Online Appendix for "Size Discovery"

Darrell Duffie* Haoxiang Zhu†

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1 Proof of the Continuous-Time Version of Sequential Double Auctions

In this appendix we prove Proposition 7 in Appendix B of Duffie and Zhu (2016), the continuous-time version of the sequential double auction model.

Recall that the HJB equation of trader i is

$$0 = \sup_D \left[-D(\Phi(D + \mathcal{D}_{-i}(\,\cdot\,; z, Z))\Phi(D + \mathcal{D}_{-i}(\,\cdot\,; z, Z)) + V'(z)D(\Phi(D + \mathcal{D}_{-i}(\,\cdot\,; z, Z))\right] - \gamma z^2 + r(vz - V(z)), \tag{1}$$

where the continuation value function is

$$V(z) = v\frac{Z}{n} - \frac{\gamma}{r}\left(\frac{Z}{n}\right)^2 + \left(v - 2\frac{\gamma}{r}\frac{Z}{n}\right)\left(z - \frac{Z}{n}\right) - \frac{\gamma}{r}\frac{1}{n-1}\left(z - \frac{Z}{n}\right)^2. \tag{2}$$

With (2), the HJB equation, applied to trader i at time t, is equivalent to solving, for each outcome of Z, the optimal demand

$$\sup_x \left[-x\mathcal{D}_{-i}^{-1}(-x; z_{it}, Z) + V'(z_{it})x\right], \tag{3}$$

where $\mathcal{D}_{-i}^{-1}(q; z_{it}, Z)$ is the inverse total demand of the other agents at any quantity q, meaning that price p for which

$$q = (n-1)a(v-p) - \frac{2a\gamma}{r}(Z - z_{it}).$$

*Graduate School of Business, Stanford University. Email: duffie@stanford.edu.
†MIT Sloan School of Management. Email: zhuh@mit.edu.

Solving,

$$\mathcal{D}_{-i}^{-1}(q; z_{it}, Z) = v - \frac{1}{a(n-1)}\left[q + \frac{2a\gamma}{r}(Z - z_{it})\right].$$

Thus, the demand problem of agent i is

$$\sup_x \left[-x\left(v - \frac{1}{a(n-1)}\left[-x + \frac{2a\gamma}{r}(Z - z_{it})\right]\right) + V'(z_{it})x\right]. \tag{4}$$

The first-order necessary condition for optimality of x^* is

$$-v + \frac{2\gamma}{r(n-1)}(Z - z_{it}) + V'(z_{it}) - 2x^*\frac{1}{a(n-1)} = 0,$$

where

$$V'(z_{it}) = v - 2\frac{\gamma}{r}\frac{Z}{n} - 2\frac{\gamma}{r}\frac{1}{n-1}\left(z_{it} - \frac{Z}{n}\right).$$

The unique solution x^* of this first-order condition also satisfies the second-order sufficiency condition, and is given by

$$x^* = \frac{2a\gamma}{r}\left(\frac{Z}{n} - z_{it}\right).$$

The associated market clearing price is

$$p^* = \mathcal{D}_{-i}^{-1}(-x^*; z_{it}, Z) = v - \frac{1}{a(n-1)}\left[-x^* + \frac{2a\gamma}{r}(Z - z_{it})\right] = v - 2\frac{\gamma}{r}\frac{Z}{n}. \tag{5}$$

We now verify that the postulated demand function D_{it} for agent i achieves the above demand x^*, regardless of the outcome of Z. We have

$$D_{it}(p^*) = a\left(v - p^* - \frac{2\gamma}{r}z_{it}\right) = a\left(v - \left(v - 2\frac{\gamma}{r}\frac{Z}{n}\right) - \frac{2\gamma}{r}z_{it}\right) = \frac{2a\gamma}{r}\left(\frac{Z}{n} - z_{it}\right),$$

which is indeed equal to the optimal demand x^*.

In order to prove that the proposed indirect utility function V satisfies the HJB equation, we substitute our expressions for $V(z)$, p^*, and $D_{it}(p^*)$ into the right-hand-side of the HJB equation (1). To confirm that (1) is satisfied, we must show that for all real z and Z,

$$0 = -\frac{2a\gamma}{r}\left(\frac{Z}{n} - z\right)\left(v - 2\frac{\gamma}{r}\frac{Z}{n}\right) + V'(z)\frac{2a\gamma}{r}\left(\frac{Z}{n} - z\right) + r(vz - V(z)) - \gamma z^2. \tag{6}$$

To see that (6) holds, note that

$$V'(z)\frac{2a\gamma}{r}\left(\frac{Z}{n} - z\right) = v\frac{2a\gamma}{r}\left(\frac{Z}{n} - z\right) - 2\frac{\gamma}{r}\frac{Z}{n}\frac{2a\gamma}{r}\left(\frac{Z}{n} - z\right) + 2\frac{\gamma}{r}\frac{1}{n-1}\frac{2a\gamma}{r}\left(\frac{Z}{n} - z\right)^2$$

and that

$$r(vz - V(z)) = rvz - rv\frac{Z}{n} + r\frac{\gamma}{r}\left(\frac{Z}{n}\right)^2 - r\left(v - \frac{2\gamma}{r}\frac{Z}{n}\right)\left(z - \frac{Z}{n}\right) + r\frac{\gamma}{r(n-1)}\left(z - \frac{Z}{n}\right)^2$$
$$= r\frac{\gamma}{r}\left(\frac{Z}{n}\right)^2 + r\frac{2\gamma}{r}\frac{Z}{n}\left(z - \frac{Z}{n}\right) + r\frac{\gamma}{r(n-1)}\left(z - \frac{Z}{n}\right)^2. \tag{7}$$

The right-hand side of (6) is thus computed as

$$-\frac{2a\gamma}{r}\left(\frac{Z}{n} - z\right)\left(v - 2\frac{\gamma}{r}\frac{Z}{n}\right) + V'(z)\frac{2a\gamma}{r}\left(\frac{Z}{n} - z\right) + r(vz - V(z)) - \gamma z^2$$

$$= \frac{4a\gamma}{r}\left(\frac{Z}{n} - z\right)\frac{\gamma}{r}\frac{Z}{n} + -2\frac{\gamma}{r}\frac{Z}{n}\frac{2a\gamma}{r}\left(\frac{Z}{n} - z\right) + 2\frac{\gamma}{r}\frac{1}{n-1}\frac{2a\gamma}{r}\left(\frac{Z}{n} - z\right)^2$$
$$+\gamma\left(\frac{Z}{n}\right)^2 + 2\gamma\frac{Z}{n}\left(z - \frac{Z}{n}\right) + \frac{\gamma}{n-1}\left(z - \frac{Z}{n}\right)^2 - \gamma z^2.$$

Substituting $a = (n-2)r^2/4\gamma$, we have

$$-\frac{2a\gamma}{r}\left(\frac{Z}{n} - z\right)\left(v - 2\frac{\gamma}{r}\frac{Z}{n}\right) + V'(z)\frac{2a\gamma}{r}\left(\frac{Z}{n} - z\right) + r(vz - V(z)) - \gamma z^2$$

$$= (n-2)\gamma\left(\frac{Z}{n} - z\right)\frac{Z}{n} + -(n-2)\gamma\frac{Z}{n}\left(\frac{Z}{n} - z\right) + \frac{(n-2)\gamma}{n-1}\left(\frac{Z}{n} - z\right)^2$$
$$+\gamma\left(\frac{Z}{n}\right)^2 + 2\gamma\frac{Z}{n}\left(z - \frac{Z}{n}\right) + \frac{\gamma}{n-1}\left(z - \frac{Z}{n}\right)^2 - \gamma z^2.$$

So, V satisfies the HJB equation because

$$\frac{(n-2)}{n-1}\left(\frac{Z}{n} - z\right)^2 + \left(\frac{Z}{n}\right)^2 + 2\frac{Z}{n}\left(z - \frac{Z}{n}\right) + \frac{1}{n-1}\left(z - \frac{Z}{n}\right)^2 - z^2 = 0.$$

Thus, using the fact that the demand function D_{it} solves the maximization problem of the HJB equation, and using the fact that V solves the HJB equation, an application of Ito's formula to the process J defined by $J(t) = V(z_{it})$ for $t < T$, and by $J(t) = \pi z_{iT}$ for $t \geq T$ implies that

$$V(z_{i0}) = E\left[z_{iT}(T)\pi - \int_0^T \left[\gamma z_{it}^2 + D_{it}\left[\Phi\left(D_{it} + D_{-i,t}\right)\right]\Phi\left(D_{it} + D_{-it}\right)\right]\, dt\right].$$

For any other demand function D for agent i, the HJB equation and Ito's formula implies that implies that

$$V(z_{i0}) \geq E\left[z_i^D(T)\pi - \int_0^T \left[\gamma z_i^D(t)^2 + D_t\left[\Phi\left(D_t + D_{-i,t}\right)\right]\Phi\left(D_t + D_{-it}\right)\right]\, dt\right].$$

Thus D_i is indeed optimal for trader i given D_{-i}, and $V(z)$ is indeed the indirect utility of any agent with inventory z. This proves Proposition 7 of Duffie and Zhu (2016).

References

Duffie, D. and H. Zhu (2016): "Size Discovery," NBER working paper 21696.