{E}\left(\tilde{\rho}^{3n}\right)\right)\right|$ is also infinitesimal. Then, we obtain that

$$\frac{K}{\Delta n'}\sum_{r=n}^{n+\Delta n'-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3r+1}\right)\right) - {}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3n}\right)\right)\right| \simeq 0.$$

By the Spillover Principle, we know that for any $\epsilon \in \mathbb{R}_{++}$, there exists $\delta \in \mathbb{R}_{++}$ such that for any $\Delta n \in \mathbb{T}_0$ with $\mathrm{st}\left(\frac{\Delta n}{M}\right) < \delta$, the standard part of

$$\frac{K}{\Delta n}\sum_{r=n}^{n+\Delta n-1}\left|{}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3r+1}\right)\right) - {}^*\theta_{kl}\left(\mathbb{E}\left(\hat{\rho}^{3n}\right)\right)\right|$$

is less than $\epsilon$. Therefore, we can claim that

$$\left|\sum_{r=n}^{n+\Delta n-1}\sum_{l\in S}(1-\hat{\xi}_{kl})\varsigma_{kl}(k')\hat{q}_{kl}\left(U_1^{3r+1}\left(\tilde{\rho}^0\right)\right) - \sum_{l\in S}(1-\hat{\xi}_{kl})\varsigma_{kl}(k')\theta_{kl}\left(\check{p}_t\right)\Delta t\right| = o(\Delta t).$$

Hence, Equation (E.43) implies that

$$\begin{aligned} & P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{N}_i^{3n+3\Delta n} - \hat{N}_i^{3n} = 1, \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\ = \; & \sum_{l\in S}(1-\xi_{kl})\varsigma_{kl}(k')\theta_{kl}\left(\check{p}_t\right)\Delta t + o(\Delta t). \end{aligned} \tag{E.44}$$

By Equations (E.35) and (E.44), we can obtain that

$$\begin{aligned} & P\left(\beta_i(t+\Delta t) = (k', J) \,\middle|\, \beta_i(t) = (k, J)\right) \\ = \; & P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{H}_i^{3n+3\Delta n} - \hat{H}_i^{3n} = 1, \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) \\ & \quad + P_0\left(\hat{\beta}_i^{3n+3\Delta n} = (k', J), \hat{N}_i^{3n+3\Delta n} - \hat{N}_i^{3n} = 1, \hat{X}_i^{3n+3\Delta n} - \hat{X}_i^{3n} = 1 \,\middle|\, \hat{\beta}_i^{3n} = (k, J)\right) + O(\Delta t^2) \\ = \; & \eta_{kk'}\Delta t + \sum_{l\in S}(1-\xi_{kl})\varsigma_{kl}(k')\theta_{kl}\left(\check{p}_t\right)\Delta t + o(\Delta t), \end{aligned}$$

which implies that agent $i$'s transition intensity for her expanded types from $(k, J)$ to $(k', J)$ at time $t$ is indeed $Q_{(k,J)(k',J)}(\check{p}(t))$, as given in Case 4 of Table 1.

## E.5 Proofs of Lemmas E.1 – E.10

The proof of Lemma E.1 is given in Subsection E.5.1. In order to prove Lemmas E.2 – E.10, some additional lemmas are presented in Subsection E.5.2. Lemmas E.2 – E.10 are then proved in Subsections E.5.3 – E.5.11 respectively.

### E.5.1 Proof of Lemma E.1

The proof consists of three steps. In the first step, we (randomly) choose a set $A_{kl}$ of agents among the type-$k$ single agents, which is to be matched with type-$l$ agents. We require that the cardinality $|A_{kl}|$ of $A_{kl}$ is even and $|A_{kl}| = |A_{lk}|$, which allow the agents in $A_{kl}$ and $A_{lk}$ to be matched. The second step is to randomly match the agents in $A_{kl}$ and $A_{lk}$. In the third step, the random matching obtained by combining the match of agents in those groups is shown to satisfy Lemma E.1 (i), (ii) and (iii).

**Step 1:** For each $k \in S$, let $I_k = \{i \in I : \alpha^0(i) = k, \pi^0(i) = i\}$ be the set of type-$k$ agents who are initially unmatched. Let

$$\begin{aligned} \Omega_0 \;=\; & \big\{(A_{kl})_{k,l\in S} : \forall k,l,l' \in S, A_{kl} \subseteq I_k,\ |A_{kl}| \text{ is the largest even integer} \\ & \text{less than or equal to } |I_k|q_{kl}, A_{kl} \text{ and } A_{kl'} \text{ are disjoint for different } l \text{ and } l'.\big\} \end{aligned}$$

Note that $\hat{\rho}_{kJ}$ is the proportion of agents of type $k$ who are initially unmatched, which implies that $|I_k| = \hat{M}\hat{\rho}_{kJ}$. Hence, we have $|I_k|q_{kl} = \hat{M}\hat{\rho}_{kJ}q_{kl} = \hat{M}\hat{\rho}_{lJ}q_{lk} = |I_l|q_{lk}$. Then for any $(A_{kl})_{k,l\in S} \in \Omega_0$, $|A_{kl}| = |A_{lk}|$ for any $k,l \in S$. Let $\mu_0$ be the counting probability measure on $(\Omega_0, \mathcal{A}_0)$, where $\mathcal{A}_0$ is the power set of $\Omega_0$.

**Step 2:** For any fixed $\omega_0 = (A_{kl})_{k,l\in S} \in \Omega_0$, we consider partial matchings on $I$ that match agents from $A_{kl}$ to $A_{lk}$. We only need to consider those sets $A_{kl}$ which are nonempty. For each $k \in S$, let $\Omega_{kk}^{\omega_0}$ be the set of all the full matchings on $A_{kk}$, and $\mu_{kk}^{\omega_0}$ the counting probability measure on $\Omega_{kk}^{\omega_0}$. For $k,l \in S$ with $k < l$, let $\Omega_{kl}^{\omega_0}$ be the set of all the bijections from $A_{kl}$ to $A_{lk}$, and $\mu_{kl}^{\omega_0}$ the counting probability measure on $\Omega_{kl}^{\omega_0}$. Let $\Omega_1$ be the set of all the partial matchings from $I$ to $I$. Define $\Omega_1^{\omega_0}$ to be the set of $\phi \in \Omega_1$, with

(i) the restriction $\phi|_H = \pi^0|_H$, where $H$ is the set $\{i : \pi^0(i) \neq i\}$ of initially matched agents;

(ii) $\{i \in I_k : \phi(i) = i\} = I_k \backslash \left(\cup_{l=1}^K A_{kl}\right)$ for each $k \in S$;

(iii) the restriction $\phi|_{A_{kk}} \in \Omega_{kk}^{\omega_0}$ for $k \in S$;

(iv) for $k,l \in S$ with $k < l$, $\phi|_{A_{kl}} \in \Omega_{kl}^{\omega_0}$.

(i) means that initially matched agents remain matched with the same partners. The rest is clear.

Define a probability measure $\mu_1^{\omega_0}$ on $\Omega_1$ such that such that

(i) for $\phi \in \Omega_1^{\omega_0}$,

$$\mu_1^{\omega_0}(\phi) = \prod_{1\le k\le l\le K, A_{kl}\neq\emptyset} \mu_{kl}^{\omega_0}(\phi|_{A_{kl}});$$

(ii) $\phi \notin \Omega_1^{\omega_0}$, $\mu_1^{\omega_0}(\phi) = 0$.

The purpose of introducing the space $\Omega_1^{\omega_0}$ and the probability measure $\mu_1^{\omega_0}$ is to match the agents in $A_{kl}$ to the agents in $A_{lk}$ randomly. The probability measure $\mu_1^{\omega_0}$ is trivially extended to the common sample space $\Omega_1$.

Define a probability measure $P_0$ on $\Omega = \Omega_0 \times \Omega_1$ with the power set $\mathcal{F}_0$ by letting

$$P_0\left((\omega_0, \omega_1)\right) = \mu_0(\omega_0) \times \mu_1^{\omega_0}(\omega_1).$$

For $(i, \omega) \in I \times \Omega$, let $\pi(i, (\omega_0, \omega_1)) = \omega_1(i)$, and $g(i, \omega) = \begin{cases} \alpha^0(\pi(i,\omega)) & \text{if } \pi(i,\omega) \neq i \\ J & \text{if } \pi(i,\omega) = i. \end{cases}$

Denote the set $\{(\omega_0, \omega_1) \in \Omega : \omega_0 \in \Omega_0,\, \omega_1 \in \Omega_1^{\omega_0}\}$ by $\hat{\Omega}$. The definition of $P_0$ indicates that $P_0\left(\hat{\Omega}\right) = 1$.

**Step 3:** It is clear that $\pi$ is a random matching and satisfies part (i) of the lemma. For any $k, l \in S$ and $\omega \in \Omega$, we have $\lambda_0\left(\{i \in I : \alpha^0(i) = k, g^0(i) = J, g(i,\omega) = l\}\right) = \frac{|A_{kl}|}{\hat{M}}$. Since $|A_{kl}|$ is the largest even integer less than or equal to $|I_k| q_{kl}$, we have $|\,|A_{kl}| - |I_k| q_{kl}\,| \leq 2$. Hence,

$$\left|\lambda_0\left(\{i \in I : \alpha^0(i) = k, g^0(i) = J, g(i,\omega) = l\}\right) - \hat{\rho}_{kJ} q_{kl}\right| = \left|\frac{|A_{kl}|}{\hat{M}} - \frac{|I_k|}{\hat{M}} q_{kl}\right| \leq \frac{2}{\hat{M}},$$

which implies part (iii) of the lemma.

It remains to prove part (ii). Fix any $i, j \in I$ with $i \neq j$, $\pi^0(i) = i$ and $\pi^0(j) = j$; denote $\alpha^0(i)$ and $\alpha^0(j)$ by $k_1$ and $k_2$ respectively.

We start with the first inequality in part (ii). By the construction above, we have

$$P_0\left(\pi_i = j\right) = P_0\left(\{((A_{kl})_{k,l \in S}, \omega_1) : i \in A_{k_1 k_2},\, j \in A_{k_2 k_1},\, \omega_1(i) = j\}\right).$$

Let $\bar{A} = \{(A_{kl})_{k,l \in S} : i \in A_{k_1 k_2}, j \in A_{k_2 k_1}\}$. Then, the definition of $P_0$ implies that

$$P_0\left(\pi_i = j\right) = \sum_{(A_{kl})_{k,l \in S} \in \bar{A}} \mu_0\left((A_{kl})_{k,l \in S}\right) \mu_1^{(A_{kl})_{k,l \in S}}(\omega_1(i) = j).$$

When $k_1 \neq k_2$, for any $(A_{kl})_{k,l \in S} \in \bar{A}$, we know that

$$\mu_1^{(A_{kl})_{k,l \in S}}(\omega_1(i) = j) = \frac{1}{|A_{k_1 k_2}|}.$$

When $k_1 = k_2$, for any $(A_{kl})_{k,l \in S} \in \bar{A}$, we have

$$\mu_1^{(A_{kl})_{k,l \in S}}(\omega_1(i) = j) = \frac{1}{|A_{k_1 k_2}| - 1} \leq \frac{2}{|A_{k_1 k_2}|},$$

since $|A_{k_1 k_2}| \geq 2$ for any $(A_{kl})_{k,l \in S} \in \bar{A}$. Then, it is clear that $\mu_1^{(A_{kl})_{k,l \in S}}(\omega_1(i) = j) \leq \frac{2}{|A_{k_1 k_2}|}$ always holds for any $(A_{kl})_{k,l \in S} \in \bar{A}$. Therefore, we can obtain that

$$\begin{aligned} P_0\left(\pi_i = j\right) &\leq \sum_{(A_{kl})_{k,l \in S} \in \bar{A}} \mu_0\left((A_{kl})_{k,l \in S}\right) \frac{2}{|A_{k_1 k_2}|} \\ &= \frac{2}{|A_{k_1 k_2}|} \mu_0\left(\{(A_{kl})_{k,l \in S} : i \in A_{k_1 k_2},\, j \in A_{k_2 k_1}\}\right) \\ &\leq \frac{2}{|A_{k_1 k_2}|} \mu_0\left(\{(A_{kl})_{k,l \in S} : i \in A_{k_1 k_2}\}\right). \end{aligned}$$

Let $M_k$ and $m_{kl}$ be the cardinality of $I_k$ and $A_{kl}$ respectively. Let $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ denote the binomial coeffiecient. Then we have

$$P_0\left(\pi_i = j\right) \quad \leq \quad \frac{2}{m_{k_1k_2}} \frac{\binom{M_{k_1}-1}{m_{k_1k_2}-1}}{\binom{M_{k_1}}{m_{k_1k_2}}} = \frac{2}{m_{k_1k_2}} \frac{m_{k_1k_2}}{M_{k_1}} = \frac{2}{M_{k_1}} = \frac{2}{\hat{M}\hat{\rho}_{k_1J}},$$

where the last identity follows from the fact that $\hat{M}\hat{\rho}_{k_1J} = |I_k| = M_{k_1}$.

Next, we prove the second inequality in part (ii). Assume that $\hat{\rho}_{k_1J} \geq \frac{1}{\hat{M}^{\frac{1}{3}}}$. We have

$$P_0(g(i) = l_1) = \mu_0\left(\{(A_{kl})_{k,l\in S} : i \in A_{k_1l_1}\}\right) = \frac{\binom{M_{k_1}-1}{m_{k_1l_1}-1}}{\binom{M_{k_1}}{m_{k_1l_1}}} = \frac{m_{k_1l_1}}{M_{k_1}}.$$

It is clear that $P_0(g(i) = l_1) \leq \frac{M_{k_1}q_{k_1l_1}}{M_{k_1}} = q_{k_1l_1}$. Note that

$$\begin{aligned} P_0(g(i) = l_1) \quad &\geq \quad \frac{M_{k_1}q_{k_1l_1} - 2}{M_{k_1}} = q_{k_1l_1} - \frac{2}{M_{k_1}} \\ &= \quad q_{k_1l_1} - \frac{2}{\hat{M}\hat{\rho}_{k_1J}} \geq q_{k_1l_1} - \frac{2}{\hat{M}^{\frac{2}{3}}}. \end{aligned}$$

Then, we have

$$q_{k_1l_1} - \frac{2}{\hat{M}^{\frac{2}{3}}} \leq P_0(g(i) = l_1) \leq q_{k_1l_1}. \tag{E.45}$$

It remains to prove the third inequality in part (ii). We make the further assumption that $\hat{\rho}_{k_2J} \geq \frac{1}{\hat{M}^{\frac{1}{3}}}$. When $k_1 \neq k_2$, we obtain that

$$\begin{aligned} P_0(g(i) = l_1, g(j) = l_2) \quad &= \quad \mu_0(\{(A_{kl})_{k,l\in S} : i \in A_{k_1l_1}, j \in A_{k_2l_2}\}) \\ &= \quad \frac{\binom{M_{k_1}-1}{m_{k_1l_1}-1}}{\binom{M_{k_1}}{m_{k_1l_1}}} \frac{\binom{M_{k_2}-1}{m_{k_2l_2}-1}}{\binom{M_{k_2}}{m_{k_2l_2}}} = P_0(g(i) = l_1)P_0(g(j) = l_2). \end{aligned}$$

Equation (E.45) implies the following inequalities:

$$q_{k_1l_1}q_{k_2l_2} \geq P_0(g(i) = l_1, g(j) = l_2) \geq (q_{k_1l_1} - \frac{2}{\hat{M}^{\frac{2}{3}}})(q_{k_2l_2} - \frac{2}{\hat{M}^{\frac{2}{3}}}) \geq q_{k_1l_1}q_{k_2l_2} - \frac{4}{\hat{M}^{\frac{2}{3}}}. \tag{E.46}$$

When $k_1 = k_2$ but $l_1 \neq l_2$, we have

$$P_0(g(i) = l_1, g(j) = l_2) = \mu_0(\{(A_{kl})_{k,l\in S} : i \in A_{k_1l_1}, j \in A_{k_1l_2}\}) = \frac{\binom{M_{k_1}-2}{m_{k_1l_1}-1, m_{k_1l_2}-1}}{\binom{M_{k_1}}{m_{k_1l_1}, m_{k_1l_2}}},$$

where $\binom{a}{b,c} = \frac{a!}{b!c!(a-b-c)!}$ is the multinomial coefficient. It is clear that

$$\begin{aligned} P_0(g(i) = l_1, g(j) = l_2) \quad &= \quad \frac{m_{k_1l_1}m_{k_1l_2}}{M_{k_1}(M_{k_1}-1)} \leq \frac{m_{k_1l_1}(m_{k_1l_2}+1)}{M_{k_1}^2} \\ &\leq \quad q_{k_1l_1}q_{k_1l_2} + q_{k_1l_1}\frac{1}{M_{k_1}} \leq q_{k_1l_1}q_{k_1l_2} + \frac{1}{M_{k_1}} \\ &= \quad q_{k_1l_1}q_{k_1l_2} + \frac{1}{\hat{M}\hat{\rho}_{k_1J}} \leq q_{k_1l_1}q_{k_1l_2} + \frac{1}{\hat{M}^{\frac{2}{3}}}. \end{aligned}$$

On the other hand, we can obtain that

$$
\begin{aligned}
& \frac{m_{k_1l_1}m_{k_1l_2}}{M_{k_1}(M_{k_1}-1)} \\
\geq\ & \frac{(M_{k_1}q_{k_1l_1}-2)}{M_{k_1}}\frac{(M_{k_1}q_{k_1l_2}-2)}{M_{k_1}} \\
\geq\ & q_{k_1l_1}q_{k_1l_2}-\frac{2}{M_{k_1}}q_{k_1l_1}-\frac{2}{M_{k_1}}q_{k_1l_2} \\
\geq\ & q_{k_1l_1}q_{k_1l_2}-\frac{4}{M_{k_1}} \\
=\ & q_{k_1l_1}q_{k_1l_2}-\frac{4}{\hat{M}\hat{\rho}_{k_1J}} \\
\geq\ & q_{k_1l_1}q_{k_1l_2}-\frac{4}{\hat{M}^{\frac{2}{3}}}.
\end{aligned}
$$

By combining the above inequalities, we have

$$
q_{k_1l_1}q_{k_1l_2}-\frac{4}{\hat{M}^{\frac{2}{3}}}\leq P_0(g(i)=l_1,g(j)=l_2)\leq q_{k_1l_1}q_{k_1l_2}+\frac{1}{\hat{M}^{\frac{2}{3}}}. \tag{E.47}
$$

When $k_1=k_2$ and $l_1=l_2$, we can obtain that

$$
P_0(g(i)=l_1,g(j)=l_1)=\mu_0(\{(A_{kl})_{k,l\in S}:i,j\in A_{k_1l_1}\})=\frac{\binom{M_{k_1}-2}{m_{k_1l_1}-2}}{\binom{M_{k_1}}{m_{k_1l_1}}}.
$$

It is clear that

$$
P_0(g(i)=l_1,g(j)=l_1)=\frac{(m_{k_1l_1})(m_{k_1l_1}-1)}{M_{k_1}(M_{k_1}-1)}\leq\frac{m_{k_1l_1}^2}{M_{k_1}^2}\leq q_{k_1l_1}^2.
$$

On the other hand,

$$
\begin{aligned}
&\frac{(m_{k_1l_1})(m_{k_1l_1}-1)}{M_{k_1}(M_{k_1}-1)}\geq\frac{(M_{k_1}q_{k_1l_1}-2)}{M_{k_1}}\frac{(M_{k_1}q_{k_1l_1}-3)}{M_{k_1}} \\
&\geq q_{k_1l_1}^2-\frac{5}{M_{k_1}}q_{k_1l_1}\geq q_{k_1l_1}^2-\frac{5}{\hat{M}^{\frac{2}{3}}}.
\end{aligned}
$$

Therefore, we obtain that

$$
q_{k_1l_1}^2-\frac{5}{\hat{M}^{\frac{2}{3}}}\leq P_0(g(i)=l_1,g(j)=l_2)\leq q_{k_1l_1}^2. \tag{E.48}
$$

By combining Equations (E.46), (E.47) and (E.48), we know that for any $(k_1,l_1),(k_2,l_2)\in S^2$,

$$
q_{k_1l_1}q_{k_2l_2}-\frac{5}{\hat{M}^{\frac{2}{3}}}\leq P_0(g(i)=l_1,g(j)=l_2)\leq q_{k_1l_1}q_{k_2l_2}+\frac{1}{\hat{M}^{\frac{2}{3}}}.
$$

### E.5.2 Some additional lemmas

The following lemma demonstrates the identity of $\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}^{3n-2}$ and $T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)$.

**Lemma E.11.** *For any $n \in \mathbb{T}_0$, $\omega^{3n-3} \in \Omega^{3n-3}$, we have*

$$\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}^{3n-2} = T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right).$$

*Proof.* Fix any $n \in \mathbb{T}_0$, $\omega^{3n-3} \in \Omega^{3n-3}$ and $(k,l,r) \in \tilde{S}$. For any $(k',l',r') \in \tilde{S}$, let

$$B^{\omega^{3n-3}}_{k'l'r'} = \{i \in I : \tilde{\beta}^{3n-3}_i(\omega^{3n-3}) = (k',l',r')\}.$$

It follows from the definition of $\tilde{\rho}^{3n-2}$ that

$$\begin{aligned}
\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}^{3n-2}_{klr} &= \int_{\Omega_{3n-2}} \tilde{\rho}^{3n-2}_{klr}(\omega^{3n-2}) dQ^{\omega^{3n-3}}_{3n-2} \\
&= \int_{\Omega_{3n-2}} \frac{1}{\hat{M}} \sum_{i \in I} \mathbf{1}_{klr}(\tilde{\beta}^{3n-2}_i(\omega^{3n-2})) dQ^{\omega^{3n-3}}_{3n-2} \\
&= \frac{1}{\hat{M}} \sum_{(k',l',r') \in \tilde{S}} \sum_{i \in B^{\omega^{3n-3}}_{k'l'r'}} \int_{\Omega_{3n-2}} \mathbf{1}_{klr}(\tilde{\beta}^{3n-2}_i(\omega^{3n-2})) dQ^{\omega^{3n-3}}_{3n-2} \\
&= \frac{1}{\hat{M}} \sum_{(k',l',r') \in \tilde{S}} \sum_{i \in B^{\omega^{3n-3}}_{k'l'r'}} Q^{\omega^{3n-3}}_{3n-2}\left(\tilde{\beta}^{3n-2}_i(\omega^{3n-2}) = (k,l,r)\right).
\end{aligned}$$

When $l \in S$ and $r = 0$, we have

$$\begin{aligned}
\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}^{3n-2}_{kl0} &= \frac{1}{\hat{M}} \sum_{k',l' \in S} \sum_{i \in B^{\omega^{3n-3}}_{k'l'0}} Q^{\omega^{3n-3}}_{3n-2}\left(\tilde{\beta}^{3n-2}_i(\omega^{3n-2}) = (k,l,0)\right) \\
&= \frac{1}{\hat{M}} \sum_{k',l' \in S} \sum_{i \in B^{\omega^{3n-3}}_{k'l'0}} \hat{\eta}_{kk'}\hat{\eta}_{ll'} \\
&= \sum_{k',l' \in S} \tilde{\rho}^{3n-3}_{k'l'0}(\omega^{3n-3})\hat{\eta}_{kk'}\hat{\eta}_{ll'} \\
&= \left[T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)\right]_{kl0}.
\end{aligned} \tag{E.49}$$

When $l = J$ and $r = 1$, we have

$$\begin{aligned}
\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}^{3n-2}_{kJ1} &= \frac{1}{\hat{M}} \sum_{k' \in S} \sum_{i \in B^{\omega^{3n-3}}_{k'J1}} Q^{\omega^{3n-3}}_{3n-2}\left(\tilde{\beta}^{3n-2}_i(\omega^{3n-2}) = (k,J,1)\right) \\
&= \frac{1}{\hat{M}} \sum_{k' \in S} \sum_{i \in B^{\omega^{3n-3}}_{k'J1}} \hat{\eta}_{kk'} \\
&= \sum_{k' \in S} \tilde{\rho}^{3n-3}_{k'J1}(\omega^{3n-3})\hat{\eta}_{kk'} \\
&= \left[T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)\right]_{kJ1}.
\end{aligned} \tag{E.50}$$

By the construction of the mutation step and the definition of $\tilde{\beta}^{3n-2}$, it is clear that

$$\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}^{3n-2}_{kl1} = 0 = \left[T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)\right]_{kl1}, \tag{E.51}$$

$$\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}^{3n-2}_{kJ0} = 0 = \left[T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)\right]_{kJ0}. \tag{E.52}$$

The identity $\mathbb{E}^{\omega^{3n-3}}\tilde{\rho}^{3n-2} = T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)$ then follows from Equations (E.49), (E.50), (E.51) and (E.52) ∎

The following lemma shows the relationship between $\mathbb{E}^{\omega^{3n-2}}\tilde{\rho}^{3n-1}$ and $T_2\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)$.

**Lemma E.12.** *For any $n \in \mathbb{T}_0$, $\omega^{3n-1} \in \Omega^{3n-1}$, we have*

$$||\tilde{\rho}^{3n-1}(\omega^{3n-1}) - T_2\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)||_\infty \leq \frac{2K}{\hat{M}}.$$

*Proof.* Fix any $n \in \mathbb{T}_0$, $\omega^{3n-2} \in \Omega^{3n-2}$ and $(k,l,r) \in \tilde{S}$. When $l \in S$ and $r = 0$, it is clear that for any $\omega^{3n-1} \in \Omega^{3n-1}$ with $\omega^{3n-1} = \left(\omega^{3n-2}, \omega_{3n-1}\right)$,

$$\tilde{\rho}^{3n-1}_{kl0}(\omega^{3n-1}) = \tilde{\rho}^{3n-2}_{kl0}(\omega^{3n-2}) = \left[T_2\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)\right]_{kl0}. \tag{E.53}$$

When $l \in S$ and $r = 1$, it follows from Lemma E.1 (iii) that for any $\omega_{3n-1} \in \Omega_{3n-1}$,

$$\begin{aligned} & \left|\tilde{\rho}^{3n-1}_{kl1}(\omega^{3n-1}) - \left[T_2(\tilde{\rho}^{3n-2}(\omega^{3n-2}))\right]_{kl1}\right| \\ = & \left|\tilde{\rho}^{3n-1}_{kl1}(\omega^{3n-1}) - \tilde{\rho}^{3n-2}_{kJ1}(\omega^{3n-2})\hat{q}_{kl}\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)\right| \\ \leq & \frac{2}{\hat{M}}. \end{aligned} \tag{E.54}$$

When $l = J$ and $r = 1$, we have for any $\omega_{3n-1} \in \Omega_{3n-1}$,

$$\begin{aligned} & \left|\tilde{\rho}^{3n-1}_{kJ1}(\omega^{3n-1}) - \left[T_2(\tilde{\rho}^{3n-2}(\omega^{3n-2}))\right]_{kJ1}\right| \\ = & \left|\sum_{l'\in S}\tilde{\rho}^{3n-1}_{kl'1}(\omega^{3n-1}) - \sum_{l'\in S}\left[T_2(\tilde{\rho}^{3n-2}(\omega^{3n-2}))\right]_{kl'1}\right| \\ \leq & \frac{2K}{\hat{M}}. \end{aligned} \tag{E.55}$$

When $l \in S$ and $r = 0$, by the construction of the matching step and the definition of $\tilde{\beta}^{3n-1}$, it is clear that for any $\omega_{3n-1} \in \Omega_{3n-1}$,

$$\tilde{\rho}^{3n-1}_{kJ0}(\omega_{3n-1}) = 0 = \left[T_2\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)\right]_{kJ0}. \tag{E.56}$$

By Equations (E.53), (E.54), (E.55) and (E.56), we have,

$$||\tilde{\rho}^{3n-1}(\omega^{3n-1}) - T_2\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)||_\infty \leq \frac{2K}{\hat{M}}$$

for any $\omega^{3n-1} \in \Omega^{3n-1}$. ∎

The identity of $\mathbb{E}^{\omega^{3n-1}}\tilde{\rho}^{3n}$ and $T_1\left(\tilde{\rho}^{3n-1}(\omega^{3n-1})\right)$ is proved in the next lemma.

**Lemma E.13.** *For any $n \in \mathbb{T}_0$, $\omega^{3n-1} \in \Omega^{3n-1}$, we have*

$$\mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n} = T_3\left(\tilde{\rho}^{3n-1}(\omega^{3n-1})\right).$$

*Proof.* Fix any $n \in \mathbb{T}_0$, $\omega^{3n-1} \in \Omega^{3n-1}$ and $(k, l, r) \in \tilde{S}$. For any $(k', l', r') \in \tilde{S}$, let

$$B^{\omega^{3n-1}}_{k'l'r'} = \{i \in I : \tilde{\beta}^{3n-1}_i(\omega^{3n-1}) = (k', l', r')\}.$$

It follows from the definition of $\tilde{\rho}^{3n}$ that

$$\begin{aligned}
\mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n}_{klr} &= \int_{\Omega_{3n}} \tilde{\rho}^{3n}_{klr}(\omega^{3n}) dQ^{\omega^{3n-1}}_{3n} \\
&= \int_{\Omega_{3n}} \frac{1}{\hat{M}} \sum_{i \in I} \mathbf{1}_{klr}(\tilde{\beta}^{3n}_i) dQ^{\omega^{3n-1}}_{3n} \\
&= \frac{1}{\hat{M}} \sum_{(k',l',r') \in \tilde{S}} \sum_{i \in B^{\omega^{3n-1}}_{k'l'r'}} \int_{\Omega_{3n}} \mathbf{1}_{klr}(\tilde{\beta}^{3n}_i) dQ^{\omega^{3n-1}}_{3n} \\
&= \frac{1}{\hat{M}} \sum_{(k',l',r') \in \tilde{S}} \sum_{i \in B^{\omega^{3n-1}}_{k'l'r'}} Q^{\omega^{3n-1}}_{3n}\left(\tilde{\beta}^{3n}_i = (k, l, r)\right).
\end{aligned}$$

When $l \in S$ and $r = 0$, we have

$$\begin{aligned}
\mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n}_{kl0} &= \frac{1}{\hat{M}} \sum_{i \in B^{\omega^{3n-1}}_{kl0}} Q^{\omega^{3n-1}}_{3n}\left(\tilde{\beta}^{3n}_i = (k, l, 0)\right) \\
&\quad + \frac{1}{\hat{M}} \sum_{k',l' \in S} \sum_{i \in B^{\omega^{3n-1}}_{k'l'1}} Q^{\omega^{3n-1}}_{3n}\left(\tilde{\beta}^{3n}_i = (k, l, 0)\right) \\
&= \tilde{\rho}^{3n-1}_{kl0}(\omega^{3n-1})(1 - \hat{\vartheta}_{kl}) + \sum_{k',l' \in S} \tilde{\rho}^{3n-1}_{k'l'1}(\omega^{3n-1}) \hat{\xi}_{k'l'} \hat{\sigma}_{k'l'}(k, l) \\
&= \left[T_3\left(\tilde{\rho}^{3n-1}(\omega^{3n-1})\right)\right]_{kl0}. \qquad \text{(E.57)}
\end{aligned}$$

When $l = J$ and $r = 1$, we obtain that

$$\begin{aligned}
\mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n}_{kJ1} &= \frac{1}{\hat{M}} \sum_{i \in B^{\omega^{3n-1}}_{kJ1}} Q^{\omega^{3n-1}}_{3n}\left(\tilde{\beta}^{3n}_i = (k, J, 1)\right) \\
&\quad + \frac{1}{\hat{M}} \sum_{k',l' \in S} \sum_{i \in B^{\omega^{3n-1}}_{k'l'0}} Q^{\omega^{3n-1}}_{3n}\left(\tilde{\beta}^{3n}_i = (k, J, 1)\right) \\
&\quad + \frac{1}{\hat{M}} \sum_{k',l' \in S} \sum_{i \in B^{\omega^{3n-1}}_{k'l'1}} Q^{\omega^{3n-1}}_{3n}\left(\tilde{\beta}^{3n}_i = (k, J, 1)\right) \\
&= \tilde{\rho}^{3n-1}_{kJ1}(\omega^{3n-1}) + \tilde{\rho}^{3n-1}_{kl0}(\omega^{3n-1})(1 - \hat{\vartheta}_{kl}) + \sum_{k',l' \in S} \tilde{\rho}^{3n-1}_{k'l'1}(\omega^{3n-1}) \hat{\xi}_{k'l'} \hat{\sigma}_{k'l'}(k, l) \\
&= \left[T_3\left(\tilde{\rho}^{3n-1}(\omega^{3n-1})\right)\right]_{kJ1}. \qquad \text{(E.58)}
\end{aligned}$$

By the construction of the type changing with break-up step, and the definition of $\tilde{\beta}^{3n}$, it is clear that

$$\mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n}_{kl1} = 0 = \left[T_3\left(\tilde{\rho}^{3n-1}(\omega^{3n})\right)\right]_{kl1}, \tag{E.59}$$

$$\mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n}_{kJ0} = 0 = \left[T_3\left(\tilde{\rho}^{3n-1}(\omega^{3n})\right)\right]_{kJ0}. \tag{E.60}$$

The identity $\mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n} = T_3\left(\tilde{\rho}^{3n-1}(\omega^{3n-1})\right)$ is then implied by Equations (E.57), (E.58), (E.59) and (E.60). ■

The following lemma shows that the cross-sectional expanded type distribution $\tilde{\rho}^m$ at the end of step $m$ can be approximated by $U_1^m(\tilde{\rho}^0)$.

**Lemma E.14.** *Let* $\epsilon_0 = \frac{3M^2K(K+1)}{\hat{M}^{\frac{1}{3}}}$. *For any* $m \in \{1, 2, \dots, 3M^2\}$, *let*

$$V^m = \{\omega^m \in \Omega^m : \|\tilde{\rho}^m(\omega^m) - U_1^m(\tilde{\rho}^0)\|_\infty > \xi_0\},$$

*where* $\xi_0$ *is defined in Lemma E.2. Then, for any* $m \in \{1, 2, \dots, 3M^2\}$, *we have* $Q^m(V^m) \leq \epsilon_0$.

*Proof.* Fix any $n \in \mathbb{T}_0$. For the mutation step in period $n$, fix any $\omega^{3n-3} \in \Omega^{3n-3}$. Let

$$C^{\omega^{3n-3}} = \{(i,j) \in I \times I : i < j,\, \hat{\pi}^{3n-3}(i, \omega^{3n-3}) = j\}.$$

For any $i, j \in I$ such that $i \neq j$ and $\hat{\pi}^{3n-3}(i, \omega^{3n-3}) \neq j$, and any $(k,l,r) \in \tilde{S}$, it is clear that $\mathbf{1}_{klr}(\tilde{\beta}^{3n-2}_i)$ and $\mathbf{1}_{klr}(\tilde{\beta}^{3n-2}_j)$ are independent on $(\Omega_{3n-2}, \mathcal{E}_{3n-2}, Q^{\omega^{3n-3}}_{3n-2})$. Therefore, such independence and the definition of $C^{\omega^{3n-3}}$ imply that

$$\begin{aligned}
\mathrm{Var}^{\omega^{3n-3}} \tilde{\rho}^{3n-2}_{klr} &= \mathrm{Var}^{\omega^{3n-3}} \frac{1}{\hat{M}} \sum_{i \in I} \mathbf{1}_{klr}(\tilde{\beta}^{3n-2}_i) \\
&= \frac{1}{\hat{M}^2} \sum_{i \in I} \mathrm{Var}^{\omega^{3n-3}} \mathbf{1}_{klr}(\tilde{\beta}^{3n-2}_i) + \frac{2}{\hat{M}^2} \sum_{(i,j) \in C^{\omega^{3n-3}}} \mathrm{Cov}\left(\mathbf{1}_{klr}(\tilde{\beta}^{3n-2}_i), \mathbf{1}_{klr}(\tilde{\beta}^{3n-2}_j)\right) \\
&\leq \frac{1}{\hat{M}^2} \hat{M} \frac{1}{4} + \frac{2}{\hat{M}^2} \frac{\hat{M}}{2} \frac{1}{4} \\
&= \frac{1}{2\hat{M}}.
\end{aligned}$$

It follows from the Chebyshev Inequality and Lemma E.11 that

$$\begin{aligned}
& Q^{\omega^{3n-3}}_{3n-2}\left(\|\tilde{\rho}^{3n-2} - T_1\left(\tilde{\rho}^{3n-3}\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right) \\
= {} & Q^{\omega^{3n-3}}_{3n-2}\left(\|\tilde{\rho}^{3n-2} - \mathbb{E}^{\omega^{3n-3}} \tilde{\rho}^{3n-2}\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right) \\
\leq {} & \sum_{(k,l,r) \in \tilde{S}} Q^{\omega^{3n-3}}_{3n-2}\left(\left|\tilde{\rho}^{3n-2}_{klr} - \mathbb{E}^{\omega^{3n-3}} \tilde{\rho}^{3n-2}_{klr}\right| \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right) \\
\leq {} & 2K(K+1) \frac{\frac{1}{2\hat{M}}}{\frac{1}{\hat{M}^{\frac{2}{3}}}} = \frac{K(K+1)}{\hat{M}^{\frac{1}{3}}}.
\end{aligned}$$

Let $W^{3n-2} = \{\omega^{3n-2} \in \Omega^{3n-2} : \|\tilde{\rho}^{3n-2}(\omega^{3n-2}) - T_1\left(\tilde{\rho}^{3n-3}(\omega^{3n-3})\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\}$. It is clear that

$$\begin{aligned} Q^{3n-2}(W^{3n-2}) &= \int_{\Omega^{3n-3}} Q^{\omega^{3n-3}}_{3n-2}\left(\|\tilde{\rho}^{3n-2} - T_1\left(\tilde{\rho}^{3n-3}\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right) dQ^{3n-3} \\ &\leq \frac{K(K+1)}{\hat{M}^{\frac{1}{3}}}. \end{aligned} \tag{E.61}$$

For the random matching step in period $n$, Lemma E.12 indicates that for any $\omega^{3n-1} \in \Omega^{3n-1}$,

$$||\tilde{\rho}^{3n-1}(\omega^{3n-1}) - T_2\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)||_\infty \leq \frac{2K}{\hat{M}}.$$

It is then clear that the set

$$W^{3n-1} = \{\omega^{3n-1} \in \Omega^{3n-1} : \|\tilde{\rho}^{3n-1}(\omega^{3n-1}) - T_2\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\}$$

is empty. Hence, we have

$$Q^{3n-1}(W^{3n-1}) = \int_{\Omega^{3n-2}} Q^{\omega^{3n-1}}_{3n-2}\left(\|\tilde{\rho}^{3n-1} - T_1\left(\tilde{\rho}^{3n-2}\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right) dQ^{3n-2} = 0. \tag{E.62}$$

For the type changing with break-up step in period $n$, fix any $\omega^{3n-1} \in \Omega^{3n-1}$. Let

$$C^{\omega^{3n-1}} = \{(i,j) \in I \times I : i < j,\, \hat{\pi}^{3n-1}(i, \omega^{3n-1}) = j\}.$$

For any $i, j \in I$ such that $i \neq j$ and $\hat{\pi}^{3n-1}(i, \omega^{3n-1}) \neq j$, and any $(k,l,r) \in \tilde{S}$, it is clear that $\mathbf{1}_{klr}(\tilde{\beta}^{3n}_i)$ and $\mathbf{1}_{klr}(\tilde{\beta}^{3n}_j)$ are independent on $(\Omega_{3n}, \mathcal{E}_{3n}, Q^{\omega^{3n-1}}_{3n})$. Therefore, we have

$$\begin{aligned} \mathrm{Var}^{\omega^{3n-1}} \tilde{\rho}^{3n}_{klr} &= \mathrm{Var}^{\omega^{3n-1}} \frac{1}{\hat{M}} \sum_{i \in I} \mathbf{1}_{klr}(\tilde{\beta}^{3n}_i) \\ &= \frac{1}{\hat{M}^2} \sum_{i \in I} \mathrm{Var}^{\omega^{3n-1}} \mathbf{1}_{klr}(\tilde{\beta}^{3n}_i) + \frac{2}{\hat{M}^2} \sum_{(i,j) \in C^{\omega^{3n-1}}} \mathrm{Cov}\left(\mathbf{1}_{klr}(\tilde{\beta}^{3n}_i), \mathbf{1}_{klr}(\tilde{\beta}^{3n}_j)\right) \\ &\leq \frac{1}{\hat{M}^2} \sum_{i \in I} \frac{1}{4} + \frac{2}{\hat{M}^2} \frac{\hat{M}}{2} \frac{1}{4} \\ &= \frac{1}{2\hat{M}}. \end{aligned}$$

It follows from the Chebyshev Inequality and Lemma E.13 that

$$\begin{aligned} & Q^{\omega^{3n-1}}_{3n}\left(\|\tilde{\rho}^{3n} - T_3\left(\tilde{\rho}^{3n-1}\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right) \\ = \;& Q^{\omega^{3n-1}}_{3n}\left(\|\tilde{\rho}^{3n} - \mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n}\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right) \\ \leq \;& \sum_{(k,l,r) \in \tilde{S}} Q^{\omega^{3n-1}}_{3n}\left(\left|\tilde{\rho}^{3n}_{klr} - \mathbb{E}^{\omega^{3n-1}} \tilde{\rho}^{3n}_{klr}\right| \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right) \\ \leq \;& 2K(K+1) \frac{\frac{1}{2\hat{M}}}{\frac{1}{\hat{M}^{\frac{2}{3}}}} = \frac{K(K+1)}{\hat{M}^{\frac{1}{3}}}. \end{aligned}$$

Let $W^{3n} = \{\omega^{3n} \in \Omega^{3n} : \|\tilde{\rho}^{3n}(\omega^{3n}) - T_3\left(\tilde{\rho}^{3n-1}(\omega^{3n-1})\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\}$. It is clear that

$$\begin{aligned} Q^{3n}(W^{3n}) &= \int_{\Omega^{3n-1}} Q_{3n}^{\omega^{3n-1}}\left(\|\tilde{\rho}^{3n} - T_3\left(\tilde{\rho}^{3n-1}\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right) dQ^{3n-1} \\ &\leq \frac{K(K+1)}{\hat{M}^{\frac{1}{3}}}. \end{aligned} \tag{E.63}$$

For any $m \in \mathbb{T}_0$, let

$$\overline{W}^m = \{\omega^m \in \Omega^m : \omega^{m'} \in W^{m'} \text{ for some } m' \text{ between 1 and } m\}.$$

By Equations (E.61), (E.62) and (E.63), we have

$$Q^m(\overline{W}^m) \leq \sum_{m'=0}^{m} Q^{m'}(W^{m'}) \leq \frac{3M^2K(K+1)}{\hat{M}^{\frac{1}{3}}} = \epsilon_0.$$

Fix any $m \in \{0, 1, \ldots, 3M^2\}$ and $\omega^m \notin \overline{W}^m$. We have

$$\begin{aligned} & \|\tilde{\rho}^m(\omega^m) - U_1^m(\tilde{\rho}^0)\|_\infty \\ \leq\; & \|\tilde{\rho}^m(\omega^m) - U_m^m(\tilde{\rho}^{m-1}(\omega^{m-1}))\|_\infty + \|U_m^m(\tilde{\rho}^{m-1}(\omega^{m-1})) - U_1^m(\tilde{\rho}^0)\|_\infty \\ \leq\; & \sum_{j=1}^{m} \|U_{j+1}^m(\tilde{\rho}^j(\omega^j)) - U_j^m(\tilde{\rho}^{j-1}(\omega^{j-1}))\|_\infty \\ =\; & \sum_{j=1}^{m} \left\|U_{j+1}^m(\tilde{\rho}^j(\omega^j)) - U_{j+1}^m\left(U_j^j(\tilde{\rho}^{j-1}(\omega^{j-1}))\right)\right\|_\infty. \end{aligned}$$

By the definition of $\hat{M}$, we know that $\frac{1}{\hat{M}^{\frac{1}{3}}} \leq \xi_{3M^2+1}$. The fact that $\omega^m \notin \overline{W}^m$ leads to $\omega^j \notin W^j$ for any $j \in \{0, 1, \ldots, m\}$, which implies that

$$\|\tilde{\rho}^j(\omega^j) - U_j^j(\tilde{\rho}^{j-1}(\omega^{j-1}))\|_\infty < \frac{1}{\hat{M}^{\frac{1}{3}}} \leq \xi_{3M^2+1}.$$

By Lemma E.2, we have

$$\begin{aligned} & \|\tilde{\rho}^m(\omega^m) - U_1^m(\tilde{\rho}^0)\|_\infty \\ \leq\; & \sum_{j=0}^{m-1} \xi_{3M^2+1-j} \\ \leq\; & \sum_{j=0}^{m-1} \frac{1}{3M^2}\xi_0 \leq \xi_0, \end{aligned}$$

which implies that $\omega^m \notin V^m = \{\omega^m \in \Omega^m : \|\tilde{\rho}^m(\omega^m) - U_1^m(\tilde{\rho}^0)\|_\infty > \xi_0\}$. Since $\omega^m$ is an arbitrarily fixed element in $\Omega^m \setminus \overline{W}^m$, the fact that $\omega^m \notin V^m$ implies that $V^m \subseteq \overline{W}^m$. Therefore, we have $Q^m(V^m) \leq \epsilon_0$. ■

The following lemma shows that, when $\omega^{3n-2}$ is not in $V^{3n-2}$, $\hat{M}^{-\frac{1}{15}}$ is a lower bound for the population of single type-$k$ agents after step $3n-2$.

**Lemma E.15.** *For any* $n \in \{1, 2, \ldots, M^2\}$, $\omega^{3n-2} \notin V^{3n-2}$ *and* $k \in S$*, we have* $\tilde{\rho}^{3n-2}_{kJ1}(\omega^{3n-2}) \geq \hat{M}^{-\frac{1}{15}}$*.*

*Proof.* : Fix any $n \in \mathbb{T}_0 = \{1, 2, \ldots, M^2\}$. When $n = 1$, our convention is that $U_1^0$ is the identity mapping, and hence we have

$$\sum_{k \in S} \left[U_1^{3n-3}(\tilde{\rho}^0)\right]_{kJ1} = \sum_{k \in S} \tilde{\rho}^0_{kJ1} = \sum_{k \in S} \hat{\rho}^0_{kJ}.$$

Note that $\hat{\rho}^0_{kJ} \geq \frac{1}{M^2}$ for any $k \in S$. Therefore, it is clear that $\sum_{k \in S} \left[U_1^{3n-3}(\tilde{\rho}^0)\right]_{kJ1} \geq \frac{1}{M^2}$.

When $n \geq 2$, the definition of $T_3$ implies the following identities:

$$\begin{aligned}
& \sum_{k \in S} \left[U_1^{3n-3}(\tilde{\rho}^0)\right]_{kJ1} = \sum_{k \in S} \left[T_3\left(U_1^{3n-4}(\tilde{\rho}^0)\right)\right]_{kJ1} \\
= & \sum_{k,k',l' \in S} (1 - \hat{\xi}_{k'l'})\, \hat{\varsigma}_{k'l'}(k) [U_1^{3n-4}(\tilde{\rho}^0)]_{k'l'1} + \sum_{k,k',l' \in S} \hat{\vartheta}_{k'l'}\, \hat{\varsigma}_{k'l'}(k) [U_1^{3n-4}(\tilde{\rho}^0)]_{k'l'0} \\
& \quad + \sum_{k \in S} [U_1^{3n-4}(\tilde{\rho}^0)]_{kJ1} \\
= & \sum_{k',l' \in S} (1 - \hat{\xi}_{k'l'}) [U_1^{3n-4}(\tilde{\rho}^0)]_{k'l'1} + \sum_{k',l' \in S} \hat{\vartheta}_{k'l'} [U_1^{3n-4}(\tilde{\rho}^0)]_{k'l'0} + \sum_{k \in S} [U_1^{3n-4}(\tilde{\rho}^0)]_{kJ1}.
\end{aligned}$$

By the definition of $\hat{\vartheta}$ and $\hat{\xi}$, we know that $\hat{\vartheta}_{kl} \geq \frac{1}{M^2}$ and $\hat{\xi}_{kl} \leq 1 - \frac{1}{M^2}$ for any $k, l \in S$. Then, we can obtain that

$$\begin{aligned}
& \sum_{k \in S} \left[U_1^{3n-3}(\tilde{\rho}^0)\right]_{kJ1} \\
\geq & \frac{1}{M^2} \left( \sum_{k',l' \in S} [U_1^{3n-4}(\tilde{\rho}^0)]_{k'l'1} + \sum_{k',l' \in S} [U_1^{3n-4}(\tilde{\rho}^0)]_{k'l'0} + \sum_{k \in S} [U_1^{3n-4}(\tilde{\rho}^0)]_{kJ1} \right) \\
= & \frac{1}{M^2}.
\end{aligned}$$

Therefore, $\sum_{k \in S} \left[U_1^{3n-3}(\tilde{\rho}^0)\right]_{kJ1} \geq \frac{1}{M^2}$ for any $n \in \mathbb{T}_0$.

Note that $\hat{\eta}_{kl} \geq \frac{1}{M^2}$ for any $k, l \in S$ by its definition. The definition of $T_1$ implies that for any $k \in S$,

$$\begin{aligned}
& \left[U_1^{3n-2}(\tilde{\rho}^0)\right]_{kJ1} = \left[T_1\left(U_1^{3n-3}(\tilde{\rho}^0)\right)\right]_{kJ1} \\
= & \sum_{l \in S} \left[U_1^{3n-3}(\tilde{\rho}^0)\right]_{lJ1} \hat{\eta}_{lk} \geq \frac{1}{M^2} \sum_{l \in S} \left[U_1^{3n-3}(\tilde{\rho}^0)\right]_{lJ1} \geq \frac{1}{M^4}.
\end{aligned}$$

Fix any $\omega^{3n-2} \notin V^{3n-2}$. We have $\|\tilde{\rho}^{3n-2}(\omega^{3n-2}) - U_1^{3n-2}(\tilde{\rho}^0)\|_\infty \leq \xi_0$, which implies that

$$\tilde{\rho}^{3n-2}_{kJ1}(\omega^{3n-2}) \geq [U_1^{3n-2}(\tilde{\rho}^0)]_{kJ1} - \xi_0 \geq \frac{1}{M^4} - \xi_{-1}.$$

Note that $M \geq 3$, $\xi_{3M^2+1} < \xi_{-1} = \frac{1}{M^{M^M}}$ and $\hat{M} > \left(\frac{1}{\xi_{3M^2+1}}\right)^3 \geq \left(\frac{1}{\xi_{-1}}\right)^3 \geq M^{3M^M}$. It is clear that $\xi_{-1} \leq \frac{1}{M^{27}} \leq \frac{1}{2M^4}$ and $\hat{M}^{-\frac{1}{15}} \leq \frac{1}{M^{\frac{81}{15}}} < \frac{1}{M^5} \leq \frac{1}{3M^4} < \frac{1}{2M^4}$. Therefore, we have

$$\tilde{\rho}_{kJ1}^{3n-2}(\omega^{3n-2}) \geq \frac{1}{M^4} - \frac{1}{2M^4} = \frac{1}{2M^4} \geq \frac{1}{\hat{M}^{\frac{1}{15}}},$$

which is the required inequality in the lemma. ■

The following lemma provides an approximation of the matching probabilities at step $3n-1$ using parameter $\hat{q}$.

**Lemma E.16.** *For any $i, j \in I$ with $i \neq j$, $\omega^{3n-2} \notin V^{3n-2}$ and $k_1, l_1, k_2, l_2 \in S$, if $\tilde{\beta}_i^{3n-2}\left(\omega^{3n-2}\right) = (k_1, J, 1)$ and $\tilde{\beta}_j^{3n-2}\left(\omega^{3n-2}\right) = (k_2, J, 1)$, then*

$$\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1) - \hat{q}_{k_1 l_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}},$$
$$\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = J) - \hat{q}_{k_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}},$$
$$\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1, \hat{g}_j^{3n-1} = l_2) - \hat{q}_{k_1 l_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2 l_2}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}},$$
$$\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = J, \hat{g}_j^{3n-1} = l_2) - \hat{q}_{k_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2 l_2}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}},$$
$$\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = J, \hat{g}_j^{3n-1} = J) - \hat{q}_{k_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \leq \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

*Proof.* Fix any $i, j \in I$ with $i \neq j$, $\omega^{3n-2} \notin V^{3n-2}$ and $k_1, l_1, k_2, l_2 \in S$. Assume that $\tilde{\beta}_i^{3n-2}\left(\omega^{3n-2}\right) = (k_1, J, 1)$ and $\tilde{\beta}_j^{3n-2}\left(\omega^{3n-2}\right) = (k_2, J, 1)$.

By Lemma E.15, we have $\tilde{\rho}_{k_1 J1}^{3n-2}\left(\omega^{3n-2}\right) \geq \frac{1}{\hat{M}^{\frac{1}{15}}} > \frac{1}{\hat{M}^{\frac{1}{3}}}$, and $\tilde{\rho}_{k_2 J1}^{3n-2}\left(\omega^{3n-2}\right) \geq \frac{1}{\hat{M}^{\frac{1}{15}}} > \frac{1}{\hat{M}^{\frac{1}{3}}}$. It follows from Lemma E.1 that

$$\begin{aligned}&\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1) - \hat{q}_{k_1 l_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\ \leq\ & \frac{2}{\hat{M}^{\frac{2}{3}}} < \frac{2}{\hat{M}^{\frac{5}{9}}}\frac{1}{\hat{M}^{\frac{1}{9}}} \leq \frac{1}{\hat{M}^{\frac{1}{9}}}, \text{ and}\end{aligned}$$

$$\begin{aligned}&\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1)Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_j^{3n-1} = l_2) - \hat{q}_{k_1 l_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2 l_2}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\ \leq\ & \frac{5}{\hat{M}^{\frac{2}{3}}} < \frac{5}{\hat{M}^{\frac{5}{9}}}\frac{1}{\hat{M}^{\frac{1}{9}}} \leq \frac{1}{\hat{M}^{\frac{1}{9}}}.\end{aligned}$$

Next, we consider the case that agent $i$ is not matched. We can obtain

$$\begin{aligned}&\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = J) - \hat{q}_{k_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\ =\ & \left|\sum_{l_1 \in S} Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1) - \sum_{l_1 \in S}\hat{q}_{k_1 l_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\ \leq\ & \frac{2K}{\hat{M}^{\frac{2}{3}}} < \frac{2K}{\hat{M}^{\frac{5}{9}}}\frac{1}{\hat{M}^{\frac{1}{9}}} < \frac{2K}{\hat{M}^{\frac{1}{3}}}\frac{1}{\hat{M}^{\frac{1}{9}}} \leq \frac{2K}{M^{M^M}}\frac{1}{\hat{M}^{\frac{1}{9}}} \leq \frac{2K}{(\max\{K\bar{a}, 3\})^{27}}\frac{1}{\hat{M}^{\frac{1}{9}}} \leq \frac{1}{\hat{M}^{\frac{1}{9}}}, \text{ and}\end{aligned}$$

$$\begin{aligned}
& \left| Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=J,\hat{g}_j^{3n-1}=l_2\right)-\hat{q}_{k_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2l_2}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\
= & \left|\sum_{l'\in S} Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=l',\hat{g}_j^{3n-1}=l_2\right)-\sum_{l'\in S}\hat{q}_{k_1l'}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2l_2}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\
\le & \frac{5K}{\hat{M}^{\frac{2}{3}}}<\frac{5K}{\hat{M}^{\frac{5}{9}}}\frac{1}{\hat{M}^{\frac{1}{9}}}<\frac{5K}{\hat{M}^{\frac{1}{3}}}\frac{1}{\hat{M}^{\frac{1}{9}}}\le\frac{5K}{M^{M^M}}\frac{1}{\hat{M}^{\frac{1}{9}}}\le\frac{5K}{(\max\{K\bar{a},3\})^{27}}\frac{1}{\hat{M}^{\frac{1}{9}}}\le\frac{1}{\hat{M}^{\frac{1}{9}}}.
\end{aligned}$$

It remains to consider the case that agents $i$ and $j$ are not matched. We have

$$\begin{aligned}
& \left| Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=J,\hat{g}_j^{3n-1}=J\right)-\hat{q}_{k_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\
= & \left|\sum_{l'\in S} Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=J,\hat{g}_j^{3n-1}=l'\right)-\sum_{l'\in S}\hat{q}_{k_1}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2l'}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\
\le & \frac{5K^2}{\hat{M}^{\frac{2}{3}}}<\frac{5K^2}{\hat{M}^{\frac{5}{9}}}\frac{1}{\hat{M}^{\frac{1}{9}}}<\frac{5K^2}{\hat{M}^{\frac{1}{3}}}\frac{1}{\hat{M}^{\frac{1}{9}}}\le\frac{5K^2}{M^{M^M}}\frac{1}{\hat{M}^{\frac{1}{9}}}\le\frac{5K^2}{(\max\{K\bar{a},3\})^{27}}\frac{1}{\hat{M}^{\frac{1}{9}}}\le\frac{1}{\hat{M}^{\frac{1}{9}}}.
\end{aligned}$$

The proof is thus completed. ∎

The following lemma strengthens Lemma E.4 by providing a sharper bound, which will be used in the proof of Lemma E.18 below.

**Lemma E.17.** *For any $i\in I$, $k,l\in S$, $n\in\mathbb{T}_0$, and $F^{3n-2}\in\mathcal{F}^{3n-2}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-2}=(k,J,1)\right)\cap F^{3n-2}\right)>0$, we have*

$$\begin{aligned}
& \left|P_0\left(\tilde{\beta}_i^{3n-1}=(k,l,1)\,\middle|\,\left(\tilde{\beta}_i^{3n-2}=(k,J,1)\right)\cap F^{3n-2}\right)-\hat{q}_{kl}\left(U_1^{3n-2}(\tilde{\rho}^0)\right)\right| \\
\le & \frac{2\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2}=(k,J,1)\right)\cap F^{3n-2}\right)}+\frac{1}{\hat{M}^{\frac{1}{9}}}+\xi_{-1}.
\end{aligned}$$

*Proof.* Fix any $i\in I$, $k,l\in S$, $n\in\mathbb{T}_0$, and $F^{3n-2}\in\mathcal{F}^{3n-2}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-2}=(k,J,1)\right)\cap F^{3n-2}\right)>0$. Let $a=(k,J,1)$, $b=(k,l,1)$, and

$$A=\left\{\omega^{3n-2}\in\Omega^{3n-2}:\tilde{\beta}_i^{3n-2}(\omega^{3n-2})=(k,J,1)\right\}\cap F^{3n-2}.$$

We know that $Q^{3n-2}(A)=P_0(A)>0$. It is clear that

$$\begin{aligned}
P_0\left(\tilde{\beta}_i^{3n-1}=b\,\middle|\,\left(\tilde{\beta}_i^{3n-2}=a\right)\cap F^{3n-2}\right) &= \frac{1}{Q^{3n-2}(A)}\int_A Q_{3n-1}^{\omega^{3n-2}}\left(\tilde{\beta}_i^{3n-1}=b\right)dQ^{3n-2} \\
&= \frac{1}{Q^{3n-2}(A)}\int_A Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=l\right)dQ^{3n-2}.
\end{aligned}$$

By Lemmas E.14 and E.16, we have

$$\begin{aligned}
& \left|P_0\left(\tilde{\beta}_i^{3n-1}=b \,\middle|\, \left(\tilde{\beta}_i^{3n-2}=a\right)\cap F^{3n-2}\right)-\frac{1}{Q^{3n-2}(A)}\int_A \hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)dQ^{3n-2}\right| \\
\leq\ & \frac{1}{Q^{3n-2}(A)}\int_{A\cap V^{3n-2}}\left|Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=l\right)-\hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right|dQ^{3n-2} \\
& +\frac{1}{Q^{3n-2}(A)}\int_{A\backslash V^{3n-2}}\left|Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=l\right)-\hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right|dQ^{3n-2} \\
\leq\ & \frac{Q^{3n-2}(V^{3n-2})}{Q^{3n-2}(A)}+\frac{Q^{3n-2}(A\backslash V^{3n-2})}{Q^{3n-2}(A)}\frac{1}{\hat{M}^{\frac{1}{9}}} \\
\leq\ & \frac{\epsilon_0}{Q^{3n-2}(A)}+\frac{1}{\hat{M}^{\frac{1}{9}}}. \qquad \text{(E.64)}
\end{aligned}$$

By the definition of $V^{3n-2}$ in Lemma E.14, we know that $\|\tilde{\rho}^{3n-2}(\omega^{3n-2})-U_1^{3n-2}(\tilde{\rho}^0)\|_\infty\leq\xi_0$ for any $\omega^{3n-2}\notin V^{3n-2}$. By Lemma E.2, we have $\left|\hat{q}_{kl}\left(\tilde{\rho}^{3n-2}(\omega^{3n-2})\right)-\hat{q}_{kl}\left(U_1^{3n-2}(\tilde{\rho}^0)\right|\leq\xi_{-1}$ for any $\omega^{3n-2}\notin V^{3n-2}$. It follows from Lemma E.14 that

$$\begin{aligned}
& \left|\hat{q}_{kl}\left(U_1^{3n-2}(\tilde{\rho}^0)\right)-\frac{1}{Q^{3n-2}(A)}\int_A \hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)dQ^{3n-2}\right| \\
\leq\ & \frac{1}{Q^{3n-2}(A)}\int_{A\cap V^{3n-2}}\left|q_{kl}\left(U_1^{3n-2}(\tilde{\rho}^0)\right)-\hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right|dQ^{3n-2} \\
& +\frac{1}{Q^{3n-2}(A)}\int_{A\backslash V^{3n-2}}\left|\hat{q}_{kl}\left(U_1^{3n-2}(\tilde{\rho}^0)\right)-\hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)\right|dQ^{3n-2} \\
\leq\ & \frac{Q^{3n-2}(V^{3n-2})}{Q^{3n-2}(A)}+\frac{Q^{3n-2}(A\backslash V^{3n-2})}{Q^{3n-2}(A)}\xi_{-1} \\
\leq\ & \frac{\epsilon_0}{Q^{3n-2}(A)}+\xi_{-1}. \qquad \text{(E.65)}
\end{aligned}$$

By Equations (E.64) and (E.65), we have

$$\begin{aligned}
& \left|P_0\left(\tilde{\beta}_i^{3n-1}=(k,l,1) \,\middle|\, \left(\tilde{\beta}_i^{3n-2}=(k,J,1)\right)\cap F^{3n-2}\right)-\hat{q}_{kl}\left(U_1^{3n-2}(\tilde{\rho}^0)\right)\right| \\
=\ & \left|P_0\left(\tilde{\beta}_i^{3n-1}=b \,\middle|\, \left(\tilde{\beta}_i^{3n-2}=a\right)\cap F^{3n-2}\right)-\frac{1}{Q^{3n-2}(A)}\int_A \hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)dQ^{3n-2}\right| \\
+\ & \left|\hat{q}_{kl}\left(U_1^{3n-2}(\tilde{\rho}^0)\right)-\frac{1}{Q^{3n-2}(A)}\int_A \hat{q}_{kl}\left(\tilde{\rho}^{3n-2}\left(\omega^{3n-2}\right)\right)dQ^{3n-2}\right| \\
\leq\ & \frac{2\epsilon_0}{Q^{3n-2}(A)}+\frac{1}{\hat{M}^{\frac{1}{9}}}+\xi_{-1} \\
\leq\ & \frac{2\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2}=(k,J,1)\right)\cap F^{3n-2}\right)}+\frac{1}{\hat{M}^{\frac{1}{9}}}+\xi_{-1},
\end{aligned}$$

which completes the proof. ∎

The following lemma improves the upper bound in Part (ii) of Lemma E.5.

**Lemma E.18.** *For any $i \in I$, $n \in \mathbb{T}_0$, $a, b \in \tilde{S}$, and $F^{3n-2} \in \mathcal{F}^{3n-2}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) > 0$, we have*

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) - P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \tilde{\beta}_i^{3n-2} = a\right)\right| \\ \leq \; & \frac{4K\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right)} + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}. \end{aligned}$$

*Proof.* Fix any $i \in I$, $n \in \mathbb{T}_0$, $a \in \tilde{S}$, and $F^{3n-2} \in \mathcal{F}^{3n-2}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) > 0$. When $\tilde{\beta}_i^{3n-2} = (k, l, 0)$ for some $k, l \in S$, agent $i$ is already matched at the mutation step of $3n-2$. Thus, her expanded type at step $3n-1$ does not change with probability one. In other words, we have for any $b \in \tilde{S}$,

$$P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) = P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \tilde{\beta}_i^{3n-2} = a\right), \tag{E.66}$$

which implies the inequality in the lemma. Since $P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) > 0$, it is not possible for $a$ to be $(k, J, 0)$ or $(k, l, 1)$ for any $k, l \in S$. Hence, we only need to consider $a = (k, J, 1)$. In this case, if $b$ is neither $(k, J, 1)$ nor $(k, l, 1)$, then we must have $P_0\left(\tilde{\beta}_i^{3n-1} = b\right) = 0$, which implies the identity in Equation (E.66). Therefore, the inequality in the lemma holds again. Thus, it remains to consider $b = (k, J, 1)$ or $(k, l, 1)$.

Let $b = (k, l, 1)$. It follows from Lemma E.17 that

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{3n-1} = (k, l, 1) \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right) - \hat{q}_{kl}\left(U_1^{3n-2}(\tilde{\rho}^0)\right)\right| \\ \leq \; & \frac{2\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right)} + \frac{1}{\hat{M}^{\frac{1}{9}}} + \xi_{-1}, \text{ and} \end{aligned}$$

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{3n-1} = (k, l, 1) \,\middle|\, \tilde{\beta}_i^{3n-2} = (k, J, 1)\right) - \hat{q}_{kl}\left(U_1^{3n-2}(\tilde{\rho}^0)\right)\right| \\ \leq \; & \frac{2\epsilon_0}{P_0\left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right)} + \frac{1}{\hat{M}^{\frac{1}{9}}} + \xi_{-1}. \end{aligned}$$

By combining the above two inequalities, we obtain that

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) - P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \tilde{\beta}_i^{3n-2} = a\right)\right| \\ \leq \; & \frac{4\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right)} + \frac{2}{\hat{M}^{\frac{1}{9}}} + 2\xi_{-1}, \end{aligned}$$

which implies the inequality in the lemma.

Assume $b = (k, J, 1)$. Then, we have

$$\begin{aligned} & \left| P_0\left(\tilde{\beta}_i^{3n-1} = (k, J, 1) \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) - P_0\left(\tilde{\beta}_i^{3n-1} = (k, J, 1) \,\middle|\, \tilde{\beta}_i^{3n-2} = a\right) \right| \\ = & \left| \sum_{l=1}^{K} P_0\left(\tilde{\beta}_i^{3n-1} = (k, l, 1) \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) - \sum_{l=1}^{K} P_0\left(\tilde{\beta}_i^{3n-1} = (k, l, 1) \,\middle|\, \tilde{\beta}_i^{3n-2} = a\right) \right| \\ \leq & \frac{4K\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right)} + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}. \end{aligned}$$

Hence, the proof is completed. ■

The following lemma is Part (i) of Lemma E.5.

**Lemma E.19.** *For any $i \in I$, $a, b \in \tilde{S}$, $n \in \mathbb{T}_0$, and $F^{3n-3} \in \mathcal{F}^{3n-3}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-3} = a\right) \cap F^{3n-3}\right) > 0$, we have*

$$P_0\left(\tilde{\beta}_i^{3n-2} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-3} = a\right) \cap F^{3n-3}\right) = P_0\left(\tilde{\beta}_i^{3n-2} = b \,\middle|\, \tilde{\beta}_i^{3n-3} = a\right).$$

*Proof.* Fix any $i \in I$, $a, b \in \tilde{S}$, $n \in \mathbb{T}_0$, and $F^{3n-3} \in \mathcal{F}^{3n-3}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-3} = a\right) \cap F^{3n-3}\right) > 0$. Let

$$D_1 = \{\omega^{3n-3} \in \Omega^{3n-3} : \tilde{\beta}_i^{3n-3}(\omega^{3n-3}) = a\} \cap F^{3n-3}.$$

We know that $P_0(D_1) = Q^{3n-3}(D_1) > 0$. By the construction of the mutation step at period $n$, it is easy to see that

$$\begin{aligned} & P_0\left(\tilde{\beta}_i^{3n-2} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-3} = a\right) \cap F^{3n-3}\right) \\ = & \frac{1}{Q^{3n-3}(D_1)} \int_{D_1} Q_{3n-2}^{\omega^{3n-3}}(\tilde{\beta}_i^{3n-2} = b) dQ^{3n-3} \\ = & \frac{1}{Q^{3n-3}(D_1)} \int_{D_1} B_{ab} dQ^{3n-3} \\ = & B_{ab}, \end{aligned}$$

where

$$B_{ab} = \begin{cases} \hat{\eta}_{k_1 l_1} \hat{\eta}_{k_2 l_2} & \text{if } a = (k_1, k_2, 0), b = (l_1, l_2, 0) \\ \hat{\eta}_{k_1 l_1} & \text{if } a = (k_1, J, 1), b = (l_1, J, 1) \\ 0 & \text{otherwise.} \end{cases}$$

By taking $F^{3n-3}$ to be $\Omega^{3n-3}$ , we have

$$P_0\left(\tilde{\beta}_i^{3n-2} = b \,\middle|\, \tilde{\beta}_i^{3n-3} = a\right) = B_{ab}.$$

Therefore, the identity in the lemma follows. ■

The following lemma is Part (iii) of Lemma E.5.

**Lemma E.20.** *For any $i \in I$, $a, b \in \tilde{S}$, $n \in \mathbb{T}_0$, and $F^{3n-1} \in \mathcal{F}^{3n-1}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-1} = a\right) \cap F^{3n-1}\right) > 0$, we have*

$$P_0\left(\tilde{\beta}_i^{3n} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-1} = a\right) \cap F^{3n-1}\right) = P_0\left(\tilde{\beta}_i^{3n} = b \,\middle|\, \tilde{\beta}_i^{3n-1} = a\right).$$

*Proof.* Fix any $i \in I$, $a, b \in \tilde{S}$, $n \in \mathbb{T}_0$, and $F^{3n-1} \in \mathcal{F}^{3n-1}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-1} = a\right) \cap F^{3n-1}\right) > 0$. Let

$$D_1 = \{\omega^{3n-1} \in \Omega^{3n-1} : \tilde{\beta}_i^{3n-1}(\omega^{3n-1}) = a\} \cap F^{3n-1}.$$

By the construction of the type changing with break-up step at period $n$, it is clear that

$$\begin{aligned}
& P_0\left(\tilde{\beta}_i^{3n} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-1} = a\right) \cap F^{3n-1}\right) \\
= \; & \frac{1}{Q^{3n-1}(D_1)} \int_{D_1} Q_{3n}^{\omega^{3n-1}}(\tilde{\beta}_i^{3n} = b) dQ^{3n-1} \\
= \; & \frac{1}{Q^{3n-1}(D_1)} \int_{D_1} B_{ab} dQ^{3n-1} \\
= \; & B_{ab},
\end{aligned}$$

where

$$B_{ab} = \begin{cases} 1 - \hat{\vartheta}_{k_1 k_2} & \text{if } a = (k_1, k_2, 0), b = (k_1, k_2, 0) \\ \hat{\vartheta}_{k_1 k_2} \varsigma_{k_1 k_2}(l_1) & \text{if } a = (k_1, k_2, 0), b = (l_1, J, 1) \\ \hat{\xi}_{k_1 k_2} \hat{\sigma}_{k_1 k_2}(l_1, l_2) & \text{if } a = (k_1, k_2, 1), b = (l_1, l_2, 0) \\ (1 - \hat{\xi}_{k_1 k_2}) \hat{\varsigma}_{k_1 k_2}(l_1) & \text{if } a = (k_1, k_2, 1), b = (l_1, J, 1) \\ 1 & \text{if } a = (k_1, J, 1), b = (k_1, J, 1) \\ 0 & \text{otherwise.} \end{cases}$$

By taking $F^{3n-1}$ to be $\Omega^{3n-1}$ , we have

$$P_0\left(\tilde{\beta}_i^{3n} = b \,\middle|\, \tilde{\beta}_i^{3n-1} = a\right) = B_{ab}.$$

Hence, we obtain the identity in the lemma. ■

The following lemma provides an upper bounded for the probability with which two agents are matched at the $m$-th step.

**Lemma E.21.** *For any $i, j \in I$ with $i \neq j$ and $\pi_i^0 \neq j$, and $m \in \{0, 1, \ldots, 3M^2\}$, we have*

$$P_0\left(\hat{\pi}_i^m = j\right) \leq m\epsilon_0 + \frac{2m}{\hat{M}^{\frac{14}{15}}}.$$

*Proof.* Fix any $i, j \in I$ with $i \neq j$. It is clear that the inequality holds when $m = 0$. Suppose the inequality holds when $m = m'$. It is easy to see that

$$\begin{aligned}
P_0\left(\hat{\pi}_i^{m'+1} = j\right) &= P_0\left(\hat{\pi}_i^{m'+1} = j, \hat{\pi}_i^{m'} = j\right) + P_0\left(\hat{\pi}_i^{m'+1} = j, \hat{\pi}_i^{m'} \neq j\right) \\
&\leq P_0\left(\hat{\pi}_i^{m'} = j\right) + P_0\left(\hat{\pi}_i^{m'+1} = j, \hat{\pi}_i^{m'} \neq j\right). \qquad \text{(E.67)}
\end{aligned}$$

If $m' = 3n-3$ or $3n-1$ for some $n \in \mathbb{T}_0$, it is clear that $P_0\left(\hat{\pi}_i^{m'+1} = j, \hat{\pi}_i^{m'} \neq j\right) = 0$. Then, Equation (E.67) and the induction hypothesis imply that

$$P_0\left(\hat{\pi}_i^{m'+1} = j\right) \leq P_0\left(\hat{\pi}_i^{m'} = j\right) \leq (m'+1)\epsilon_0 + \frac{2m'+2}{\hat{M}^{\frac{14}{15}}}.$$

If $m' = 3n-2$ for some $n \in \mathbb{T}_0$, we have

$$P_0\left(\hat{\pi}_i^{m'+1} = j, \hat{\pi}_i^{m'} \neq j\right) = P_0\left(\hat{\pi}_i^{m'+1} = j, \hat{\pi}_i^{m'} = i\right).$$

Let $A^{3n-2} = \{\omega^{3n-2} \in \Omega^{3n-2} : \hat{\pi}_i^{3n-2}(\omega^{3n-2}) = i\}$. Then, we obtain that

$$\begin{aligned}
& P_0\left(\hat{\pi}_i^{m+1} = j, \hat{\pi}_i^{m} \neq j\right) \\
= & \int_{A^{3n-2}} Q_{3n-1}^{\omega^{3n-2}}(\hat{\pi}_i^{3n-1} = j) dQ^{3n-2} \\
= & \int_{A^{3n-2}\cap V^{3n-2}} Q_{3n-1}^{\omega^{3n-2}}(\hat{\pi}_i^{3n-1} = j) dQ^{3n-2} + \int_{A^{3n-2}\setminus V^{3n-2}} Q_{3n-1}^{\omega^{3n-2}}(\hat{\pi}_i^{3n-1} = j) dQ^{3n-2} \\
\leq & Q^{3n-2}\left(A^{3n-2}\cap V^{3n-2}\right) + \int_{A^{3n-2}\setminus V^{3n-2}} Q_{3n-1}^{\omega^{3n-2}}(\hat{\pi}_i^{3n-1} = j) dQ^{3n-2}.
\end{aligned}$$

For any $\omega^{3n-2} \in A^{3n-2}$, if $\hat{\pi}_j^{3n-2}(\omega^{3n-2}) \neq j$, it is clear that $Q_{3n-1}^{\omega^{3n-2}}(\hat{\pi}_i^{3n-1} = j) = 0$; if $\hat{\pi}_j^{3n-2}(\omega^{3n-2}) = j$, Lemma E.1 (ii) implies that $Q_{3n-1}^{\omega^{3n-2}}(\hat{\pi}_i^{3n-1} = j) \leq \frac{2}{\hat{M}\hat{\rho}_{\alpha_i^{3n-2}(\omega^{3n-2})J}}$. It follows from Lemma E.15 that for any $\omega^{3n-2} \notin V^{3n-2}$ and $k \in S$, we have $\tilde{\rho}_{kJ1}^{3n-2}(\omega^{3n-2}) \geq \hat{M}^{-\frac{1}{15}}$. Therefore, Lemma E.14 leads to

$$\begin{aligned}
& P_0\left(\hat{\pi}_i^{m'+1} = j, \hat{\pi}_i^{m'} \neq j\right) \\
\leq & Q^{3n-2}\left(A^{3n-2}\cap V^{3n-2}\right) + Q^{3n-2}\left(A^{3n-2}\setminus V^{3n-2}\right)\frac{2}{\hat{M}\hat{M}^{-\frac{1}{15}}} \\
\leq & \epsilon_0 + \frac{2}{\hat{M}^{\frac{14}{15}}}.
\end{aligned}$$

The above inequality and Equation (E.67) together with the induction hypothesis imply that

$$P_0\left(\hat{\pi}_i^{m'+1} = j\right) \leq P_0\left(\hat{\pi}_i^{m'} = j\right) + \epsilon_0 + \frac{2}{\hat{M}^{\frac{14}{15}}} \leq (m'+1)\epsilon_0 + \frac{2m'+2}{\hat{M}^{\frac{14}{15}}}.$$

By induction, we have

$$P_0\left(\hat{\pi}_i^{m} = j\right) \leq m\epsilon_0 + \frac{2m}{\hat{M}^{\frac{14}{15}}}$$

for any $m \in \{0, 1, \ldots, 3M^2\}$. ■

### E.5.3 Proof of Lemma E.2

First, we work with $T_1$. Since $T_1$ is continuous on $\tilde{\Delta}$, there exists a strictly increasing continuous bijection $v_1$ on $\mathbb{R}_+$ with $v_1(0) = 0$ such that $||T_1(\tilde{\rho}) - T_1(\tilde{\rho}')||_\infty \leq v_1(||\tilde{\rho} - \tilde{\rho}'||_\infty)$. for any $\tilde{\rho}, \tilde{\rho}' \in \tilde{\Delta}$

(which is called a modulus of continuity of the function $T_1$).[7]

For $T_2$, $T_3$, $\hat{q}$, we can derive their modulus of continuity in the same way. By taking the maximum, we can get a strictly increasing bijection $v$ on $\mathbb{R}_+$ which is a common modulus of continuity for all these mappings.

Let $\xi_{-1} = \frac{1}{M^{M^M}}$ and $w$ be the inverse function $v^{-1}$ on $\mathbb{R}_+$. Let $\xi_0 = \min(w(\xi_{-1}), \xi_{-1})$, $\xi_m = \min\left(w(\xi_{m-1}), \frac{\xi_0}{3M^2}\right)$ for any $m \in \{1, \ldots, 3M^2\}$. Hence, it is clear that $3M^2\xi_m \le \xi_0 \le \xi_{-1}$ for any $m \in \{1, 2, \ldots, 3M^2+1\}$.

Fix any $m \in \{-1, 0, \ldots, 3M^2\}$, and $\tilde{\rho}, \tilde{\rho}' \in \tilde{\Delta}$ with $\|\tilde{\rho} - \tilde{\rho}'\|_\infty \le \xi_{m+1}$. Then, we know that $\|\tilde{\rho} - \tilde{\rho}'\|_\infty \le w(\xi_m)$. The fact that $v$ is a strictly increasing bijection on $\mathbb{R}_+$ implies that $v(\|\tilde{\rho} - \tilde{\rho}'\|_\infty) \le \xi_m$. Since $v$ is a common modulus of continuity for $T_1$, $T_2$, $T_3$ and $\hat{q}$, we obtain that for any $r \in \{1, 2, 3\}$,

$$\|T_r(\tilde{\rho}) - T_r(\tilde{\rho}')\|_\infty \le \xi_m, \text{ and } \|\hat{q}(\tilde{\rho}) - \hat{q}(\tilde{\rho}')\|_\infty \le \xi_m,$$

which completes the proof.

#### E.5.4 Proof of Lemma E.3

Fix any $m \in \{0, 1, \ldots, 3M^2\}$. Let $V^m = \{\omega^m \in \Omega^m : \|\tilde{\rho}^m(\omega^m) - U_1^m(\tilde{\rho}^0)\|_\infty > \xi_0\}$. By Lemma E.14, we know that $P_0(V^m) \le \epsilon_0$.

Then we can obtain that

$$\begin{aligned}
& \|\mathbb{E}(\tilde{\rho}^m) - U_1^m(\tilde{\rho}^0)\|_\infty \\
= & \left\| \int_{\Omega^m} \left(\tilde{\rho}^m - U_1^m(\tilde{\rho}^0)\right) dQ^m \right\|_\infty \\
\le & \int_{\Omega^m} \|\tilde{\rho}^m - U_1^m(\tilde{\rho}^0)\|_\infty dQ^m \\
\le & \int_{V^m} \|\tilde{\rho}^m - U_1^m(\tilde{\rho}^0)\|_\infty dQ^m + \int_{\Omega^m \setminus V^m} \|\tilde{\rho}^m - U_1^m(\tilde{\rho}^0)\|_\infty dQ^m \\
\le & \epsilon_0 + \xi_0 \le \epsilon_0 + \xi_{-1} = \frac{3M^2K(K+1)}{\hat{M}^{\frac{1}{3}}} + \frac{1}{M^{M^M}} \\
\le & \frac{3M^2K(K+1)}{M^{M^M}} + \frac{1}{M^{M^M}} \le \frac{3M^2(K+1)^2}{M^{M^M}}.
\end{aligned}$$

Let $B_1(M) = \frac{3M^2(K+1)^2}{M^{M^M}}$. It is clear that $\lim_{M\to\infty} B_1(M) = 0$.

[7] Given a continuous function $f$ from a compact metric space $(X, d_X)$ to a metric space $(Y, d_Y)$, $f$ admits a (global) modulus of continuity $\omega$ in the sense that $\omega$ is a function from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $\lim_{t\to 0}\omega(t) = \omega(0) = 0$, and for any $x, x' \in X$, $d_Y(f(x), f(x')) \le \omega(d_X(x, x'))$. Since the range of $f$ is compact, we can assume with loss of generality that $\omega$ is a bounded function on $\mathbb{R}_+$. Following the wikipedia entry "Modulus of continuity" (https://en.wikipedia.org/wiki/Modulus_of_continuity), let $\omega'(t) := \frac{1}{t}\int_t^{2t}\left[\sup_{0\le s'\le s}\omega(s')\right] ds$ for $t > 0$ and $\omega'(0) = 0$. Then, it is easy to verify that $\omega'$ is increasing and continuous on $\mathbb{R}_+$. Let $\hat{\omega}(t) := \omega'(t) + t$ for any $t \in \mathbb{R}_+$, which is a modulus of continuity for $f$ that is a strictly increasing continuous bijection on $\mathbb{R}_+$.

### E.5.5 Proof of Lemma E.4

By Lemma E.17, for any $i \in I$, $k, l \in S$, $n \in \mathbb{T}_0$ and $F^{3n-2} \in \mathcal{F}^{3n-2}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right) > 0$, we have

$$\begin{aligned}
& \left| P_0\left(\tilde{\beta}_i^{3n-1} = (k, l, 1) \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right) - \hat{q}_{k_2 k_1}\left(U_1^{3n-2}(\tilde{\rho}^0)\right) \right| \\
\leq\ & \frac{2\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right)} + \frac{1}{\hat{M}^{\frac{1}{9}}} + \xi_{-1}.
\end{aligned}$$

Note that $2\epsilon_0 = \frac{6M^2K(K+1)}{\hat{M}^{\frac{1}{3}}} < \frac{1}{M^3}$ and $\frac{1}{\hat{M}^{\frac{1}{9}}} + \xi_{-1} = \frac{1}{\hat{M}^{\frac{1}{9}}} + \frac{1}{M^{M^M}} < \frac{1}{M^2}$. It is then clear that

$$\begin{aligned}
& \left| P_0\left(\tilde{\beta}_i^{3n-1} = (k, l, 1) \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right) - \hat{q}_{k_2 k_1}\left(U_1^{3n-2}(\tilde{\rho}^0)\right) \right| \\
\leq\ & \frac{1}{M^3 P_0\left(\left(\tilde{\beta}_i^{3n-2} = (k, J, 1)\right) \cap F^{3n-2}\right)} + \frac{1}{M^2},
\end{aligned}$$

which is the required inequality in Lemma E.4.

### E.5.6 Proof of Lemma E.5

Parts (i) and (iii) of Lemma E.5 have been shown in Lemmas E.19 and E.20 respectively.

Fix any $i \in I$, $a, b \in \tilde{S}$, $n \in \{1, 2, \ldots, M^2\}$, and $F^{3n-2} \in \mathcal{F}^{3n-2}$ such that $P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) > 0$. Lemma E.18 indicates that

$$\begin{aligned}
& \left| P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) - P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \tilde{\beta}_i^{3n-2} = a\right) \right| \\
\leq\ & \frac{4K\epsilon_0}{P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right)} + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}.
\end{aligned}$$

Note that $M \geq \max\{K\bar{a}, 3\}$, $\xi_{-1} = \frac{1}{M^{M^M}}$, $\epsilon_0 = \frac{3M^2K(K+1)}{\hat{M}^{\frac{1}{3}}}$ and $\hat{M} > \left(\frac{1}{\xi_{3M^2+1}}\right)^3 \geq \left(\frac{1}{\xi_{-1}}\right)^3 \geq M^{3M^M}$. Then we can obtain that

$$4K\epsilon_0 = \frac{12M^2K^2(K+1)}{\hat{M}^{\frac{1}{3}}} \leq \frac{24M^5}{\hat{M}^{\frac{1}{3}}} \leq \frac{24M^5}{M^{M^M}} \leq \frac{24M^5}{M^{27}} < \frac{1}{M^3},$$

$$\frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1} \leq \frac{2M}{M^{\frac{1}{3}M^M}} + \frac{2M}{M^{M^M}} \leq \frac{4M}{M^{\frac{1}{3}M^M}} \leq \frac{4M}{M^9} < \frac{1}{M^2}.$$

Therefore, we have

$$\begin{aligned}
& \left| P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right) - P_0\left(\tilde{\beta}_i^{3n-1} = b \,\middle|\, \tilde{\beta}_i^{3n-2} = a\right) \right| \\
\leq\ & \frac{1}{M^3 P_0\left(\left(\tilde{\beta}_i^{3n-2} = a\right) \cap F^{3n-2}\right)} + \frac{1}{M^2},
\end{aligned}$$

which is Part (ii) of Lemma E.5.

### E.5.7 Proof of Lemma E.6

For a random variable $f$ and an event $G$ on $\Omega$, we shall use (from now onwards) the simplified notation $(f = a, G)$ to represent the event $(f = a) \cap G$ that event $G$ happens while $f$ takes value $a$.

We need to provide a sequence of estimations $\{c_m\}_{1 \le m \le 3M^2}$ such that for any $i \in I$, any $m, m_1 \in \{0, 1, \ldots, 3M^2\}$ with $m > m_1$, any expanded types $a, a_1 \in \tilde{S}$, and any $F_i^{m_1-1} \in \mathcal{F}_i^{m_1-1}$, we have

$$\begin{aligned}&\Big| P_0\left(\tilde{\beta}_i^m = a, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) \\ &\quad - P_0\left(\tilde{\beta}_i^m = a, \tilde{\beta}_i^{m_1} = a_1\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right)\Big| \le c_m.\end{aligned}$$

Fix any $i \in I$ and $m \in \{1, 2, \ldots, 3M^2\}$. When $m = 1$, it is clear that $c_1$ can be taken to be 0. Suppose that we have already defined $c_m$, we need to define $c_{m+1}$ using $c_m$.

Fix any $m_1, \boldsymbol{m_2}$ with $m+1 > m_1 > \boldsymbol{m_2}$, and expanded types $a, a_1, \boldsymbol{a_2}$. We first consider the case when $m > m_1$. It is clear that

$$\begin{aligned}& P_0\left(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) \\ = & \sum_{b \in \tilde{S}} P_0\left(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right).\end{aligned}$$

Let $A = \left\{b \in \tilde{S} : P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) > 0\right\}$. We have[8]

$$\begin{aligned}& P_0\left(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) \\ = & \sum_{b \in A} P_0\left(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) \\ = & \sum_{b \in A} P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) \\ & \qquad P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right).\end{aligned}$$

Let $B = \sum_{b \in A} P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1\right) P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right)$. We can obtain that

$$\begin{aligned}& \left| P_0\left(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = a_2\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) - B\right| \\ = & \sum_{b \in A} \left| P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) - P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1\right)\right| \\ & \quad \times P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right).\end{aligned}$$

[8]When $A$ is empty, we follow the convention that summation over an empty set is zero.

By Lemmas E.18, E.19 and E.20, we know that for any $m \in \{1, 2, \dots, 3M^2\}$, and $b \in A$,

$$\begin{aligned}
& \left| P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1 - 1}\right) - P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b\right)\right| \\
\leq \;& K\left(\frac{4\epsilon_0}{P_0(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1})} + 2\xi_{-1} + \frac{2}{\hat{M}^{\frac{1}{9}}}\right), \text{ and}
\end{aligned}$$

$$\begin{aligned}
& \left| P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1\right) - P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b\right)\right| \\
\leq \;& K\left(\frac{4\epsilon_0}{P_0(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1)} + 2\xi_{-1} + \frac{2}{\hat{M}^{\frac{1}{9}}}\right).
\end{aligned}$$

It follows from the above inequalities that

$$\begin{aligned}
& \left| P_0\left(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m_1} = a_1, \tilde{\beta}_i^{\boldsymbol{m_2}} = a_2\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) - B\right| \\
= \;& \sum_{b \in A}\left| P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) - P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b\right)\right. \\
& \quad \left. + P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b\right) - P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1\right)\right| \\
& \quad \times P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) \\
\leq \;& \sum_{b \in A} K\left(\frac{4\epsilon_0}{P_0(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1})} + 2\xi_{-1} + \frac{2}{\hat{M}^{\frac{1}{9}}} + \frac{4\epsilon_0}{P_0(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1)} + 2\xi_{-1} + \frac{2}{\hat{M}^{\frac{1}{9}}}\right) \\
& \quad \times P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) \\
\leq \;& \sum_{b \in A} K\left(4\epsilon_0 + 2\xi_{-1} + \frac{2}{\hat{M}^{\frac{1}{9}}} + 4\epsilon_0 + 2\xi_{-1} + \frac{2}{\hat{M}^{\frac{1}{9}}}\right) \\
\leq \;& 2K^2(K+1)\left(8\epsilon_0 + 4\xi_{-1} + \frac{4}{\hat{M}^{\frac{1}{9}}}\right). \qquad \text{(E.68)}
\end{aligned}$$

The induction hypothesis implies that for any $b \in \tilde{S}$,

$$\left| P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right) - P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right)\right| \leq c_m.$$

Then, we can obtain that

$$\begin{aligned}
& \left| B - P_0\left(\tilde{\beta}_i^{m+1} = a, \tilde{\beta}_i^{m_1} = a_1\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right)\right| \\
= \;& \left| B - \sum_{b \in A} P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1\right) P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right)\right| \\
\leq \;& \sum_{b \in A} P_0\left(\tilde{\beta}_i^{m+1} = a \,\middle|\, \tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1\right) \left| P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1\right)\right. \\
& \quad \left. - P_0\left(\tilde{\beta}_i^m = b, \tilde{\beta}_i^{m_1} = a_1\right) P_0\left(\tilde{\beta}_i^{m_1} = a_1, F_i^{m_1-1}\right)\right| \\
\leq \;& 2K(K+1)c_m \leq 2K^2(K+1)c_m. \qquad \text{(E.69)}
\end{aligned}$$

By Equations (E.68) and (E.69), we have

$$\begin{aligned}
&\Big| P_0\left(\tilde{\beta}_i^{m+1}=a,\tilde{\beta}_i^{m_1}=a_1,F_i^{m_1-1}\right)P_0\left(\tilde{\beta}_i^{m_1}=a_1\right)\\
&-P_0\left(\tilde{\beta}_i^{m+1}=a,\tilde{\beta}_i^{m_1}=a_1\right)P_0\left(\tilde{\beta}_i^{m_1}=a_1,F_i^{m_1-1}\right)\Big|\\
\leq\ & 2K^2(K+1)\left(4\xi_{-1}+8\epsilon_0+\frac{4}{\hat{M}^{\frac{1}{9}}}+c_m\right).
\end{aligned} \tag{E.70}$$

If $m=m_1$ and $P_0\left(\tilde{\beta}_i^{m}=a_1,F_i^{m_1-1}\right)=0$, then it is clear that

$$\begin{aligned}
&P_0\left(\tilde{\beta}_i^{m+1}=a,\tilde{\beta}_i^{m_1}=a_1,F_i^{m_1-1}\right)P_0\left(\tilde{\beta}_i^{m_1}=a_1\right)\\
=\ &P_0\left(\tilde{\beta}_i^{m+1}=a,\tilde{\beta}_i^{m_1}=a_1\right)P_0\left(\tilde{\beta}_i^{m_1}=a_1,F_i^{m_1-1}\right).
\end{aligned} \tag{E.71}$$

If $m=m_1$ and $P_0\left(\tilde{\beta}_i^{m}=a_1,F_i^{m_1-1}\right)>0$, then it follows from Lemma E.18, E.19 and E.20 that

$$\begin{aligned}
&\left|P_0\left(\tilde{\beta}_i^{m+1}=a\,\middle|\,\tilde{\beta}_i^{m_1}=a_1,F_i^{m_1-1}\right)-P_0\left(\tilde{\beta}_i^{m+1}=a\,\middle|\,\tilde{\beta}_i^{m_1}=a_1\right)\right|\\
\leq\ &\frac{4K\epsilon_0}{P_0\left(\tilde{\beta}_i^{m_1}=a_1,F_i^{m_1-1}\right)}+\frac{2K}{\hat{M}^{\frac{1}{9}}}+2K\xi_{-1}.
\end{aligned}$$

Therefore, we can obtain that

$$\begin{aligned}
&\Big|P_0\left(\tilde{\beta}_i^{m+1}=a,\tilde{\beta}_i^{m_1}=a_1,F_i^{m_1-1}\right)P_0\left(\tilde{\beta}_i^{m_1}=a_1\right)\\
&\quad-P_0\left(\tilde{\beta}_i^{m+1}=a,\tilde{\beta}_i^{m_1}=a_1\right)P_0\left(\tilde{\beta}_i^{m_1}=a_1,F_i^{m_1-1}\right)\Big|\\
=\ &\left|P_0\left(\tilde{\beta}_i^{m+1}=a\,\middle|\,\tilde{\beta}_i^{m_1}=a_1,F_i^{m_1-1}\right)-P_0\left(\tilde{\beta}_i^{m+1}=a\,\middle|\,\tilde{\beta}_i^{m_1}=a_1\right)\right|\\
&\quad P_0\left(\tilde{\beta}_i^{m_1}=a_1,F_i^{m_1-1}\right)P_0\left(\tilde{\beta}_i^{m_1}=a_1\right)\\
\leq\ &\left(\frac{4K\epsilon_0}{P_0\left(\tilde{\beta}_i^{m_1}=a_1,F_i^{m_1-1}\right)}+\frac{2K}{\hat{M}^{\frac{1}{9}}}+2K\xi_{-1}\right)P_0\left(\tilde{\beta}_i^{m_1}=a_1,F_i^{m_1-1}\right)P_0\left(\tilde{\beta}_i^{m_1}=a_1\right)\\
\leq\ &4K\epsilon_0+\frac{2K}{\hat{M}^{\frac{1}{9}}}+2K\xi_{-1}.
\end{aligned} \tag{E.72}$$

By Equations (E.70), (E.71) and (E.72), we can define $c_{m+1}$ to be

$$2K^2(K+1)\left(4\xi_{-1}+8\epsilon_0+\frac{4}{\hat{M}^{\frac{1}{9}}}+c_m\right)$$

so that $c_{m+1}$ has the desired inductive property.

Next, we use induction again to prove that for any $m\in\{1,2,\ldots,3M^2\}$,

$$c_m\leq 2^{2m}K^{4m}(K+1)^{2m}\left(4\xi_{-1}+8\epsilon_0+\frac{4}{\hat{M}^{\frac{1}{9}}}\right). \tag{E.73}$$

Note that $c_1 = 0$. It is clear that this inequality holds for $m = 1$. Suppose that this inequality holds for $m = m' \geq 1$. Then we have

$$\begin{aligned} c_{m'+1} &= 2K^2(K+1)\left(4\xi_{-1} + 8\epsilon_0 + \frac{4}{\hat{M}^{\frac{1}{9}}} + c_{m'}\right) \\ &\leq 2K^2(K+1)\left(4\xi_{-1} + 8\epsilon_0 + \frac{4}{\hat{M}^{\frac{1}{9}}}\right) + 2^{2m'+1}K^{4m'+2}(K+1)^{2m'+1}\left(4\xi_{-1} + 8\epsilon_0 + \frac{4}{\hat{M}^{\frac{1}{9}}}\right) \\ &\leq 2^{2m'+2}K^{4m'+4}(K+1)^{2m'+2}\left(4\xi_{-1} + 8\epsilon_0 + \frac{4}{\hat{M}^{\frac{1}{9}}}\right). \end{aligned}$$

Hence, Equation (E.73) holds. Let $B_2(M) = 2^{6M^2}K^{12M^2}(K+1)^{6M^2}\left(4\xi_{-1} + 8\epsilon_0 + \frac{4}{\hat{M}^{\frac{1}{9}}}\right)$. Since $\xi_{-1} = \frac{1}{M^{M^M}}$, $\epsilon_0 = \frac{3M^2K(K+1)}{\hat{M}^{\frac{1}{3}}}$ and $\hat{M} > \left(\frac{1}{\xi_{3M^2+1}}\right)^3 \geq \left(\frac{1}{\xi_{-1}}\right)^3 \geq M^{3M^M}$, it is clear that $\lim_{M\to\infty} B_2(M) = 0$.

### E.5.8 Proof of Lemma E.7

Fix any $i, j \in I$ with $\hat{\pi}_i^0 \neq j$. It follows from Lemma E.21 that for any $m \in \{0, 1, \ldots, 3M^2\}$, $P_0\left(\hat{\pi}_i^m = j\right) \leq m\epsilon_0 + \frac{2m}{\hat{M}^{\frac{14}{15}}}$. Recall that $M \geq \max\{K\bar{a}, 3\}$, $\hat{M} \geq M^{3M^M}$ and $\epsilon_0 = \frac{3M^2K(K+1)}{\hat{M}^{\frac{1}{3}}}$. We can obtain the following estimation

$$\begin{aligned} P_0\left(\hat{\pi}_i^m = j\right) &\leq m\epsilon_0 + \frac{2m}{\hat{M}^{\frac{14}{15}}} \\ &\leq 3M^2\frac{3M^2K(K+1)}{\hat{M}^{\frac{1}{3}}} + \frac{6M^2}{\hat{M}^{\frac{14}{15}}} \\ &\leq 3M^2\frac{3M^2M2M}{\hat{M}^{\frac{1}{3}}} + \frac{6M^2}{\hat{M}^{\frac{1}{3}}} \\ &\leq \frac{24M^6}{\hat{M}^{\frac{1}{3}}} = \frac{24M^6}{\hat{M}^{\frac{1}{6}}}\frac{1}{\hat{M}^{\frac{1}{6}}} \\ &\leq \frac{24M^6}{M^{\frac{27}{2}}}\frac{1}{\hat{M}^{\frac{1}{6}}} = \frac{24}{M^{\frac{15}{2}}}\frac{1}{\hat{M}^{\frac{1}{6}}} < \frac{1}{\hat{M}^{\frac{1}{6}}}. \end{aligned}$$

For any $m \in \{0, 1, \ldots, 3M^2\}$, let $F_{ij}^m = \{\omega^m \in \Omega^m : \hat{\pi}_i^m(\omega) = j\}$; then we have $P_0\left(F_{ij}^m\right) \leq \frac{1}{\hat{M}^{\frac{1}{6}}}$.

We need to provide a sequence of estimations $\{d_m\}_{0 \leq m \leq 3M^2}$ such that for any $m \in \{0, 1, \ldots, 3M^2\}$, $a_1, a_2 \in \tilde{S}$, and $F_i^{m-1} \in \mathcal{F}_i^{m-1}$ and $F_j^{m-1} \in \mathcal{F}_j^{m-1}$, we have

$$\begin{aligned} &\Big| P_0\left(\tilde{\beta}_i^m = a_1, \tilde{\beta}_j^m = a_2, F_i^{m-1}, F_j^{m-1}\right) \\ &\quad - P_0\left(\tilde{\beta}_i^m = a_1, F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^m = a_2, F_j^{m-1}\right)\Big| \leq d_m. \end{aligned} \tag{E.74}$$

Fix any $m \in \{0, 1, \ldots, 3M^2\}$. When $m = 0$, we can take $d_0$ to be 0. Suppose that we have already defined $d_m$, we need to define $d_{m+1}$ using $d_m$.

Fix any $a_1, a_2, b_1, b_2 \in \tilde{S}$, $F_i^{m-1} \in \mathcal{F}_i^{m-1}$ and $F_j^{m-1} \in \mathcal{F}_j^{m-1}$. We first estimate the following difference

$$\Big| P_0 \left( \tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1} \right) \\ -P_0 \left( \tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, F_i^{m-1} \right) P_0 \left( \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1} \right) \Big| .$$

For notational simplicity, we let

$$\begin{aligned} A &= \left\{ \omega^m \in \Omega^m : \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2 \right\} \cap F_i^{m-1} \cap F_j^{m-1}, \\ A' &= \{ \omega^m \in \Omega^m : \tilde{\beta}_i^m = b_1 \} \cap F_i^{m-1}, \\ A'' &= \{ \omega^m \in \Omega^m : \tilde{\beta}_j^m = b_2 \} \cap F_j^{m-1}. \end{aligned}$$

We can obtain that

$$\begin{aligned} & P_0 \left( \tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1} \right) \\ = & \int_A Q_{m+1}^{\omega^m} \left( \tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2 \right) dQ^m. \end{aligned}$$

We first consider the case when $m = 3n - 2$ for some $n \in \mathbb{T}_0$, that is, the $m + 1$ step is the matching step in the $n$-th period. We start by assuming $b_1 = (k, l, 0)$. When $\tilde{\beta}_i^m = (k, l, 0)$ for some $k, l \in S$, agent $i$ is already matched at the mutation step in the $n$-th period. By the construction of the finite-agent dynamic matching model, paired agents do not change their expanded types in the matching step. When $a_1 \neq (k, l, 0)$, it is clear that

$$\begin{aligned} & P_0 \left( \tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = (k, l, 0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1} \right) \\ = & P_0 \left( \tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = (k, l, 0), F_i^{m-1} \right) P_0 \left( \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1} \right) = 0. \end{aligned} \tag{E.75}$$

When $a_1 = (k, l, 0)$, and $P_0 \left( \tilde{\beta}_i^m = (k, l, 0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1} \right) = 0$, the inductive hypothesis implies that

$$P_0 \left( \tilde{\beta}_i^m = (k, l, 0), F_i^{m-1} \right) P_0 \left( \tilde{\beta}_j^m = b_2, F_j^{m-1} \right) \leq d_m.$$

We can then obtain

$$\begin{aligned} & \Big| P_0 \left( \tilde{\beta}_i^{m+1} = (k, l, 0), \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = (k, l, 0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1} \right) \\ & \quad -P_0 \left( \tilde{\beta}_i^{m+1} = (k, l, 0), \tilde{\beta}_i^m = (k, l, 0), F_i^{m-1} \right) P_0 \left( \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1} \right) \Big| \\ = & P_0 \left( \tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = (k, l, 0), F_i^{m-1} \right) P_0 \left( \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1} \right) \\ \leq & P_0 \left( \tilde{\beta}_i^m = (k, l, 0), F_i^{m-1} \right) P_0 \left( \tilde{\beta}_j^m = b_2, F_j^{m-1} \right) \leq d_m. \end{aligned} \tag{E.76}$$

When $a_1 = (k,l,0)$, $P_0\left(\tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) > 0$, we have

$$\begin{aligned}
& P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\
= \; & P_0\left(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\
= \; & P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\
& \quad P_0\left(\tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right).
\end{aligned}$$

It follows from Lemma E.18 that

$$\begin{aligned}
& \left| P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) - P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_j^m = b_2\right) \right| \\
\leq \; & \frac{4K\epsilon_0}{P_0\left(\tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right)} + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
& \left| P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_j^m = b_2, F_j^{m-1}\right) - P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_j^m = b_2\right) \right| \\
\leq \; & \frac{4K\epsilon_0}{P_0\left(\tilde{\beta}_j^m = b_2, F_j^{m-1}\right)} + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}.
\end{aligned}$$

Then, the above inequalities imply that

$$\begin{aligned}
& \Big| P_0\left(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\
& \quad - P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_j^m = b_2\right) P_0\left(\tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \Big| \\
= \; & \left| P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) - P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_j^m = b_2\right) \right| \\
& \quad P_0\left(\tilde{\beta}_i^m = (k,l,0), \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\
\leq \; & 4K\epsilon_0 + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}, \text{ and}
\end{aligned}$$

$$\begin{aligned}
& \left| P_0\left(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1}\right) - P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_j^m = b_2\right) P_0\left(\tilde{\beta}_j^m = b_2, F_j^{m-1}\right) \right| \\
= \; & \left| P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_j^m = b_2, F_j^{m-1}\right) - P_0\left(\tilde{\beta}_j^{m+1} = a_2 \,|\, \tilde{\beta}_j^m = b_2\right) \right| P_0\left(\tilde{\beta}_j^m = b_2, F_j^{m-1}\right) \\
\leq \; & 4K\epsilon_0 + \frac{2K}{\hat{M}^{\frac{1}{9}}} + 2K\xi_{-1}.
\end{aligned}$$

By the above two inequalities and the inductive hypothesis, we can obtain that

$$\begin{aligned}
& \Big|P_0\left(\tilde{\beta}_i^{m+1}=(k,l,0),\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=(k,l,0),\tilde{\beta}_j^{m}=b_2,F_i^{m-1},F_j^{m-1}\right) \\
& \qquad -P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=(k,l,0),F_i^{m-1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,F_j^{m-1}\right)\Big| \\
= \;& \Big|P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=(k,l,0),\tilde{\beta}_j^{m}=b_2,F_i^{m-1},F_j^{m-1}\right) \\
& \qquad -P_0\left(\tilde{\beta}_j^{m+1}=a_2\,|\,\tilde{\beta}_j^{m}=b_2\right)P_0\left(\tilde{\beta}_i^{m}=(k,l,0),\tilde{\beta}_j^{m}=b_2,F_i^{m-1},F_j^{m-1}\right)\Big| \\
& \quad +\Big|P_0\left(\tilde{\beta}_j^{m+1}=a_2\,|\,\tilde{\beta}_j^{m}=b_2\right)P_0\left(\tilde{\beta}_i^{m}=(k,l,0),\tilde{\beta}_j^{m}=b_2,F_i^{m-1},F_j^{m-1}\right) \\
& \qquad -P_0\left(\tilde{\beta}_j^{m+1}=a_2\,|\,\tilde{\beta}_j^{m}=b_2\right)P_0\left(\tilde{\beta}_i^{m}=(k,l,0),F_i^{m-1}\right)P_0\left(\tilde{\beta}_j^{m}=b_2,F_j^{m-1}\right)\Big| \\
& \quad +\Big|P_0\left(\tilde{\beta}_j^{m+1}=a_2\,|\,\tilde{\beta}_j^{m}=b_2\right)P_0\left(\tilde{\beta}_i^{m}=(k,l,0),F_i^{m-1}\right)P_0\left(\tilde{\beta}_j^{m}=b_2,F_j^{m-1}\right) \\
& \qquad -P_0\left(\tilde{\beta}_i^{m}=(k,l,0),F_i^{m-1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,F_j^{m-1}\right)\Big| \\
\le \;& 4K\epsilon_0+\frac{2K}{\hat{M}^{\frac{1}{9}}}+2K\xi_{-1}+d_m+4K\epsilon_0+\frac{2K}{\hat{M}^{\frac{1}{9}}}+2K\xi_{-1} \\
= \;& 8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{9}}}+4K\xi_{-1}+d_m. \qquad \text{(E.77)}
\end{aligned}$$

Next, we assume that $b_1=(k,J,0)$ or $(k,l,1)$ for some $k,l\in S$. It is clear that $P_0\left(\tilde{\beta}_i^{m}=b_1\right)=0$. Then we have

$$\begin{aligned}
& P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_j^{m}=b_2,F_i^{m-1},F_j^{m-1}\right) \\
= \;& P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=b_1,F_i^{m-1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,F_j^{m-1}\right) \\
= \;& 0. \qquad \text{(E.78)}
\end{aligned}$$

One can exchange the positions of $i$ and $j$ to obtain exactly the same estimations as in Equations (E.75), (E.76), (E.77) and (E.78), when $i$ is replaced by $j$, and the conditions on $b_1$ are restated on $b_2$. Thus, for the matching step, it remains to consider $b_1=(k_1,J,1)$ and $b_2=(k_2,J,1)$ for some $k_1,k_2\in S$. In this case, if $a_1$ is neither $(k_1,J,1)$ nor $(k_1,l,1)$ $(l\in S)$, then we must have $P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=b_1,F_i^{m-1}\right)=0$, which implies the identities in Equation (E.78). By the same reason, Equation (E.78) also holds when $a_2$ is neither $(k_2,J,1)$ nor $(k_2,l,1)$ $(l\in S)$. Hence, we only need to consider $a_1=(k_1,l_1,1)$ and $a_2=(k_2,l_2,1)$ for some $l_1,l_2\in S\cup\{J\}$.

In this paragraph, we work with $a_1=(k_1,l_1,1)$, $a_2=(k_2,l_2,1)$, $b_1=(k_1,J,1)$, and $b_2=(k_2,J,1)$ for some $k_1,k_2\in S$, and $l_1,l_2\in S\cup\{J\}$. The inequalities in Lemma E.16 give symmetric treatment the cases for $l\in S$ or $l=J$. For the simplicity of applying this lemma, we introduce the notation $\hat{q}_{kJ}$ to represent $\hat{q}_k$ in the rest of the proof for Lemma E.7. By Lemmas

E.14 and E.16, we have

$$\begin{aligned}
&\Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_j^{m}=b_2,F_i^{m-1},F_j^{m-1}\right)\\
&\qquad-\int_A \hat{q}_{k_1l_1}(\tilde{\rho}^m)\hat{q}_{k_2l_2}(\tilde{\rho}^m)dQ^m\Big|\\
=&\ \left|\int_A\left(Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2)-\hat{q}_{k_1l_1}(\tilde{\rho}^m)\hat{q}_{k_2l_2}(\tilde{\rho}^m)\right)dQ^m\right|\\
\leq&\ \int_{A\backslash V^m}\left|Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2)-\hat{q}_{k_1l_1}(\tilde{\rho}^m)\hat{q}_{k_2l_2}(\tilde{\rho}^m)\right|dQ^m+Q^m(V^m)\\
\leq&\ \frac{1}{\hat{M}^{\frac{1}{9}}}+\epsilon_0.
\end{aligned}\tag{E.79}$$

Next, we estimate the difference

$$\begin{aligned}
&\left|\int_A \hat{q}_{k_1l_1}(\tilde{\rho}^m)\hat{q}_{k_2l_2}(\tilde{\rho}^m)dQ^m-P_0(A)\hat{q}_{k_1l_1}\left(U_1^m(\tilde{\rho}^0)\right)\hat{q}_{k_2l_2}\left(U_1^m(\tilde{\rho}^0)\right)\right|\\
\leq&\ \int_A\left|\hat{q}_{k_1l_1}(\tilde{\rho}^m)\hat{q}_{k_2l_2}(\tilde{\rho}^m)-\hat{q}_{k_1l_1}\left(U_1^m(\tilde{\rho}^0)\right)\hat{q}_{k_2l_2}\left(U_1^m(\tilde{\rho}^0)\right)\right|dQ^m\\
\leq&\ \int_{A\backslash V^m}\left|\hat{q}_{k_1l_1}(\tilde{\rho}^m)\hat{q}_{k_2l_2}(\tilde{\rho}^m)-\hat{q}_{k_1l_1}\left(U_1^m(\tilde{\rho}^0)\right)\hat{q}_{k_2l_2}\left(U_1^m(\tilde{\rho}^0)\right)\right|dQ^m+Q^m(V^m)\\
=&\ \int_{A\backslash V^m}\left|\hat{q}_{k_1l_1}(\tilde{\rho}^m)\hat{q}_{k_2l_2}(\tilde{\rho}^m)-\hat{q}_{k_1l_1}\left(U_1^m(\tilde{\rho}^0)\right)\hat{q}_{k_2l_2}(\tilde{\rho}^m)\right|dQ^m\\
&+\int_{A\backslash V^m}\left|\hat{q}_{k_1l_1}\left(U_1^m(\tilde{\rho}^0)\right)\hat{q}_{k_2l_2}(\tilde{\rho}^m)-\hat{q}_{k_1l_1}\left(U_1^m(\tilde{\rho}^0)\right)\hat{q}_{k_2l_2}\left(U_1^m(\tilde{\rho}^0)\right)\right|dQ^m+Q^m(V^m)\\
\leq&\ \int_{A\backslash V^m}\left|\hat{q}_{k_1l_1}(\tilde{\rho}^m)-\hat{q}_{k_1l_1}\left(U_1^m(\tilde{\rho}^0)\right)\right|dQ^m\\
&+\int_{A\backslash V^m}\left|\hat{q}_{k_2l_2}(\tilde{\rho}^m)-\hat{q}_{k_2l_2}\left(U_1^m(\tilde{\rho}^0)\right|dQ^m+Q^m(V^m).
\end{aligned}$$

By Lemma E.14, for any $\omega^m\notin V^m$, we have $||\tilde{\rho}^m(\omega^m)-U_1^m(\tilde{\rho}^0)||_\infty\leq\xi_0$. Lemma E.2 implies that for any $\omega^m\notin V^m$,

$$\left|\hat{q}_{k_1l_1}(\tilde{\rho}^m(\omega^m))-\hat{q}_{k_1l_1}\left(U_1^m(\tilde{\rho}^0)\right)\right|\leq\xi_{-1}\text{ and }\left|\hat{q}_{k_2l_2}(\tilde{\rho}^m(\omega^m))-\hat{q}_{k_2l_2}\left(U_1^m(\tilde{\rho}^0)\right)\right|\leq\xi_{-1}.$$

It follows from the above inequalities and Lemma E.14 that

$$\begin{aligned}
&\left|\int_A \hat{q}_{k_1l_1}(\tilde{\rho}^m)\hat{q}_{k_2l_2}(\tilde{\rho}^m)dQ^m-P_0(A)\hat{q}_{k_1l_1}\left(U_1^m(\tilde{\rho}^0)\right)\hat{q}_{k_2l_2}\left(U_1^m(\tilde{\rho}^0)\right)\right|\\
\leq&\ 2\xi_{-1}+\epsilon_0
\end{aligned}\tag{E.80}$$

By Equations (E.79) and (E.80), we have

$$\begin{aligned}
&\Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_j^{m}=b_2,F_i^{m-1},F_j^{m-1}\right)\\
&\qquad-P_0(A)\hat{q}_{k_1l_1}\left(U_1^m(\tilde{\rho}^0)\right)\hat{q}_{k_2l_2}\left(U_1^m(\tilde{\rho}^0)\right)\Big|\leq 2\epsilon_0+\frac{1}{\hat{M}^{\frac{1}{9}}}+2\xi_{-1}.
\end{aligned}\tag{E.81}$$

It follows from Lemmas E.14 and E.16 that

$$\begin{aligned}
& \left| P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, F_i^{m-1}\right) - \int_{A'} \hat{q}_{k_1 l_1}(\tilde{\rho}^m) dQ^m \right| \\
= & \left| \int_{A'} \left( Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1) - \hat{q}_{k_1 l_1}(\tilde{\rho}^m) \right) dQ^m \right| \\
\leq & \int_{A' \setminus V^m} \left| Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1) - \hat{q}_{k_1 l_1}(\tilde{\rho}^m) \right| dQ^m + Q^m(V^m) \\
\leq & \frac{1}{\hat{M}^{\frac{1}{9}}} + \epsilon_0. \qquad \text{(E.82)}
\end{aligned}$$

Next, we estimate the difference

$$\begin{aligned}
& \left| \int_{A'} \hat{q}_{k_1 l_1}(\tilde{\rho}^m) dQ^m - P_0(A') \hat{q}_{k_1 l_1}\left(U_1^m(\tilde{\rho}^0)\right) \right| \\
\leq & \int_{A'} \left| \hat{q}_{k_1 l_1}(\tilde{\rho}^m) - \hat{q}_{k_1 l_1}\left(U_1^m(\tilde{\rho}^0)\right) \right| dQ^m \\
\leq & \int_{A' \setminus V^m} \left| \hat{q}_{k_1 l_1}(\tilde{\rho}^m) - \hat{q}_{k_1 l_1}\left(U_1^m(\tilde{\rho}^0)\right) \right| dQ^m + Q^m(V^m).
\end{aligned}$$

It follows from Lemma E.14 that for any $\omega^m \notin V^m$, $||\tilde{\rho}^m(\omega^m) - U_1^m(\tilde{\rho}^0)||_\infty \leq \xi_0$. Lemma E.2 implies that for any $\omega^m \notin V^m$, $\left|\hat{q}_{k_1 l_1}(\tilde{\rho}^m)(\omega^m) - \hat{q}_{k_1 l_1}\left(U_1^m(\tilde{\rho}^0)\right)\right| \leq \xi_{-1}$. It is then obvious that

$$\begin{aligned}
& \left| \int_{A'} \hat{q}_{k_1 l_1}(\tilde{\rho}^m) - P_0(A') \hat{q}_{k_1 l_1}\left(U_1^m(\tilde{\rho}^0)\right) \right| \\
\leq & \xi_{-1} + \epsilon_0 \qquad \text{(E.83)}
\end{aligned}$$

By combining Equations (E.82) and (E.83), we have

$$\begin{aligned}
& \left| P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, F_i^{m-1}\right) - P_0(A') \hat{q}_{k_1 l_1}\left(U_1^m(\tilde{\rho}^0)\right) \right| \\
\leq & 2\epsilon_0 + \frac{1}{\hat{M}^{\frac{1}{9}}} + \xi_{-1}. \qquad \text{(E.84)}
\end{aligned}$$

Equation (E.84) states an inequality for a general agent $i$, which can be restated for agent $j$ as follows:

$$\begin{aligned}
& \left| P_0\left(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1}\right) - P_0(A'') \hat{q}_{k_2 l_2}\left(U_1^m(\tilde{\rho}^0)\right) \right| \\
\leq & 2\epsilon_0 + \frac{1}{\hat{M}^{\frac{1}{9}}} + \xi_{-1}, \qquad \text{(E.85)}
\end{aligned}$$

Based on Equations (E.84) and (E.85), we can obtain that

$$\begin{aligned}
& \Big| P_0\left(\tilde{\beta}_i^{m+1}=a_1, \tilde{\beta}_i^m=b_1, F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_j^m=b_2, F_j^{m-1}\right) \\
& \quad -P_0(A') P_0(A'') \hat{q}_{k_1 l_1}\left(U_1^m(\tilde{\rho}^0)\right) \hat{q}_{k_2 l_2}\left(U_1^m(\tilde{\rho}^0)\right) \big| \\
\leq\ & \Big| P_0(\tilde{\beta}_i^{m+1}=a_1, \tilde{\beta}_i^m=b_1, F_i^{m-1}) P_0(\tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_j^m=b_2, F_j^{m-1}) \\
& \quad -P_0(A') \hat{q}_{k_1 l_1}\left(U_1^m(\tilde{\rho}^0)\right) P_0(\tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_j^m=b_2, F_j^{m-1}) \Big| \\
& +\Big| P_0(A') \hat{q}_{k_1 l_1}(U_1^m(\tilde{\rho}^0)) P_0(\tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_j^m=b_2, F_j^{m-1}) \\
& \quad -P_0(A') P_0(A'') \hat{q}_{k_1 l_1}(U_1^m(\tilde{\rho}^0)) \hat{q}_{k_2 l_2}(U_1^m(\tilde{\rho}^0))\ | \\
\leq\ & \Big| P_0(\tilde{\beta}_i^{m+1}=a_1, \tilde{\beta}_i^m=b_1, F_i^{m-1})-P_0(A') \hat{q}_{k_1 l_1}\left(U_1^m(\tilde{\rho}^0)\right) \Big| \\
& +\Big| P_0(\tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_j^m=b_2, F_j^{m-1})-P_0(A'') \hat{q}_{k_2 l_2}(U_1^m(\tilde{\rho}^0))\ | \\
\leq\ & 4\epsilon_0+\frac{2}{\hat{M}^{\frac{1}{9}}}+2\xi_{-1},
\end{aligned} \tag{E.86}$$

The induction hypothesis indicates that $|P_0(A)-P_0(A')P_0(A'')| \leq d_m$. By Equations (E.81) and (E.86), we have

$$\begin{aligned}
& \Big| P_0\left(\tilde{\beta}_i^{m+1}=a_1, \tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_i^m=b_1, \tilde{\beta}_j^m=b_2, F_i^{m-1}, F_j^{m-1}\right) \\
& \quad -P_0\left(\tilde{\beta}_i^{m+1}=a_1, \tilde{\beta}_i^m=b_1, F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_j^m=b_2, F_j^{m-1}\right) \Big| \\
\leq\ & \left| P_0(A) \hat{q}_{k_1 l_1}(U_1^m(\tilde{\rho}^0)) \hat{q}_{k_2 l_2}(U_1^m(\tilde{\rho}^0))-P_0(A') P_0(A'') \hat{q}_{k_1 l_1}(U_1^m(\tilde{\rho}^0)) \hat{q}_{k_2 l_2}(U_1^m(\tilde{\rho}^0)) \right| \\
& \quad +6\epsilon_0+\frac{3}{\hat{M}^{\frac{1}{9}}}+4\xi_{-1} \\
\leq\ & \left| P_0(A)-P_0(A') P_0(A'') \right|+6\epsilon_0+\frac{3}{\hat{M}^{\frac{1}{9}}}+4\xi_{-1} \\
\leq\ & 6\epsilon_0+\frac{3}{\hat{M}^{\frac{1}{9}}}+4\xi_{-1}+d_m.
\end{aligned} \tag{E.87}$$

By Equations (E.76), (E.77) and (E.87), we know that for $m=3n-2$, and for any $a_1, a_2, b_1, b_2 \in \tilde{S}$, $F_i^{m-1} \in \mathcal{F}_i^{m-1}$ and $F_j^{m-1} \in \mathcal{F}_j^{m-1}$,

$$\begin{aligned}
& \Big| P_0\left(\tilde{\beta}_i^{m+1}=a_1, \tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_i^m=b_1, \tilde{\beta}_j^m=b_2, F_i^{m-1}, F_j^{m-1}\right) \\
& \quad -P_0\left(\tilde{\beta}_i^{m+1}=a_1, \tilde{\beta}_i^m=(k, l, 0), F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^{m+1}=a_2, \tilde{\beta}_j^m=b_2, F_j^{m-1}\right) \Big| \\
\leq\ & 8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{9}}}+4K\xi_{-1}+d_m.
\end{aligned} \tag{E.88}$$

Next, we consider the case when $m=3n-3$ for some $n \in \mathbb{T}_0$, that is, the $m+1$ step is the mutation step in the $n$-th period. Let

$$B_{ab}=\begin{cases} \hat{\eta}_{k_1 l_1} \hat{\eta}_{k_2 l_2} & \text{if } a=(k_1, k_2, 0), b=(l_1, l_2, 0) \\ \hat{\eta}_{k_1 l_1} & \text{if } a=(k_1, J, 1), b=(l_1, J, 1) \\ 0 & \text{otherwise.} \end{cases}$$

By the construction of the mutation step in the finite-agent dynamic matching model, we know that if $\hat{\pi}_i^m(\omega^m) \neq j$, $\tilde{\beta}_i^m(\omega^m) = b_1$, $\tilde{\beta}_j^m(\omega^m) = b_2$,

$$Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2) = Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1) Q_{m+1}^{\omega^m}(\tilde{\beta}_j^{m+1} = a_2) = B_{b_1 a_1} B_{b_2 a_2}.$$

Recall from the beginning of this proof that $F_{ij}^m = \{\omega^m \in \Omega^m : \hat{\pi}_i^m(\omega) = j\}$ and $P_0\left(F_{ij}^m\right) \leq \frac{1}{\hat{M}^{\frac{1}{11}}}$. It then follows from Lemma E.14 that

$$\begin{aligned}
& \left| P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) - P_0(A) B_{b_1 a_1} B_{b_2 a_2} \right| \\
\leq\; & \int_A \left| Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2) - B_{b_1 a_1} B_{b_2 a_2} \right| dQ^m \\
=\; & \int_{A \setminus (F_{ij}^m \cup V^m)} \left| B_{b_1 a_1} B_{b_2 a_2} - B_{b_1 a_1} B_{b_2 a_2} \right| dQ^m \\
& \quad + \int_{A \cap (F_{ij}^m \cup V^m)} \left| Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2) - B_{b_1 a_1} B_{b_2 a_2} \right| dQ^m \\
\leq\; & P_0(F_{ij}^m \cup V^m) \leq P_0(F_{ij}^m) + P_0(V^m) \leq \frac{1}{\hat{M}^{\frac{1}{6}}} + \epsilon_0. \qquad \text{(E.89)}
\end{aligned}$$

By the construction of the mutation step in the finite-agent dynamic matching model again, we have

$$\begin{aligned}
& P_0(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, F_i^{m-1}) = P_0(A') B_{b_1 a_1}, \text{ and} \\
& P_0(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1}) = P_0(A'') B_{b_2 a_2}.
\end{aligned}$$

It follows from the above identities, the induction hypothesis $\left| P_0(A) - P_0(D_{b_1}^i) P_0(D_{b_2}^j) \right| \leq d_m$, and Equation (E.89) that

$$\begin{aligned}
& \Big| P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_i^m = b_1, \tilde{\beta}_j^m = b_2, F_i^{m-1}, F_j^{m-1}\right) \\
& \qquad - P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_i^m = b_1, F_i^{m-1}\right) P_0\left(\tilde{\beta}_j^{m+1} = a_2, \tilde{\beta}_j^m = b_2, F_j^{m-1}\right) \Big| \\
\leq\; & \left| P_0(A) B_{b_1 a_1} B_{b_2 a_2} - P_0(A') B_{b_1 a_1} P_0(A'') B_{b_2 a_2} \right| + \frac{1}{\hat{M}^{\frac{1}{11}}} + \epsilon_0 \\
=\; & \left| P_0(A) - P_0(A') P_0(A'') \right| B_{b_1 a_1} B_{b_2 a_2} + \frac{1}{\hat{M}^{\frac{1}{11}}} + \epsilon_0 \\
\leq\; & \epsilon_0 + \frac{1}{\hat{M}^{\frac{1}{6}}} + d_m. \qquad \text{(E.90)}
\end{aligned}$$

It remains to consider the case when $m = 3n - 1$ for some $n \in \mathbb{T}_0$, that is, the $m + 1$ step is the type changing with break-up step in the $n$-th period. Though the proof of this part is similar to the case of the mutation step, we present a full proof for the sake of completeness.

Let

$$C_{ab}=\begin{cases}1-\hat{\vartheta}_{k_1k_2} & \text{if } a=(k_1,k_2,0), b=(k_1,k_2,0)\\ \hat{\vartheta}_{k_1k_2}\varsigma_{k_1k_2}(l_1) & \text{if } a=(k_1,k_2,0), b=(l_1,J,1)\\ \hat{\xi}_{k_1k_2}\hat{\sigma}_{k_1k_2}(l_1,l_2) & \text{if } a=(k_1,k_2,1), b=(l_1,l_2,0)\\ (1-\hat{\xi}_{k_1k_2})\hat{\varsigma}_{k_1k_2}(l_1) & \text{if } a=(k_1,k_2,1), b=(l_1,J,1)\\ 1 & \text{if } a=(k_1,J,1), b=(k_1,J,1)\\ 0 & \text{otherwise.}\end{cases}$$

By the construction of the type changing with break-up step in the finite-agent dynamic matching model, we know that if $\hat{\pi}_i^m(\omega^m)\neq j$, $\tilde{\beta}_i^m(\omega^m)=b_1$, $\tilde{\beta}_j^m(\omega^m)=b_2$,

$$Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2)=Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1}=a_1)Q_{m+1}^{\omega^m}(\tilde{\beta}_j^{m+1}=a_2)=C_{b_1a_1}C_{b_2a_2}.$$

Lemma E.14 implies that

$$\begin{aligned}
&\left|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^m=b_1,\tilde{\beta}_j^m=b_2,F_i^{m-1},F_j^{m-1}\right)-P_0(A)C_{b_1a_1}C_{b_2a_2}\right|\\
\leq\ & \int_A\left|Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2)-C_{b_1a_1}C_{b_2a_2}\right|dQ^m\\
=\ & \int_{A\setminus(F_{ij}^m\cup V^m)}\left|C_{b_1a_1}C_{b_2a_2}-C_{b_1a_1}C_{b_2a_2}\right|dQ^m\\
&\quad+\int_{A\cap(F_{ij}^m\cup V^m)}\left|Q_{m+1}^{\omega^m}(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2)-C_{b_1a_1}C_{b_2a_2}\right|dQ^m\\
\leq\ & P_0(F_{ij}^m\cup V^m)\leq P_0(F_{ij}^m)+P_0(V^m)\leq\frac{1}{\hat{M}^{\frac{1}{6}}}+\epsilon_0.
\end{aligned}\tag{E.91}$$

By the construction of the type changing with break-up step in the finite-agent dynamic matching model again, we have

$$\begin{aligned}
&P_0(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^m=b_1,F_i^{m-1})=P_0(A')C_{b_1a_1},\ \text{and}\\
&P_0(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^m=b_2,F_j^{m-1})=P_0(A'')C_{b_2a_2}.
\end{aligned}$$

By the above identities, the induction hypothesis $\left|P_0(A)-P_0(D_{b_1}^i)P_0(D_{b_2}^j)\right|\leq d_m$, and Equation (E.91), we obtain that

$$\begin{aligned}
&\Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^m=b_1,\tilde{\beta}_j^m=b_2,F_i^{m-1},F_j^{m-1}\right)\\
&\qquad-P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^m=b_1,F_i^{m-1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^m=b_2,F_j^{m-1}\right)\Big|\\
\leq\ & \left|P_0(A)C_{b_1a_1}C_{b_2a_2}-P_0(A')C_{b_1a_1}\,P_0(A'')C_{b_2a_2}\right|+\frac{1}{\hat{M}^{\frac{1}{6}}}+\epsilon_0\\
=\ & \left|P_0(A)-P_0(A')P_0(A'')\right|C_{b_1a_1}C_{b_2a_2}+\frac{1}{\hat{M}^{\frac{1}{6}}}+\epsilon_0\\
\leq\ & \epsilon_0+\frac{1}{\hat{M}^{\frac{1}{6}}}+d_m.
\end{aligned}\tag{E.92}$$

By combining Equations (E.88), (E.90), (E.92), we obtain that

$$\begin{aligned}
&\Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_j^{m}=b_2,F_i^{m-1},F_j^{m-1}\right)\\
&-P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=b_1,F_i^{m-1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,F_j^{m-1}\right)\Big|\\
\leq\;& 8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{9}}}+4K\xi_{-1}+d_m. \qquad \text{(E.93)}
\end{aligned}$$

Fix any $F_i^m\in\mathcal{F}_i^m$. There exist $F_{ib}^{m-1}\in\mathcal{F}_i^{m-1}$ for $b\in\tilde{S}$ such that $F_i^m=\bigcup_{b\in\tilde{S}}\left(\left(\tilde{\beta}_i^m=b\right)\cap F_{ib}^{m-1}\right)$. Similarly, for any fixed $F_j^m\in\mathcal{F}_j^m$, there exist $F_{jb}^{m-1}\in\mathcal{F}_j^{m-1}$ for $b\in\tilde{S}$ such that $F_j^m=\bigcup_{b\in\tilde{S}}\left(\left(\tilde{\beta}_j^m=b\right)\cap F_{jb}^{m-1}\right)$. Therefore, by Equation (E.93), we can obtain that

$$\begin{aligned}
&\Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,F_i^{m},F_j^{m}\right)\\
&\quad-P_0\left(\tilde{\beta}_i^{m+1}=a_1,F_i^{m}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,F_j^{m}\right)\Big|\\
\leq\;&\sum_{b_1,b_2\in\tilde{S}}\Big|P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_i^{m}=b_1,\tilde{\beta}_j^{m}=b_2,F_{ib_1}^{m-1},F_{jb_2}^{m-1}\right)\\
&\quad-P_0\left(\tilde{\beta}_i^{m+1}=a_1,\tilde{\beta}_i^{m}=b_1,F_{ib_1}^{m-1}\right)P_0\left(\tilde{\beta}_j^{m+1}=a_2,\tilde{\beta}_j^{m}=b_2,F_{jb_2}^{m-1}\right)\Big|\\
\leq\;&4K^2(K+1)^2\left(8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{9}}}+4K\xi_{-1}+d_m\right). \qquad \text{(E.94)}
\end{aligned}$$

Thus, we can define $d_{m+1}$ to be $4K^2(K+1)^2\left(8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{9}}}+4K\xi_{-1}+d_m\right)$.

Next, we prove that for any $m\in\{0,2,\ldots,3M^2\}$,

$$d_m\leq 2^{4m}K^{4m}(K+1)^{4m}\left(8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{9}}}+4K\xi_{-1}\right). \qquad \text{(E.95)}$$

Since $d_0=0$, it is clear that Equation (E.95) holds for $m=0$. Suppose that Equation (E.95) holds for $m=m'$. Then

$$\begin{aligned}
&d_{m'+1}=4K^2(K+1)^2\left(8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{9}}}+4K\xi_{-1}+d_{m'}\right)\\
\leq\;&4K^2(K+1)^2\left(8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{9}}}+4K\xi_{-1}\right)+2^{4m'+2}K^{4m'+2}(K+1)^{4m'+2}\left(8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{9}}}+4K\xi_{-1}\right)\\
\leq\;&2^{4m'+4}K^{4m'+4}(K+1)^{4m'+4}\left(8K\epsilon_0+\frac{4K}{\hat{M}^{\frac{1}{9}}}+4K\xi_{-1}\right).
\end{aligned}$$

Therefore, Equation (E.95) holds for any $m\in\{0,2,\ldots,3M^2\}$ by mathematical induction.

Fix any $F_i^m \in \mathcal{F}_i^m$ and $F_j^m \in \mathcal{F}_j^m$. Equation (E.94) implies that

$$\begin{aligned}
& \left|P_0\left(F_i^m \cap F_j^m\right) - P_0\left(F_i^m\right) P_0\left(F_j^m\right)\right| \\
= & \left|\sum_{a_1,a_2\in\tilde{S}} P_0\left(\tilde{\beta}_i^{m+1} = a_1, \tilde{\beta}_j^{m+1} = a_2, F_i^m, F_j^m\right)\right. \\
& \left. - \sum_{a_1,a_2\in\tilde{S}} P_0\left(\tilde{\beta}_i^{m+1} = a_1, F_i^m\right) P_0\left(\tilde{\beta}_j^{m+1} = a_2, F_j^m\right)\right| \\
\leq & 4K^2(K+1)^2 d_{m+1} \\
\leq & 4K^2(K+1)^2 2^{12M^2+4} K^{12M^2+4} (K+1)^{12M^2+4} \left(8K\epsilon_0 + \frac{4K}{\hat{M}^{\frac{1}{9}}} + 4K\xi_{-1}\right) \\
\leq & 2^{12M^2+6} K^{12M^2+6} (K+1)^{12M^2+6} \left(8K\epsilon_0 + \frac{4K}{\hat{M}^{\frac{1}{9}}} + 4K\xi_{-1}\right).
\end{aligned}$$

Let $B_3(M) = 2^{12M^2+6} K^{12M^2+6} (K+1)^{12M^2+6} \left(8K\epsilon_0 + \frac{4K}{\hat{M}^{\frac{1}{9}}} + 4K\xi_{-1}\right)$. Since $\xi_{-1} = \frac{1}{M^{M^M}}$, $\epsilon_0 = \frac{3M^2K(K+1)}{\hat{M}^{\frac{1}{3}}}$ and $\hat{M} > \left(\frac{1}{\xi_{3M^2+1}}\right)^3 \geq \left(\frac{1}{\xi_{-1}}\right)^3 \geq M^{3M^M}$, it is clear that $\lim_{M\to\infty} B_3(M) = 0$. Hence, Lemma E.7 is proved.

### E.5.9 Proof of Lemma E.8

Fix any $i \in I$, $m, \Delta m \in \{0, \ldots, 3M^2\}$ and $F^m \in \mathcal{F}^m$ such that $m + \Delta m \leq 3M^2$, $\frac{\Delta m}{M}$ is finite, and $P_0(F^m) > 0$.

We first consider the case when $m+\Delta m = 3n-2$ for some $n \in \mathbb{T}_0$. Fix any $\omega^{3n-3} \in \Omega^{3n-3}$. Denote $\hat{\alpha}_i^{3n-3}(\omega^{3n-3})$ and $\hat{g}_i^{3n-3}(\omega^{3n-3})$ by $k \in S$ and $l \in S \cup \{J\}$ respectively. If $l \neq J$, by the construction of the mutation step in the finite-agent dynamic matching model, we have

$$Q_{3n-2}^{\omega^{3n-3}}\left(\hat{X}_i^{3n-2} = \hat{X}_i^{3n-3}\right) = \hat{\eta}_{kk}\hat{\eta}_{ll} \geq \left(1 - \frac{K\bar{a}}{M}\right)^2.$$

If $l = J$, we have

$$Q_{3n-2}^{\omega^{3n-3}}\left(\hat{X}_i^{3n-2} = \hat{X}_i^{3n-3}\right) = \hat{\eta}_{kk} \geq 1 - \frac{K\bar{a}}{M} \geq \left(1 - \frac{K\bar{a}}{M}\right)^2.$$

Let $A^{3n-3} = \left(\hat{X}_i^{3n-3} = \hat{X}_i^m\right) \cap F^m$. If $P_0(A^{3n-3}) > 0$, then

$$\begin{aligned}
& P_0(\hat{X}_i^{3n-2} = \hat{X}_i^{3n-3} | \hat{X}_i^{3n-3} = \hat{X}_i^m, F^m) \\
= & \frac{\int_{A^{3n-3}} Q_{3n-2}^{\omega^{3n-3}}\left(\hat{X}_i^{3n-2} = \hat{X}_i^{3n-3}\right) dQ^{3n-3}}{P_0(A^{3n-3})} \\
\geq & \frac{\int_{A^{3n-3}} \left(1 - \frac{K\bar{a}}{M}\right)^2 dQ^{3n-3}}{P_0(A^{3n-3})} \\
= & \left(1 - \frac{K\bar{a}}{M}\right)^2.
\end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} & P_0(\hat{X}_i^{3n-2} = \hat{X}_i^m | F^m) \\ = \;& P_0(\hat{X}_i^{3n-3} = \hat{X}_i^m | F^m) P_0(\hat{X}_i^{3n-2} = \hat{X}_i^{3n-3} | \hat{X}_i^{3n-3} = \hat{X}_i^m, F^m) \\ \geq \;& P_0(\hat{X}_i^{3n-3} = \hat{X}_i^m | F^m) \left(1 - \frac{K\bar{a}}{M}\right)^2. \end{aligned} \tag{E.96}$$

If $P_0(\hat{X}_i^{3n-3} = \hat{X}_i^m, F^m) = 0$, then the above inequality is trivially satisfied.

Next, we consider the case when $m + \Delta m = 3n - 1$ for some $n \in \mathbb{T}_0$. Fix any $\omega^{3n-2} \in \Omega^{3n-2}$. Denote $\hat{\alpha}_i^{3n-2}(\omega^{3n-2})$ and $\hat{g}_i^{3n-2}(\omega^{3n-2})$ by $k \in S$ and $l \in S \cup \{J\}$ respectively. The construction of the matching step in the finite-agent dynamic matching model and Lemma E.1 allows us to claim that

$$Q_{3n-1}^{\omega^{3n-2}} \left(\hat{X}_i^{3n-1} = \hat{X}_i^{3n-2}\right) = 1, \quad \text{if } l \neq J,$$

$$Q_{3n-1}^{\omega^{3n-2}} \left(\hat{X}_i^{3n-1} > \hat{X}_i^{3n-2}\right) \leq \sum_{l' \in S} \hat{q}_{kl'} \leq \frac{K\bar{a}}{M}, \quad \text{if } l = J.$$

It is then clear that

$$Q_{3n-1}^{\omega^{3n-2}} \left(\hat{X}_i^{3n-1} = \hat{X}_i^{3n-2}\right) \geq 1 - \sum_{l' \in S} \hat{q}_{kl'} \geq 1 - \frac{K\bar{a}}{M},$$

Let $A^{3n-2} = \left(\hat{X}_i^{3n-2} = \hat{X}_i^m\right) \cap F^m$. If $P_0(A^{3n-2}) > 0$, then

$$\begin{aligned} & P_0(\hat{X}_i^{3n-1} = \hat{X}_i^{3n-2} | \hat{X}_i^{3n-2} = \hat{X}_i^m, F^m) \\ = \;& \frac{\int_{A^{3n-2}} Q_{3n-1}^{\omega^{3n-2}} \left(\hat{X}_i^{3n-1} = \hat{X}_i^{3n-2}\right) dQ^{3n-2}}{P_0(A^{3n-2})} \\ \geq \;& \frac{\int_{A^{3n-2}} \left(1 - \frac{K\bar{a}}{M}\right) dQ^{3n-2}}{P_0(A^{3n-2})} \\ = \;& \left(1 - \frac{K\bar{a}}{M}\right). \end{aligned}$$

Hence, we can derive the following estimation

$$\begin{aligned} & P_0(\hat{X}_i^{3n-1} = \hat{X}_i^m | F^m) \\ = \;& P_0(\hat{X}_i^{3n-2} = \hat{X}_i^m | F^m) P_0(\hat{X}_i^{3n-1} = \hat{X}_i^{3n-2} | \hat{X}_i^{3n-2} = \hat{X}_i^m, F^m) \\ \geq \;& P_0(\hat{X}_i^{3n-2} = \hat{X}_i^m | F^m) \left(1 - \frac{K\bar{a}}{M}\right). \end{aligned} \tag{E.97}$$

If $P_0(\hat{X}_i^{3n-2} = \hat{X}_i^m, F^m) = 0$, then the above inequality is trivially satisfied.

It remains to consider the case when $m + \Delta m = 3n$ for some $n \in \mathbb{T}_0$. Fix any $\omega^{3n-1} \in \Omega^{3n-1}$. Denote $\hat{\alpha}_i^{3n-1}(\omega^{3n-1})$ and $\hat{g}_i^{3n-1}(\omega^{3n-1})$ and $\hat{h}_i^{3n-1}(\omega^{3n-1})$ by $k \in S$, $l \in S \cup \{J\}$

and $r \in \{0,1\}$ respectively. The construction of the type changing and break-up step in the finite-agent dynamic matching model says that

$$Q_{3n}^{\omega^{3n-1}}\left(\hat{X}_i^{3n} = \hat{X}_i^{3n-1}\right) = 1, \quad \text{if } l = J \text{ or } r = 1,$$

$$Q_{3n}^{\omega^{3n-1}}\left(\hat{X}_i^{3n} = \hat{X}_i^{3n-1}\right) = \left(1 - \hat{\vartheta}_{kl}\right) \geq 1 - \frac{\bar{a}}{M}, \quad \text{if } l \neq J \text{ and } r = 0.$$

Let $A^{3n-1} = \left(\hat{X}_i^{3n-1} = \hat{X}_i^m\right) \cap F^m$. If $P_0(A^{3n-1}) > 0$, then

$$\begin{aligned} & P_0(\hat{X}_i^{3n} = \hat{X}_i^{3n-1} | \hat{X}_i^{3n-1} = \hat{X}_i^m, F^m) \\ = & \frac{\int_{A^{3n-1}} Q_{3n}^{\omega^{3n-1}}\left(\hat{X}_i^{3n} = \hat{X}_i^{3n-1}\right) dQ^{3n-1}}{P_0(A^{3n-1})} \\ \geq & \frac{\int_{A^{3n-1}} \left(1 - \frac{\bar{a}}{M}\right) dQ^{3n-1}}{P_0(A^{3n-1})} \\ = & \left(1 - \frac{\bar{a}}{M}\right). \end{aligned}$$

Therefore, we obtain that

$$\begin{aligned} & P_0(\hat{X}_i^{3n} = \hat{X}_i^m | F^m) \\ = & P_0(\hat{X}_i^{3n-1} = \hat{X}_i^m | F^m) P_0(\hat{X}_i^{3n} = \hat{X}_i^{3n-1} | \hat{X}_i^{3n-1} = \hat{X}_i^m, F^m) \\ \geq & P_0(\hat{X}_i^{3n-1} = \hat{X}_i^m | F^m) \left(1 - \frac{\bar{a}}{M}\right). \end{aligned} \tag{E.98}$$

If $P_0(\hat{X}_i^{3n-1} = \hat{X}_i^m, F^m) = 0$, then the above inequality is trivially satisfied.

By Equations (E.96), (E.97) and (E.98), we can derive

$$\begin{aligned} & P_0(\hat{X}_i^{m+\Delta m} = \hat{X}_i^m | F^m) \\ \geq & P_0(\hat{X}_i^{m+\Delta m-1} = \hat{X}_i^m | F^m) \left(1 - \frac{K\bar{a}}{M}\right)^2 \\ \geq & P_0(\hat{X}_i^m = \hat{X}_i^m | F^m) \prod_{m'=m+1}^{m+\Delta m} \left(1 - \frac{K\bar{a}}{M}\right)^2 \\ \geq & \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m}, \end{aligned}$$

which is the required inequality in Lemma E.8.

### E.5.10 Proof of Lemma E.9

Fix any $i \in I$, $m, \Delta m \in \{0, \ldots, 3M^2\}$ such that $m + \Delta m \leq 3M^2$, $\frac{\Delta m}{M}$ is finite and $P_0(F^m) > 0$. It is clear that

$$\begin{aligned} & P_0\left(\hat{X}_i^{m+\Delta m} - \hat{X}_i^m \geq 2 \,\middle|\, F^m\right) \\ = & \sum_{r=m+1}^{m+\Delta m-1} P_0\left(\hat{X}^{m+\Delta m} - \hat{X}_i^r \geq 1, \hat{X}_i^r = \hat{X}_i^{r-1} + 1, \hat{X}^{r-1} = \hat{X}^m \,\middle|\, F^m\right). \end{aligned} \tag{E.99}$$

Fix any $r \in \{m+1, m+2, \ldots, m+\Delta m-1\}$. Assume that $P_0\left(\hat{X}_i^r = \hat{X}_i^{r-1}+1, \hat{X}^{r-1} = \hat{X}^m, F^m\right) > 0$. By Lemma E.8, we can obtain that

$$\begin{aligned} & P_0\left(\hat{X}^{m+\Delta m} = \hat{X}_i^r \,\middle|\, \hat{X}_i^r = \hat{X}_i^{r-1}+1, \hat{X}^{r-1} = \hat{X}^m, F^m\right) \\ \geq \;& \left(1 - \frac{K\bar{a}}{M}\right)^{2(m+\Delta m-r)} \geq \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m}, \end{aligned}$$

which implies that

$$\begin{aligned} & P_0\left(\hat{X}^{m+\Delta m} - \hat{X}_i^r \geq 1 \,\middle|\, \hat{X}_i^r = \hat{X}_i^{r-1}+1, \hat{X}^{r-1} = \hat{X}^m, F^m\right) \\ \leq \;& 1 - \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m}. \end{aligned}$$

It follows from the above inequality that

$$\begin{aligned} & P_0\left(\hat{X}^{m+\Delta m} - \hat{X}_i^r \geq 1, \hat{X}_i^r = \hat{X}_i^{r-1}+1, \hat{X}^{r-1} = \hat{X}^m \,\middle|\, F^m\right) \\ = \;& P_0\left(\hat{X}^{m+\Delta m} - \hat{X}_i^r \geq 1 \,\middle|\, \hat{X}_i^r = \hat{X}_i^{r-1}+1, \hat{X}^{r-1} = \hat{X}^m, F^m\right) \\ & \quad P_0\left(\hat{X}_i^r = \hat{X}_i^{r-1}+1, \hat{X}^{r-1} = \hat{X}^m \,\middle|\, F^m\right) \\ \leq \;& \left(1 - \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m}\right) P_0\left(\hat{X}_i^r = \hat{X}_i^{r-1}+1, \hat{X}^{r-1} = \hat{X}^m \,\middle|\, F^m\right). \end{aligned} \tag{E.100}$$

When $P_0\left(\hat{X}_i^r = \hat{X}_i^{r-1}+1, \hat{X}^{r-1} = \hat{X}^m, F^m\right) = 0$, the above inequality is trivially satisfied. Hence, Equations (E.99) and (E.100) together with Lemma E.8 imply that

$$\begin{aligned} & P_0\left(\hat{X}_i^{m+\Delta m} - \hat{X}_i^m \geq 2 \,\middle|\, F^m\right) \\ \leq \;& \left(1 - \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m}\right) \sum_{r=m+1}^{m+\Delta m} P_0\left(\hat{X}_i^r = \hat{X}_i^{r-1}+1, \hat{X}^{r-1} = \hat{X}^m \,\middle|\, F^m\right) \\ = \;& \left(1 - \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m}\right) P_0\left(\hat{X}_i^{m+\Delta m} \geq \hat{X}^m + 1 \,\middle|\, F^m\right) \\ = \;& \left(1 - \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m}\right) \left(1 - P_0\left(\hat{X}_i^{m+\Delta m} = \hat{X}^m \,\middle|\, F^m\right)\right) \\ \leq \;& \left(1 - \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m}\right)^2, \end{aligned}$$

which is the required inequality in Lemma E.9.

### E.5.11 Proof of Lemma E.10

Fix any $(k,l,r) \in \tilde{S}$. By the definition of $\tilde{\rho}$, we obtain that

$$\begin{aligned} & \left|\mathbb{E}\left(\tilde{\rho}_{klr}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}_{klr}^{m}\right)\right| \\ = & \left|\mathbb{E}\left(\frac{1}{\hat{M}}\sum_{i\in I}\mathbf{1}_{klr}\left(\tilde{\beta}_i^{m+\Delta m}\right)\right) - \mathbb{E}\left(\frac{1}{\hat{M}}\sum_{i\in I}\mathbf{1}_{klr}\left(\tilde{\beta}_i^{m}\right)\right)\right| \\ \leq & \frac{1}{\hat{M}}\sum_{i\in I}\mathbb{E}\left|\mathbf{1}_{klr}\left(\tilde{\beta}_i^{m+\Delta m}\right) - \mathbf{1}_{klr}\left(\tilde{\beta}_i^{m}\right)\right| \\ = & \frac{1}{\hat{M}}\sum_{i\in I}P_0\left(\left|\mathbf{1}_{klr}\left(\tilde{\beta}_i^{m+\Delta m}\right) - \mathbf{1}_{klr}\left(\tilde{\beta}_i^{m}\right)\right| = 1\right). \end{aligned} \tag{E.101}$$

For any $\omega \in \Omega$, if $\left|\mathbf{1}_{klr}\left(\tilde{\beta}_i^{m+\Delta m}(\omega)\right) - \mathbf{1}_{klr}\left(\tilde{\beta}_i^{m}(\omega)\right)\right| = 1$ , then $\hat{X}_i^{m+\Delta m}(\omega) > \hat{X}_i^{m}(\omega)$. Thus, we can obtain from Equation (E.101) that

$$\left|\mathbb{E}\left(\tilde{\rho}_{klr}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}_{klr}^{m}\right)\right| \leq \frac{1}{\hat{M}}\sum_{i\in I}P_0\left(\hat{X}_i^{m+\Delta m} > \hat{X}_i^{m}\right).$$

By Lemma E.8, we have

$$P_0\left(\hat{X}_i^{m+\Delta m} > \hat{X}_i^{m}\right) \leq 1 - \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m}.$$

Therefore, we can obtain that

$$\left|\mathbb{E}\left(\tilde{\rho}_{klr}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}_{klr}^{m}\right)\right| \leq 1 - \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m},$$

which implies that

$$\|\mathbb{E}\left(\tilde{\rho}^{m+\Delta m}\right) - \mathbb{E}\left(\tilde{\rho}^{m}\right)\|_\infty \leq 1 - \left(1 - \frac{K\bar{a}}{M}\right)^{2\Delta m}.$$

Hence, Lemma E.10 is proven.

## F Proof of Results in Section 2

The continuous-time random matching model with immediate break-up described in Section 2 can be treated as a special case of the model of random matching with enduring partnerships in Appendix A by taking the enduring probabilities $\xi_{kl}$ to be 0 for any $k, l \in S$. It is natural to define other parameters for the random matching model with enduring partnerships. For any $k, l, k', l' \in S$, extend $\theta_{kl}$ from its domain $\Delta$ to $\hat{\Delta}$ by letting $\theta_{kl}(\hat{p}) = \theta_{kl}(p)$, $\sigma_{kl}\left((k', l')\right) = \delta_k(k')\delta_l(l')$, $\vartheta_{kl} = 1$, and $\eta_{kl}$ and $\varsigma_{kl}$ remain the same.

In Section 2, at any given time $t$, any agent $i$ has no partner with probability one. It means that the process $g$ is the constant $J$ with probability one. Hence, we can obtain the properties and results on the type process $\alpha$ in Section 2 directly from the corresponding properties and results on the extended type process $(\alpha, g)$ in Appendix A. In particular, the transition intensity matrix $R$ corresponds to Case 4 in Table 1. Properties 1, 2, 3 and 4 of the random matching model $\mathbb{D}$ in Section 2, are also special cases of the corresponding properties for the random matching model $\hat{\mathbb{D}}$ in Appendix A, while Parts (1), (2), (4), (5) and (6) of Theorem 2.1 are direct implications of Theorem A.1, Propositions A.1 and A.2. It remains to verify Property 5 of the dynamical matching model $\mathbb{D}$ and to prove Theorem 2.1 (3).

To check Property 5 of the dynamical matching model $\mathbb{D}$, we shall need to study the properties of agents' last partners. Let $M$ be an unlimited hyperfinite integer in ${}^*\mathbb{N}_\infty$ as in Subsection E.4. Suppose that the hyperfinite dynamic matching model transferred from Section E.2 has been constructed with $\hat{\pi}^0(i) = i$ for each $i \in I$. Fix any agent $i \in I$ and any standard natural number $n \in \mathbb{N}$. For any $\omega \in \Omega$, let $\hat{d}_i^n(\omega)$ be the $n$-th matching period of agent $i$. That is, $\hat{d}_i^n(\omega)$-th period is the period when agent $i$'s $n$-th partner arrives. If the total number of matching periods is less than $n$, we let $\hat{d}_i^n(\omega)$ be $J$; otherwise $1 \le \hat{d}_i^n(\omega) \le M^2$. The real time for the $n$-th matching of agent $i$ is defined by

$$d_i^n(\omega) = \begin{cases} \mathrm{st}\left(\frac{\hat{d}_i^n(\omega)}{M}\right) & \text{if } \hat{d}_i^n(\omega) \neq J \text{ and } \frac{\hat{d}_i^n(\omega)}{M} \text{ is limited} \\ \infty & \text{if } \hat{d}_i^n(\omega) = J \text{ or } \frac{\hat{d}_i^n(\omega)}{M} \text{ is unlimited} \end{cases}$$

Recall that $[tM]$ denotes the hyperinteger part of $tM$. For any $\omega \in \Omega$ and $t \in \mathbb{R}_+$, agent $i$'s last matching period up to time $t$ is define by

$$\hat{\tau}_i^t(\omega) = \max\{n' \in \mathbb{T}_0 : \hat{\pi}_i^{3n'-1}(\omega^{3n'-1}) \neq i \text{ and } n' \le [tM]\}$$

when the set $\{n' \in \mathbb{T}_0 : \hat{\pi}_i^{3n'-1}(\omega^{3n'-1}) \neq i \text{ and } n' \le [tM]\}$ is nonempty; otherwise, $\hat{\tau}_i^t(\omega)$ is defined to be $J$.

Next, we define the process $\varphi$ for agents' last partners. Fix any $i \in I$. For any $\omega \in \Omega$ and $t \in \mathbb{R}_+$, let

$$\varphi_i'(\omega, t) = \begin{cases} \hat{\pi}_i^{3\hat{\tau}_i^t(\omega)-1}(\omega) & \text{if } \hat{\tau}_i^t(\omega) \neq J \\ i & \text{if } \hat{\tau}_i^t(\omega) = J. \end{cases}$$

Then, $\varphi_i'(\omega, t)$ is agent $i$'s last partner up to the $[tM]$-th period. Since $\hat{\tau}_i^t(\omega) = J$ means that agent $i$ has not been matched up to the $[tM]$-th period, agent $i$'s last partner is simply defined to be herself in this case. Note that $\varphi_i'(t)$ may not be RCLL. Recall that the set

$$A_i = \{\omega' \in \Omega : \hat{X}_i^m(\omega') \text{ is finite for any positive hyperinteger } m \text{ such that } \frac{m}{M} \text{ is finite}\}$$

has probability one as shown in the proof of Part 1 in Subsection E.4. For any $\omega \notin A_i$ and $t \in \mathbb{R}_+$, define $\varphi_i(\omega, t)$ to be $i$; it is obvious that $\varphi_i(\omega, t)$ is RCLL in $t$. Fix any $\omega \in A_i$ and

$t \in \mathbb{R}_+$. By the definition of $A_i$, we know that agent $i$ matches finitely many times up to the $[(t+1)M]$-th period. For any $t'$ in the real time interval $[0, t+1]$, since $\varphi_i'(\omega, t')$ is agent $i$'s last partner up to the $[t'M]$-th period, we know that there exists $j, j' \in I$ and $\epsilon \in \mathbb{R}_{++}$ such that $\varphi_i'(\omega, t')$ is $j$ on $(t, t+\epsilon)$ and $j'$ on $(t-\epsilon, t)$. Define $\varphi_i(\omega, t)$ to be $j$. For any $t' \in (t, t+\epsilon)$, we know that $\varphi_i'(\omega, t'')$ is $j$ for $t'' \in (t', t+\epsilon')$. According to the definition of $\varphi$, we obtain that $\varphi_i(\omega, t')$ is still $j$ for $t' \in (t, t+\epsilon)$. Therefore, $\varphi_i(\omega, t')$ is right continuous at real time $t$. Similarly, for any $t' \in (t-\epsilon, t)$, $\varphi_i'(\omega, t'')$ is $j'$ for $t'' \in (t', t)$. The definition of $\varphi$ implies that $\varphi_i(\omega, t')$ is $j'$ for $t' \in (t-\epsilon, t)$. Therefore, the left limit of $\varphi_i(\omega, t')$ exists at time $t$. For simplicity, let $\varphi_i(\omega, \infty) = i$ for any $\omega \in \Omega$.

For any $i \in I$, let $B(i) = \{\omega \in \Omega : \varphi\left(\varphi\left(i, \omega, d_i^n(\omega)\right), \omega, d_i^n(\omega)\right) = i\}$. It is clear that $\{\omega \in \Omega : d_i^n(\omega) = \infty\} \subseteq B(i)$. We are going to show that $P(B(i)) = 1$. For any $N \in \mathbb{T}_0$, let

$$B_N(i) = \{\omega \in \Omega : \quad \hat{d}_i^n(\omega) \neq J,\ \hat{\pi}^{3n'-1}(i, \omega) = i,\ \hat{\pi}^{3n'-1}\left(\hat{\pi}^{3\hat{d}_i^n(\omega)-1}(i, \omega), \omega\right) = \hat{\pi}^{3\hat{d}_i^n-1}(i, \omega)$$
$$\text{for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n(\omega) < n' \leq \hat{d}_i^n(\omega) + N\}.$$

Then $B_N(i)$ is the event that agent $i$ and her $n$-th partner do not match in period $\hat{d}_i^n + 1, \hat{d}_i^n + 2, \ldots, \hat{d}_i^n + N$. The following lemma shows a relationship between $B_N(i)$ and $B(i)$.

**Lemma F.1.** *For any $N \in \mathbb{T}_0$ such that $\frac{N}{M}$ is limited and* $\operatorname{st}\left(\frac{N}{M}\right) > 0$, *$B_N(i) \subseteq B(i)$.*

*Proof.* Fix any $N \in \mathbb{T}_0$ such that $\frac{N}{M}$ is limited and $\operatorname{st}\left(\frac{N}{M}\right) > 0$, and any $\omega \in B_N(i)$. If $\omega \notin A_i$, by the definition of $\varphi$, $\varphi\left(i, \omega, d_i^n(\omega)\right) = i$. It is clear that

$$\varphi\left(\varphi\left(i, \omega, d_i^n(\omega)\right), \omega, d_i^n(\omega)\right) = i.$$

If $\frac{\hat{d}_i^n(\omega)}{M}$ is unlimited, we have $d_i^n(\omega) = \infty$. By the definition of $\varphi$, we have

$$\varphi\left(\varphi\left(i, \omega, d_i^n(\omega)\right), \omega, d_i^n(\omega)\right) = i.$$

Next, we consider the case when $\omega \in A_i$ and $\frac{\hat{d}_i^n(\omega)}{M}$ is limited. Since $\omega \in B_N(i)$, we have $\hat{d}_i^n(\omega) \neq J, \quad \hat{\pi}^{3n'-1}(i, \omega) = i$,

$$\hat{\pi}^{3n'-1}\left(\hat{\pi}^{3\hat{d}_i^n(\omega)-1}(i, \omega), \omega\right) = \hat{\pi}^{3\hat{d}_i^n(\omega)-1}(i, \omega)$$

for any $n' \in \mathbb{T}_0$ such that $\hat{d}_i^n(\omega) < n' \leq \hat{d}_i^n(\omega) + N$. Therefore, for any $t' \in \left(d_i^n(\omega), d_i^n(\omega) + \operatorname{st}\left(\frac{N}{M}\right)\right)$, $\varphi'(i, \omega, t') = \hat{\pi}^{3\hat{d}_i^n(\omega)-1}(i, \omega)$ and $\varphi'(\hat{\pi}^{3\hat{d}_i^n(\omega)-1}(i, \omega), \omega, t') = i$. By the definition of $\varphi$, we have $\varphi(i, \omega, d_i^n(\omega)) = \hat{\pi}^{3\hat{d}_i^n(\omega)-1}(i, \omega)$ and $\varphi(\hat{\pi}^{3\hat{d}_i^n(\omega)-1}(i), \omega, d_i^n) = i$, which implies

$$\varphi\left(\varphi\left(i, \omega, d_i^n(\omega)\right), \omega, d_i^n(\omega)\right) = i.$$

Hence, we have $\omega \in B(i)$. By the arbitrary choice of $\omega$ in $B_N(i)$, we know that $B_N(i)$ is a subset of $B(i)$. ∎

The following lemma verifies Property 5 of the dynamical matching model $\mathbb{D}$ in Section 2 , which says that for any agent $i$, her partner's partner at her $n$-th matching time $d_i^n$ is agent $i$ herself with probability one.

**Lemma F.2.** *For any $i \in I$ and $n \in \mathbb{N}$, we have $\varphi\left(\varphi\left(i, d_i^n\right), d_i^n\right) = i$ $P$-almost surely.*

*Proof.* Fix any $i \in I$, $n \in \mathbb{N}$, and any $N \in \mathbb{T}_0$ such that $\frac{N}{M}$ is limited and $\operatorname{st}\left(\frac{N}{M}\right) > 0$. It follows from the definition of $B_N(i)$ that

$$\begin{aligned} P_0\left(B_N(i)\right) &= \sum_{j\in I}\sum_{r\in\mathbb{T}_0} P_0\Big(\hat{d}_i^n = r, \hat{\pi}^{3r-1}(i) = j, \hat{\pi}^{3n'-1}(i) = i, \hat{\pi}^{3n'-1}(j) = j \\ &\qquad \text{for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n < n' \leq \hat{d}_i^n + N\Big) \\ &= \sum_{j\in I}\sum_{r\in\mathbb{T}_0} P_0\left(\hat{d}_i^n = r, \hat{\pi}^{3r-1}(i) = j\right) \\ &\qquad P_0\Big(\hat{\pi}^{3n'-1}(i) = i, \hat{\pi}^{3n'-1}(j) = j \\ &\qquad \text{for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n < n' \leq \hat{d}_i^n + N \,\Big|\, \hat{d}_i^n = r, \hat{\pi}^{3r-1}(i) = j\Big). \end{aligned}$$

It follows from Lemma E.8 that

$$P_0\left(\hat{\pi}^{3n'-1}(i) = i \text{ for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n < n' \leq \hat{d}_i^n + N \,\Big|\, \hat{d}_i^n = r, \hat{\pi}^{3r-1}(i) = j\right) \gtrsim e^{-\frac{6K\bar{a}N}{M}},$$

$$P_0\left(\hat{\pi}^{3n'-1}(j) = j \text{ for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n < n' \leq \hat{d}_i^n + N \,\Big|\, \hat{d}_i^n = r, \hat{\pi}^{3r-1}(i) = j\right) \gtrsim e^{-\frac{6K\bar{a}N}{M}}.$$

Then, we can obtain that

$$\begin{aligned} & P_0\Big(\hat{\pi}^{3n'-1}(i) = i, \hat{\pi}^{3n'-1}(j,\omega) = j \\ &\qquad \text{for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n < n' \leq \hat{d}_i^n + N \,\Big|\, \hat{d}_i^n = r, \hat{\pi}^{3r-1}(i) = j\Big) \\ \geq\; & P_0\left(\hat{\pi}^{3n'-1}(i) = i \text{ for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n < n' \leq \hat{d}_i^n + N \,\Big|\, \hat{d}_i^n = r, \hat{\pi}^{3r-1}(i) = j\right) \\ & + P_0\left(\hat{\pi}^{3n'-1}(j) = j \text{ for any } n' \in \mathbb{T}_0 \text{ such that } \hat{d}_i^n < n' \leq \hat{d}_i^n + N \,\Big|\, \hat{d}_i^n = r, \hat{\pi}^{3r-1}(i) = j\right) - 1 \\ \gtrsim\; & 2e^{-\frac{6K\bar{a}N}{M}} - 1. \end{aligned}$$

Therefore, we can derive that

$$\begin{aligned} P_0\left(B_N(i)\right) &\gtrsim \sum_{j\in I}\sum_{r\in\mathbb{T}_0} P_0\left(\hat{d}_i^n = r, \hat{\pi}^{3r-1}(i) = j\right)\left(2e^{-\frac{6K\bar{a}N}{M}} - 1\right) \\ &= P_0\left(\hat{d}_i^n \neq J\right)\left(2e^{-\frac{6K\bar{a}N}{M}} - 1\right). \end{aligned}$$

It is clear that

$$\begin{aligned} P_0\left(B_N(i) \cup \left(\hat{d}_i^n = J\right)\right) &= P_0\left(B_N(i)\right) + P_0\left(\hat{d}_i^n = J\right) \\ &\gtrsim P_0\left(\hat{d}_i^n \neq J\right)\left(2e^{-\frac{6K\bar{a}N}{M}} - 1\right) + P_0\left(\hat{d}_i^n = J\right). \qquad \text{(F.1)} \end{aligned}$$

Since $\left(\hat{d}_i^n = J\right) \subseteq \{\omega \in \Omega : d_i^n(\omega) = \infty\}$ and $\{\omega \in \Omega : d_i^n(\omega) = \infty\} \subseteq B(i)$, we know that $\left(\hat{d}_i^n = J\right) \subseteq B(i)$. Hence, Lemma F.1 implies that $B_N(i) \cup \left(\hat{d}_i^n = J\right) \subseteq B(i)$. Therefore, by Equation (F.1), we obtain that

$$P_0\left(B(i)\right) \gtrsim P_0\left(\hat{d}_i^n \neq J\right)\left(2e^{-\frac{6K\bar{a}N}{M}} - 1\right) + P_0\left(\hat{d}_i^n = J\right). \tag{F.2}$$

If $\mathrm{st}\left(\frac{N}{M}\right) \to 0$, then $\left(2e^{-\frac{6K\bar{a}N}{M}} - 1\right) \to 1$, which implies that the right hand side of Equation (F.2) tends to $P_0\left(\hat{d}_i^n \neq J\right) + P_0\left(\hat{d}_i^n = J\right) = 1$. Therefore, we can claim that $P\left(B(i)\right) = 1$, which implies that $\varphi\left(\varphi\left(i, d_i^n\right), d_i^n\right) = i$ $P$-almost surely. ■

For any $k, l \in S$, and $1 \leq m \leq 3M^2$, the number of matches by agent $i$ up to the $m$-step, when of type $k$, to an agent of type $l$ is defined to be

$$\hat{N}_{ikl}^m(\omega) = |\{n \in \mathbb{T}_0 : \ \hat{\alpha}_i^{3n-1}(\omega) = k, \, \hat{\pi}_i^{3n-1}(\omega) \neq i, \, \hat{g}_i^{3n-1}(\omega) = l, 3n-1 \leq m\}|.$$

The following defines the counting process for the number of matches by agent $i$, when of type $k$, to an agent of type $l$:

$$N_{ikl}(\omega, t) = \begin{cases} \hat{N}_{ikl}^{3[tM]}(\omega) & \text{if } \omega \in A_i \\ 0 & \text{if } \omega \notin A_i. \end{cases}$$

Recall that $\Theta_{kl}(t) = \int_I N_{ikl}(\omega, t)\, d\lambda(i)$ denotes the cumulative total quantity of matches of agents of any given type $k$ with agents of another given type $l$, by time $t$.

Finally, we are ready to prove Part (3) of Theorem 2.1.

**Proof of Theorem 2.1 (3):** Fix any $t \in \mathbb{R}_+$, $k, l \in S$, and non-negative standard integers $n$ and $n'$. For any $i, j \in I$ with $i \neq j$ (and thus $\hat{\pi}^0(i) = i \neq j$), it is clear that the events $\left(\hat{N}_{ikl}^{3[tM]} = n\right)$ and $\left(\hat{N}_{jkl}^{3[tM]} = n'\right)$ are in $\mathcal{F}_i^{3[tM]}$ and $\mathcal{F}_j^{3[tM]}$ respectively. It follows from Lemma E.7 that

$$P\left(\hat{N}_{ikl}^{3[tM]} = n, \, \hat{N}_{jkl}^{3[tM]} = n'\right) = P\left(\hat{N}_{ikl}^{3[tM]} = n\right) P\left(\hat{N}_{jkl}^{3[tM]} = n'\right).$$

Since $A_i$ has probability one, it is obvious that the events $\left(\hat{N}_{ikl}(t) = n\right)$ and $\left(\hat{N}_{jkl}(t) = n'\right)$ are independent. By the arbitrary choices of $n$ and $n'$, we know that the random variables $N_{ikl}(t)$ and $N_{jkl}(t)$ are independent. By the exact law of large numbers (Corollary 2.10 in Sun (2006)), we have $\Theta_{kl}(\omega, t) = \mathbb{E}\Theta_{kl}(t)$ for $P$-almost all $\omega \in \Omega$.

Fix any $\Delta t \in \mathbb{R}_+$. Let $n$ and $n + \Delta n$ be the hyperinteger parts of $tM$ and $(t + \Delta t)M$

respectively. It follows from the Fubini property and the definition of $\Theta_{kl}$ that

$$\begin{aligned}
&\frac{1}{\Delta t}\left(\mathbb{E}\Theta_{kl}(t+\Delta t)-\mathbb{E}\Theta_{kl}(t)\right)\\
=\ &\frac{1}{\Delta t}\int_I\left(\mathbb{E}\hat{N}_{ikl}(t+\Delta t)-\mathbb{E}\hat{N}_{ikl}(t)\right)d\lambda\\
\simeq\ &\frac{1}{\Delta t}\int_I\left(\mathbb{E}\hat{N}_{ikl}^{3(n+\Delta n)}-\mathbb{E}\hat{N}_{ikl}^{3n}\right)d\lambda_0\\
=\ &\frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_I\mathbb{E}\left(\hat{N}_{ikl}^{3n'}-\hat{N}_{ikl}^{3(n'-1)}\right)d\lambda_0\\
=\ &\frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_I\mathbb{E}\left(\hat{N}_{ikl}^{3n'-1}-\hat{N}_{ikl}^{3n'-2}\right)d\lambda_0\\
=\ &\frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_I P_0\left(\hat{N}_{ikl}^{3n'-1}-\hat{N}_{ikl}^{3n'-2}=1\right)d\lambda_0\\
=\ &\frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_I\int_{\Omega^{3n'-2}}Q_{3n'-1}^{\omega^{3n'-2}}\left(\hat{N}_{ikl}^{3n'-1}-\hat{N}_{ikl}^{3n'-2}=1\right)dQ^{3n'-2}d\lambda_0\\
=\ &\frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_{\Omega^{3n'-2}}\int_I Q_{3n'-1}^{\omega^{3n'-2}}\left(\hat{N}_{ikl}^{3n'-1}-\hat{N}_{ikl}^{3n'-2}=1\right)d\lambda_0 dQ^{3n'-2}. \qquad \text{(F.3)}
\end{aligned}$$

For any $i\in I$ and $\omega^{3n'-2}\in\Omega^{3n'-2}\setminus V^{3n'-2}$, if $\hat{\beta}_i^{3n'-2}(\omega^{3n'-2})=(k,J)$ (i.e., $\tilde{\beta}_i^{3n'-2}(\omega^{3n'-2})=(k,J,1)$), then Lemma E.16 (1) implies that

$$Q_{3n'-1}^{\omega^{3n'-2}}\left(\hat{N}_{ikl}^{3n'-1}-\hat{N}_{ikl}^{3n'-2}=1\right)=Q_{3n'-1}^{\omega^{3n'-2}}\left(\hat{g}_i^{3n'-1}=l\right)\simeq\hat{q}_{kl}\left(\tilde{\rho}^{3n'-2}\left(\omega^{3n'-2}\right)\right);$$

if $\hat{\beta}_i^{3n'-2}(\omega^{3n'-2})\neq(k,J)$, then the construction of the dynamic matching model implies that $Q_{3n'-1}^{\omega^{3n'-2}}\left(\hat{N}_{ikl}^{3n'-1}-\hat{N}_{ikl}^{3n'-2}=1\right)=0$. By Equation (F.3) and Lemma E.14, we can obtain that

$$\begin{aligned}
&\frac{1}{\Delta t}\left(\mathbb{E}\Theta_{kl}(t+\Delta t)-\mathbb{E}\Theta_{kl}(t)\right)\\
\simeq\ &\frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_{\Omega^{3n'-2}\setminus V^{3n'-2}}\int_I Q_{3n'-1}^{\omega^{3n'-2}}\left(\hat{N}_{ikl}^{3n'-1}-\hat{N}_{ikl}^{3n'-2}=1\right)d\lambda_0 dQ^{3n'-2}\\
\simeq\ &\frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_{\Omega^{3n'-2}\setminus V^{3n'-2}}\hat{\rho}_{kJ}^{3n'-2}\left(\omega^{3n'-2}\right)\hat{q}_{kl}\left(\tilde{\rho}^{3n'-2}\left(\omega^{3n'-2}\right)\right)dQ^{3n'-2}\\
\simeq\ &\frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\int_{\Omega^{3n'-2}}\hat{\rho}_{kJ}^{3n'-2}\left(\omega^{3n'-2}\right)\hat{q}_{kl}\left(\tilde{\rho}^{3n'-2}\left(\omega^{3n'-2}\right)\right)dQ^{3n'-2}\\
=\ &\frac{1}{\Delta t}\sum_{n'=n+1}^{n+\Delta n}\mathbb{E}\left[\hat{\rho}_{kJ}^{3n'-2}\left(\omega^{3n'-2}\right)\hat{q}_{kl}\left(\tilde{\rho}^{3n'-2}\left(\omega^{3n'-2}\right)\right)\right]\\
\simeq\ &\frac{1}{\Delta n}\sum_{n'=n+1}^{n+\Delta n}\mathbb{E}\left[\hat{\rho}_{kJ}^{3n'-2}\left(\omega^{3n'-2}\right){}^*\theta_{kl}\left(\tilde{\rho}^{3n'-2}\left(\omega^{3n'-2}\right)\right)\right]. \qquad \text{(F.4)}
\end{aligned}$$

Fix any $\Delta n' \in \mathbb{T}_0$ such that $\frac{\Delta n'}{M}$ is infinitesimal. For any $\tilde{p} \in \tilde{\Delta}$, let $\hat{p}$ be the marginal probability distribution of $\tilde{p}$ on $\hat{\Delta}$. Let $f$ be a real valued function on on $\tilde{\Delta}$ such that $f(\tilde{p}) = \hat{p}_{kJ}\theta_{kl}(\hat{p})$ for any $\tilde{p} \in \tilde{\Delta}$. Then, it is clear that $f$ is continuous on $\tilde{\Delta}$.

It follows from Lemmas E.3 and E.14 that for any $\omega^{3n} \in \Omega^{3n}\backslash V^{3n}$, $\tilde{\rho}^{3n}(\omega^{3n}) \simeq U_1^{3n}(\tilde{\rho}^0) \simeq \mathbb{E}\tilde{\rho}^{3n}$. Since $f$ is continuous on the compact set $\tilde{\Delta}$, ${}^*f\left(\tilde{\rho}^{3n}(\omega^{3n})\right) \simeq {}^*f\left(\mathbb{E}\tilde{\rho}^{3n}\right)$ for any $\omega^{3n} \in \Omega^{3n}\backslash V^{3n}$. Since $f$ is continuous on the compact set $\tilde{\Delta}$, it is bounded. Then, by Lemma E.14, we have

$$\begin{aligned}
& \left|\mathbb{E}\,{}^*f(\tilde{\rho}^{3n}) - {}^*f(\mathbb{E}\tilde{\rho}^{3n})\right| \\
= & \left|\int_{\Omega^{3n}} \left({}^*f(\tilde{\rho}^{3n}) - {}^*f(\mathbb{E}\tilde{\rho}^{3n})\right) dQ^{3n}\right| \\
= & \left|\int_{\Omega^{3n}\backslash V^{3n}} \left({}^*f(\tilde{\rho}^{3n}) - {}^*f(\mathbb{E}\tilde{\rho}^{3n})\right) dQ^{3n}\right| + \left|\int_{V^{3n}} \left({}^*f(\tilde{\rho}^{3n}) - {}^*f(\mathbb{E}\tilde{\rho}^{3n})\right) dQ^{3n}\right| \\
\simeq & \left|\int_{\Omega^{3n}\backslash V^{3n}} \left({}^*f(\tilde{\rho}^{3n}) - {}^*f(\mathbb{E}\tilde{\rho}^{3n})\right) dQ^{3n}\right| \simeq 0.
\end{aligned}$$

Fix any $n'$ between $n+1$ and $n+\Delta n'$. The above equation implies that

$$\mathbb{E}\,{}^*f\left(\tilde{\rho}^{3n'-2}\right) - \mathbb{E}\,{}^*f\left(\tilde{\rho}^{3n}\right) \simeq {}^*f\left(\mathbb{E}\tilde{\rho}^{3n'-2}\right) - {}^*f\left(\mathbb{E}\tilde{\rho}^{3n}\right).$$

By Lemma E.10, $||\mathbb{E}\tilde{\rho}^{3n'-2} - \mathbb{E}\tilde{\rho}^{3n}||_\infty$ is infinitesimal. Since $f$ is continuous on the compact set $\tilde{\Delta}$, we know that ${}^*f\left(\mathbb{E}\tilde{\rho}^{3n'-2}\right) - {}^*f\left(\mathbb{E}\tilde{\rho}^{3n}\right)$ is infinitesimal, which implies $\mathbb{E}\,{}^*f\left(\tilde{\rho}^{3n'-2}\right) - \mathbb{E}\,{}^*f\left(\tilde{\rho}^{3n}\right)$ is also infinitesimal. By the definition of $f$, we can obtain that

$$\frac{1}{\Delta n'}\sum_{n'=n+1}^{n+\Delta n'} \mathbb{E}\left[\hat{\rho}_{kJ}^{3n'-2}\,{}^*\theta_{kl}\left(\hat{\rho}^{3n'-2}\right)\right] \simeq \frac{1}{\Delta n'}\sum_{n'=n+1}^{n+\Delta n'} \mathbb{E}\left[\hat{\rho}_{kJ}^{3n}\,{}^*\theta_{kl}\left(\hat{\rho}^{3n}\right)\right] = \mathbb{E}\left[\hat{\rho}_{kJ}^{3n}\,{}^*\theta_{kl}\left(\hat{\rho}^{3n}\right)\right]. \text{(F.5)}$$

As noted above, Lemmas E.3 and E.14 imply that for $P$-almost all $\omega^{3n} \in \Omega^{3n}$,

$$\tilde{\rho}^{3n}\left(\omega^{3n}\right) \simeq U_1^{3n}\left(\tilde{\rho}^0\right) \simeq \mathbb{E}\tilde{\rho}^{3n},$$

which implies that $\hat{\rho}^{3n}\left(\omega^{3n}\right) \simeq \mathbb{E}\hat{\rho}^{3n}$ for $P$-almost all $\omega^{3n} \in \Omega^{3n}$. Since $\hat{p}_{kJ}\theta_{kl}(\hat{p})$ is also continuous on $\hat{\Delta}$, Equation (F.5) implies that

$$\frac{1}{\Delta n'}\sum_{n'=n+1}^{n+\Delta n'} \mathbb{E}\left[\hat{\rho}_{kJ}^{3n'-2}\,{}^*\theta_{kl}\left(\hat{\rho}^{3n'-2}\right)\right] \simeq \left(\mathbb{E}\hat{\rho}_{kJ}^{3n}\right)\,{}^*\theta_{kl}\left(\mathbb{E}\hat{\rho}^{3n}\right).$$

It follows from Equation (E.8) that $\check{p}(t) = \mathbb{E}\hat{p}(t) \simeq \mathbb{E}\hat{\rho}^{3n}$. Hence, we have

$$\frac{1}{\Delta n'}\sum_{n'=n+1}^{n+\Delta n'} \mathbb{E}\left[\hat{\rho}_{kJ}^{3n'-2}\,{}^*\theta_{kl}\left(\hat{\rho}^{3n'-2}\right)\right] \simeq \check{p}_{kJ}(t)\,\theta_{kl}\left(\check{p}(t)\right).$$

Note that $\check{p}_{kJ}(t) = \bar{p}_k(t)$ and $\theta_{kl}\left(\check{p}(t)\right) = \theta_{kl}\left(\bar{p}(t)\right)$. By the Spillover Principle and Equation (F.4), we obtain that

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t}\left(\mathbb{E}\Theta_{kl}(t+\Delta t) - \mathbb{E}\Theta_{kl}(t)\right) = \bar{p}_k(t)\theta_{kl}\left(\bar{p}(t)\right),$$

which implies that $\mathbb{E}\Theta_{kl}(t)$ is differentiable and $\frac{d\mathbb{E}\Theta_{kl}(t)}{dt} = \bar{p}_k(t)\theta_{kl}\left(\bar{p}(t)\right).$ ■

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