Supplement to "Continuous Time Random Matching"

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APPENDIX B: PROOFS FOR THE PROPERTIES OF FINITE MATCHING MODELS

The supplement provides proofs of Lemmas 1–11 in Section 3 (concerning properties of finite matching models) and Lemmas A.1–A.3 in Appendix A (about cumulating correlations in a general class of stochastic processes) as follows. Proofs of Lemmas 1–4 are given in Subsections B.1–B.4 respectively. In order to prove Lemmas 5, some additional lemmas are presented in Subsection B.5. Lemma 5 is then proved in Subsection B.6. Lemmas A.1–A.3 are proved in Subsection B.7. Lemmas 6 – 11 are then proved in Subsections B.8 – B.12 respectively (with both Lemmas 9 and 10 proved in Subsection B.11).

B.1. Proof of Lemma 1 in Section 3.1. The proof consists of three steps. In the first step, we (randomly) choose a set A_{kl} of agents among the type-k agents, which is to be matched with type-l agents. We require that the cardinality $|A_{kl}|$ of A_{kl} is even and $|A_{kl}| = |A_{lk}|$, which allow the agents in A_{kl} and A_{lk} to be matched. The second step is to randomly match the agents in A_{kl} and A_{lk}. In the third step, the random matching obtained by combining the match of agents in those groups is shown to satisfy Lemma 1 (i) and (ii).

Step 1: For each $k \in S$, let $I_k = \{i \in I : \hat{\alpha}(i) = k\}$ be the set of type-k agents. Let

$$\Omega_0 = \Big\{ (A_{kl})_{k,l \in S} : \forall k, l, l' \in S, A_{kl} \subseteq I_k,\ |A_{kl}| \text{ is the largest even integer} \\ \text{less than or equal to } |I_k|\hat{q}_{kl}, A_{kl} \text{ and } A_{kl'} \text{ are disjoint for different } l \text{ and } l' \Big\}.$$

Note that ρ_k is the proportion of agents of type k, which implies that $|I_k| = \hat{M}\rho_k$. Hence, we have $|I_k|\hat{q}_{kl} = \hat{M}\rho_k\hat{q}_{kl} = \hat{M}\rho_l\hat{q}_{lk} = |I_l|\hat{q}_{lk}$. Then for any $(A_{kl})_{k,l\in S} \in \Omega_0$, $|A_{kl}| = |A_{lk}|$ for any $k, l \in S$. Let μ_0 be the counting probability measure on $(\Omega_0, \mathcal{A}_0)$, where $\mathcal{A}_0$ is the power set of Ω_0.

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Step 2: For any fixed $\omega_0 = (A_{kl})_{k,l\in S} \in \Omega_0$, we consider partial matchings on I that match agents from A_{kl} to A_{lk}. We only need to consider those sets A_{kl} which are nonempty. For each $k \in S$, let $\Omega_{kk}^{\omega_0}$ be the set of all the full matchings on A_{kk}, and $\mu_{kk}^{\omega_0}$ the counting probability measure on $\Omega_{kk}^{\omega_0}$. For $k,l \in S$ with $k<l$, let $\Omega_{kl}^{\omega_0}$ be the set of all the bijections from A_{kl} to A_{lk}, and $\mu_{kl}^{\omega_0}$ the counting probability measure on $\Omega_{kl}^{\omega_0}$. Let Ω_1 be the set of all the partial matchings from I to I. Define $\Omega_1^{\omega_0}$ to be the set of $\phi \in \Omega_1$ that satisfy:

(i) $\{i \in I_k : \phi(i) = i\} = I_k \backslash \left(\cup_{l=1}^{K} A_{kl}\right)$ for each $k \in S$;
(ii) the restriction $\phi|_{A_{kk}} \in \Omega_{kk}^{\omega_0}$ for $k \in S$;
(iii) for $k,l \in S$ with $k<l$, $\phi|_{A_{kl}} \in \Omega_{kl}^{\omega_0}$.

Define a probability measure $\mu_1^{\omega_0}$ on Ω_1 such that such that

(i) for $\phi \in \Omega_1^{\omega_0}$,

$$\mu_1^{\omega_0}(\phi) = \prod_{1\le k\le l\le K, A_{kl}\neq\emptyset} \mu_{kl}^{\omega_0}(\{\phi|_{A_{kl}}\});$$

(ii) $\phi \notin \Omega_1^{\omega_0}$, $\mu_1^{\omega_0}(\phi) = 0$, where $\mu_1^{\omega_0}(\phi)$ is a simplified notation for the probability of the singleton set $\{\phi\}$ under μ.

The purpose of introducing the space $\Omega_1^{\omega_0}$ and the probability measure $\mu_1^{\omega_0}$ is to match the agents in A_{kl} to the agents in A_{lk} randomly. The probability measure $\mu_1^{\omega_0}$ is trivially extended to the common sample space Ω_1.

Define a probability measure P_0 on $\Omega = \Omega_0 \times \Omega_1$ with the power set $\mathcal{F}_0$ by letting

$$P_0\left((\omega_0,\omega_1)\right) = \mu_0(\omega_0) \times \mu_1^{\omega_0}(\omega_1).$$

For $(i,\omega) \in I \times \Omega$, let $\hat{\alpha}(i,(\omega_0,\omega_1)) = \omega_1(i)$, and $\hat{g}(i,\omega) = \begin{cases} \hat{\alpha}(\hat{\alpha}(i,\omega)) & \text{if } \hat{\alpha}(i,\omega) \neq i \\ J & \text{if } \hat{\alpha}(i,\omega) = i. \end{cases}$

Denote the set $\{(\omega_0,\omega_1) \in \Omega : \omega_0 \in \Omega_0,\ \omega_1 \in \Omega_1^{\omega_0}\}$ by $\hat{\Omega}$. The definition of P_0 indicates that $P_0\left(\hat{\Omega}\right) = 1$.

Step 3: For any $k,l \in S$ and $\omega \in \hat{\Omega}$, we have $\lambda_0\left(\{i \in I : \hat{\alpha}(i) = k, \hat{g}(i,\omega) = l\}\right) = \frac{|A_{kl}|}{\hat{M}}$. Since $|A_{kl}|$ is the largest even integer less than or equal to $|I_k|\hat{q}_{kl}$, we have $0 \le |I_k|\hat{q}_{kl} - |A_{kl}| \le 2$. Hence,

$$\left|\lambda_0\left(\{i \in I : \hat{\alpha}(i) = k, \hat{g}(i,\omega) = l\}\right) - \rho_k \hat{q}_{kl}\right| = \left|\frac{|A_{kl}|}{\hat{M}} - \frac{|I_k|}{\hat{M}}\hat{q}_{kl}\right| \le \frac{2}{\hat{M}},$$

which implies Part (ii) of the lemma.

To prove Part (i), fix any $i,j \in I$ with $i \neq j$, denote $\hat{\alpha}^0(i)$ and $\hat{\alpha}^0(j)$ by k_1 and k_2 respectively.

We start with the first inequality in Part (i). By the construction above, we have

$$P_0\left(\hat{\alpha}_i = j\right) = P_0\left(\{((A_{kl})_{k,l\in S},\omega_1) \in \Omega : i \in A_{k_1k_2},\ j \in A_{k_2k_1},\ \omega_1(i) = j\}\right).$$

Let $\bar{A} = \{(A_{kl})_{k,l\in S} \in \Omega_0 : i \in A_{k_1k_2}, j \in A_{k_2k_1}\}$. Then, the definition of P_0 implies that

$$P_0(\hat{\alpha}_i = j) = \sum_{(A_{kl})_{k,l\in S}\in\bar{A}} \mu_0((A_{kl})_{k,l\in S})\, \mu_1^{(A_{kl})_{k,l\in S}}(\omega_1(i) = j),$$

where $\mu_1^{(A_{kl})_{k,l\in S}}(\omega_1(i) = j)$ is the measure of the set $\{\omega_1 \in \Omega_1 : \omega_1(i) = j\}$. When $k_1 \neq k_2$, we know that for any $(A_{kl})_{k,l\in S} \in \bar{A}$,

$$\mu_1^{(A_{kl})_{k,l\in S}}(\omega_1(i) = j) = \frac{1}{|A_{k_1k_2}|}.$$

When $k_1 = k_2$, we have for any $(A_{kl})_{k,l\in S} \in \bar{A}$,

$$\mu_1^{(A_{kl})_{k,l\in S}}(\omega_1(i) = j) = \frac{1}{|A_{k_1k_2}| - 1} \leq \frac{2}{|A_{k_1k_2}|},$$

since $|A_{k_1k_2}| \geq 2$ for any $(A_{kl})_{k,l\in S} \in \bar{A}$. Then, $\mu_1^{(A_{kl})_{k,l\in S}}(\omega_1(i) = j) \leq \frac{2}{|A_{k_1k_2}|}$ always holds for any $(A_{kl})_{k,l\in S} \in \bar{A}$ whether or not $k_1 = k_2$. Therefore, we can obtain that

$$\begin{aligned} P_0(\hat{\alpha}_i = j) &\leq \sum_{(A_{kl})_{k,l\in S}\in\bar{A}} \mu_0((A_{kl})_{k,l\in S}) \frac{2}{|A_{k_1k_2}|} \\ &= \frac{2}{|A_{k_1k_2}|}\mu_0(\{(A_{kl})_{k,l\in S} \in \Omega_0 : i \in A_{k_1k_2}, j \in A_{k_2k_1}\}) \\ &\leq \frac{2}{|A_{k_1k_2}|}\mu_0(\{(A_{kl})_{k,l\in S} \in \Omega_0 : i \in A_{k_1k_2}\}). \end{aligned}$$

Let M_k and m_{kl} be the cardinality of I_k and A_{kl} respectively. Let $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ denote the binomial coefficient. Then we have

$$P_0(\hat{\alpha}_i = j) \leq \frac{2}{m_{k_1k_2}} \frac{\binom{M_{k_1}-1}{m_{k_1k_2}-1}}{\binom{M_{k_1}}{m_{k_1k_2}}} = \frac{2}{m_{k_1k_2}} \frac{m_{k_1k_2}}{M_{k_1}} = \frac{2}{M_{k_1}} = \frac{2}{\hat{M}\rho_{k_1}},$$

where the last identity follows from the fact that $\hat{M}\rho_{k_1} = |I_k| = M_{k_1}$.

Next, we prove the second inequality in Part (i). We have

$$P_0(\hat{g}(i) = l_1) = \mu_0(\{(A_{kl})_{k,l\in S} \in \Omega_0 : i \in A_{k_1l_1}\}) = \frac{\binom{M_{k_1}-1}{m_{k_1l_1}-1}}{\binom{M_{k_1}}{m_{k_1l_1}}} = \frac{m_{k_1l_1}}{M_{k_1}}.$$

It is clear that $P_0(\hat{g}(i) = l_1) \leq \frac{M_{k_1}\hat{q}_{k_1l_1}}{M_{k_1}} = \hat{q}_{k_1l_1}$. To show the lower bound for $P_0(\hat{g}(i) = l_1)$ as stated in the lemma, we assume that $\rho_{k_1} \geq \frac{1}{\hat{M}^{\frac{1}{3}}}$. Note that

$$\begin{aligned} P_0(\hat{g}(i) = l_1) &\geq \frac{M_{k_1}\hat{q}_{k_1l_1} - 2}{M_{k_1}} = \hat{q}_{k_1l_1} - \frac{2}{M_{k_1}} \\ &= \hat{q}_{k_1l_1} - \frac{2}{\hat{M}\rho_{k_1}} \geq \hat{q}_{k_1l_1} - \frac{2}{\hat{M}^{\frac{2}{3}}}. \end{aligned}$$

Then, we have

$$\hat{q}_{k_1l_1} - \frac{2}{\hat{M}^{\frac{2}{3}}} \leq P_0(\hat{g}(i) = l_1) \leq \hat{q}_{k_1l_1}. \tag{1}$$

It remains to prove the third inequality in Part (i). We make the further assumption that $\rho_{k_2} \geq \frac{1}{\hat{M}^{\frac{1}{3}}}$. When $k_1 \neq k_2$, we obtain that

$$\begin{aligned} P_0(\hat{g}(i) = l_1, \hat{g}(j) = l_2) &= \mu_0(\{(A_{kl})_{k,l \in S} \in \Omega_0 : i \in A_{k_1 l_1}, j \in A_{k_2 l_2}\}) \\ &= \frac{\binom{M_{k_1}-1}{m_{k_1 l_1}-1}}{\binom{M_{k_1}}{m_{k_1 l_1}}} \frac{\binom{M_{k_2}-1}{m_{k_2 l_2}-1}}{\binom{M_{k_2}}{m_{k_2 l_2}}} = P_0(\hat{g}(i) = l_1) P_0(\hat{g}(j) = l_2). \end{aligned}$$

Equation (1) implies the following inequalities:

$$\begin{aligned} \hat{q}_{k_1 l_1} \hat{q}_{k_2 l_2} &\geq P_0(\hat{g}(i) = l_1, \hat{g}(j) = l_2) \\ &\geq (\hat{q}_{k_1 l_1} - \frac{2}{\hat{M}^{\frac{2}{3}}})(\hat{q}_{k_2 l_2} - \frac{2}{\hat{M}^{\frac{2}{3}}}) \geq \hat{q}_{k_1 l_1} \hat{q}_{k_2 l_2} - \frac{4}{\hat{M}^{\frac{2}{3}}}. \end{aligned} \tag{2}$$

When $k_1 = k_2$ but $l_1 \neq l_2$, we have

$$P_0(\hat{g}(i) = l_1, \hat{g}(j) = l_2) = \mu_0(\{(A_{kl})_{k,l \in S} \in \Omega_0 : i \in A_{k_1 l_1}, j \in A_{k_1 l_2}\}) = \frac{\binom{M_{k_1}-2}{m_{k_1 l_1}-1, m_{k_1 l_2}-1}}{\binom{M_{k_1}}{m_{k_1 l_1}, m_{k_1 l_2}}},$$

where $\binom{a}{b,c} = \frac{a!}{b!c!(a-b-c)!}$ is the multinomial coefficient. It is clear that

$$\begin{aligned} P_0(\hat{g}(i) = l_1, \hat{g}(j) = l_2) = \frac{m_{k_1 l_1} m_{k_1 l_2}}{M_{k_1}(M_{k_1} - 1)} &\leq \frac{m_{k_1 l_1}(m_{k_1 l_2} + 1)}{M_{k_1}^2} \\ &\leq \hat{q}_{k_1 l_1} \hat{q}_{k_1 l_2} + \hat{q}_{k_1 l_1} \frac{1}{M_{k_1}} \leq \hat{q}_{k_1 l_1} \hat{q}_{k_1 l_2} + \frac{1}{M_{k_1}} \\ &= \hat{q}_{k_1 l_1} \hat{q}_{k_1 l_2} + \frac{1}{\hat{M} \rho_{k_1}} \leq \hat{q}_{k_1 l_1} \hat{q}_{k_1 l_2} + \frac{1}{\hat{M}^{\frac{2}{3}}}. \end{aligned}$$

On the other hand, we can obtain that

$$\begin{aligned} &\frac{m_{k_1 l_1} m_{k_1 l_2}}{M_{k_1}(M_{k_1} - 1)} \\ \geq\ &\frac{(M_{k_1} \hat{q}_{k_1 l_1} - 2)}{M_{k_1}} \frac{(M_{k_1} \hat{q}_{k_1 l_2} - 2)}{M_{k_1}} \\ \geq\ &\hat{q}_{k_1 l_1} \hat{q}_{k_1 l_2} - \frac{2}{M_{k_1}} \hat{q}_{k_1 l_1} - \frac{2}{M_{k_1}} \hat{q}_{k_1 l_2} \\ \geq\ &\hat{q}_{k_1 l_1} \hat{q}_{k_1 l_2} - \frac{4}{M_{k_1}} \\ =\ &\hat{q}_{k_1 l_1} \hat{q}_{k_1 l_2} - \frac{4}{\hat{M} \rho_{k_1}} \\ \geq\ &\hat{q}_{k_1 l_1} \hat{q}_{k_1 l_2} - \frac{4}{\hat{M}^{\frac{2}{3}}}. \end{aligned}$$

By combining the above inequalities, we have

$$\hat{q}_{k_1 l_1} \hat{q}_{k_1 l_2} - \frac{4}{\hat{M}^{\frac{2}{3}}} \leq P_0(\hat{g}(i) = l_1, \hat{g}(j) = l_2) \leq \hat{q}_{k_1 l_1} \hat{q}_{k_1 l_2} + \frac{1}{\hat{M}^{\frac{2}{3}}}. \tag{3}$$

When $k_1 = k_2$ and $l_1 = l_2$, we can obtain that

$$P_0(\hat{g}(i) = l_1, \hat{g}(j) = l_1) = \mu_0(\{(A_{kl})_{k,l \in S} \in \Omega_0 : i, j \in A_{k_1 l_1}\}) = \frac{\binom{M_{k_1}-2}{m_{k_1 l_1}-2}}{\binom{M_{k_1}}{m_{k_1 l_1}}}.$$

It is clear that

$$P_0(\hat{g}(i) = l_1, \hat{g}(j) = l_1) = \frac{(m_{k_1 l_1})(m_{k_1 l_1} - 1)}{M_{k_1}(M_{k_1} - 1)} \leq \frac{m_{k_1 l_1}^2}{M_{k_1}^2} \leq q_{k_1 l_1}^2.$$

On the other hand,

$$\frac{(m_{k_1 l_1})(m_{k_1 l_1} - 1)}{M_{k_1}(M_{k_1} - 1)} \geq \frac{(M_{k_1}\hat{q}_{k_1 l_1} - 2)}{M_{k_1}} \frac{(M_{k_1}\hat{q}_{k_1 l_1} - 3)}{M_{k_1}}$$
$$\geq q_{k_1 l_1}^2 - \frac{5}{M_{k_1}} \hat{q}_{k_1 l_1} \geq q_{k_1 l_1}^2 - \frac{5}{M_{k_1}} = q_{k_1 l_1}^2 - \frac{5}{\hat{M}\rho_{k1}} \geq q_{k_1 l_1}^2 - \frac{5}{\hat{M}^{\frac{2}{3}}}.$$

Therefore, we obtain that

$$q_{k_1 l_1}^2 - \frac{5}{\hat{M}^{\frac{2}{3}}} \leq P_0(\hat{g}(i) = l_1, \hat{g}(j) = l_2) \leq q_{k_1 l_1}^2. \tag{4}$$

By combining Equations (2), (3) and (4), we know that for any $(k_1, l_1), (k_2, l_2) \in S^2$,

$$\hat{q}_{k_1 l_1}\hat{q}_{k_2 l_2} - \frac{5}{\hat{M}^{\frac{2}{3}}} \leq P_0(\hat{g}(i) = l_1, \hat{g}(j) = l_2) \leq \hat{q}_{k_1 l_1}\hat{q}_{k_2 l_2} + \frac{1}{\hat{M}^{\frac{2}{3}}}.$$

B.2. Proof of Lemma 2 in Section 3.3. First, we work with T_1. Since T_1 is continuous on Δ, there exists a strictly increasing continuous bijection v_1 on $\mathbb{R}_+$ with $v_1(0) = 0$ such that $||T_1(\rho) - T_1(\rho')||_\infty \leq v_1(||\rho - \rho'||_\infty)$ for any $\rho, \rho' \in \Delta$ (which is called a modulus of continuity of the function T_1).[1]

For T_2, T_3 and $\{\hat{q}_{kl}\}_{k,l \in S}$, we can derive their modulus of continuity in the same way. By taking the maximum, we can get a strictly increasing bijection v on $\mathbb{R}_+$ which is a common modulus of continuity for all these mappings.

Recall that K denotes the number of types. Let $\xi_0 = \frac{1}{K^{M^3}}$ and w be the inverse function v^{-1} on $\mathbb{R}_+$. Let $\xi_1 = \min(w(\xi_0), \xi_0)$, $\xi_m = \min\left(w(\xi_{m-1}), \frac{\xi_1}{3M^2K}\right)$ for any $m \in \{2, 3, \ldots, 3M^2\}$. Hence, it is clear that $3M^2K\xi_m \leq \xi_1 \leq \xi_0$ for any $m \in \{2, 3, \ldots, 3M^2+1\}$.

Fix any $m \in \{0, 1, \ldots, 3M^2\}$, and $\rho, \rho' \in \Delta$ with $\|\rho - \rho'\|_\infty \leq \xi_{m+1}$. Then, we know that $\|\rho - \rho'\|_\infty \leq w(\xi_m)$. The fact that v is a strictly increasing bijection on $\mathbb{R}_+$ implies

[1] Given a continuous function f from a compact metric space (X, d_X) to a metric space (Y, d_Y), f admits a (global) modulus of continuity ω in the sense that ω is a function from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $\lim_{t\to 0}\omega(t) = \omega(0) = 0$, and for any $x, x' \in X$, $d_Y\left(f(x), f(x')\right) \leq \omega\left(d_X(x, x')\right)$. Since the range of f is compact, we can assume with loss of generality that ω is a bounded function on $\mathbb{R}_+$. Following the wikipedia entry "Modulus of continuity" (https://en.wikipedia.org/wiki/Modulus_of_continuity), let $\omega'(t) := \frac{1}{t}\int_t^{2t}\left[\sup_{0\leq s'\leq s}\omega(s')\right]ds$ for $t > 0$ and $\omega'(0) = 0$. Then, it is easy to verify that ω' is increasing and continuous on $\mathbb{R}_+$. Let $\hat{\omega}(t) := \omega'(t) + t$ for any $t \in \mathbb{R}_+$, which is a modulus of continuity for f that is a strictly increasing continuous bijection on $\mathbb{R}_+$.

that $v\left(\|\rho-\rho'\|_{\infty}\right) \leq \xi_m$. Since v is a common modulus of continuity for T_1, T_2, T_3 and $\{\hat{q}_{kl}\}_{k,l\in S}$, we obtain that for any $r \in \{1,2,3\}$ and $k, l \in S$,

$$\|T_r(\rho) - T_r(\rho')\|_{\infty} \leq \xi_m,$$

$$|\hat{q}_{kl}(\rho) - \hat{q}_{kl}(\rho')| \leq \xi_m,$$

which completes the proof.

B.3. Proof of Lemma 3 in Section 3.3. Recall that $\mathbb{T}_0 = \{n\}_{n=0}^{M^2}$. Fix any $n \in \mathbb{T}_0$ and $k \in S$. For any $\omega^{3n-3} \in \Omega^{3n-3}$, we show that

$$\mathbb{E}^{\omega^{3n-3}} \rho^{3n-2} = T_1\left(\rho^{3n-3}(\omega^{3n-3})\right). \tag{5}$$

For any $k' \in S$, let $B_{k'}^{\omega^{3n-3}} = \{i \in I : \hat{\alpha}_i^{3n-3}(\omega^{3n-3}) = k'\}$. It follows from the definition of ρ^{3n-2} that

$$\begin{aligned}
\mathbb{E}^{\omega^{3n-3}} \rho_k^{3n-2} &= \int_{\Omega_{3n-2}} \rho_k^{3n-2}(\omega^{3n-2}) dQ_{3n-2}^{\omega^{3n-3}} = \int_{\Omega_{3n-2}} \frac{1}{\hat{M}} \sum_{i\in I} \mathbf{1}_k(\hat{\alpha}_i^{3n-2}(\omega^{3n-2})) dQ_{3n-2}^{\omega^{3n-3}} \\
&= \frac{1}{\hat{M}} \sum_{k'\in S} \sum_{i\in B_{k'}^{\omega^{3n-3}}} \int_{\Omega_{3n-2}} \mathbf{1}_k(\hat{\alpha}_i^{3n-2}(\omega^{3n-2})) dQ_{3n-2}^{\omega^{3n-3}} \\
&= \frac{1}{\hat{M}} \sum_{k'\in S} \sum_{i\in B_{k'}^{\omega^{3n-3}}} Q_{3n-2}^{\omega^{3n-3}}\left(\hat{\alpha}_i^{3n-2}(\omega^{3n-2}) = k\right) \\
&= \frac{1}{\hat{M}} \sum_{k'\in S} \sum_{i\in B_{k'}^{\omega^{3n-3}}} \hat{\eta}_{k'k} = \sum_{k'\in S} \rho_{k'}^{3n-3}(\omega^{3n-3}) \hat{\eta}_{k'k} = \left[T_1\left(\rho^{3n-3}(\omega^{3n-3})\right)\right]_k,
\end{aligned}$$

where $\mathbf{1}_k$ is the indicator function of the singleton set $\{k\}$. Thus, Equation (5) is verified.

Next, we prove that for P_0-almost all $\omega^{3n-1} \in \Omega^{3n-1}$, we have

$$||\mathbb{E}^{\omega^{3n-1}} \rho^{3n} - T_3\left(\rho^{3n-1}(\omega^{3n-1})\right)||_{\infty} \leq \frac{2K(K+1)}{\hat{M}}. \tag{6}$$

For any $\omega^{3n-1} \in \Omega^{3n-1}$, $k' \in S$ and $l' \in S \cup \{J\}$, let

$$B_{k'l'}^{\omega^{3n-1}} = \{i \in I : \hat{\alpha}_i^{3n-1}(\omega^{3n-1}) = k', \hat{g}_i^{3n-1}(\omega^{3n-1}) = l'\}.$$

It follows from the definition of ρ^{3n} that for any $\omega^{3n-1} \in \Omega^{3n-1}$,

$$\begin{aligned}
\mathbb{E}^{\omega^{3n-1}} \rho_k^{3n} &= \int_{\Omega_{3n}} \rho_k^{3n}(\omega^{3n}) dQ_{3n}^{\omega^{3n-1}} = \int_{\Omega_{3n}} \frac{1}{\hat{M}} \sum_{i\in I} \mathbf{1}_k\left(\hat{\alpha}_i^{3n}(\omega_{3n})\right) dQ_{3n}^{\omega^{3n-1}} \\
&= \frac{1}{\hat{M}} \sum_{k'\in S, l'\in S\cup\{J\}} \sum_{i\in B_{k'l'}^{\omega^{3n-1}}} \int_{\Omega_{3n}} \mathbf{1}_k(\hat{\alpha}_i^{3n}) dQ_{3n}^{\omega^{3n-1}} \\
&= \frac{1}{\hat{M}} \sum_{k'\in S, l'\in S\cup\{J\}} \sum_{i\in B_{k'l'}^{\omega^{3n-1}}} Q_{3n}^{\omega^{3n-1}}\left(\hat{\alpha}_i^{3n} = k\right)
\end{aligned}$$

$$= \frac{1}{\hat{M}} \sum_{k',l' \in S} \sum_{i \in B_{k'l'}^{\omega^{3n-1}}} Q_{3n}^{\omega^{3n-1}} \left(\hat{\alpha}_i^{3n} = k\right) + \frac{1}{\hat{M}} \sum_{k' \in S} \sum_{i \in B_{k'J}^{\omega^{3n-1}}} Q_{3n}^{\omega^{3n-1}} \left(\hat{\alpha}_i^{3n} = k\right)$$

$$= \frac{1}{\hat{M}} \sum_{k',l' \in S} \sum_{i \in B_{k'l'}^{\omega^{3n-1}}} \hat{\varsigma}_{k'l'}(k) + \frac{1}{\hat{M}} \sum_{i \in B_{kJ}^{\omega^{3n-1}}} \sum_{k' \in S} \mathbf{1}_k(k')$$

$$= \sum_{k',l' \in S} \lambda_0 \left(B_{k'l'}^{\omega^{3n-1}}\right) \hat{\varsigma}_{k'l'}(k) + \lambda_0 \left(B_{kJ}^{\omega^{3n-1}}\right). \tag{7}$$

By Part (ii) of Lemma 1, we know that for any $k', l' \in S$ and P_0-almost all $\omega \in \Omega$,

$$\left|\lambda_0 \left(B_{k'l'}^{\omega^{3n-1}}\right) - \rho_{k'}^{3n-1}(\omega^{3n-1}) \hat{q}_{k'l'}(\rho^{3n-1}(\omega^{3n-1}))\right|$$

$$= \left|\lambda_0 \left(\{i \in I : \hat{\alpha}_i^{3n-1}(\omega^{3n-1}) = k', \hat{g}_i^{3n-1}(\omega^{3n-1}) = l'\}\right)\right.$$

$$\left. -\rho_{k'}^{3n-1}(\omega^{3n-1}) \hat{q}_{k'l'}(\rho^{3n-1}(\omega^{3n-1}))\right|$$

$$\leq \frac{2}{\hat{M}}. \tag{8}$$

It follows from the above estimation that for P_0-almost all $\omega \in \Omega$,

$$\left|\lambda_0 \left(B_{kJ}^{\omega^{3n-1}}\right) - \rho_k^{3n-1}(\omega^{3n-1}) \hat{q}_k(\rho^{3n-1})\right|$$

$$= \left|\rho_k^{3n-1}(\omega^{3n-1}) - \sum_{l \in S} \lambda_0 \left(B_{kl}^{\omega^{3n-1}}\right) - \rho_k^{3n-1}(\omega^{3n-1})(1 - \sum_{l \in S} \hat{q}_{kl}(\rho^{3n-1}(\omega^{3n-1})))\right|$$

$$= \left|\sum_{l \in S} \left(\lambda_0 \left(B_{kl}^{\omega^{3n-1}}\right) - \rho_k^{3n-1}(\omega^{3n-1}) \hat{q}_{kl}(\rho^{3n-1}(\omega^{3n-1}))\right)\right| \leq \frac{2K}{\hat{M}}. \tag{9}$$

By combining Equations (7), (8) and (9), we can obtain that for P_0-almost all $\omega \in \Omega$,

$$\left|\mathbb{E}^{\omega^{3n-1}} \rho_k^{3n} - \left[T_3\left(\rho^{3n-1}(\omega^{3n-1})\right)\right]_k\right|$$

$$= \left|\sum_{k',l' \in S} \lambda_0 \left(B_{k'l'}^{\omega^{3n-1}}\right) \hat{\varsigma}_{k'l'}(k) + \lambda_0 \left(B_{kJ}^{\omega^{3n-1}}\right)\right.$$

$$\left. - \sum_{k',l' \in S} \rho_{k'}^{3n-1}(\omega^{3n-1}) \hat{q}_{k'l'}(\rho^{3n-1}(\omega^{3n-1})) \hat{\varsigma}_{k'l'}(k) - \rho_k^{3n-1}(\omega^{3n-1}) \hat{q}_k(\rho^{3n-1}(\omega^{3n-1}))\right|$$

$$\leq \sum_{k',l' \in S} \frac{2}{\hat{M}} \hat{\varsigma}_{k'l'}(k) + \frac{2K}{\hat{M}} \leq \frac{2K^2}{\hat{M}} + \frac{2K}{\hat{M}} = \frac{2K(K+1)}{\hat{M}},$$

which is Equation (6).

We divide the proof for the estimation on V^m into three steps. For the mutation step in period n, fix any $\omega^{3n-3} \in \Omega^{3n-3}$. For any $i, j \in I$ with $i \neq j$, it is clear that $\mathbf{1}_k(\hat{\alpha}_i^{3n-2})$ and $\mathbf{1}_k(\hat{\alpha}_j^{3n-2})$ are independent on $(\Omega_{3n-2}, \mathcal{E}_{3n-2}, Q_{3n-2}^{\omega^{3n-3}})$. Therefore, we can obtain that

$$\mathrm{Var}^{\omega^{3n-3}} \rho_k^{3n-2} = \mathrm{Var}^{\omega^{3n-3}} \frac{1}{\hat{M}} \sum_{i \in I} \mathbf{1}_k(\hat{\alpha}_i^{3n-2}) = \frac{1}{\hat{M}^2} \sum_{i \in I} \mathrm{Var}^{\omega^{3n-3}} \mathbf{1}_k(\hat{\alpha}_i^{3n-2}) \leq \frac{1}{\hat{M}^2} \hat{M} \frac{1}{4} = \frac{1}{4\hat{M}}.$$

It follows from the Chebyshev Inequality and Equation (5) that

$$Q^{\omega^{3n-3}}_{3n-2}\left(\|\rho^{3n-2}-T_1\left(\rho^{3n-3}\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right) = Q^{\omega^{3n-3}}_{3n-2}\left(\|\rho^{3n-2}-\mathbb{E}^{\omega^{3n-3}}\rho^{3n-2}\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right)$$

$$\leq \sum_{k\in S} Q^{\omega^{3n-3}}_{3n-2}\left(\left|\rho^{3n-2}_k-\mathbb{E}^{\omega^{3n-3}}\rho^{3n-2}_k\right| \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right) \leq K\frac{1/4\hat{M}}{1\hat{M}^{\frac{2}{3}}} = \frac{K}{4\hat{M}^{\frac{1}{3}}}.$$

Let $W^{3n-2}=\{\omega^{3n-2}\in\Omega^{3n-2}: \|\rho^{3n-2}(\omega^{3n-2})-T_1\left(\rho^{3n-3}(\omega^{3n-3})\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\}$. It is clear that

$$Q^{3n-2}(W^{3n-2}) = \int_{\Omega^{3n-3}} Q^{\omega^{3n-3}}_{3n-2}(\|\rho^{3n-2}-T_1(\rho^{3n-3})\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}})dQ^{3n-3} \leq \frac{K}{4\hat{M}^{\frac{1}{3}}}. \tag{10}$$

For the random matching step in period n,

$$\rho^{3n-1}(\omega^{3n-1}) = \rho^{3n-2}(\omega^{3n-2}) = T_2\left(\rho^{3n-2}(\omega^{3n-2})\right)$$

since the agents do not change their types at the random matching step. It is then clear that the set

$$W^{3n-1}=\{\omega^{3n-1}\in\Omega^{3n-1}: \|\rho^{3n-1}(\omega^{3n-1})-T_2\left(\rho^{3n-2}(\omega^{3n-2})\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\}$$

is empty. Hence, we have

$$Q^{3n-1}(W^{3n-1}) = \int_{\Omega^{3n-2}} Q^{\omega^{3n-1}}_{3n-2}\left(\|\rho^{3n-1}-T_2\left(\rho^{3n-2}\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\right)dQ^{3n-2} = 0. \tag{11}$$

For the type changing step in period n, fix any $\omega^{3n-1}\in\Omega^{3n-1}$. For any $i,j\in I$ with $i\neq j$, it is clear that $\mathbf{1}_k(\hat{\alpha}^{3n}_i)$ and $\mathbf{1}_k(\hat{\alpha}^{3n}_j)$ are independent on $(\Omega_{3n},\mathcal{E}_{3n},Q^{\omega^{3n-1}}_{3n})$. Therefore, we have

$$\mathrm{Var}^{\omega^{3n-1}}\rho^{3n}_k = \mathrm{Var}^{\omega^{3n-1}}\frac{1}{\hat{M}}\sum_{i\in I}\mathbf{1}_k(\hat{\alpha}^{3n}_i) = \frac{1}{\hat{M}^2}\sum_{i\in I}\mathrm{Var}^{\omega^{3n-1}}\mathbf{1}_k(\hat{\alpha}^{3n}_i) \leq \frac{1}{\hat{M}^2}\sum_{i\in I}\frac{1}{4} = \frac{1}{4\hat{M}}.$$

It follows from the Chebyshev Inequality that

$$Q^{\omega^{3n-1}}_{3n}\left(\|\rho^{3n}-\mathbb{E}^{\omega^{3n-1}}\rho^{3n}\|_\infty \geq \frac{1}{2\hat{M}^{\frac{1}{3}}}\right)$$

$$\leq \sum_{k\in S} Q^{\omega^{3n-1}}_{3n}\left(\left|\rho^{3n}_k-\mathbb{E}^{\omega^{3n-1}}\rho^{3n}_k\right| \geq \frac{1}{2\hat{M}^{\frac{1}{3}}}\right) \leq K\frac{1/4\hat{M}}{1/4\hat{M}^{\frac{2}{3}}} = \frac{K}{\hat{M}^{\frac{1}{3}}}.$$

Since $\hat{M} > \left(\frac{1}{\xi_{3M^2+1}}\right)^9 \geq \left(\frac{1}{\xi_0}\right)^9 = K^{9M^3}$, it is clear that $\frac{2K(K+1)}{\hat{M}} < \frac{1}{2\hat{M}^{\frac{1}{3}}}$. Equation (6) implies that for P_0-almost all $\omega\in\Omega$, if $\|\rho^{3n}(\omega^{3n})-\mathbb{E}^{\omega^{3n-1}}\rho^{3n}\|_\infty < \frac{1}{2\hat{M}^{\frac{1}{3}}}$ holds, then

$$\|\rho^{3n}(\omega^{3n})-T_3\left(\rho^{3n-1}(\omega^{3n-1})\right)\|_\infty$$

$$\leq \|\rho^{3n}(\omega^{3n})-\mathbb{E}^{\omega^{3n-1}}\rho^{3n}\|_\infty + \|\mathbb{E}^{\omega^{3n-1}}\rho^{3n}-T_3\left(\rho^{3n-1}(\omega^{3n-1})\right)\|_\infty$$

$$< \frac{1}{2\hat{M}^{\frac{1}{3}}} + \frac{2K(K+1)}{\hat{M}} < \frac{1}{2\hat{M}^{\frac{1}{3}}} + \frac{1}{2\hat{M}^{\frac{1}{3}}} = \frac{1}{\hat{M}^{\frac{1}{3}}}.$$

Let $W^{3n} = \{\omega^{3n} \in \Omega^{3n} : \|\rho^{3n}(\omega^{3n}) - T_3\left(\rho^{3n-1}(\omega^{3n-1})\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}}\}$. It is clear that

$$
\begin{aligned}
Q^{3n}(W^{3n}) &= \int_{\Omega^{3n-1}} Q_{3n}^{\omega^{3n-1}} \left(\|\rho^{3n} - T_3\left(\rho^{3n-1}\right)\|_\infty \geq \frac{1}{\hat{M}^{\frac{1}{3}}} \right) dQ^{3n-1} \\
&\leq \int_{\Omega^{3n-1}} Q_{3n}^{\omega^{3n-1}} \left(\|\rho^{3n} - \mathbb{E}^{\omega^{3n-1}} \rho^{3n}\|_\infty \geq \frac{1}{2\hat{M}^{\frac{1}{3}}} \right) dQ^{3n-1} \leq \frac{K}{\hat{M}^{\frac{1}{3}}}. \qquad (12)
\end{aligned}
$$

For any $m \in \mathbb{T}_0$, let

$$
\overline{W}^m = \{\omega^m \in \Omega^m : \omega^{m'} \in W^{m'} \text{ for some } m' \text{ between } 1 \text{ and } m\}.
$$

By Equations (10), (11) and (12), we have

$$
Q^m(\overline{W}^m) \leq \sum_{m'=0}^{m} Q^{m'}(W^{m'}) \leq \frac{3M^2K}{\hat{M}^{\frac{1}{3}}}.
$$

The definition of $\hat{M}$ implies that $\frac{1}{\hat{M}^{\frac{1}{3}}} \leq \frac{1}{\hat{M}^{\frac{1}{9}}} \leq \xi_{3M^2+1} \leq \frac{1}{3M^2K}\xi_1$. Then, $Q^m(\overline{W}^m) \leq \xi_1$.

Fix any $m \in \{0, 1, \ldots, 3M^2\}$ and $\omega^m \notin \overline{W}^m$. We have

$$
\begin{aligned}
&\|\rho^m(\omega^m) - U_1^m(\rho^0)\|_\infty \\
&\leq \|\rho^m(\omega^m) - U_m^m(\rho^{m-1}(\omega^{m-1}))\|_\infty + \|U_m^m(\rho^{m-1}(\omega^{m-1})) - U_1^m(\rho^0)\|_\infty \\
&\leq \sum_{j=1}^{m} \|U_{j+1}^m(\rho^j(\omega^j)) - U_j^m(\rho^{j-1}(\omega^{j-1}))\|_\infty \\
&= \sum_{j=1}^{m} \left\| U_{j+1}^m(\rho^j(\omega^j)) - U_{j+1}^m\left(U_j^j(\rho^{j-1}(\omega^{j-1}))\right) \right\|_\infty.
\end{aligned}
$$

The fact that $\omega^m \notin \overline{W}^m$ leads to $\omega^j \notin W^j$ for any $j \in \{0, 1, \ldots, m\}$. The definition of W^j implies that

$$
\|\rho^j(\omega^j) - U_j^j(\rho^{j-1}(\omega^{j-1}))\|_\infty < \frac{1}{\hat{M}^{\frac{1}{3}}} \leq \xi_{3M^2+1}.
$$

By Lemma 2, we have

$$
\|\rho^m(\omega^m) - U_1^m(\rho^0)\|_\infty \leq \sum_{j=0}^{m-1} \xi_{3M^2+1-j} \leq \sum_{j=0}^{m-1} \frac{1}{3M^2K}\xi_1 \leq \xi_1,
$$

which implies that $\omega^m \notin V^m = \{\omega^m \in \Omega^m : \|\rho^m(\omega^m) - U_1^m(\rho^0)\|_\infty > \xi_1\}$. Since ω^m is an arbitrarily fixed element in $\Omega^m \setminus \overline{W}^m$, the fact that $\omega^m \notin V^m$ implies that $V^m \subseteq \overline{W}^m$. Therefore, we have $Q^m(V^m) \leq \xi_1$.

B.4. Proof of Lemma 4 in Section 3.3. Fix any $m \in \{0, 1, \ldots, 3M^2\}$. Recall that $V^m = \{\omega^m \in \Omega^m : \|\rho^m(\omega^m) - U_1^m(\rho^0)\|_\infty > \xi_1\}$. By Lemma 3, we know that $P_0(V^m) \leq \xi_1$. Then, we can obtain that

$$
\|\mathbb{E}(\rho^m) - U_1^m(\rho^0)\|_\infty = \left\| \int_{\Omega^m} \left(\rho^m - U_1^m(\rho^0)\right) dQ^m \right\|_\infty \leq \int_{\Omega^m} \|\rho^m - U_1^m(\rho^0)\|_\infty dQ^m
$$

$$\leq \int_{V^m} \|\rho^m - U_1^m(\rho^0)\|_\infty dQ^m + \int_{\Omega^m \setminus V^m} \|\rho^m - U_1^m(\rho^0)\|_\infty dQ^m$$

$$\leq \xi_1 + \xi_1 \leq 2\xi_0 = \frac{2}{K^{M^3}}.$$

Let $B_1(M) = \frac{2}{K^{M^3}}$. It is clear that $\lim_{M\to\infty} B_1(M) = 0$.

B.5. Additional Lemmas B.1–B.3. This subsection presents three additional lemmas. Lemma B.1 (Lemma B.2) is used to prove Lemma B.2 (Lemma B.3), while Lemma B.3 is used in the proof of Lemmas 5 and 6 of the main text.

The following lemma shows that $\frac{1}{M^2}$ is a lower bound for $\left[U_1^{3n-2}(\rho^0)\right]_k$.

LEMMA B.1. *For any $n \in \{1, 2, \ldots, M^2\}$ and $k \in S$, we have $\left[U_1^{3n-2}(\rho^0)\right]_k \geq \frac{1}{M^2}$.*

PROOF. Note that $\hat{\eta}_{kl} \geq \frac{1}{M^2}$ for any $k, l \in S$ by its definition. The definition of T_1 implies that for any $k \in S$,

$$\left[U_1^{3n-2}(\rho^0)\right]_k = \left[T_1\left(U_1^{3n-3}(\rho^0)\right)\right]_k$$
$$= \sum_{l\in S}\left[U_1^{3n-3}(\rho^0)\right]_l \hat{\eta}_{lk} \geq \frac{1}{M^2}\sum_{l\in S}\left[U_1^{3n-3}(\rho^0)\right]_l = \frac{1}{M^2},$$

which is the required inequality in the lemma. ∎

The following lemma shows that, if ω^{3n-2} is not in V^{3n-2}, then $\frac{1}{2M^2}$ is a lower bound for the population of type-k agents after step $3n-2$.

LEMMA B.2. *For any $n \in \{1, 2, \ldots, M^2\}$, $\omega^{3n-2} \notin V^{3n-2}$ and $k \in S$, we have $\rho_k^{3n-2}(\omega^{3n-2}) \geq \frac{1}{2M^2}$.*

PROOF. Fix any $n \in \mathbb{T}_0 = \{1, 2, \ldots, M^2\}$ and $\omega^{3n-2} \notin V^{3n-2}$. By the definition of V^{3n-2} in Lemma 3, we know that $\|\rho^{3n-2}(\omega^{3n-2}) - U_1^{3n-2}(\rho^0)\|_\infty \leq \xi_1$. It follows from Lemma B.1 that

$$\rho_k^{3n-2}(\omega^{3n-2}) \geq [U_1^{3n-2}(\rho^0)]_k - \xi_1 \geq \frac{1}{M^2} - \xi_0.$$

Note that $\xi_0 = \frac{1}{K^{M^3}}$. It is clear that $\xi_0 \leq \frac{1}{2M^2}$. Therefore, we have

$$\rho_k^{3n-2}(\omega^{3n-2}) \geq \frac{1}{M^2} - \frac{1}{2M^2} = \frac{1}{2M^2},$$

which is the required inequality in the lemma. ∎

The following lemma provides an approximation of the matching probabilities at step $3n-1$ using parameter $\hat{q}$.

LEMMA B.3. *For any $i, j \in I$ with $i \neq j$, any $\omega^{3n-2} \notin V^{3n-2}$ and any $k_1, l_1, k_2, l_2 \in S$, if $\hat{\alpha}_i^{3n-2}\left(\omega^{3n-2}\right) = k_1$ and $\hat{\alpha}_j^{3n-2}\left(\omega^{3n-2}\right) = k_2$, then*

$$\left| Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1) - \hat{q}_{k_1 l_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right| < \frac{1}{\hat{M}^{\frac{1}{9}}},$$

$$\left| Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = J) - \hat{q}_{k_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right| < \frac{1}{\hat{M}^{\frac{1}{9}}},$$

$$\left| Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1, \hat{g}_j^{3n-1} = l_2) - \hat{q}_{k_1 l_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2 l_2}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right| < \frac{1}{\hat{M}^{\frac{1}{9}}},$$

$$\left| Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = J, \hat{g}_j^{3n-1} = l_2) - \hat{q}_{k_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2 l_2}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right| < \frac{1}{\hat{M}^{\frac{1}{9}}},$$

$$\left| Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = J, \hat{g}_j^{3n-1} = J) - \hat{q}_{k_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right| < \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

PROOF. Fix any $i, j \in I$ with $i \neq j$, $\omega^{3n-2} \notin V^{3n-2}$ and $k_1, l_1, k_2, l_2 \in S$. Assume that $\hat{\alpha}_i^{3n-2}\left(\omega^{3n-2}\right) = k_1$ and $\hat{\alpha}_j^{3n-2}\left(\omega^{3n-2}\right) = k_2$. Since $\hat{M} > \left(\frac{1}{\xi_{3M^2+1}}\right)^9 \geq \left(\frac{1}{\xi_0}\right)^9 = K^{9M^3}$, it is clear that $5K^2 + 2K < \hat{M}^{\frac{5}{9}}$. By Lemma B.2, we have $\rho_{k_1}^{3n-2}\left(\omega^{3n-2}\right) \geq \frac{1}{2M^2} > \frac{1}{\hat{M}^{\frac{1}{3}}}$, and $\rho_{k_2}^{3n-2}\left(\omega^{3n-2}\right) \geq \frac{1}{2M^2} > \frac{1}{\hat{M}^{\frac{1}{3}}}$. It follows from Lemma 1 that

$$\left| Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1) - \hat{q}_{k_1 l_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \leq \frac{2}{\hat{M}^{\frac{2}{3}}} = \frac{2}{\hat{M}^{\frac{5}{9}}}\frac{1}{\hat{M}^{\frac{1}{9}}} < \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

Lemma 1 also implies that

$$\left| Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1} = l_1, \hat{g}_j^{3n-1} = l_2\right) - \hat{q}_{k_1 l_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2 l_2}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right|$$
$$\leq \frac{5}{\hat{M}^{\frac{2}{3}}} = \frac{5}{\hat{M}^{\frac{5}{9}}}\frac{1}{\hat{M}^{\frac{1}{9}}} < \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

Next, we consider the case when agent i is not matched. We can obtain

$$\left| Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = J) - \hat{q}_{k_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right|$$
$$= \left| \sum_{l_1 \in S} Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1} = l_1) - \sum_{l_1 \in S} \hat{q}_{k_1 l_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right|$$
$$\leq \frac{2K}{\hat{M}^{\frac{2}{3}}} = \frac{2K}{\hat{M}^{\frac{5}{9}}}\frac{1}{\hat{M}^{\frac{1}{9}}} < \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

Similarly, we can prove that

$$\left| Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1} = J, \hat{g}_j^{3n-1} = l_2\right) - \hat{q}_{k_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2 l_2}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right|$$
$$= \left| Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_j^{3n-1} = l_2\right) - \sum_{l' \in S} Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1} = l', \hat{g}_j^{3n-1} = l_2\right) - \right.$$
$$\left. - \hat{q}_{k_2 l_2}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right) + \sum_{l' \in S} \hat{q}_{k_1 l'}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2 l_2}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right|$$
$$\leq \frac{5K+2}{\hat{M}^{\frac{2}{3}}} = \frac{5K+2}{\hat{M}^{\frac{5}{9}}}\frac{1}{\hat{M}^{\frac{1}{9}}} < \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

It remains to consider the case when agents i and j are not matched. We have

$$
\begin{aligned}
&\left|Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=J,\hat{g}_j^{3n-1}=J\right)-\hat{q}_{k_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\
=&\left|Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=J\right)-\sum_{l'\in S}Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1}=J,\hat{g}_j^{3n-1}=l'\right)\right. \\
&\left.-\hat{q}_{k_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)+\sum_{l'\in S}\hat{q}_{k_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\hat{q}_{k_2l'}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\
\le&\frac{5K^2+2K}{\hat{M}^{\frac{2}{3}}}=\frac{5K^2+2K}{\hat{M}^{\frac{5}{9}}}\frac{1}{\hat{M}^{\frac{1}{9}}}<\frac{1}{\hat{M}^{\frac{1}{9}}}.
\end{aligned}
$$

The proof is thus completed.

B.6. Proof of Lemma 5 in Section 3.3. By Lemma B.3 and the fact that $\frac{1}{\hat{M}^{\frac{1}{9}}}<\frac{1}{M^2}$, it is clear that

$$
\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{g}_i^{3n-1}=l)-\hat{q}_{kl}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right|\le\frac{1}{\hat{M}^{\frac{1}{9}}}<\frac{1}{M^2}
$$

for any $i\in I$, $\omega^{3n-2}\notin V^{3n-2}$ and $k,l\in S$ with $\hat{\alpha}_i^{3n-2}\left(\omega^{3n-2}\right)=k$.

B.7. Proofs of Lemmas A.1–A.3 in Appendix A.

B.7.1. *Proof of Lemma A.1.* Fix any $i\in\bar{I}$. We first show that for any $m\in\{0,1,,\dots,\bar{N}-1\}$, $a,b\in\bar{X}$ and $F^m\in\bar{\mathcal{F}}^m$,

$$
\left|\bar{P}_0\left(f_i^{m+1}=b,f_i^m=a,F^m\right)-\bar{\vartheta}_{ab}^m\bar{P}_0\left(f_i^m=a,F^m\right)\right|\le 2\bar{\varepsilon}_1.
$$

Let

$$
A=\left\{\omega^m\in\bar{\Omega}^m:f_i^m(\omega^m)=a\right\}\cap F^m.
$$

It is clear that

$$
\bar{P}_0\left(f_i^{m+1}=b,f_i^m=a,F^m\right)=\int_{\omega^m\in A}\bar{Q}_{m+1}^{\omega^m}\left(f_i^{m+1}=b\right)d\bar{Q}^m.
$$

Recall that $\bar{Q}^m\left(\bar{C}^m(\bar{\varepsilon}_1)\right)<\bar{\varepsilon}_1$ and for any $\omega^m\in A\backslash\bar{C}^m(\bar{\varepsilon}_1)$,

$$
\left|\bar{Q}_{m+1}^{\omega^m}\left(f_i^{m+1}=b\right)-\bar{\vartheta}_{ab}^m\right|\le\bar{\varepsilon}_1.
$$

We can then obtain that

$$
\begin{aligned}
&\left|\bar{P}_0\left(f_i^{m+1}=b,f_i^m=a,F^m\right)-\bar{\vartheta}_{ab}^m\bar{P}_0\left(f_i^m=a,F^m\right)\right| \\
=&\left|\int_A\left(\bar{Q}_{m+1}^{\omega^m}\left(f_i^{m+1}=b\right)-\bar{\vartheta}_{ab}^m\right)d\bar{Q}^m\right|
\end{aligned}
$$

$$\leq \int_{A\cap\bar{C}^m(\bar{\varepsilon}_1)} \left|\bar{Q}^{\omega^m}_{m+1}\left(f_i^{m+1}=b\right)-\bar{\vartheta}^m_{ab}\right| d\bar{Q}^m$$
$$+\int_{A\backslash\bar{C}^m(\bar{\varepsilon}_1)} \left|\bar{Q}^{\omega^m}_{m+1}\left(f_i^{m+1}=b\right)-\bar{\vartheta}^m_{ab}\right| d\bar{Q}^m$$
$$\leq \bar{Q}^m(\bar{C}^m(\bar{\varepsilon}_1))+\bar{Q}^m(A\backslash\bar{C}^m(\bar{\varepsilon}_1))\bar{\varepsilon}_1$$
$$\leq \bar{\varepsilon}_1+\bar{\varepsilon}_1=2\bar{\varepsilon}_1\leq 2|\bar{X}|\bar{\varepsilon}_1. \tag{13}$$

Proceeding inductively, assume that for a positive integer m',

$$\left|\bar{P}_0\left(f_i^{m+m'}=b,f_i^m=a,F^m\right)-\bar{\Theta}^{m+m'-1}_m(a,b)\bar{P}_0\left(f_i^m=a,F^m\right)\right|\leq(2|\bar{X}|)^{m'}\bar{\varepsilon}_1 \tag{14}$$

for any $a,b\in\bar{X}$, $m\in\{0,1,\ldots,\bar{N}-m'\}$, and $F^m\in\bar{\mathcal{F}}^m$. Note that

$$\begin{aligned}
&\bar{P}_0\left(f_i^{m+m'+1}=b,f_i^m=a,F^m\right)\\
&=\sum_{c\in\bar{X}}\bar{P}_0\left(f_i^{m+m'+1}=b,f_i^{m+m'}=c,f_i^m=a,F^m\right)\\
&=\sum_{c\in\bar{X}}\bar{P}_0\left(f_i^{m+m'+1}=b\,\middle|\,f_i^{m+m'}=c,f_i^m=a,F^m\right)\\
&\qquad\times\bar{P}_0\left(f_i^{m+m'}=c,f_i^m=a,F^m\right)
\end{aligned}$$

and

$$\bar{\Theta}^{m+m'}_m(a,b)=\sum_{c\in\bar{X}}\bar{\Theta}^{m+m'-1}_m(a,c)\ \bar{\vartheta}^{m+m'}(c,b).$$

We can then obtain that

$$\begin{aligned}
&\left|\bar{P}_0\left(f_i^{m+m'+1}=b,f_i^m=a,F^m\right)-\bar{\Theta}^{m+m'}_m(a,b)\bar{P}_0\left(f_i^m=a,F^m\right)\right|\\
\leq&\left|\bar{P}_0\left(f_i^{m+m'+1}=b,f_i^m=a,F^m\right)-\sum_{c\in\bar{X}}\bar{\vartheta}^{m+m'}_{c,b}\bar{P}_0\left(f_i^{m+m'}=c,f_i^m=a,F^m\right)\right|\\
&+\left|\sum_{c\in\bar{X}}\bar{\vartheta}^{m+m'}_{c,b}\bar{P}_0\left(f_i^{m+m'}=c,f_i^m=a,F^m\right)-\sum_{c\in\bar{X}}\bar{\vartheta}^{m+m'}_{c,b}\bar{\Theta}^{m+m'-1}_m(a,c)\bar{P}_0\left(f_i^m=a,F^m\right)\right|\\
\leq&\sum_{c\in\bar{X}}\left|\bar{P}_0\left(f_i^{m+m'+1}=b\,\middle|\,f_i^{m+m'}=c,f_i^m=a,F^m\right)-\bar{\vartheta}^{m+m'}_{c,b}\right|\bar{P}_0\left(f_i^{m+m'}=c,f_i^m=a,F^m\right)\\
&+\sum_{c\in\bar{X}}\vartheta^{m+m'}_{c,b}\left|\bar{P}_0\left(f_i^{m+m'}=c,f_i^m=a,F^m\right)-\bar{\Theta}^{m+m'-1}_m(a,c)\bar{P}_0\left(f_i^m=a,F^m\right)\right|.
\end{aligned}$$

By Equations (13) and (14), we know that

$$\begin{aligned}
&\left|\bar{P}_0\left(f_i^{m+m'+1}=b,f_i^m=a,F^m\right)-\bar{\Theta}^{m+m'}_m(a,b)\bar{P}_0\left(f_i^m=a,F^m\right)\right|\\
&\leq 2|\bar{X}|\bar{\varepsilon}_1+|\bar{X}|(2|\bar{X}|)^{m'}\bar{\varepsilon}_1\leq(2|\bar{X}|)^{m'+1}\bar{\varepsilon}_1.
\end{aligned}$$

Hence, the first inequality of the lemma holds.

For the second inequality of the lemma, we only need to consider the case when $\bar{P}_0\left(f_i^m = a, F^m\right) > 0$. The first inequality implies that

$$\left|\bar{P}_0\left(f_i^{m+m'} = b \,\middle|\, f_i^m = a, F^m\right) - \bar{\Theta}_m^{m+m'-1}(a,b)\right| \le \frac{(2|\bar{X}|)^{m'}\bar{\varepsilon}_1}{\bar{P}_0\left(f_i^m = a, F^m\right)},$$

$$\left|\bar{P}_0\left(f_i^{m+m'} = b \,\middle|\, f_i^m = a\right) - \bar{\Theta}_m^{m+m'-1}(a,b)\right| \le \frac{(2|\bar{X}|)^{m'}\bar{\varepsilon}_1}{\bar{P}_0\left(f_i^m = a\right)}.$$

By the triangle inequality, we know that

$$\left|\bar{P}_0\left(f_i^{m+m'} = b \,\middle|\, f_i^m = a, F^m\right) - \bar{P}_0\left(f_i^{m+m'} = b \,\middle|\, f_i^m = a\right)\right|$$
$$\le \frac{(2|\bar{X}|)^{m'}\bar{\varepsilon}_1}{\bar{P}_0\left(f_i^m = a, F^m\right)} + \frac{(2|\bar{X}|)^{m'}\bar{\varepsilon}_1}{\bar{P}_0\left(f_i^m = a\right)},$$

which implies that

$$\left|\bar{P}_0\left(f_i^{m+m'} = b, f_i^m = a, F^m\right)\bar{P}_0\left(f_i^m = a\right) - \bar{P}_0\left(f_i^{m+m'} = b, f_i^m = a\right)\bar{P}_0\left(f_i^m = a, F^m\right)\right|$$
$$\le (2|\bar{X}|)^{m'}\bar{\varepsilon}_1\left(\bar{P}_0\left(f_i^m = a, F^m\right) + \bar{P}_0\left(f_i^m = a\right)\right)$$
$$\le 2(2|\bar{X}|)^{\bar{N}}\bar{\varepsilon}_1 \le (2|\bar{X}|)^{|\bar{T}|}\bar{\varepsilon}_1.$$

Hence,the second inequality of the lemma holds.

B.7.2. *Proof of Lemma A.2.* When $|\bar{X}| = 1$, the inequality in the lemma holds trivially. Without loss of generality, assume that $|\bar{X}| \ge 2$ in rest of this proof. We need to provide a sequence of estimations $\{d_m\}_{m\in\bar{T}}$ such that for any $m \in \bar{T}$, $a, b \in \bar{X}$ and $F_i^{m-1} \in \bar{\mathcal{F}}_i^{m-1}$ and $F_j^{m-1} \in \bar{\mathcal{F}}_j^{m-1}$, we have

$$\text{(15)} \quad \left|\bar{P}_0(f_i^m = a, F_i^{m-1}, f_j^m = b, F_j^{m-1}) - \bar{P}_0(f_i^m = a, F_i^{m-1})\bar{P}_0(f_j^m = b, F_j^{m-1})\right| \le d_m.$$

Fix any $m \in \bar{T}$. When $m = 0$, we can take d_0 to be 0. Suppose that we have already defined d_m, we need to define d_{m+1} using d_m.

Fix any $a_1, a_2, b_1, b_2 \in \bar{X}$, $F_i^{m-1} \in \bar{\mathcal{F}}_i^{m-1}$ and $F_j^{m-1} \in \bar{\mathcal{F}}_j^{m-1}$. We need to estimate the following difference

$$\left|\bar{P}_0\left(f_i^{m+1} = b_1, f_j^{m+1} = b_2, f_i^m = a_1, f_j^m = a_2, F_i^{m-1}, F_j^{m-1}\right)\right.$$
$$\left. -\bar{P}_0\left(f_i^{m+1} = b_1, f_i^m = a_1, F_i^{m-1}\right)\bar{P}_0\left(f_j^{m+1} = b_2, f_j^m = a_2, F_j^{m-1}\right)\right|.$$

For notational simplicity, let

$$B = \left\{\omega^m \in \bar{\Omega}^m : f_i^m = a_1, f_j^m = a_2\right\} \cap F_i^{m-1} \cap F_j^{m-1},$$

$$B' = \{\omega^m \in \bar{\Omega}^m : f_i^m = a_1\} \cap F_i^{m-1},$$

$$B'' = \{\omega^m \in \bar{\Omega}^m : f_j^m = a_2\} \cap F_j^{m-1}.$$

We can obtain that

$$\bar{P}_0\left(f_i^{m+1}=b_1, f_j^{m+1}=b_2, f_i^m=a_1, f_j^m=a_2, F_i^{m-1}, F_j^{m-1}\right)$$
$$=\int_B \bar{Q}_{m+1}^{\omega^m}\left(f_i^{m+1}=b_1, f_j^{m+1}=b_2\right) d\bar{Q}^m.$$

Since f_i and f_j are not $\bar{\varepsilon}_2$-correlated, the probability of the event

$$\{\omega^m \in \bar{\Omega}^m : f_i \text{ and } f_j \text{ are } \bar{\varepsilon}_2\text{-correlated at } \omega^m\}$$

is less than or equal to $\bar{\varepsilon}_2$. Let

$$D^m = \{\omega^m \in \bar{\Omega}^m : \text{there exists } a, b \in \bar{X} \text{ such that}$$
$$\left|\bar{Q}_{m+1}^{\omega^m}\left(f_i^{m+1}=a, f_j^{m+1}=b\right) - \bar{Q}_{m+1}^{\omega^m}\left(f_i^{m+1}=a\right)\bar{Q}_{m+1}^{\omega^m}\left(f_j^{m+1}=b\right)\right| \geq \bar{\varepsilon}_2\}$$

It is clear that $\bar{Q}^m(D^m) \leq \bar{\varepsilon}_2$. Then, we have

$$\left|\bar{P}_0\left(f_i^{m+1}=b_1, f_j^{m+1}=b_2, f_i^m=a_1, f_j^m=a_2, F_i^{m-1}, F_j^{m-1}\right)\right.$$
$$\left.-\int_B \bar{Q}_{m+1}^{\omega^m}\left(f_i^{m+1}=b_1\right)\bar{Q}_{m+1}^{\omega^m}\left(f_j^{m+1}=b_2\right) d\bar{Q}^m\right|$$
$$=\left|\int_B \left(\bar{Q}_{m+1}^{\omega^m}(f_i^{m+1}=b_1, f_j^{m+1}=b_2) - \bar{Q}_{m+1}^{\omega^m}\left(f_i^{m+1}=b_1\right)\bar{Q}_{m+1}^{\omega^m}\left(f_j^{m+1}=b_2\right)\right) d\bar{Q}^m\right|$$
$$\leq \int_{B\setminus D^m}\left|\bar{Q}_{m+1}^{\omega^m}(f_i^{m+1}=b_1, f_j^{m+1}=b_2) - \bar{Q}_{m+1}^{\omega^m}\left(f_i^{m+1}=b_1\right)\bar{Q}_{m+1}^{\omega^m}\left(f_j^{m+1}=b_2\right)\right| d\bar{Q}^m$$
$$+\bar{Q}^m(D^m)$$
$$\leq \bar{\varepsilon}_2 + \bar{\varepsilon}_2 = 2\bar{\varepsilon}_2. \tag{16}$$

Next, we estimate the difference

$$\left|\int_B \bar{Q}_{m+1}^{\omega^m}\left(f_i^{m+1}=b_1\right)\bar{Q}_{m+1}^{\omega^m}\left(f_j^{m+1}=b_2\right) d\bar{Q}^m - \bar{P}_0(B)\bar{\vartheta}_{a_1b_1}^m\bar{\vartheta}_{a_2b_2}^m\right|.$$

Recall the definition of $\bar{C}^m(\bar{\varepsilon}_1)$ above Equation (33) in the main text. that

$$\bar{C}^m(\bar{\varepsilon}_1) = \{\omega^m \in \bar{\Omega}^m : \left|\bar{Q}_{m+1}^{\omega^m}(f_i^{m+1}=a) - \bar{\vartheta}^m(f_i^m(\omega^m), a)\right| > \bar{\varepsilon}_1 \text{ for some } i \in \bar{I} \text{ and } a \in \bar{X}\}.$$

By Equation (33) in the main text, $\bar{Q}^m(\bar{C}^m(\bar{\varepsilon}_1)) \leq \bar{\varepsilon}_1$. Then, we can obtain that

$$\left|\int_B \bar{Q}_{m+1}^{\omega^m}\left(f_i^{m+1}=b_1\right)\bar{Q}_{m+1}^{\omega^m}\left(f_j^{m+1}=b_2\right) d\bar{Q}^m - \bar{P}_0(B)\bar{\vartheta}_{a_1b_1}^m\bar{\vartheta}_{a_2b_2}^m\right|$$
$$\leq \int_B \left|\bar{Q}_{m+1}^{\omega^m}\left(f_i^{m+1}=b_1\right)\bar{Q}_{m+1}^{\omega^m}\left(f_j^{m+1}=b_2\right) - \bar{\vartheta}_{a_1b_1}^m\bar{\vartheta}_{a_2b_2}^m\right| d\bar{Q}^m$$
$$\leq \int_{B\setminus\bar{C}^m(\bar{\varepsilon}_1)} \left|\bar{Q}_{m+1}^{\omega^m}\left(f_i^{m+1}=b_1\right)\bar{Q}_{m+1}^{\omega^m}\left(f_j^{m+1}=b_2\right) - \bar{\vartheta}_{a_1b_1}^m\bar{\vartheta}_{a_2b_2}^m\right| d\bar{Q}^m + \bar{Q}^m(\bar{C}^m(\bar{\varepsilon}_1))$$
$$= \int_{B\setminus\bar{C}^m(\bar{\varepsilon}_1)} \left|\bar{Q}_{m+1}^{\omega^m}\left(f_i^{m+1}=b_1\right)\bar{Q}_{m+1}^{\omega^m}\left(f_j^{m+1}=b_2\right) - \bar{\vartheta}_{a_1b_1}^m\bar{Q}_{m+1}^{\omega^m}\left(f_j^{m+1}=b_2\right)\right| d\bar{Q}^m$$

$$
\begin{aligned}
&+\int_{B\setminus\bar{C}^m(\bar{\varepsilon}_1)}\left|\bar{\vartheta}^m_{a_1b_1}\bar{Q}^{\omega^m}_{m+1}\left(f^{m+1}_j=b_2\right)-\bar{\vartheta}^m_{a_1b_1}\bar{\vartheta}^m_{a_2b_2}\right|d\bar{Q}^m+\bar{\varepsilon}_1\\
\le&\int_{B\setminus\bar{C}^m(\bar{\varepsilon}_1)}\left|\bar{Q}^{\omega^m}_{m+1}\left(f^{m+1}_i=b_1\right)-\bar{\vartheta}^m_{a_1b_1}\right|d\bar{Q}^m\\
&+\int_{B\setminus\bar{C}^m(\bar{\varepsilon}_1)}\left|\bar{Q}^{\omega^m}_{m+1}\left(f^{m+1}_j=b_2\right)-\bar{\vartheta}^m_{a_2b_2}\right|d\bar{Q}^m+\bar{\varepsilon}_1\\
\le&\ \bar{\varepsilon}_1+\bar{\varepsilon}_1+\bar{\varepsilon}_1=3\bar{\varepsilon}_1.
\end{aligned}
\tag{17}
$$

By Equations (16) and (17), we have

$$
\begin{aligned}
&\left|\bar{P}_0\left(f^{m+1}_i=b_1,f^{m+1}_j=b_2,f^m_i=a_1,f^m_j=a_2,F^{m-1}_i,F^{m-1}_j\right)\right.\\
&\qquad\left.-\bar{P}_0(B)\bar{\vartheta}^m_{a_1b_1}\bar{\vartheta}^m_{a_2b_2}\right|\le 3\bar{\varepsilon}_1+2\bar{\varepsilon}_2.
\end{aligned}
\tag{18}
$$

It follows from the definition of $\bar{C}^m(\bar{\varepsilon}_1)$ and Equation (33) in the main text again that

$$
\begin{aligned}
&\left|\bar{P}_0\left(f^{m+1}_i=b_1,f^m_i=a_1,F^{m-1}_i\right)-\bar{P}_0(B')\bar{\vartheta}^m_{a_1b_1}\right|\\
=&\left|\int_{B'}\left(\bar{Q}^{\omega^m}_{m+1}(f^{m+1}_i=b_1)-\bar{\vartheta}^m_{a_1b_1}\right)d\bar{Q}^m\right|\\
\le&\int_{B'\setminus\bar{C}^m(\bar{\varepsilon}_1)}\left|\bar{Q}^{\omega^m}_{m+1}(f^{m+1}_i=b_1)-\bar{\vartheta}^m_{a_1b_1}\right|d\bar{Q}^m+\bar{Q}^m(\bar{C}^m(\bar{\varepsilon}_1))\\
\le&\ \bar{\varepsilon}_1+\bar{\varepsilon}_1=2\bar{\varepsilon}_1.
\end{aligned}
\tag{19}
$$

Equation (19) states an inequality for a general index i, which can be restated for index j as follows:

$$
\left|\bar{P}_0\left(f^{m+1}_j=b_2,f^m_j=a_2,F^{m-1}_j\right)-\bar{P}_0(B'')\bar{\vartheta}^m_{a_2b_2}\right|\le 2\bar{\varepsilon}_1.
\tag{20}
$$

Based on Equations (19) and (20), we can obtain that

$$
\begin{aligned}
&\left|\bar{P}_0\left(f^{m+1}_i=b_1,f^m_i=a_1,F^{m-1}_i\right)\bar{P}_0\left(f^{m+1}_j=b_2,f^m_j=a_2,F^{m-1}_j\right)\right.\\
&\qquad\left.-\bar{P}_0(B')\bar{P}_0(B'')\bar{\vartheta}^m_{a_1b_1}\bar{\vartheta}^m_{a_2b_2}\right|\\
\le&\left|\bar{P}_0(f^{m+1}_i=b_1,f^m_i=a_1,F^{m-1}_i)\bar{P}_0(f^{m+1}_j=b_2,f^m_j=a_2,F^{m-1}_j)\right.\\
&\qquad\left.-\bar{P}_0(B')\bar{\vartheta}^m_{a_1b_1}\bar{P}_0(f^{m+1}_j=b_2,f^m_j=a_2,F^{m-1}_j)\right|\\
&\quad+\left|\bar{P}_0(B')\bar{\vartheta}^m_{a_1b_1}\bar{P}_0(f^{m+1}_j=b_2,f^m_j=a_2,F^{m-1}_j)-\bar{P}_0(B')\bar{P}_0(B'')\bar{\vartheta}^m_{a_1b_1}\bar{\vartheta}^m_{a_2b_2}\right|\\
\le&\left|\bar{P}_0(f^{m+1}_i=b_1,f^m_i=a_1,F^{m-1}_i)-\bar{P}_0(B')\bar{\vartheta}^m_{a_1b_1}\right|\\
&\qquad+\left|\bar{P}_0(f^{m+1}_j=b_2,f^m_j=a_2,F^{m-1}_j)-\bar{P}_0(B'')\bar{\vartheta}^m_{a_2b_2}\right|\\
\le&\ 4\bar{\varepsilon}_1.
\end{aligned}
\tag{21}
$$

The induction hypothesis indicates that $\left|\bar{P}_0(B) - \bar{P}_0(B')\bar{P}_0(B'')\right| \le d_m$. By Equations (18) and (21), we have

$$\begin{aligned}
&\Big|\bar{P}_0\left(f_i^{m+1} = b_1, f_j^{m+1} = b_2, f_i^m = a_1, f_j^m = a_2, F_i^{m-1}, F_j^{m-1}\right) \\
&\qquad - \bar{P}_0\left(f_i^{m+1} = b_1, f_i^m = a_1, F_i^{m-1}\right)\bar{P}_0\left(f_j^{m+1} = b_2, f_j^m = a_2, F_j^{m-1}\right)\Big| \\
&\le \left|\bar{P}_0(B)\bar{\vartheta}^m_{a_1 b_1}\bar{\vartheta}^m_{a_2 b_2} - \bar{P}_0(B')\bar{P}_0(B'')\bar{\vartheta}^m_{a_1 b_1}\bar{\vartheta}^m_{a_2 b_2}\right| + +7\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2 \\
&\le \left|\bar{P}_0(B) - \bar{P}_0(B')\bar{P}_0(B'')\right| + 7\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2 \le 7\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2 + d_m. \qquad (22)
\end{aligned}$$

Fix any $F_i^m \in \bar{\mathcal{F}}_i^m$. There exists $F_{ic}^{m-1} \in \bar{\mathcal{F}}_i^{m-1}$ for any $c \in \bar{X}$ such that

$$F_i^m = \bigcup_{c \in \bar{X}} \left((f_i^m = c) \cap F_{ic}^{m-1}\right).$$

Similarly, for any fixed $F_j^m \in \bar{\mathcal{F}}_j^m$, there exists $F_{jc}^{m-1} \in \bar{\mathcal{F}}_j^{m-1}$ for any $c \in \bar{X}$ such that

$$F_j^m = \bigcup_{c \in \bar{X}} \left((f_j^m = c) \cap F_{jc}^{m-1}\right).$$

Therefore, it follows from Equation (22) that

$$\begin{aligned}
&\left|\bar{P}_0\left(f_i^{m+1} = b_1, f_j^{m+1} = b_2, F_i^m, F_j^m\right) - \bar{P}_0\left(f_i^{m+1} = b_1, F_i^m\right)\bar{P}_0\left(f_j^{m+1} = b_2, F_j^m\right)\right| \\
&\le \sum_{c_1, c_2 \in \bar{X}} \Big|\bar{P}_0\left(f_i^{m+1} = b_1, f_j^{m+1} = b_2, f_i^m = c_1, f_j^m = c_2, F_{ic_1}^{m-1}, F_{jc_2}^{m-1}\right) \\
&\qquad - \bar{P}_0\left(f_i^{m+1} = b_1, f_i^m = c_1, F_{ic_1}^{m-1}\right)\bar{P}_0\left(f_j^{m+1} = b_2, f_j^m = c_2, F_{jc_2}^{m-1}\right)\Big| \\
&\le |\bar{X}|^2\left(7\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2 + d_m\right). \qquad (23)
\end{aligned}$$

Thus, we can define d_{m+1} to be $|\bar{X}|^2\left(7\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2 + d_m\right)$.

Next, we prove that for any $m \in \bar{T}$,

$$d_m \le |\bar{X}|^{3m}\left(7\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2\right). \qquad (24)$$

Since $d_0 = 0$, it is clear that Equation (24) holds for $m = 0$. Suppose that Equation (24) holds for $m = m'$. Then, we have

$$\begin{aligned}
d_{m'+1} &= |\bar{X}|^2\left(7\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2 + d_{m'}\right) \\
&\le |\bar{X}|^2\left(7\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2\right) + |\bar{X}|^{3m+2}\left(7\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2\right) \\
&\le |\bar{X}|^{3m+3}\left(7\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2\right).
\end{aligned}$$

Therefore, Equation (24) holds for any $m \in \bar{T}$ by mathematical induction.

Fix any $F_i^m \in \bar{\mathcal{F}}_i^m$ and $F_j^m \in \bar{\mathcal{F}}_j^m$. Equations (15) and (24) imply that

$$\left|\bar{P}_0\left(F_i^m \cap F_j^m\right) - \bar{P}_0\left(F_i^m\right)\bar{P}_0\left(F_j^m\right)\right|$$

$$= \left| \sum_{b_1,b_2\in\bar{X}} \bar{P}_0\left(f_i^m = b_1, f_j^m = b_2, F_{ib_1}^{m-1}, F_{jb_2}^{m-1}\right) \right.$$

$$\left. - \sum_{b_1,b_2\in\bar{X}} \bar{P}_0\left(f_i^m = b_1, F_{ib_1}^{m-1}\right) \bar{P}_0\left(f_j^m = b_2, F_{jb_2}^{m-1}\right) \right|$$

$$\le |\bar{X}|^2 d_m \le |\bar{X}|^{3m+2}\left(7\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2\right) \le |\bar{X}|^{3\bar{N}+2}\left(7\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2\right)$$

$$\le |\bar{X}|^{3|\bar{T}|}\left(7\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2\right),$$

which is the required inequality in the lemma.

B.7.3. *Proof of Lemmas A.3.* Fix any $m \in \bar{T}$. For any $m' \in \{m, m+1, \ldots, \bar{N}-1\}$, let

$$B^{m'} = \{\omega^{m'} \in \Omega^{m'} : Y^{m'}(\omega^{m'}) = Y^m(\omega^{m'})\} \cap F^m.$$

If $\bar{P}_0(B^{m'}) > 0$, then it follows from Equation (35) in the main text that

$$\bar{P}_0(Y^{m'+1} = Y^{m'} | Y^{m'} = Y^m, F^m) = \frac{\int_{B^{m'}} \bar{Q}_{m'+1}^{\omega^{m'}}\left(Y^{m'+1} = Y^{m'}\right) d\bar{Q}^{m'}}{\bar{P}_0(B^{m'})}$$

$$\ge \frac{\int_{B^{m'}} (1-\bar{\varepsilon}_3)\, d\bar{Q}^{m'}}{\bar{P}_0(B^{m'})} = (1-\bar{\varepsilon}_3).$$

If $\bar{P}_0(Y^{m'} = Y^m, F^m) > 0$, then

$$\bar{P}_0(Y^{m'+1} = Y^m | F^m) = \bar{P}_0(Y^{m'+1} = Y^{m'} | Y^{m'} = Y^m, F^m)\bar{P}_0(Y^{m'} = Y^m | F^m)$$

$$\ge (1-\bar{\varepsilon}_3)\,\bar{P}_0(Y^{m'} = Y^m | F^m). \tag{25}$$

If $\bar{P}_0(Y^{m'} = Y^m, F^m) = 0$, then the above inequality is trivially satisfied.

By Equation (25), we can derive

$$\bar{P}_0(Y^{m+\Delta m} = Y^m | F^m) \ge \bar{P}_0(Y_i^{m+\Delta m-1} = Y_i^m | F^m)\,(1-\bar{\varepsilon}_3) \ge (1-\bar{\varepsilon}_3)^{\Delta m}.$$

One can easily prove by induction that for any $x \in [0,1]$ and $n \in \mathbb{N}$, $(1-x)^n \ge 1 - nx$. Recall that $\bar{\varepsilon}_3 \in [0,1]$. It is then clear that

$$\bar{P}_0(Y^{m+\Delta m} = Y^m | F^m) \ge 1 - \bar{\varepsilon}_3 \Delta m,$$

which is the first inequality in the lemma.

To prove the second inequality in the lemma, we first note that

$$\bar{P}_0\left(Y^{m+\Delta m} - Y^m \ge 2 \,\middle|\, F^m\right)$$

$$= \sum_{m'=m+1}^{m+\Delta m-1} \bar{P}_0\left(Y^{m+\Delta m} - Y^{m'} \ge 1, Y^{m'} = Y^{m'-1} + 1, Y^{m'-1} = Y^m \,\middle|\, F^m\right). \tag{26}$$

Fix any $m' \in \{m+1, m+2, \ldots, m+\Delta m - 1\}$. Assume that

$$\bar{P}_0\left(Y^{m'} = Y^{m'-1} + 1, Y^{m'-1} = Y^m, F^m\right) > 0.$$

By the first inequality in the lemma, we can obtain that

$$\begin{aligned}
&\bar{P}_0\left(Y^{m+\Delta m} = Y^{m'} \,\middle|\, Y^{m'} = Y^{m'-1} + 1, Y^{m'-1} = Y^m, F^m\right) \\
&\geq 1 - \bar{\varepsilon}_3(m + \Delta m - m') \geq 1 - \bar{\varepsilon}_3 \Delta m,
\end{aligned}$$

which implies that

$$\bar{P}_0\left(Y^{m+\Delta m} - Y^{m'} \geq 1 \,\middle|\, Y^{m'} = Y^{m'-1} + 1, Y^{m'-1} = Y^m, F^m\right) \leq \bar{\varepsilon}_3 \Delta m.$$

It follows from the above inequality that

$$\begin{aligned}
&\bar{P}_0\left(Y^{m+\Delta m} - Y^{m'} \geq 1, Y^{m'} = Y^{m'-1} + 1, Y^{m'-1} = Y^m \,\middle|\, F^m\right) \\
=& \bar{P}_0\left(Y^{m+\Delta m} - Y^{m'} \geq 1 \,\middle|\, Y^{m'} = Y^{m'-1} + 1, Y^{m'-1} = Y^m, F^m\right) \\
&\times \bar{P}_0\left(Y^{m'} = Y^{m'-1} + 1, Y^{m'-1} = Y^m \,\middle|\, F^m\right) \\
\leq& \bar{\varepsilon}_3 \Delta m\, \bar{P}_0\left(Y^{m'} = Y^{m'-1} + 1, Y^{m'-1} = Y^m \,\middle|\, F^m\right). \qquad (27)
\end{aligned}$$

When $\bar{P}_0\left(Y^{m'} = Y^{m'-1} + 1, Y^{m'-1} = Y^m, F^m\right) = 0$, the inequality in Equation (27) is trivially satisfied. Hence, Equations (26) and (27) together with the first inequality in the lemma imply that

$$\begin{aligned}
&\bar{P}_0\left(Y^{m+\Delta m} - Y^m \geq 2 \,\middle|\, F^m\right) \\
\leq& \bar{\varepsilon}_3 \Delta m \sum_{m'=m+1}^{m+\Delta m - 1} \bar{P}_0\left(Y^{m'} = Y^{m'-1} + 1, Y^{m'-1} = Y^m \,\middle|\, F^m\right) \\
\leq& \bar{\varepsilon}_3 \Delta m \bar{P}_0\left(Y^{m+\Delta m} \geq Y^m + 1 \,\middle|\, F^m\right) \\
=& \bar{\varepsilon}_3 \Delta m \left(1 - \bar{P}_0\left(Y^{m+\Delta m} = Y^m \,\middle|\, F^m\right)\right) \\
\leq& \bar{\varepsilon}_3^2 (\Delta m)^2,
\end{aligned}$$

which is the second inequality in the lemma.

B.8. Proof of Lemma 6 in Section 3.3. The statement of Lemma 6 provides an estimate on the conditional probability for an agent to change type in the next period. Since each period has a random matching step, we will need to consider the joint process $(\hat{\alpha}^m, \hat{g}^m)$. For notational simplicity, we denote $S \times (S \cup \{J\})$ and $(\hat{\alpha}^m, \hat{g}^m)$ by $\hat{S}$ and $\hat{\beta}^m$ respectively.

We need to apply Lemma A.1 in the main text to the case that f is the function $\hat{\beta} : I \times \Omega \times \{0, 1, \ldots, 3M^2\} \to \hat{S}$ by choosing suitable parameters (including $\bar{I}$, $\bar{\Omega}$, $\bar{T}$, $\bar{P}_0$, $\bar{X}$, $\bar{\vartheta}^m$ and $\bar{\varepsilon}_1$), which are introduced in Appendix A before Lemma A.1. Apparently, we should take $\bar{I}$, $\bar{\Omega}$, $\bar{T}$, $\bar{P}_0$, $\bar{X}$ to be I, Ω, $\{0,1,\ldots,3M^2\}$, P_0, $\hat{S}$ respectively. It remains to define $\bar{\vartheta}^m$ and $\bar{\varepsilon}_1$.

We first consider the case when $m = 3n-3$ for some $n \in \mathbb{T}_0 \setminus \{0\}$. Then, the $(m+1)$-th step is the mutation step in the n-th period. Let

$$\bar{\vartheta}^{3n-3}(a,b) = \begin{cases} \hat{\eta}_{kl} & \text{if } a = (k,J) \text{ and } b = (l,J) \\ 0 & \text{otherwise.} \end{cases}$$

By the construction of the mutation step in the finite-agent dynamic matching model in Subsection 3.2, for any $\omega^{3n-3} \in \Omega^{3n-3}$ and any $b \in \hat{S}$

$$Q^{\omega^{3n-3}}_{3n-2}(\hat{\beta}^{3n-2}_i = b) = \bar{\vartheta}^{3n-3}(\hat{\beta}^{3n-3}_i(\omega^{3n-3}), b). \tag{28}$$

Next, we consider the case when $m = 3n-1$ for some $n \in \mathbb{T}_0 \setminus \{0\}$. Then, the $(m+1)$-th step is the type-changing step in the n-th period.

$$\bar{\vartheta}^{3n-1}(a,b) = \begin{cases} \hat{\varsigma}_{kl}(k') & \text{if } a = (k,l), b = (k',J) \\ 1 & \text{if } a = (k,J), b = (k,J) \\ 0 & \text{otherwise.} \end{cases}$$

By the construction of the type-changing step in the finite-agent dynamic matching model in Subsection 3.2, for any $\omega^{3n-1} \in \Omega^{3n-1}$ and any $b \in \hat{S}$

$$Q^{\omega^{3n-1}}_{3n}(\hat{\beta}^{3n}_i = b) = \bar{\vartheta}^{3n-1}(\hat{\beta}^{3n-1}_i(\omega^{3n-1}), b). \tag{29}$$

It remains to consider the case when $m = 3n-2$ for some $n \in \mathbb{T}_0 \setminus \{0\}$. Then, the m-th step and the $(m+1)$-th step are the mutation and matching steps in the n-th period respectively.

$$\bar{\vartheta}^{3n-2}(a,b) = \begin{cases} \bar{q}^{3n-2}_{kl} & \text{if } a = (k,J) \text{ and } b = (k,l) \\ \bar{q}^{3n-2}_{k} & \text{if } a = (k,J) \text{ and } b = (k,J) \\ 0 & \text{otherwise,} \end{cases}$$

where $\bar{q}^{3n-2}_{kl}$ and $\bar{q}^{3n-2}_{k}$ are introduced three lines above Lemma 3. Recall that agents do not match at the mutation step. We can focus on the case when $\hat{\beta}^{3n-2}(\omega^{3n-2}) = (k,J)$ (i.e., $\hat{\alpha}^{3n-2}(\omega^{3n-2}) = k$) for some $k \in S$. By Lemma B.3, if $\omega^{3n-2} \notin V^{3n-2}$ and $\hat{\beta}^{3n-2}(\omega^{3n-2}) = (k,J)$,

$$\left| Q^{\omega^{3n-2}}_{3n-1}(\hat{g}^{3n-1}_i = l) - \hat{q}_{kl}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right) \right| < \frac{1}{\hat{M}^{\frac{1}{9}}},$$

$$\left| Q^{\omega^{3n-2}}_{3n-1}(\hat{g}^{3n-1}_i = J) - \hat{q}_{k}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right) \right| < \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

By the definition of V^{3n-2} in Lemma 3, we know that for any $\omega^{3n-2} \notin V^{3n-2}$, $\|\rho^{3n-2}(\omega^{3n-2}) - U^{3n-2}_1(\rho^0)\|_\infty \le \xi_1$. Then, Lemma 2 implies that for any $\omega^{3n-2} \notin V^{3n-2}$,

$$\left|\hat{q}_{kl}\left(\rho^{3n-2}(\omega^{3n-2})\right) - \bar{q}^{3n-2}_{kl}\right| = \left|\hat{q}_{kl}\left(\rho^{3n-2}(\omega^{3n-2})\right) - \hat{q}_{kl}\left(U^{3n-2}_1(\rho^0)\right)\right| \le \xi_0,$$

$$\left|\hat{q}_k\left(\rho^{3n-2}(\omega^{3n-2})\right) - \bar{q}_k^{3n-2}\right| = \left|\sum_{l\in S}\left(\hat{q}_{kl}\left(\rho^{3n-2}(\omega^{3n-2})\right) - \hat{q}_{kl}\left(U_1^{3n-2}(\rho^0)\right)\right)\right| \le K\xi_0.$$

Therefore, for any $k \in S$, $\omega^{3n-2} \notin V^{3n-2}$ with $\hat{\beta}^{3n-2}(\omega^{3n-2}) = (k, J)$, if $a = (k, l)$ for some $l \in S$, then

$$\begin{aligned}&\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{\beta}_i^{3n-1} = a) - \bar{\vartheta}^{3n-2}(\hat{\beta}_i^{3n-2}(\omega^{3n-2}), a)\right| \\ = &\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{\beta}_i^{3n-1} = (k,l)) - \bar{q}_{kl}^{3n-2}\right| \le \xi_0 + \frac{1}{\hat{M}^{\frac{1}{9}}};\end{aligned}$$

if $a = (k, J)$, then

$$\begin{aligned}&\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{\beta}_i^{3n-1} = a) - \bar{\vartheta}^{3n-2}(\hat{\beta}_i^{3n-2}(\omega^{3n-2}), a)\right| \\ = &\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{\beta}_i^{3n-1} = (k,J)) - \bar{q}_{k}^{3n-2}\right| \le K\xi_0 + \frac{1}{\hat{M}^{\frac{1}{9}}};\end{aligned}$$

if $a \notin \{k\} \times (S \cup \{J\})$,

$$\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{\beta}_i^{3n-1} = a) - \bar{\vartheta}^{3n-2}(\hat{\beta}_i^{3n-2}(\omega^{3n-2}), a)\right| = |0 - 0| = 0.$$

To sum up, for any $i \in I$, any $\omega^{3n-2} \notin V^{3n-2}$ and any $a \in \hat{S}$,

$$\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{\beta}_i^{3n-1} = a) - \bar{\vartheta}^{3n-2}(\hat{\beta}_i^{3n-2}(\omega^{3n-2}), a)\right| \le K\xi_0 + \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

In other words, if

$$\left|Q_{3n-1}^{\omega^{3n-2}}(\hat{\beta}_i^{3n-1} = a) - \bar{\vartheta}^{3n-2}(\hat{\beta}_i^{3n-2}(\omega^{3n-2}), a)\right| > K\xi_0 + \frac{1}{\hat{M}^{\frac{1}{9}}}. \tag{30}$$

for some $i \in I$ and $a \in \hat{S}$, then $\omega^{3n-2} \in V^{3n-2}$.

For $m \in \{1, 2, \ldots, 3M^2 - 1\}$ and $\varepsilon > 0$, recall the definition of $\bar{C}^m(\varepsilon)$ above Equation (33) in the main text that

$$\bar{C}^m(\varepsilon) = \{\omega^m \in \Omega^m : \left|Q_{\omega^m}^{m+1}(\hat{\beta}_i^{m+1} = a) - \bar{\vartheta}^m(\hat{\beta}_i^m(\omega^m), a)\right| > \varepsilon \text{ for some } i \in I \text{ and } a \in \hat{S}\}.$$

It follows from Equation (30) that $\bar{C}^{3n-2}\left(K\xi_0 + \frac{1}{\hat{M}^{\frac{1}{9}}}\right) \subseteq V^{3n-2}$, which implies that

$$Q^m\left(\bar{C}^m\left(K\xi_0 + \frac{1}{\hat{M}^{\frac{1}{9}}}\right)\right) \le Q^m(V^m) \tag{31}$$

holds for $m = 3n-2$. By Equations (28) and (29), Equation (31) holds trivially for $m = 3n-3$ or $3n-1$ since the left side of the inequality is zero. By Lemma 3, $Q^m(V^m) \le \xi_1$. It is clear that

$$Q^m\left(\bar{C}^m\left(K\xi_0 + \frac{1}{\hat{M}^{\frac{1}{9}}}\right)\right) \le \xi_1 \le \xi_0 < K\xi_0 + \frac{1}{\hat{M}^{\frac{1}{9}}}.$$

Then, we can take $\bar{\varepsilon}_1$ as introduced in Equation (33) in the main text to be $K\xi_0 + \frac{1}{\hat{M}^{\frac{1}{9}}}$.

It follows from Lemma A.1 for the case $m=3n-3$, $m'=3$, $a=(k,J)$ and $b=(r,J)$ that

$$\begin{aligned}
&\left|P_0\left(\hat{\alpha}_i^{3n}=r \,\middle|\, \hat{\alpha}_i^{3n-3}=k, F^{3n-3}\right)-\bar{\Theta}_{3n-3}^{3n-1}((k,J),(r,J))\right| \\
&=\left|P_0\left(\hat{\beta}_i^{3n}=(r,J) \,\middle|\, \hat{\beta}_i^{3n-3}=(k,J), F^{3n-3}\right)-\bar{\Theta}_{3n-3}^{3n-1}((k,J),(r,J))\right| \\
&\leq \frac{\left(2|\bar{X}|\right)^3 \bar{\varepsilon}_1}{P_0\left(\hat{\beta}_i^{3n-3}=(k,J), F^{3n-3}\right)}=\frac{\left(2|\hat{S}|\right)^3 \bar{\varepsilon}_1}{P_0\left(\hat{\alpha}_i^{3n-3}=k, F^{3n-3}\right)}.
\end{aligned}$$

It is straightforward to verify that

$$\begin{aligned}
&\bar{\Theta}_{3n-3}^{3n-1}((k,J),(r,J))=[\bar{\vartheta}^{3n-3}\bar{\vartheta}^{3n-2}\bar{\vartheta}^{3n-1}]((k,J),(r,J)) \\
&=\sum_{k'\in S}\sum_{l\in S}\left(\hat{\eta}_{kk'}\bar{q}_{k'l}^{3n-2}\hat{\varsigma}_{k'l}(r)\right)+\hat{\eta}_{kr}\bar{q}_r^{3n-2}.
\end{aligned}$$

Then, we can obtain that

$$\begin{aligned}
&\left|P_0\left(\hat{\alpha}_i^{3n}=r \,\middle|\, \hat{\alpha}_i^{3n-3}=k, F^{3n-3}\right)-\hat{\eta}_{kr}-\sum_{l\in S}\bar{q}_{kl}^{3n-2}\hat{\varsigma}_{kl}(r)\right| \\
&\leq\left|\bar{\Theta}_{3n-3}^{3n-1}((k,J),(r,J))-\hat{\eta}_{kr}-\sum_{l\in S}\bar{q}_{kl}^{3n-2}\hat{\varsigma}_{kl}(r)\right|+\frac{(2|\hat{S}|)^3\bar{\varepsilon}_1}{P_0\left(\hat{\alpha}_i^{3n-3}=k, F^{3n-3}\right)} \\
&=\left|\sum_{k'\in S}\sum_{l\in S}\left(\hat{\eta}_{kk'}\bar{q}_{k'l}^{3n-2}\hat{\varsigma}_{k'l}(r)\right)+\hat{\eta}_{kr}\bar{q}_r^{3n-2}-\hat{\eta}_{kr}-\sum_{l\in S}\bar{q}_{kl}^{3n-2}\hat{\varsigma}_{kl}(r)\right| \\
&\qquad+\frac{(2|\hat{S}|)^3\bar{\varepsilon}_1}{P_0\left(\hat{\alpha}_i^{3n-3}=k, F^{3n-3}\right)} \\
&\leq\left|\hat{\eta}_{kr}(1-\bar{q}_r^{3n-2})\right|+\left|\sum_{k'\neq k}\sum_{l\in S}\hat{\eta}_{kk'}\bar{q}_{k'l}^{3n-2}\hat{\varsigma}_{k'l}(r)\right|+\left|\sum_{l\in S}(1-\hat{\eta}_{kk})\bar{q}_{kl}^{3n-2}\hat{\varsigma}_{kl}(r)\right| \\
&\qquad+\frac{(2|\hat{S}|)^3\bar{\varepsilon}_1}{P_0\left(\hat{\alpha}_i^{3n-3}=k, F^{3n-3}\right)}.
\end{aligned}$$

Recall from the beginning of Subsection 3.2 that $\bar{a}=\max\{\bar{\eta},\bar{\theta}\}+1$. It is clear that

$$\begin{aligned}
&\hat{\eta}_{kl}=\frac{1}{M}\eta_{kl}+\frac{1}{M^2}\leq\frac{\bar{a}}{M}\text{ for any } k,l\in S \text{ with } k\neq l, \\
&1-\hat{\eta}_{kk}=\sum_{l\neq k}\hat{\eta}_{kl}\leq\frac{K\bar{a}}{M}\text{ for any } k\in S, \\
&\bar{q}_{kl}^{3n-2}=\frac{1}{M}\theta_{kl}\left(U_1^{3n-2}(\rho^0)\right)\leq\frac{\bar{a}}{M}\text{ for any } k,l\in S, \\
&1-\bar{q}_k^{3n-2}=\sum_{l\in S}\bar{q}_{kl}^{3n-2}\leq\frac{K\bar{a}}{M}\text{ for any } k\in S.
\end{aligned}$$

Then, we have

$$\left|P_0\left(\hat{\alpha}_i^{3n}=r\,\middle|\,\hat{\alpha}_i^{3n-3}=k,F^{3n-3}\right)-\hat{\eta}_{kr}-\sum_{l\in S}\bar{q}_{kl}^{3n-2}\hat{\varsigma}_{kl}(r)\right|$$

$$\leq\frac{K\bar{a}^2}{M^2}+\frac{K(K-1)\bar{a}^2}{M^2}+\frac{K^2\bar{a}^2}{M^2}+\frac{(2|\hat{S}|)^3\bar{\varepsilon}_1}{P_0\left(\hat{\alpha}_i^{3n-3}=k,F^{3n-3}\right)}$$

$$=\frac{2K^2\bar{a}^2}{M^2}+\frac{(2|\hat{S}|)^3\bar{\varepsilon}_1}{P_0\left(\hat{\alpha}_i^{3n-3}=k,F^{3n-3}\right)}. \tag{32}$$

It is clear that $(2|\hat{S}|)^3\bar{\varepsilon}_1=(2K(K+1))^3(K\xi_0+\frac{1}{\hat{M}^{\frac{1}{9}}})$. By Lemma 2 and the definition of $\hat{M}$, we have $\frac{1}{\hat{M}^{\frac{1}{9}}}\leq\xi_{3M^2+1}\leq\xi_0=\frac{1}{K^{M^3}}$, which implies that

$$(2|\hat{S}|)^3\bar{\varepsilon}_1\leq(2K)^3(K+1)^4\frac{1}{K^{M^3}}.$$

Since $K\geq 2$ and $M\geq 3$, we know that

$$(2K)^3(K+1))^4M^3\leq(K^2)^3(K^2)^4K^{3log_K M}\leq K^{14+3log_2 M}<K^{M^3},$$

which implies that

$$(2|\hat{S}|)^3\bar{\varepsilon}_1\leq(2K)^3(K+1)^4\frac{1}{K^{M^3}}<\frac{1}{M^3}. \tag{33}$$

Since $P_0\left(\hat{\alpha}_i^{3n-3}=k,F^{3n-3}\right)\geq\frac{1}{M}$, it follows from Equations (32) and (33) that

$$\left|P_0\left(\hat{\alpha}_i^{3n}=r\,\middle|\,\hat{\alpha}_i^{3n-3}=k,F^{3n-3}\right)-\hat{\eta}_{kr}-\sum_{l\in S}\bar{q}_{kl}^{3n-2}\hat{\varsigma}_{kl}(r)\right|$$

$$<\frac{2K^2\bar{a}^2}{M^2}+\frac{1}{M^2}<\frac{3K^2\bar{a}^2}{M^2}.$$

The proof is thus completed.

B.9. Proof of Lemma 7 in Section 3.3. We need to apply Lemma A.1 to the case that f is the function $\hat{\beta}=(\hat{\alpha},\hat{g})$. As in the proof of Lemma 6, let $\bar{I}=I$, $\bar{\Omega}=\Omega$, $\bar{T}=\{0,1,\ldots,3M^2\}$, $\bar{P}_0=P_0$, $\bar{X}=\hat{S}=S\times(S\cup\{J\})$ and $\bar{\varepsilon}_1=K\xi_0+\frac{1}{\hat{M}^{\frac{1}{9}}}$.

By the second inequality in Lemma A.1, we obtain that for any $i\in I$, any $n,n_1\in\{1,2,\ldots,M^2\}$ with $n>n_1$, any types $k,k_1\in S$, and any $F_i^{3n_1-3}\in\mathcal{F}_i^{3n_1-3}\subseteq\mathcal{F}^{3n_1-3}$,

$$\begin{aligned}&\big|P_0\left(\hat{\alpha}_i^{3n}=k,\hat{\alpha}_i^{3n_1}=k_1,F_i^{3n_1-3}\right)P_0\left(\hat{\alpha}_i^{3n_1}=k_1\right)\\&\qquad-P_0\left(\hat{\alpha}_i^{3n}=k,\hat{\alpha}_i^{3n_1}=k_1\right)P_0\left(\hat{\alpha}_i^{3n_1}=k_1,F_i^{3n_1-3}\right)\big|\\=&\Big|P_0\left(\hat{\beta}_i^{3n}=(k,J),\hat{\beta}_i^{3n_1}=(k_1,J),F_i^{3n_1-3}\right)P_0\left(\hat{\beta}_i^{3n_1}=(k_1,J)\right)\\&\qquad-P_0\left(\hat{\beta}_i^{3n}=(k,J),\hat{\beta}_i^{3n_1}=(k_1,J)\right)P_0\left(\hat{\beta}_i^{3n_1}=(k_1,J),F_i^{3n_1-3}\right)\Big|\\\leq&\,(2|\bar{X}|)^{|\bar{T}|}\bar{\varepsilon}_1=(2K(K+1))^{3M^2+1}\left(K\xi_0+\frac{1}{\hat{M}^{\frac{1}{9}}}\right).\end{aligned}$$

Let $B_2(M) = (2K(K+1))^{3M^2+1}\left(K\xi_0 + \frac{1}{\hat{M}^{\frac{1}{9}}}\right)$. Since $\frac{1}{\hat{M}^{\frac{1}{9}}} < \xi_{3M^2+1} \le \xi_0 = \frac{1}{K^{M^3}}$, we know that

$$B_2(M) \le \frac{(2K(K+1))^{3M^2+1}(K+1)}{K^{M^3}},$$

which implies that $\lim_{M\to\infty} B_2(M) = 0$. Hence, Lemma 7 is proved.

B.10. Proof of Lemma 8 in Section 3.3. We need to apply Lemma A.2 to the case that f is the function $\hat{\beta} = (\hat{\alpha}, \hat{g})$. As in the proof of Lemma 6, let $\bar{I} = I$, $\bar{\Omega} = \Omega$, $\bar{T} = \{0, 1, \ldots, 3M^2\}$, $\bar{P}_0 = P_0$, $\bar{X} = \hat{S} = S \times (S \cup \{J\})$ and $\bar{\varepsilon}_1 = K\xi_0 + \frac{1}{\hat{M}^{\frac{1}{9}}}$. We only need to choose an appropriate value for $\bar{\varepsilon}_2$.

When $m = 3n-3$ or $3n-1$, the $(m+1)$-th step is the mutation step or the type changing step. By the construction of these two steps in Subsection 3.2, agents change their types independently. Therefore, for any $n \in \mathbb{T}_0 \setminus \{0\}$, $a, b \in \hat{S}$, $\omega^{3n-3} \in \Omega^{3n-3}$ and $\omega^{3n-1} \in \Omega^{3n-1}$,

$$Q_{3n-2}^{\omega^{3n-3}}\left(\hat{\beta}_i^{3n-2} = a, \hat{\beta}_j^{3n-2} = b\right) = Q_{3n-2}^{\omega^{3n-3}}\left(\hat{\beta}_i^{3n-2} = a\right) Q_{3n-2}^{\omega^{3n-3}}\left(\hat{\beta}_j^{3n-2} = b\right), \tag{34}$$

$$Q_{3n}^{\omega^{3n-1}}\left(\hat{\beta}_i^{3n} = a, \hat{\beta}_j^{3n} = b\right) = Q_{3n}^{\omega^{3n-1}}\left(\hat{\beta}_i^{3n} = a\right) Q_{3n}^{\omega^{3n-1}}\left(\hat{\beta}_j^{3n} = b\right). \tag{35}$$

For step of random matching, we need estimate the difference

$$\left|Q_{3n-1}^{\omega^{3n-2}}\left(\hat{\beta}_i^{3n-1} = a, \hat{\beta}_j^{3n-1} = b\right) - Q_{3n-1}^{\omega^{3n-2}}\left(\hat{\beta}_i^{3n-1} = a\right) Q_{3n-1}^{\omega^{3n-2}}\left(\hat{\beta}_j^{3n-1} = b\right)\right|.$$

for any $n \in \mathbb{T}_0 \setminus \{0\}$, $a, b \in \hat{S}$, $\omega^{3n-2} \in \Omega^{3n-2}$. Fix any $k_1, k_2 \in S$ and $\omega^{3n-2} \notin V^{3n-2}$ (introduced in Lemma 3) with $\hat{\beta}_i^{3n-2}(\omega^{3n-2}) = (k_1, J)$ and $\hat{\beta}_j^{3n-2}(\omega^{3n-2}) = (k_2, J)$. The inequalities in Lemma B.3 give symmetric treatment for the cases $l \in S$ and $l = J$. For the simplicity of applying this lemma, we introduce the notation $\hat{q}_{kJ}$ to represent $\hat{q}_k$ in the rest of the proof for Lemma 8. For notational simplicity in the following displayed formula, we use $Q_{3n-2}^{ij}(a,b)$ and $Q_{3n-2}^{i}(a)$ to denote $Q_{3n-1}^{\omega^{3n-2}}\left(\hat{\beta}_i^{3n-1} = a, \hat{\beta}_j^{3n-1} = b\right)$ and $Q_{3n-1}^{\omega^{3n-2}}\left(\hat{\beta}_i^{3n-1} = a\right)$ respectively. Note that for any $l_1, l_2 \in S \cup \{J\}$,

$$\begin{aligned}
&\left|Q_{3n-2}^{ij}((k_1,l_1),(k_2,l_2)) - Q_{3n-2}^{i}((k_1,l_1))\, Q_{3n-2}^{j}((k_2,l_2))\right| \\
\le &\left|Q_{3n-2}^{ij}((k_1,l_1),(k_2,l_2)) - \hat{q}_{k_1 l_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right) \hat{q}_{k_2 l_2}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\
&+ \left|\hat{q}_{k_1 l_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right) \hat{q}_{k_2 l_2}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right) - \hat{q}_{k_1 l_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right) Q_{3n-2}^{j}((k_2,l_2))\right| \\
&+ \left|\hat{q}_{k_1 l_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right) Q_{3n-2}^{j}((k_2,l_2)) - Q_{3n-2}^{i}((k_1,l_1))\, Q_{3n-2}^{j}((k_2,l_2))\right| \\
\le &\left|Q_{3n-2}^{ij}((k_1,l_1),(k_2,l_2)) - \hat{q}_{k_1 l_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right) \hat{q}_{k_2 l_2}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right)\right| \\
&+ \left|\hat{q}_{k_2 l_2}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right) - Q_{3n-2}^{j}((k_2,l_2))\right| \\
&+ \left|\hat{q}_{k_1 l_1}\left(\rho^{3n-2}\left(\omega^{3n-2}\right)\right) - Q_{3n-2}^{i}((k_1,l_1))\right|.
\end{aligned}$$

Lemma B.3 then implies that for any $l_1, l_2 \in S \cup \{J\}$,

$$\left|Q^{ij}_{3n-2}\left((k_1,l_1),(k_2,l_2)\right) - Q^{i}_{3n-2}\left((k_1,l_1)\right) Q^{j}_{3n-2}\left((k_2,l_2)\right)\right| \le \frac{3}{\hat{M}^{\frac{1}{9}}}. \tag{36}$$

Note that for any $a \notin \{k_1\} \times (S \cup \{J\})$,

$$Q^{ij}_{3n-2}(a,b) = Q^{i}_{3n-2}(a)\, Q^{j}_{3n-2}(b) = 0 \tag{37}$$

for any $b \in \hat{S}$. By combining Equations (36) and (37), we know that for any $a, b \in \hat{S}$ and $\omega^m \notin V^{3n-2}$,

$$\left|Q^{ij}_{3n-2}(a,b) - Q^{i}_{3n-2}(a)\, Q^{j}_{3n-2}(b)\right| \le \frac{3}{\hat{M}^{\frac{1}{9}}}. \tag{38}$$

Equations (34), (35) and (38) imply that for any $m \in \{0, 1, \ldots, 3M^2 - 1\}$, $a, b \in \hat{S}$ and $\omega^m \notin V^m$,

$$\left|Q^{\omega^m}_{m+1}\left(\hat{\beta}^{m+1}_i = a, \hat{\beta}^{m+1}_j = b\right) - Q^{\omega^m}_{m+1}\left(\hat{\beta}^{m+1}_i = a\right) Q^{\omega^m}_{m+1}\left(\hat{\beta}^{m+1}_j = b\right)\right| \le \frac{3}{\hat{M}^{\frac{1}{9}}}. \tag{39}$$

Let $\bar{\varepsilon}_2 = 3\xi_0$. Since $\frac{1}{\hat{M}^{\frac{1}{9}}} < \xi_{3M^2+1} \le \xi_0 = \frac{1}{K^{M^3}}$, we have $\frac{3}{\hat{M}^{\frac{1}{9}}} \le 3\xi_0 = \bar{\varepsilon}_2$. Then, Equation (39) implies that for any $\omega^m \notin V^m$, $\hat{\beta}_i$ and $\hat{\beta}_j$ are not $\bar{\varepsilon}_2$-correlated at ω^m. Therefore, it follows from Lemma 3 that for any $m \in \{0, 1, \ldots, 3M^2 - 1\}$,

$$Q^m\left(\{\omega^m \in \Omega^m : \hat{\beta}_i \text{ and } \hat{\beta}_j \text{ are } \bar{\varepsilon}_2 \text{ -correlated at } \omega^m\}\right) \le Q^m(V^m) \le \xi_1 \le \xi_0 < \bar{\varepsilon}_2,$$

which implies that $\hat{\beta}_i$ and $\hat{\beta}_j$ are not $\bar{\varepsilon}_2$-correlated. Recall that $|\bar{X}| = |\hat{S}| = K(K+1)$ and $\bar{\varepsilon}_1 = K\xi_0 + \frac{1}{\hat{M}^{\frac{1}{9}}}$. Lemma A.2 implies that for any $m \in \{0, 1, \ldots, 3M^2\}$, $F^m_i \in \mathcal{F}^m_i$, and $F^m_j \in \mathcal{F}^m_j$,

$$\begin{aligned}
&\left|P_0\left(F^m_i \cap F^m_j\right) - P_0\left(F^m_i\right) P\left(F^m_j\right)\right| \\
\le\, & |\bar{X}|^{3|\bar{T}|}\left(7\bar{\varepsilon}_1 + 2\bar{\varepsilon}_2\right) \\
=\, & K^{9M^2+3}(K+1)^{9M^2+3}\left((7K+6)\xi_0 + \frac{7}{\hat{M}^{\frac{1}{9}}}\right).
\end{aligned}$$

Let $B_3(M) = K^{9M^2+3}(K+1)^{9M^2+3}((7K+6)\xi_0 + \frac{7}{\hat{M}^{\frac{1}{9}}})$. Since $\frac{1}{\hat{M}^{\frac{1}{9}}} \le \xi_0 = \frac{1}{K^{M^3}}$, we know that

$$B_3(M) \le \frac{(7K+13)K^{9M^2+3}(K+1)^{9M^2+3}}{K^{M^3}},$$

which implies that $\lim_{M\to\infty} B_3(M) = 0$. Hence, Lemma 8 is proved.

B.11. Proofs of Lemmas 9 and 10 in Section 3.3. We need to apply Lemma A.3 to the case that the counting process Y in the lemma is the process $\hat{X}_i$ in Lemmas 9 and 10. As in the proof of Lemma 6, let $\bar{I} = I$, $\bar{\Omega} = \Omega$, $\bar{T} = \{0, 1, \ldots, 3M^2\}$, $\bar{P}_0 = P_0$ and $\bar{X} = \hat{S} = S \times (S \cup \{J\})$. We only need to choose an appropriate value for $\bar{\varepsilon}_3$.

Fix any $i \in I$, $m \in \{0, 1, \ldots, 3M^2\}$, $k \in S$ and $\omega^m \in \Omega^m$ with $\hat{\alpha}^m_i(\omega^m) = k$.

We first consider the case when $m = 3n-3$ for some $n \in \mathbb{T}_0 \setminus \{0\}$. Then, the $(m+1)$-th step is the mutation step in the n-th period. By the construction of the mutation step in the finite-agent dynamic matching model in Subsection 3.2, we have

$$Q_{3n-2}^{\omega^{3n-3}}\left(\hat{X}_i^{3n-2} = \hat{X}_i^{3n-3} + 1\right) = \sum_{l \in S \setminus \{k\}} \hat{\eta}_{kl}.$$

Recall that

$$\hat{\eta}_{kl} = \frac{1}{M}\eta_{kl} + \frac{1}{M^2} \le \frac{\bar{a}}{M} \text{ for any } k,l \in S \text{ with } k \neq l.$$

It is clear that

$$Q_{3n-2}^{\omega^{3n-3}}\left(\hat{X}_i^{3n-2} = \hat{X}_i^{3n-3} + 1\right) < \frac{K\bar{a}}{M}. \tag{40}$$

Next, we consider the case when $m = 3n-2$ for some $n \in \mathbb{T}_0 \setminus \{0\}$. Then, the $(m+1)$-th step is the matching step in the n-th period. The construction of the matching step in the finite-agent dynamic matching model in Subsection 3.2 and Lemma 1 allows us to claim that

$$Q_{3n-1}^{\omega^{3n-2}}\left(\hat{X}_i^{3n-1} = \hat{X}_i^{3n-2} + 1\right) = \sum_{l \in S} Q_{3n-1}^{\omega^{3n-2}}\left(\hat{g}_i^{3n-1} = l\right) \le \sum_{l \in S} \hat{q}_{kl}(\rho^{3n-2}(\omega^{3n-2})).$$

Recall that

$$\hat{q}_{kl}(p) = \frac{1}{M}\theta_{kl}(p) \le \frac{\bar{a}}{M} \text{ for any } k,l \in S \text{ and } p \in \Delta.$$

It is then clear that

$$Q_{3n-1}^{\omega^{3n-2}}\left(\hat{X}_i^{3n-1} = \hat{X}_i^{3n-2} + 1\right) \le \sum_{l \in S} \hat{q}_{kl}(\rho^{3n-2}(\omega^{3n-2})) \le \frac{K\bar{a}}{M}. \tag{41}$$

It remains to consider the case when $m = 3n-1$ for some $n \in \mathbb{T}_0 \setminus \{0\}$. Then the $(m+1)$-th step is the type changing step in the n-th period. Note that $\hat{X}_i$ only counts the number of mutations and matchings. It is clear that

$$Q_{3n}^{\omega^{3n-1}}\left(\hat{X}_i^{3n} = \hat{X}_i^{3n-1} + 1\right) = 0 < \frac{K\bar{a}}{M}. \tag{42}$$

By Equations (40), (41) and (42), we can take $\bar{\varepsilon}_3$ to be $\frac{K\bar{a}}{M}$. By Lemma A.3, for any $m, \Delta m \in \{0, \ldots, 3M^2\}$ and $F^m \in \mathcal{F}^m$ such that $m + \Delta m \le 3M^2$ and $P_0(F^m) > 0$,

$$P_0(\hat{X}_i^{m+\Delta m} = \hat{X}_i^m | F^m) \ge 1 - \bar{\varepsilon}_3 \Delta m = 1 - K\bar{a}\frac{\Delta m}{M},$$

$$P_0(\hat{X}_i^{m+\Delta m} \ge \hat{X}_i^m + 2 | F^m) \le \bar{\varepsilon}_3^2 (\Delta m)^2 = (K\bar{a})^2 \left(\frac{\Delta m}{M}\right)^2,$$

which are the required inequalities in Lemmas 9 and 10.

B.12. Proof of Lemma 11 in Section 3.3. Fix any $k \in S$. By the definition of ρ^m, we obtain that

$$\begin{aligned}
&\left|\mathbb{E}\left(\rho_k^{m+\Delta m}\right) - \mathbb{E}\left(\rho_k^m\right)\right| \\
&= \left|\mathbb{E}\left(\frac{1}{\hat{M}}\sum_{i\in I}\mathbf{1}_k\left(\hat{\alpha}_i^{m+\Delta m}\right)\right) - \mathbb{E}\left(\frac{1}{\hat{M}}\sum_{i\in I}\mathbf{1}_k\left(\hat{\alpha}_i^m\right)\right)\right| \\
&\leq \frac{1}{\hat{M}}\sum_{i\in I}\mathbb{E}\left|\mathbf{1}_k\left(\hat{\alpha}_i^{m+\Delta m}\right) - \mathbf{1}_k\left(\hat{\alpha}_i^m\right)\right| \\
&= \frac{1}{\hat{M}}\sum_{i\in I}P_0\left(\left|\mathbf{1}_k\left(\hat{\alpha}_i^{m+\Delta m}\right) - \mathbf{1}_k\left(\hat{\alpha}_i^m\right)\right| = 1\right). \qquad (43)
\end{aligned}$$

For any $\omega \in \Omega$, if $\left|\mathbf{1}_k\left(\hat{\alpha}_i^{m+\Delta m}(\omega)\right) - \mathbf{1}_k\left(\hat{\alpha}_i^m(\omega)\right)\right| = 1$, then $\hat{X}_i^{m+\Delta m}(\omega) > \hat{X}_i^m(\omega)$. Thus, we can obtain from Equation (43) that

$$\left|\mathbb{E}\left(\rho_k^{m+\Delta m}\right) - \mathbb{E}\left(\rho_k^m\right)\right| \leq \frac{1}{\hat{M}}\sum_{i\in I}P_0\left(\hat{X}_i^{m+\Delta m} > \hat{X}_i^m\right).$$

By Lemma 9, we have

$$P_0\left(\hat{X}_i^{m+\Delta m} > \hat{X}_i^m\right) = 1 - P_0\left(\hat{X}_i^{m+\Delta m} = \hat{X}_i^m\right) \leq K\bar{a}\frac{\Delta m}{M}.$$

Hence, we can obtain that

$$\left|\mathbb{E}\left(\rho_k^{m+\Delta m}\right) - \mathbb{E}\left(\rho_k^m\right)\right| \leq K\bar{a}\frac{\Delta m}{M},$$

which implies that

$$\|\mathbb{E}\left(\rho^{m+\Delta m}\right) - \mathbb{E}\left(\rho^m\right)\|_\infty \leq K\bar{a}\frac{\Delta m}{M}.$$

Therefore, Lemma 11 is proved.