Continuous-Time Random Matching

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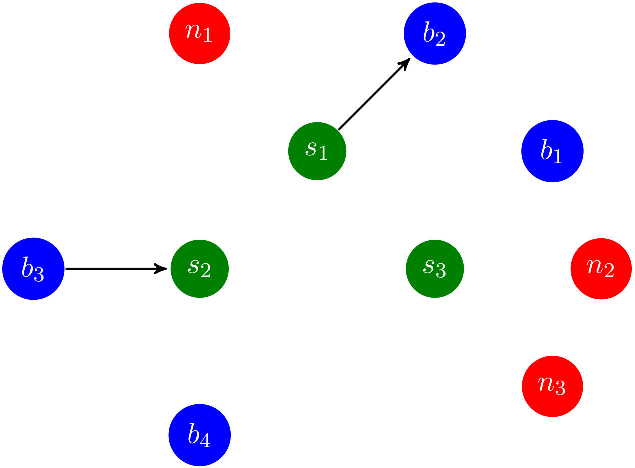
Illustrative example of an over-the-counter market



Population dynamics for an illustrative OTC market example

- The interval $[0,1]$ of agents has masses p_{bt}, p_{st}, and p_{nt} of buyers, sellers, and inactive agents, respectively.
- Each buyer or seller, at Poisson event times with intensity ν, finds an agent drawn uniformly from $[0,1]$,
- Inactive agents mutate at mean rate γ to sellers or buyers, equally likely.
- When a buyer and seller meet, they trade and become inactive.
- With cross-agent independence, the dynamic equation for the cross-sectional distribution of agent types "should be," almost surely,

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Research areas relying on continuous-time random matching

- **Monetary theory.** Hellwig (1976), Diamond-Yellin (1990), Diamond (1993), Trejos-Wright (1995), Shi (1997), Zhou (1997), Postel-Vinay-Robin (2002), Moscarini (2005).
- **Labor markets.** Pissarides (1985), Hosios (1990), Mortensen-Pissarides (1994), Acemoglu-Shimer (1999), Shimer (2005), Flinn (2006), Kiyotaki-Lagos (2007).
- **Over-the-counter financial markets.** Duffie-Gârleanu-Pedersen (2003, 2005), Weill (2008), Vayanos-Wang (2007), Vayanos-Weill (2008), Weill (2008), Lagos-Rocheteau (2009), Hugonnier-Lester-Weill (2014), Lester, Rocheteau, Weill (2015), Üslü (2016).
- **Biology (genetics, molecular dynamics, epidemiology).** Hardy-Weinberg (1908), Crow-Kimura (1970), Eigen (1971), Shashahani (1978), Schuster-Sigmund (1983), Bomze (1983).
- **Stochastic games.** Mortensen (1982), Foster-Young (1990), Binmore-Samuelson (1999), Battalio-Samuelson-Van Huycjk (2001), Burdzy-Frankel-Pauzner (2001), Benaïm-Weibull (2003), Currarini-Jackson-Pin (2009), Hofbauer-Sandholm (2007).
- **Social learning.** Börgers (1997), Hopkins (1999), Duffie-Manso (2007), Duffie-Malamud-Manso (2009).

Parameters of the basic model

1. Type space $S = \{1, \ldots, K\}$.
2. Initial cross-sectional distribution $p^0 \in \Delta(S)$ of agent types.
3. For each pair (k, ℓ) of types:
   - Mutation intensity $\eta_{k\ell}$.
   - Matching intensity $\theta_{k\ell} : \Delta(S) \to \mathbb{R}_+$ satisfying the balance identity
     $$p_k \theta_{k\ell}(p) = p_\ell \theta_{\ell k}(p).$$
     - ⋆ Technical condition: $p \mapsto p_k \theta_{kl}(p)$ is Lipschitz.
   - Match-induced type probability distribution $\gamma_{k\ell} \in \Delta(S)$.

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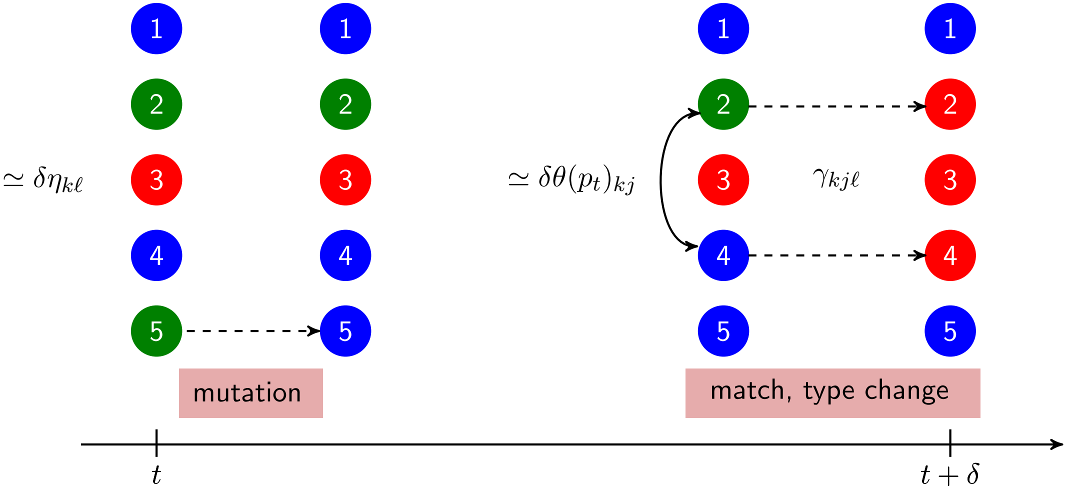
Mutation, matching, and match-induced type changes



Key solution processes

For a probability space $(\Omega, \mathcal{F}, P)$, atomless agent space $(I, \mathcal{I}, \lambda)$, and σ-algebra on $I \times \Omega \times \mathbb{R}_+$ to be specified:

- Agent type $\alpha(i, \omega, t)$, for $\alpha : I \times \Omega \times \mathbb{R}_+ \to S$.
- Latest counterparty $\pi(i, \omega, t)$, for $\pi : I \times \Omega \times \mathbb{R}_+ \to I$.
- Cross-sectional type distribution $p : \Omega \times \mathbb{R}_+ \to \Delta(S)$. That is,

  $$p(\omega, t)_k = \lambda(\{i \in I : \alpha(i, \omega, t) = k\})$$

  is the fraction of agents of type k.

Evolution of the cross-sectional distribution p_t of agent types

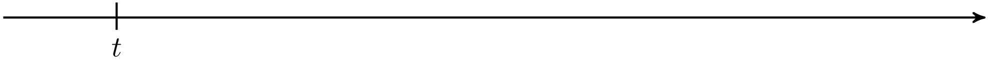
Existence of a model with independence conditions under which

$$\dot{p}_t = p_t R(p_t) \quad \text{almost surely,}$$

where $R(p_t)$ is also the agent-level Markov-chain infinitesimal generator:

$$R(p_t)_{k\ell} = \eta_{k\ell} + \sum_{j=1}^{K} \theta_{kj}(p_t)\gamma_{kj\ell}$$

$$R(p_t)_{kk} = -\sum_{\ell \neq k}^{K} R_{k\ell}(p_t).$$



A Fubini extension

Agent-level independence is impossible on the product measure space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \times P)$, except in the trivial case (Doob, 1953).

So, we use a Fubini extension $(I \times \Omega, \mathcal{W}, Q)$ of the product space, defined by the property that any real-valued integrable function f satisfies

$$\int_I \left(\int_\Omega f(i, \omega)\, dP(\omega) \right) d\lambda(i) = \int_\Omega \left(\int_I f(i, \omega)\, d\lambda(i) \right) dP(\omega).$$

We show the existence of a Fubini extension satisfying the cross-agent independence properties that we need for an exact law of large numbers.

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The exact law of large numbers

Suppose $(I \times \Omega, \mathcal{W}, Q)$ is a Fubini extension and some measurable $f : (I \times \Omega, \mathcal{W}, Q) \to \mathbb{R}$ is pairwise independent.

That is, for every pair (i, j) of distinct agents, the agent-level random variables $f(i) = f(i, \cdot)$ and $f(j)$ are independent.

The cross-sectional distribution G of f at $x \in \mathbb{R}$ in state ω is $G(x, \omega) = \lambda(\{i : f(i, \omega) \leq x\})$.

Proposition (Sun, 2006)

For P-almost every ω,

$$G(x, \omega) = \int_I P(f(i) \leq x) \, d\lambda(i).$$

In particular, if the probability distribution F of $f(i)$ does not depend on i, then the cross-sectional distribution G is equal to F almost surely.

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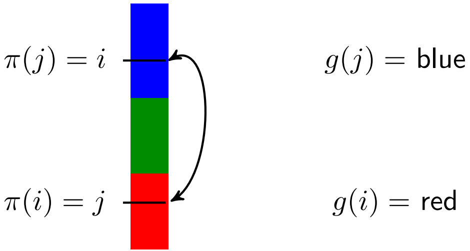
Random matching

- A *random matching* $\pi : I \times \Omega \to I$ assigns a unique agent $\pi(i)$ to agent i, with $\pi(\pi(i)) = i$. If $\pi(i) = i$, agent i is not matched.
- Let $g(i) = \alpha(\pi(i))$ be the type of the agent to whom i is matched. (If i is not matched, let $g(i) = J$.)



Independent random matching with given probabilities

- Given: A measurable type assignment $\alpha : I \to S$ with distribution $p \in \Delta(S)$ and matching probabilities $(q_{k\ell})$ satisfying $p_k q_{k\ell} = p_\ell q_{\ell k}$.
- A random matching π is said to be independent with parameters (p, q) if the counterparty type g is $\mathcal{W}$-measurable and essentially pairwise independent with

$$P(g(i) = \ell) = q_{\alpha(i),\ell} \quad \lambda\text{–}a.e.$$

- In this case, the exact law of large numbers implies, for any k and ℓ, that

$$\lambda(\{i : \alpha(i) = k, g(i) = \ell\}) = p_k q_{k\ell} \quad a.s.$$

Proposition (Duffie, Qiao, and Sun, 2015)

For any given (p, q), there exists an independent matching π.

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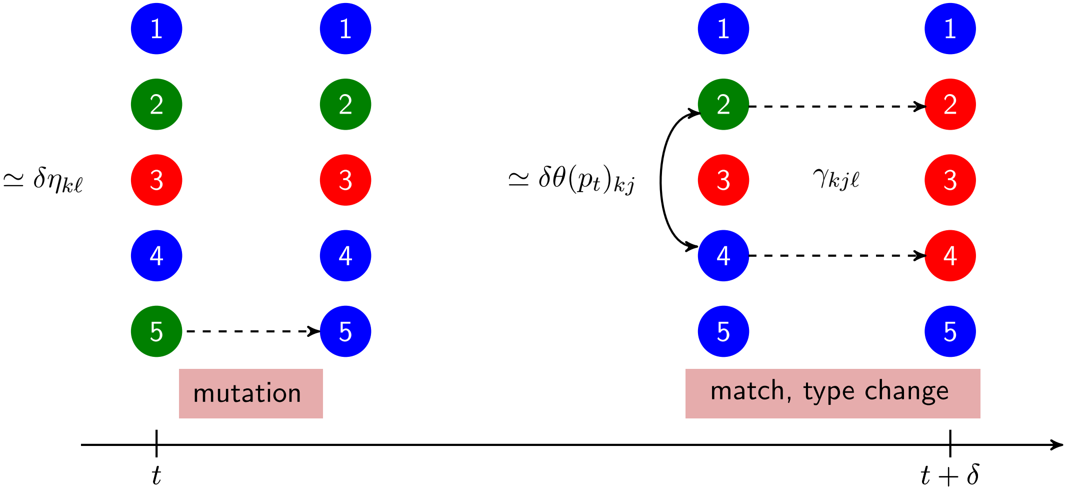
Recursive construction of the dynamic model



Continuous-time random matching

Theorem

For any parameters $(p^0, \eta, \theta, \gamma)$, there exists a Fubini extension $(I \times \Omega, \mathcal{W}, Q)$ on which there is a continuous-time system (α, π) of agent type and last-counterparty processes such that:

1. *The agent type process α and last-counterparty type process $g = \alpha \circ \pi$ are measurable with respect to $\mathcal{W} \otimes \mathcal{B}(\mathbb{R}_+)$ and pairwise independent.*
2. *The cross-sectional type distribution process $\{p_t : t \geq 0\}$ satisfies $\dot{p}_t = p_t R(p_t)$ almost surely.*
3. *For λ-almost every agent i, the type process $\alpha(i)$ is a Markov chain with infinitesimal generator $\{R(p_t) : t \geq 0\}$.*
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Stationary case

Proposition

For any (η, θ, γ), there is an initial type distribution p^0 such that the continuous-time system (α, π) associated with parameters $(p^0, \eta, \theta, \gamma)$ has constant cross-sectional type distribution $p_t = p^0$.

If the initial agent types $\{\alpha_0(i) : i \in I\}$ are pairwise independent with probability distribution p^0, then the probability distribution of the agent type $\alpha_t(i)$ is also constant and equal to p^0, for λ-a.e. agent.

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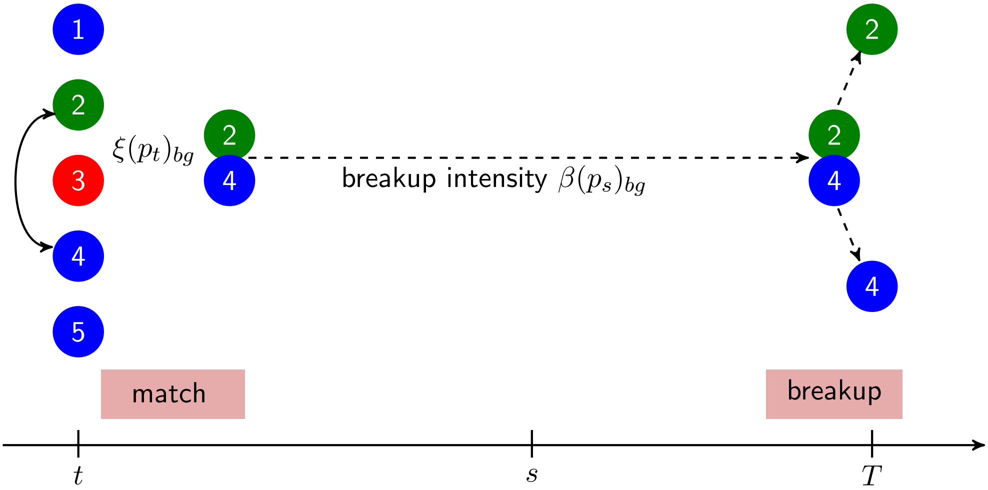
Proposition

For any (η, θ, γ)*, there is an initial type distribution* p^0 *such that the continuous-time system* (α, π) *associated with parameters* $(p^0, \eta, \theta, \gamma)$ *has constant cross-sectional type distribution* $p_t = p^0$*.*

If the initial agent types $\{\alpha_0(i) : i \in I\}$ *are pairwise independent with probability distribution* p^0*, then the probability distribution of the agent type* $\alpha_t(i)$ *is also constant and equal to* p^0*, for* λ*-a.e. agent.*

With enduring match probability $\xi(p_t)$



Further generality

- When agents of types k and ℓ form an enduring match at time t, their new types are drawn with a given joint probability distribution $\sigma(p_t)_{k\ell} \in \Delta(S \times S)$.
- While enduringly matched, the mutation parameters of an agent may depend on both the agent's own type and the counterparty's type.
- Time-dependent parameters $(\eta_t, \theta_t, \gamma_t, \xi_t, \beta_t, \sigma_t)$, subject to continuity.
- The agent type space can be infinite, for example $S = \mathbb{Z}_+$ or $S = [0, 1]^m$.