Chapter 31

THE THEORY OF VALUE IN SECURITY MARKETS

DARRELL DUFFIE*

Graduate School of Business, Stanford University

Contents

1. Introduction

General equilibrium theory, as summarized for example in Debreu's (1959) *Theory of Value*, can be applied wholesale to obtain a theory of value for security markets, as shown by Arrow (1953). The modern theory of value for security markets, however, elaborates or extends general equilibrium theory in at least the following major ways:

(1) It explicitly treats general multi-period trading opportunities under uncertainty and in incomplete markets.

(2) It investigates, in remarkable depth, implications of the law of one price, that is, of arbitrage-free prices.

(3) In order to represent security returns in convenient and testable ways, it places strong restrictions on preferences and exploits a great deal of probability theory, especially the theories of Markov processes and stochastic integration, separately and together.

By looking in these and other directions, finance theory has been a catalyst for further developments of the general equilibrium model, particularly for equilibrium existence theorems with incomplete markets, infinite-dimensional consumption spaces or asymmetric information. In the other direction, general equilibrium theory has offered financial economists a benchmark for market behavior that was missing before Arrow (1953) and Arrow and Debreu (1954).

As financial market theory grows, it laps over the boundaries of the general equilibrium paradigm in order to focus on the process of price formation. The "microstructure" of security markets has come under increasing scrutiny; the theory of specialist market makers, for example, is gradually being filled out. The need to address asymmetric information, in particular, has led to strategic models of investment behavior.

For conventional purposes such as asset pricing, however, general competitive equilibrium models are still the norm. Indeed, it may be argued that the Walrasian notion of price-taking suits large financial exchanges better than most other markets. Thus, despite the diverse aims of financial economic theory, in this chapter we summarize developments in finance that rest or build on general equilibrium theory, emphasizing the valuation of financial assets.

This chapter is organized into four sections: Early milestones, Basic asset pricing techniques, Continuous-time general equilibrium and Derivative asset pricing. The chapter concludes with further notes to the literature on these topics and references to related topics.

2. Early milestones

We review in this section some of the major milestones along the path of early theoretical developments to models of security market equilibrium: (i) Arrow's "Role of Securities" paper, the central paradigm of financial market equilibrium theory, (ii) the Modigliani–Miller theorem on the irrelevance of corporate financial policy, and (iii) the Capital Asset Pricing Model (CAPM) of Sharpe (1964), Lintner (1965) and Black (1972).

2.1. Arrow's "Role of Securities" paper

The first major milestone is Arrow's (1953) paper, "The Role of Securities in the Optimal Allocation of Risk Bearing," still a standard reading requirement for doctoral finance students. Among others, Hicks (1939) had earlier worked toward general models incorporating markets for claims to future value, that is, securities. Arrow (1953), however, had the first general closed model of equilibrium for markets in which both spot commodities and securities are traded. This milestone preceded even the presentation by Arrow and Debreu (1954) as well as McKenzie (1954) of techniques suitable for demonstrating the existence of equilibrium in such a model.

We follow here with a reprise of Arrow's model in a slightly extended form.

There are S possible states of the economy, one of which will be revealed as true. Before the true state is revealed, n securities are traded. Security number j is a vector $d_j \in \mathbb{R}^S$, representing a claim to d_{sj} units of account ("dividends") in state s, for each $s \in \{1, \ldots, S\}$. A portfolio $\theta \in \mathbb{R}^n$ of securities thereby lays claim to $\theta \cdot d_s = \sum_{j=1}^n \theta_j d_{sj}$ units of account in a given state $s \in \{1, \ldots, S\}$. A portfolio θ is budget-feasible given a vector $q \in \mathbb{R}^n$ of security prices if $\theta \cdot q \leq 0$. After the true state is revealed, agents collect their dividends and trade on spot commodity markets. In each state s, each agent $i \in \{1, \ldots, m\}$ is endowed with a bundle $e_s^i \in \mathbb{R}^l$ of l commodities, whose respective unit prices are given by some vector $p_s \in \mathbb{R}^l$.

A *consumption plan* is a vector c in $\mathbb{R}_+^{lS}$, written $\{c_s \in \mathbb{R}_+^l : 1 \leq s \leq S\}$ for convenience, with c_s denoting the planned consumption vector in state s. (In looking at consumption sets defined only by non-negativity, we are letting generality slip through our fingers in orders to focus on a few main ideas, and will continue to do so.) Given security prices $q \in \mathbb{R}^n$ and spot commodity prices $p \in \mathbb{R}_+^{lS}$, a pair (θ, c) is a budget-feasible plan for agent i if θ is a budget-feasible portfolio and c is a consumption plan satisfying

$$p_s \cdot (c_s - e_s^i) \leq \theta \cdot d_s , \quad s \in \{1, \ldots, S\} .$$

Each agent i has a utility function $U_i : \mathbb{R}_+^{IS} \to \mathbb{R}$. Given prices $(q, p) \in \mathbb{R}^n \times \mathbb{R}_+^{IS}$, a pair $(\theta, c) \in \mathbb{R}^n \times \mathbb{R}_+^{IS}$ is an optimal plan for agent i if it is budget-feasible and if there is no budget-feasible plan (θ', c') such that $U_i(c') > U_i(c)$. A *security-spot market equilibrium* for the *economy* $((U_i, e^i), (d_j))$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, is a collection $((q, p), (\theta^i, c^i))$, $i \in \{1, \ldots, m\}$, satisfying:

(1) for each $i \in \{1, \ldots, m\}$, (θ^i, c^i) is an optimal plan for agent i given the security-spot price pair $(q, p) \in \mathbb{R}^n \times \mathbb{R}_+^{IS}$;

(2) markets clear: $\Sigma_i \, \theta^i = 0$ and $\Sigma_i \, (c^i - e^i) = 0$.

For purposes of comparison, a *complete contingent-commodity market equilibrium* for the economy (U_i, e^i), $i \in \{1, \ldots, m\}$, a concept also appearing for the first time in Arrow's paper, is defined as a collection $(\tilde{p}, (c^i))$, $i \in \{1, \ldots, m\}$, where $\tilde{p} \in \mathbb{R}_+^{IS}$ and, for each $i \in \{1, \ldots, m\}$, $c^i \in \arg\max_{c \in \mathbb{R}_+^{IS}} U_i(c)$ subject to $\tilde{p} \cdot c \leqslant \tilde{p} \cdot e^i$, with $\Sigma_i \, c^i - e^i = 0$. We treat $\tilde{p}_{sc}$ as the price, before the true state is revealed, of a contract promising delivery of one unit of commodity number c if state s is revealed to be true, and nothing otherwise.

It is widely felt that Arrow showed a security-spot market equilibrium to be Pareto optimal provided that securities span, in the sense described below, a first welfare theorem for security markets. This is essentially the stated goal of his paper. In fact, however, Arrow proved a second welfare theorem for security markets: if an allocation $\{c^i \in \mathbb{R}_+^{IS} : 1 \leqslant i \leqslant m\}$ is Pareto optimal, then it comprises the consumption plans of some security-spot market equilibrium, under the conditions:

(1) spanning: $\text{span}(\{d_j : 1 \leqslant j \leqslant n\}) = \mathbb{R}^S$;

(2) convexity: for each $i \in \{1, \ldots, m\}$, U_i is quasi-concave;

(3) strict monotonicity: for each $i \in \{1, \ldots, m\}$, $c > c' \Rightarrow U_i(c) > U_i(c')$.

(Arrow's specific assumptions were slightly different.) The proof is in two steps. First, given the Pareto optimality of (c^i), the usual second welfare theorem [Arrow (1951), Debreu (1951)] implies the existence of contingent commodity prices $\tilde{p} \in \mathbb{R}_+^{IS}$ supporting the allocation (c^i) in a complete contin-gent-commodity market compensated[1] equilibrium $(\tilde{p}, (c^i))$. The second step is to implement the contingent-commodity compensated equilibrium $(\tilde{p}, (c^i))$ as a security-spot market compensated equilibrium. This implementation is easily done, after choosing an arbitrary "state-price" vector $\pi \in \mathbb{R}_{++}^S$, as follows:

(1) let $q_j = \Sigma_{s=1}^S \, \pi_s d_{js}$, $j \in \{1, \ldots, n\}$;

(2) let $p_s = \tilde{p}_s \pi_s$, $s \in \{1, \ldots, S\}$;

(3) for $i \in \{1, \ldots, m-1\}$, let θ^i solve the system of linear equations

---

[1] The collection $(\tilde{p}, (c^i))$ is a compensated equilibrium if there is a vector $w \in \mathbb{R}^m$ (of wealth transfers) such that $\Sigma_i \, c^i - e^i = 0$ and, for all i, $c^i \in \arg\max_c U_i(c)$ subject to $p \cdot c \leqslant w_i + p \cdot e^i$.

$$p_s \cdot (c_s^i - e_s^i) = \sum_{j=1}^{n} \theta_j^i d_{js} , \quad 1 \le s \le S ; \tag{1}$$

(4) let $\theta^m = -\Sigma_{i=1}^{m-1} \theta^i$.

The reader can quickly verify as an exercise that $(q, p, (\theta^i, c^i))$ is a bona fide security-spot market equilibrium provided the wealth transfers making $(\tilde{p}, (c^i))$ a compensated contingent-commodity equilibrium are again applied, before security trading. The spanning condition on securities provides for the existence of solutions to the system of equations (1) defining θ^i.

Of course, there is also a First Welfare Theorem: Any security-spot market equilibrium for an economy with strictly monotone utilities and spanning securities involves a Pareto optimal allocation of contingent commodities. Arrow's (1951) simple "adding-up" argument suffices for proof; convexity is not required. Moreover, with the additional assumptions of continuous quasi-concave utility functions and interior total endowments, security-spot market equilibria exist. The existence proof proceeds by demonstrating the existence of a complete contingent-commodity market equilibrium, and then implementing the allocation in a security-spot market equilibrium by the above four-step procedure.

The theory of existence and optimality of equilibria is quite different, however, without spanning, as explained in Chapter 30. Briefly, if individual endowments are interior, in addition to the other assumptions, then equilibria always exist, but are generically inefficient without spanning. If security dividends are defined instead, in terms of commodity bundles, then equilibria may exist only generically, that is, except for a closed set of measure zero of endowments and securities, under regularity conditions on utility functions.

2.2. Modigliani–Miller's irrelevance of corporate financial structure

Modigliani and Miller (1958) demonstrated the irrelevance of corporate financial structure in the absence of such market "imperfections" as taxes, transaction costs, bankruptcy costs, asymmetric information, and so on. It is instructive to view a version of their results in the setting of Arrow's model of security and spot markets.

First, we extend Arrow's model with the addition of a firm, characterized by a production set $Y \subset \mathbb{R}^{lS}$ and a vector $\gamma \in \mathbb{R}_+^m$ of initial shareholdings of the firm among agents, normalized so that $\Sigma_{i=1}^m \gamma_i = 1$.

We take the firm's production choice $y \in Y$ as given. The firm is also free to choose a portfolio $\theta^0 \in \mathbb{R}^n$ of securities. Given a spot commodity price vector $p \in \mathbb{R}_+^{lS}$, the firm's total dividend is the vector $\delta(y, p, \theta^0) \in \mathbb{R}^S$ defined by

$$\delta(y, p, \theta^0)_s = p_s \cdot y_s + \theta_s^0 \cdot d_s , \quad s \in \{1, \ldots, S\} .$$

The firm finances its portfolio θ^0 by issuing the dividend $-\theta^0 \cdot q$ in the first period. The firm's shares are traded, like any other security, at some price v. A portfolio in this setting is a pair $(\varphi, \theta) \in \mathbb{R} \times \mathbb{R}^n$ representing φ shares of the firm and a portfolio θ of the other n securities. A budget-feasible plan for agent i, given prices $(v, q, p) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_+^{IS}$, is therefore some triple $(\varphi, \theta, c) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}_+^{IS}$ satisfying

$$(\varphi - \gamma_i)v + \varphi(-\theta^0 \cdot q) + \theta \cdot q \le 0 ,$$

$$p_s \cdot (c_s - e_s^i) \le \varphi \delta(y, p, \theta^0)_s + \sum_{j=1}^n \theta_j d_{js} , \quad s \in \{1, \ldots, S\} .$$

The budget constraint for the initial period reflects the share $-\varphi\theta^0 \cdot q$ of the firm's initial dividend allotted to the shareholder. A budget-feasible plan (φ, θ, c) is optimal for agent i if there is no budget-feasible plan (φ', θ', c') with $U_i(c') > U_i(c)$. A security-spot market equilibrium, given the firm's production and financial plan $(y, \theta^0) \in Y \times \mathbb{R}^n$, is a collection $(v, q, p, (\varphi^i, \theta^i, c^i))$, $i \in \{1, \ldots, m\}$, satisfying:

(1) for $i \in \{1, \ldots, m\}$, $(\varphi^i, \theta^i, c^i)$ is an optimal plan for agent i given the prices (v, q, p),

(2) markets clear: $\Sigma_i \varphi^i = 1$, $\Sigma_i \theta^i = 0$ and $\Sigma_i c^i - e^i = y$.

By the "irrelevance of corporate financial structure," we mean a result of the following sort, a version of Modigliani and Miller's assertions found, in a more general form, in DeMarzo (1988).

Proposition 1. *Suppose $(v, q, p, (\varphi^i, \theta^i, c^i))$, $i \in \{1, \ldots, m\}$, is a security-spot market equilibrium given the firm's choice $(y, \theta^0) \in Y \times \mathbb{R}^n$. Let $\hat{\theta}^0 \in \mathbb{R}^n$ be any other portfolio choice by the firm, and let $\hat{\theta}^i = \theta^i + \varphi^i(\theta^0 - \hat{\theta}^0)$ be revised portfolios for the agents, $i \in \{1, \ldots, m\}$. Then $(v, q, p, (\varphi^i, \hat{\theta}^i, c^i))$, $i \in \{1, \ldots, m\}$, is an equilibrium given the firm's alternate choice $(y, \hat{\theta}^0) \in Y \times \mathbb{R}^n$.*

Proof is left as an easy exercise. Interpreting, an adjustment in the financial policy of the firm causes no change in its market value v. Moreover, every shareholder is indifferent to changes in the firm's financial structure, given the ability of each shareholder to adjust his or her own portfolio strategy so as to exactly offset any changes in the firm's financial structure.

The notes in Section 6 indicate a number of directions that one can take in order to overturn the irrelevance of corporate financial policy. The notes also

introduce the literature on the role of spanning in shareholder unanimity, or lack thereof, concerning the production policy of the firm.

2.3. The capital asset pricing model (CAPM)

Due to Sharpe (1964), Lintner (1965) and, with no riskless asset, Black (1972), the Capital Asset Pricing Model (CAPM) can also be viewed in Arrow's (1953) setting. First, we suppose there is only one commodity consumed in each state. Second, we fix this spot commodity, usually called "consumption," as the numeraire, so that $p_s = 1$ in each state $s \in \{1, \ldots, S\}$. This allows us to view a security d_j as a claim to units of account or to units of consumption. Finally, we append to the state set $\Omega = \{1, \ldots, S\}$ the σ-algebra $\mathscr{F} = 2^{\Omega}$ of all subsets as well as a probability measure P on $\mathscr{F}$, which (for concreteness and fairness to the strength of the preference assumptions to follow) is taken to represent the common probability assessments[2] of the agents. In fact, for the CAPM, we could let $(\Omega, \mathscr{F}, P)$ be an arbitrary probability space, provided one restricts attention in the following calculations to random variables with finite variance.

For convenience, the utility function U_i is defined on the whole space $\mathbb{R}^S$. For each agent i, the restriction of U_i to $Z = \mathrm{span}(\{d_j : 1 \leq j \leq n\})$ is *variance-averse* if, for any c and c' in Z with $E(c) = E(c')$, we have $U_i(c) > U_i(c')$ provided $\mathrm{var}(c) < \mathrm{var}(c')$, where $\mathrm{var}(\cdot)$ denotes variance, treating a consumption vector c in $\mathbb{R}^S$ as a random variable $c : \Omega \to \mathbb{R}$. In infinite-state settings, if $\{d_j\}$ is jointly normally distributed, variance-aversion is implied by expected concave utility. This implication is generalized from normally distributed to spherically distributed dividends by Chamberlain (1983a).

We adopt the following principal assumptions.

Assumption 1 (*variance-aversion*). For all $i \in \{1, \ldots, m\}$, U_i is variance-averse when restricted to the span of security dividends, $Z = \mathrm{span}(\{d_j : 1 \leq j \leq n\})$.

Assumption 2 (*endowment-spanning*). For all $i \in \{1, \ldots, m\}$, $e^i \in Z$.

For the economy $((U_i, e^i), (d_j))$, we take as given an equilibrium $(q, (\theta^i))$, $i \in \{1, \ldots, m\}$, notationally suppressing the spot price $p_s = 1$ in each state s and the consumption plan $c^i = e^i + \sum_{j=1}^{n} \theta_j d_j$ of each agent i. Equilibrium existence results can be found in Nielson (1985, 1987, 1989a,b).

---

[2]From an axiomatic point of view, probability assessments are properties of preferences with Savage's (1954) framework, which calls for an infinite set of states, but we shall merely take probabilities as given.

Security pricing is *arbitrage-free* if $\theta \cdot q > 0$ for any portfolio θ with total dividend $\Sigma_{j=1}^{n} \theta_j d_j > 0$ and $\theta \cdot q = 0$ whenever $\Sigma_{j=1}^{n} \theta_j d_j = 0$. For example, if U_i is strictly increasing for some agent i and there is a security with non-zero non-negative dividends, then security pricing is arbitrage-free. Security pricing is *risk-neutral* if there is a constant K such that $q_j = KE(d_j)$ for all j. Risk-neutral pricing is possible, but somewhat pathological with variance-aversion,[3] so we consider the following.

Assumption 3. Security pricing is arbitrage-free and *not* risk-neutral.

The following proof of the CAPM allows for, but does not require, the presence of a *riskless asset*, some portfolio θ whose total dividend $\Sigma_j \theta_j d_j$ is a non-zero constant. The result applies to an arbitrary probability space $(\Omega, \mathcal{F}, P)$, and to a (closed) possibly infinite-dimensional linear subspace Z defining the span of security dividends. We first prove the "price form" of the CAPM, later reducing the CAPM to its return (or "beta") form.

Proposition 2 (CAPM: price form). *Under Assumptions 1–3, for any equilibrium $(q, (\theta^i))$, there are constants k and K such that the equilibrium security prices satisfy*

$$q_j = k \operatorname{cov}(e, d_j) + KE(d_j), \quad j \in \{1, \ldots, n\}, \quad (2)$$

where $e = \Sigma_i e^i$ is the aggregate consumption.

Proof. We take the case of no riskless asset, leaving the easier case with a riskless asset for the reader. Our proof is inspired by Chamberlain's (1988). We can endow $Z = \operatorname{span}(\{d_j : 1 \le j \le n\})$ with the Hilbert-space inner product $(c, c') \mapsto \operatorname{cov}(c, c')$. (Assumption 3 implies that Z is at least 2-dimensional.) Since security pricing is arbitrage-free, the linear functional $\Pi : Z \to \mathbb{R}$ defined by

$$\Pi(z) = \sum_{j=1}^{n} q_j \theta_j, \qquad z = \sum_{j=1}^{n} \theta_j d_j,$$

is represented [as in Luenberger (1969)] by a unique $\pi \in Z$ in the form

$$\Pi(z) = \operatorname{cov}(z, \pi), \quad z \in Z.$$

Likewise, let the expectation functional $E : Z \to \mathbb{R}$ be represented by $\eta \in Z$.

[3]For example, let $e^1 = d_1 = -d_2 = -e^2$ and $U_1 = U_2$, which implies risk-neutral pricing.

For any agent i, let $\hat{c}_i$ denote the orthogonal projection of c^i onto span($\{\pi, \eta\}$), and let $\hat{\theta}^i$ denote any portfolio satisfying

$$e^i + \sum_{j=1}^{n} \hat{\theta}^i_j d_j = \hat{c}^i .$$

By the definition of orthogonal projection, $\Pi(\hat{c}^i - c^i) = 0$, so $\hat{\theta}^i$ is budget-feasible. Since $E(\hat{c}^i - c^i) = 0$ and $\text{cov}(\hat{c}^i, \hat{c}^i - c^i) = 0$,

$$\text{var}(c^i) = \text{var}[\hat{c}^i + (c^i - \hat{c}^i)] = \text{var}(\hat{c}^i) + \text{var}(c^i - \hat{c}^i) \geq \text{var}(\hat{c}^i) .$$

We therefore know that $c^i = \hat{c}^i$, for otherwise $\hat{c}^i$ is strictly preferred to c^i and is achieved by the budget-feasible portfolio $\hat{\theta}^i$. Thus $e = \Sigma_i c^i \in \text{span}(\{\pi, \eta\})$. Using Assumption 3, $\pi = ke + K\eta$ for some constants k and K with $k \neq 0$. It follows that, for any $c \in Z$,

$$\Pi(c) = k \, \text{cov}(e, c) + K \, \text{cov}(\eta, c) = k \, \text{cov}(e, c) + KE(c) .$$

This produces the result in the absence of a riskless asset. With a riskless asset, the proof uses the inner product $(c, c') \mapsto E(cc')$.

The *return* of any portfolio $\theta \in \mathbb{R}^n$ of securities with non-zero market value $q \cdot \theta$ is defined as the random variable

$$R_\theta = \frac{\Sigma_{j=1}^{n} \theta_j d_j}{q \cdot \theta} .$$

By the endowment-spanning assumption, there is a portfolio $M \in \mathbb{R}^n$ of securities with total dividend $\Sigma_{j=1}^{n} M_j d_j = e = \Sigma_{i=1}^{m} c^i$, the equilibrium aggregate consumption. We refer to M as the *market portfolio*.

Assumption 4 (*non-triviality*). $q \cdot M \neq 0$ and $\text{var}(R_M) \neq 0$.

Under Assumption 4, the beta of any portfolio θ (of non-zero market value) is defined by

$$\beta_\theta = \frac{\text{cov}(R_\theta, R_M)}{\text{var}(R_M)} .$$

Corollary (CAPM: return form).   *Under Assumptions 1–4, there is at least one portfolio φ with non-zero market value and with $\beta_\varphi = 0$. Fixing φ, for any portfolio θ with non-zero market value,*

$$E(R_\theta) - E(R_\varphi) = \beta_\theta[E(R_M) - E(R_\varphi)] \ . \tag{3}$$

If there is a riskless asset with non-zero market value, we can take φ to be that riskless asset.

Proof. The fact that there exists a "zero-beta" portfolio φ follows from Assumption 3, which implies that Z is at least 2-dimensional and that some point in Z is uncorrelated with R_M and has non-zero price. Equation (3) follows from (2) and a few algebraic manipulations.

The proof of Proposition 2 also shows a "mutual fund theorem." There exist two portfolios of securities, say φ^A and φ^B, such that, for each agent i, the equilibrium consumption c^i is financed by some portfolio $a_i\varphi^A + b_i\varphi^B$ of these two mutual funds. For example, we could let φ^A and φ^B be portfolios such that $\Sigma_{j=1}^n \varphi_j^A d_j = \eta$ and $\Sigma_{j=1}^n \varphi_j^B d_j = \pi$. This follows from the fact that, for all i, $c^i \in \text{span}(\{\pi, \eta\})$. A more natural choice would be to let one of the two mutual funds be M, the market portfolio. If a riskless asset exists, it would serve as the other mutual fund.

3. Basic asset pricing techniques

In this section, we review two basic theoretical techniques used in pricing securities. The first, arbitrage pricing, is transparent in concept, but deep in its application, for example, in continuous-time settings such as those illustrated in Section 5. The second, "representative-agent pricing," is merely a simple application of the standard formula equating ratios of prices with marginal rates of substitution. We will see representative-agent asset pricing first in the two-period setting of Arrow (1953) and then in an infinite-horizon setting. In Section 4, representative-agent pricing reappears in a multi-agent continuous-time setting.

3.1. Arbitrage pricing

Still maintaining the structure of Arrow's (1953) model as outlined in the previous section, consider a security-spot market equilibrium $(q, p, (\theta^i, c^i))$ for the economy $((d_j), (U_i, e^i)), j \in \{1, \ldots, n\}, i \in \{1, \ldots, m\}$. As in the CAPM, arbitrage-free security pricing implies the existence of a unique linear functional Π on $Z = \text{span}(\{d_j : 1 \leq j \leq n\})$ by

$$\Pi(z) = \sum_{j=1}^n q_j\theta_j \ , \qquad z = \sum_{j=1}^n \theta_j d_j \ .$$

Arbitrage-free pricing also implies that Π is strictly positive, that is, any portfolio with total dividend $z > 0$ has a positive total price $\Pi(z) > 0$ (throughout, "$z > 0$" means $z \geqslant 0$ and $z \neq 0$.)

Lemma. *Any strictly positive linear functional Π on a linear subspace Z of a Euclidean space $\mathbb{R}^S$ has a strictly positive linear extension $\bar{\Pi} : \mathbb{R}^S \to \mathbb{R}$.*

This well known result, found for example in Gale (1960), can be proved by using the theorem of the alternative, and yields the following state-pricing result, which first appeared in Ross (1976c, p. 202).

Corollary (state-pricing). *If security pricing is arbitrage-free, there is some (state-price) vector $\pi \in \mathbb{R}^S_{++}$ such that*

$$\Pi(z) = \sum_{s=1}^{S} \pi_s z_s , \quad z \in Z .$$

The state-price vector π is uniquely determined if and only if $Z = \mathbb{R}^S$.

Suppose some portfolio $\bar{\theta}$ has a dividend $z = \sum_{j=1}^{n} \bar{\theta}_j d_j \gg 0$ (meaning $z_s > 0$ for all s). Assuming that (d, q) is arbitrage-free, we know that $q \cdot \bar{\theta} = \Pi(z) > 0$. We can therefore normalize prices and dividends relative to the price and dividends of $\bar{\theta}$, respectively, by defining

$$\hat{q}_j = \frac{q_j}{q \cdot \bar{\theta}} , \quad j \in \{1, \ldots, n\} ,$$

$$\hat{d}_{js} = \frac{d_{js}}{\sum_{j=1}^{n} \bar{\theta}_j d_{js}} , \quad s \in \{1, \ldots, S\}, j \in \{1, \ldots, n\} .$$

Security pricing for the normalized pair $(\hat{q}, \hat{d})$ is also arbitrage-free, implying an associated state-price vector $\hat{\pi} \in \mathbb{R}^S_{++}$ with

$$\hat{q}_j = \sum_{s=1}^{S} \hat{\pi}_s \hat{d}_{js} , \quad j \in \{1, \ldots, n\} .$$

For the portfolio $\bar{\theta}$, we have $\bar{\theta} \cdot \hat{q} = 1$ and $\sum_{j=1}^{n} \bar{\theta}_j \hat{d}_{js} = 1$ for all s. This implies that $\sum_{s=1}^{S} \hat{\pi}_s = 1$, so we may treat $\hat{\pi}$ as a vector of probability assessments of the states. Endowing $\Omega = \{1, \ldots, S\}$ with the σ-algebra $\mathcal{F}$ consisting of all subsets, and giving $(\Omega, \mathcal{F})$ the probability measure Q defined by $Q(\{s\}) = \hat{\pi}_s$, we have

$$\hat{q}_j = E^Q(\hat{d}_j) , \quad j \in \{1, \ldots, n\} , \tag{4}$$

where E^Q denotes expectations under Q. (As with the CAPM, we are treating an element of $\mathbb{R}^S$ as a random variable on Ω into $\mathbb{R}$.) In summary, by choosing an appropriate numeraire and probability assessments, one can always view the price of an asset as the expected value of its dividends.

The measure Q is called an *equivalent martingale measure* by Harrison and Kreps (1979), who extended this idea to a continuous-time setting, as explained in Section 5. There is no general infinite-dimensional result, however, guaranteeing the existence of strictly positive linear extensions, which is annoying, since many financial models are by nature infinite-dimensional. There are, however, results such as the Krein–Rutman Theorem implying (weakly) positive linear extensions of positive linear functionals on a linear subspace with a positive interior point. Ross (1978a) was the first to apply this sort of result to infinite-dimensional asset pricing. For a strictly positive linear extension, it is typical, instead, to follow the lead of Harrison and Kreps (1979) and Kreps (1981) in assuming the existence, for some agent with convex continuous strictly increasing preferences, of an optimal consumption choice in the interior of a convex consumption set. The separating hyperplane theorem then produces a satisfactory strictly positive continuous linear extension of the price functional. Because of technical issues, even a strictly positive continuous linear extension does not guarantee the existence of an equivalent martingale measure. Rather than reviewing the infinite-dimensional case in more detail here, we refer readers to Section 5.

3.2. Representative-agent pricing

The object here is a formula relating the aggregate consumption level of the economy (which is, arguably, an observable macro-economic variable) to the linear functional Π that prices securities. An example is the CAPM.

Suppose, to begin, that there is a single commodity ($l = 1$) and a single agent ($m = 1$) with a differentiable strictly monotone concave utility function $U : \mathbb{R}_+^S \to \mathbb{R}$ and a consumption endowment $e \geqslant 0$. As with the CAPM, we normalize so that the equilibrium consumption price is $p_s = 1$ in each state $s \in \{1, \ldots, S\}$, and assume that the security dividends (d_{js}) are defined in terms of this same numeraire. By inspection, an equilibrium is given by the consumption choice $c = e$, the portfolio choice $\theta = 0$, and the security price vector $q \in \mathbb{R}^n$ defined by

$$q_j = \nabla U(e) d_j , \quad j \in \{1, \ldots, n\} ,$$

where $\nabla U(e)$ denotes the vector of partial derivatives of U at e. Suppose, as previously, that $\Omega = \{1, \ldots, S\}$ is endowed with the structure of a probability

space and that vectors in $\mathbb{R}^S$ are treated as random variables. We consider the utility function U defined by $U(c) = E[u(c)]$, for some differentiable $u : \mathbb{R}_+ \to \mathbb{R}$. In this case

$$q_j = E[u'(e)d_j], \quad j \in \{1, \ldots, n\} . \tag{5}$$

Despite its simplicity, this is a basic asset pricing formula used in much of financial economics and macro-economics. A multi-period analogue, suitable for econometric analysis, is reviewed in the next subsection.

Turning to the case of heterogeneous agents, we assume spanning:

$$\text{span}(\{d_j : 1 \leq j \leq n\}) = \mathbb{R}^S .$$

As stated in Section 2, with this spanning assumption an equilibrium consumption allocation $(c^i) = \{c^i \in \mathbb{R}_+^S : 1 \leq i \leq m\}$ is Pareto optimal for the agents (U_i, e^i), $i \in \{1, \ldots, m\}$, provided, for example, that, for all i, U_i is increasing and strictly concave.

For any given "utility weights" $\lambda \in \mathbb{R}_+^m$, let $U_\lambda : \mathbb{R}_+^S \to \mathbb{R}$ be defined by

$$U_\lambda(x) = \max_{x^1 + \cdots + x^m \leq x} \sum_{i-1}^m \lambda_i U_i(x^i) .$$

By the Pareto optimality of (c^i), we can choose λ so that $U_\lambda(e) = \Sigma_{i=1}^m \lambda_i U_i(c^i)$, where $e = \Sigma_{i=1}^m e^i$. In order to give an interpretation of prices in terms of marginal utility, we want to guarantee that the equilibrium consumption allocation (c^i) is interior. For this, it is enough that $\|\nabla U_i(c)\| = \infty$ for c in the boundary of the positive cone. Pareto optimality then implies the co-linearity of $\{\nabla U_i(c^i) : 1 \leq i \leq m\}$. The implicit function theorem[4] implies that U_λ is differentiable, and the equilibrium security price vector is then given by

$$q_j = k \nabla U_\lambda(e) d_j , \quad j \in \{1, \ldots, n\} ,$$

for some constant $k > 0$. Again, we have related security prices to aggregate consumption.

In order to exploit the special case of von Neumann–Morgenstern (expected utility) preferences, we let $\Omega = \{1, \ldots, S\}$ be given the structure of a probability space $(\Omega, \mathscr{F}, P)$, and treat any $x \in \mathbb{R}^S$ as a random variable $x : \Omega \to \mathbb{R}$. We assume, for each agent i, the preference representation $U_i(x) = E[u_i(x)]$, where u_i is differentiable, increasing and strictly concave. The representative-agent utility function U_λ is then of the form

---

[4]For details and the required regularity on utility functions, see Mas-Colell (1985).

$$U_\lambda(x) = E[u_\lambda(x)] , \quad x \in \mathbb{R}^S ,$$

where $u_\lambda : \mathbb{R}_+ \to \mathbb{R}$ is defined by

$$u_\lambda(\alpha) = \max_{(a_1, \ldots, a_m) \in \mathbb{R}_+^m} \sum_{i=1}^m \lambda_i u_i(a_i) \quad \text{subject to } a_1 + \cdots + a_m \leq \alpha .$$

It follows that

$$q_j = kE[u_\lambda'(e)d_j] = k_1 E(d_j) + k_2 \operatorname{cov}[u_\lambda'(e), d_j] , \quad j \in \{1, \ldots, n\} , \qquad (6)$$

for positive constants k, k_1 and k_2. Constantinides (1982) developed a finite-dimensional multi-period version of this construction.

If u_i is locally quadratic at the equilibrium consumption level c^i, then $u_i'(c^i) = a_i + b_i c^i$ for some constants a_i and b_i, and thus $u_\lambda'(e) = a + be$ for some constants a and b. We then have

$$q_j = kE[u_\lambda'(e)d_j] = k_1 E(d_j) + k_2 b \operatorname{cov}(d_j, e) , \quad j \in \{1, \ldots, n\} ,$$

from which we recover the CAPM. Of course, we could have obtained the CAPM directly from the fact that concave quadratic expected utility is variance-averse.

3.3. Recursive representative-agent pricing

The work of LeRoy (1973) and Rubinstein (1976) on asset pricing in an infinite-horizon setting was capped off by Lucas (1978) with a simple recursive pricing relation known as the "stochastic Euler equation." As shown by Kandori (1988), few assumptions are required for the existence of equilibria with this pricing formula, given the usual outright assumption of a single agent.

In order to see this model in a simple form, let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathbb{F} = \{\mathcal{F}_t : t \in \mathbb{N}\}$ denote a sequence of sub-σ-algebras of $\mathcal{F}$ that is increasing in the sense that $t \geq s$ implies that $\mathcal{F}_s \subset \mathcal{F}_t$. The set $\mathcal{F}_t$ of events represents information available at time t. A sequence $\{X_t\}$ of random variables is *adapted* if, for all t, X_t is $\mathcal{F}_t$-measurable. Naturally, all economic processes in the model are adapted.

Let L denote the space of bounded adapted sequences, with the usual positive cone L_+. The single agent in the model is represented by an endowment sequence $e \in L$ and a utility function $U : L_+ \to \mathbb{R}$, There are n securities represented by a collection $d = (d^1, \ldots, d^n) \in L^n$ of dividend sequences. The economy is therefore completely specified by the list

$$((\Omega, \mathcal{F}, P), \mathbb{F}, (U, e), d) .$$

As in the CAPM, we take it that the securities' dividends are paid in units of the single consumption commodity. Given a vector of security price processes $S = (S^1, \ldots, S^n) \in L^n$, a budget-feasible plan for the agent is a security trading strategy $\theta = (\theta^1, \ldots, \theta^n) \in L^n$ satisfying, for all $t \in \mathbb{N}$,

$$c_t^{\theta} \equiv e_t + \theta_{t-1} \cdot d_t - S_t \cdot (\theta_t - \theta_{t-1}) \geq 0 ,$$

with $\theta_0 = (1, \ldots, 1)$. That is, θ is feasible if the associated consumption sequence c^{θ} is non-negative. A budget-feasible trading strategy θ is optimal if there is no budget-feasible strategy φ such that $U(c^{\varphi}) > U(c^{\theta})$. An *equilibrium* is a security price process $\{S_t\}$ such that the (no-trade) strategy θ^*, defined by $\theta_t^* = (1, \ldots, 1)$ for all t, is an optimal trading strategy.

For simplicity, we suppose that

$$U(c) = E\left[\sum_{t \in \mathbb{N}} \beta^t u(c_t) \right] , \quad c \in L_+ , \tag{7}$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is, say, bounded and measurable and $\beta \in (0, 1)$. Extensions are discussed at the end of this subsection.

The following proposition states that an equilibrium is defined by a separating hyperplane argument. Since there is but a single agent, there is no need to apply (as is commonly done) fixed point theory, Markovian assumptions, or Bellman's principle of dynamic programming.

Proposition 3. *Suppose u is increasing, bounded, differentiable and strictly concave. Let $c_t^* = e_t + \Sigma_j d_t^j$ define the total consumption process c^*. If $c_t^* > 0$ almost surely for all t, then equilibrium is defined by*

$$S_t = \frac{1}{u'(e_t)} E\left[\sum_{s>t} \beta^{s-t} u'(c_s^*) d_s \mid \mathcal{F}_t \right] , \quad a.s., \ t \in \mathbb{N} . \tag{8}$$

Proof. For the given price process $\{S_t\}$, we need only show optimality of the trading strategy θ^*. The associated consumption process is c^*. The proof here is the same as that used in Duffie, Geanakoplos, Mas-Colell and McLennan (1988). Let φ be an arbitrary budget-feasible policy.

The first step is to show that, for any given $T \in \mathbb{N}$,

$$U(c^*) \geq E\left[\sum_{t=1}^{T} \beta^t u(c_t^{\varphi}) \right] + E\left[\sum_{t=T+1}^{\infty} \beta^t u(c_t^*) \right]$$
$$+ \beta^T E[u'(c_T^*) S_T \cdot (\varphi_T - \theta_T^*)] . \tag{9}$$

We prove (9) by induction. For $T = 1$, (9) is true since concavity of u_i implies that $u(c_1^*) \geq u(c_1^\varphi) + u'(c_1^*)(c_1^* - c_1^\varphi)$. Next, we show, for any $\tau \in \mathbb{N}$, that if (9) is true for $T = \tau$, then (9) is true for $T = \tau + 1$. By the construction of S,

$$u'(c_\tau^*) S_\tau \cdot (\varphi_\tau - \theta_\tau) = \beta E[u'(c_{\tau+1}^*)(S_{\tau+1} + d_{\tau+1}) \mid \mathscr{F}_\tau] \cdot (\varphi_\tau - \theta_\tau), \quad \text{a.s.}$$

In addition, concavity of u implies that

$$u(c_{\tau+1}^*) \geq u(c_{\tau+1}^\varphi) + u'(c_{\tau+1}^*)(c_{\tau+1}^* - c_{\tau+1}^\varphi).$$

Then (9) follows for $T = \tau + 1$ by combining the last two relations with the identity

$$(S_{\tau+1} + d_{\tau+1}) \cdot (\varphi_\tau - \theta_\tau^*) + c_{\tau+1}^* - c_{\tau+1}^\varphi = S_{\tau+1} \cdot (\varphi_{\tau+1} - \theta_{\tau+1}^*).$$

Thus (9) follows for all T by induction. Since u and c^* are bounded and u is concave, $\{u'(c_i^*)c_i^*\}$ is bounded. Thus $\{S_t\}$ is (as presumed) bounded. Since $\{\varphi_t - \theta_t^*\}$ is also bounded, it follows that $\{u'(c_t^*)S_t \cdot (\varphi_t - \theta_t^*)\}$ is bounded. From this, $\beta^T E[u'(c_T^*)S_T \cdot (\varphi_T - \theta_T^*)] \to 0$ as $T \to \infty$. Combining this fact with (9), we have $U(c^*) \geq U(c^\varphi)$. Since φ is arbitrary, θ^* is optimal, so $\{S_t\}$ is an equilibrium. Uniqueness is shown with an argument by contradiction that we leave to the leader.

Corollary (stochastic Euler equation). *Under the same conditions, for the unique equilibrium $\{S_t\}$ and any time t,*

$$S_t = \frac{\beta}{u'(c_t^*)} E[u'(c_{t+1}^*)(S_{t+1} + d_{t+1}) \mid \mathscr{F}_t], \quad \text{a.s.}$$

Proof. This follows from substitution of the equilibrium equation (8) for S_{t+1} into the equilibrium equation (8) for S_t, and by applying the law of iterated expectations.

Just as in the previous subsection, one can extend the representative-agent asset-pricing formula shown here to economies with heterogeneous agents, provided the securities are spanning and all consumption choices are interior. Rather than pursue this here, we return to it in the continuous-time framework of the following section.

3.4. Extended recursive preference models and time consistency

The additively separable utility criterion (7) is restrictive. For example, this utility criterion cannot reflect any attitude toward the timing of the resolution

of uncertainty, as pointed out by Kreps and Porteus (1978). For settings like the present, a utility model developed by Epstein and Zin (1989a) retains the recursive structure of the additively separable model while admitting preferences for early or for late resolution of uncertainty, and for independent adjustment of intertemporal elasticity of substitution and risk aversion. The two basic primitives of the Epstein–Zin utility model are:

   (i) a certainty equivalent functional $m : \mathcal{P}(\mathbb{R}) \to \mathbb{R}$ (where $\mathcal{P}(\mathbb{R})$ denotes the probability measures on the real line) and

   (ii) an aggregator $W : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$.

The certainty equivalent m is defined so that $m(\delta_x) = x$ for any dirac measure δ_x, consistent with indifference between any distribution μ of utility in the next period and the deterministic utility $m(\mu)$. An adapted stochastic process V is by definition the utility process for a consumption process c if V uniquely satisfies, for all t,

$$V_t = W[c_t, m(\sim V_{t+1} | \mathcal{F}_t)] \, ,$$

where $\sim V_{t+1} | \mathcal{F}_t$ is the conditional distribution of V_{t+1} given $\mathcal{F}_t$. (We could also append the condition that $V_t = \lim_T V_t^T$, where V^T is the utility process for c in a T-horizon model with $V_T^T = 0$.) We then have the utility function U on L_+ defined by $U(c) = V_1$. As a special case, we can recover the additively separable criterion (7) $W(x, y) = x + \beta y$ and $m(u) = \int x \, d\mu(x)$ (expectation).

   The relaxation of the additively separable criterion (7) to general recursive utilities, such as the Epstein–Zin model, opens the way to a rich set of implications of attitudes towards risk for security pricing. For example, one can immediately study, using an appropriate certainty equivalent m, various forms of Machina's (1982) relaxation of the independence axiom of expected utility, or an alternative axiomatization of risk preferences such as that of Dekel (1986) and Chew (1989). Other extensions of the additively separable criterion (7) are cited in Section 6.6.

   In a multi-period model, one reconsider the optimality of an initially chosen strategy at intermediate dates, after the passage of time and revelation of information, setting up the issue of "time consistency" examined by Johnsen and Donaldson (1985). In treating this problem, one usually restricts attention to preferences defined at each date and each state of the world that are *time-consistent*, in the sense that: for any c and $\tilde{c}$ in L and any stopping time T, if $c_t = \tilde{c}_t$ for all $t \leqslant T$ and if the continuation of c beginning at time T is strictly preferred to the continuation of $\tilde{c}$ beginning at time T, then c is strictly preferred to $\tilde{c}$ beginning at time zero. If we denote by V^c the utility process for c under recursive preference primitives (m, W), we can then define c to be preferred to $\tilde{c}$ at time t if $V_t^c > V_t^{\tilde{c}}$ almost surely. Monotonicity conditions on m and W are then sufficient for the time-consistency of recursive preferences, including the additively separable criterion.

4. Continuous-time equilibrium in security markets

This section reviews the main concepts of general equilibrium and equilibrium asset pricing models in a continuous-time financial setting.

4.1. General equilibrium in continuous-time

The objective in this first subsection is to formulate and demonstrate general equilibria in a continuous-time setting with security markets. The approach is basically an extension of Arrow's model of Section 2. We will eventually presume that the available securities are dynamically spanning; that is, given the possibilities of continuous trading, markets are effectively complete. By using recent infinite-dimensional conditions for (static) complete contingent-commodity market equilibria discussed in Chapter 5, we can then implement a complete contingent-commodity equilibrium consumption allocation within a continuous-time security-spot market equilibrium.

The setting for uncertainty is a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ for the time set $\mathcal{T} = [0, T]$, as described in the appendix, where $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$ satisfies the usual conditions and $\mathcal{F}_0$ contains all subsets of zero probability events. The σ-algebra $\mathcal{F}_t$ represents the information available at time t. A cumulative dividend process is an integrable predictable semimartingale. For a dividend process D, the random variable D_t represents the cumulative number of units of account paid by the security in dividends up to and including time t. A semimartingale is right continuous with left limits, so $D_t = \lim_{s \downarrow t} D_s$ for all t almost surely and $D_{t-} \equiv \lim_{s \uparrow t} D_s$ exists for all t almost surely. The difference $\Delta D_t = D_t - D_{t-}$ is the jump of D at t, a lump sum dividend. Let $\mathcal{D}$ denote the space of dividend processes. If S is the stochastic price process of a security with the dividend process D, then $G_t = S_t + D_t$ represents the number of units of account at time t due to an agent holding one unit of the security from time 0 to time t. We call the process $G = S + D$ the *gain process* of this security. If one holds θ_t units of the security from time t until time τ, one realizes a dividend gain of $\theta_t(D_\tau - D_t)$ and a capital gain of $\theta_t(S_\tau - S_t)$, adding up to the total gain $\theta_t(G_\tau - G_t)$. If one varies one's holdings of the security at times $t_0, t_1, \ldots, t_k$ (with $0 = t_0 < t_1 < \cdots < t_k$), then the total gain through time t_k is

$$\sum_{j=0}^{k-1} \theta_{t_j}(G_{t_{j+1}} - G_{t_j}) .$$

Extending to the case of "continual trading," if G is a semimartingale and one chooses, as a strategy for the number of units of the security to hold at each time in $[0, T]$, some process θ from the space $L^1[G]$ (the space of predictable

processes described in the appendix), then the total gain between any times t and τ is the stochastic integral $\int_t^\tau \theta_s \, dG_s$.

One of the primitives of our economy is a vector $D = (D^0, \ldots, D^N) \in \mathscr{D}^{N+1}$ comprising $N + 1$ dividend processes. With only a small loss in generality, we take D^0 to be a unit discount bond payable at T; that is, $D_t = 0$, $t < T$ and $D_T = 1$. Letting $\mathscr{S}$ denote the space of semimartingales, a gain operator is a linear function $\Pi : \mathscr{D} \to \mathscr{S}$ mapping each dividend process D to its gain $G = \Pi(D)$. Given Π, we can define the gain process $G = (G^0, \ldots, G^N)$ by $G^n = \Pi(D^n)$. Given (Π, D), a *trading strategy* is an $\mathbb{R}^{N+1}$-valued process $\theta = (\theta^0, \ldots, \theta^N)$ in $L^1[G]$, with θ_t representing the portfolio of securities held at time t. The total gain process for $\theta \in L^1[G]$ is $\int \theta_t \, dG_t$.

For l given commodities, a consumption process is a predictable process $c : \Omega \times [0, T] \to \mathbb{R}^l$ with $E(\int_0^T c_t \cdot c_t \, dt) < \infty$. As usual, two consumption processes are treated as equivalent if they are equal almost everywhere on $\Omega \times [0, T]$. We let L denote the space of (equivalence classes of) consumption processes. For a given consumption process $c \in L$, the vector c_t represents the rate (per unit of time) at which the ℓ commodities are consumed at time t. Likewise, a spot price process is some element p of L, with p_t representing the vector of unit prices of the l commodities at time t. Given p, a consumption process c is therefore financed by paying units of account at the rate $p_t \cdot c_t$ at time t. Each agent $i \in \{1, \ldots, m\}$ is defined by an endowment e^i in the usual positive cone L_+ of L and by a utility function $U_i : L_+ \to \mathbb{R}$.

Given a gain operator Π, which defines the security price process $S = \Pi(D) - D$, and given a spot price process $p \in L$, a trading strategy θ *finances* a consumption process $c \in L$ at an initial cost of $\psi(c)$ if:

(i) $\theta_0 \cdot S_0 = \psi(c)$;
(ii) for all $t \in [0, T]$, $\theta_t \cdot (S_t + \Delta D_t) = \theta_0 \cdot S_0 + \int_0^t \theta_s \, dG_s - \int_0^t p_s \cdot c_s \, ds$;
(iii) $\theta_T \cdot (S_T + \Delta D_T) = 0$.

The cost $\psi(c)$ represents the required initial investment; the terminal constraint (iii) requires that the terminal market value of the trading strategy is zero; while the intermediate constraint (ii) requires that the interim value of the trading strategy is precisely that generated by security trading gains net of consumption purchases. If, as in the equilibria we are about to describe, $S_T = 0$, then (iii) is superfluous.

Given (Π, p), a budget-feasible plan for agent i is a pair (θ, c) consisting of a trading strategy θ and a consumption process c such that θ finances the net consumption purchase $c - e^i$ at an initial cost of zero (since there is no initial endowment of securities). A budget-feasible plan (θ, c) is optimal for agent i if there is no budget-feasible plan (θ', c') such that $U_i(c') > U_i(c)$.

A security-spot market equilibrium for the economy

$$\mathscr{E} = ((\Omega, \mathscr{F}, \mathbb{F}, P), D, (U_i, e^i)), \quad i \in \{1, \ldots, m\},$$

is a collection $(\Pi, p, (\theta^i, c^i))$, $i \in \{1, \dots, m\}$, such that, given the gain operator Π and spot price process p, for each agent $i \in \{1, \dots, m\}$, the plan (θ^i, c^i) is optimal, and markets clear: $\Sigma_{i=1}^m c^i - e^i = 0$ and $\Sigma_{i=1}^m \theta^i = 0$.

This is clearly a continuous-time analogue of Arrow (1953). Just as in that model, sufficient conditions for an equilibrium are conditions ensuring a (static) Walrasian equilibrium for the complete contingent-commodity markets economy (U_i, e^i), $i \in \{1, \dots, m\}$, as well as a spanning condition on the security dividends D.

Since L is a Hilbert lattice under the inner product $(\cdot|\cdot)$ defined by

$$(p|c) = E\left(\int_0^T p_t \cdot c_t \, dt\right),$$

we can exploit utility conditions developed by Mas-Colell (1986) for the existence of a (static) contingent-commodity market equilibrium. Let $\|c\|^2 = (c|c)$ define a topology on L, and define a utility function U to be v-proper on X, for some $v \in L_+$ and $X \subset L_+$, if there exists a scalar $\epsilon > 0$ such that, for all x in X, α in $[0, \infty)$, and z in L_+,

$$U_i(x - \alpha v + z) \geq U(x) \quad \Rightarrow \quad \|z\| \geq \alpha \epsilon \,.$$

For further details, see Chapter 34. We have the following variant of Mas-Colell's (1986) Theorem.

Theorem 1. *Let $e = \Sigma_{i=1}^m e^i$. Suppose, for each agent $i \in \{1, \dots, m\}$, that U_i is quasi-concave, continuous, locally non-satiated in the order interval $[0, e]$, and e-proper on $[0, e]$. Then (U_i, e^i) has a complete contingent-commodity market equilibrium $(\psi, (c^i))$, where $\psi : L \to \mathbb{R}$ is a continuous linear price functional and the allocation (c^i) is Pareto optimal.*

The properness assumption is satisfied, for example, if U_i has an additive representation of the form

$$U_i(c) = E\left[\int_0^T u_i(c_t, t) \, dt\right], \quad c \in L_+, \tag{10}$$

where $U_i : \mathbb{R}_+^l \times [0, T] \to \mathbb{R}$ is strictly increasing and concave such that $D_c^+ u_i(0, t)$, the right derivative of $u_i(\cdot, t)$ at zero, is bounded in t. For later purposes of pricing securities, however, we will need to work with a pointwise-interior equilibrium allocation ($c_t^i > 0$ a.s. for all t for all i), and will therefore

later cite an alternative existence result using the Inada condition $D_c^+ u_i(0, t) = +\infty$ for all t.

In order to formulate a dynamic spanning condition, we consider first the following related definition. An $\mathbb{R}^N$-valued martingale $M = (M^1, \ldots, M^N)$ is a *martingale generator* for $(\Omega, \mathscr{F}, \mathbb{F}, P)$ if, for any martingale X, there exists $\varphi \in L^1[M]$ such that for all t, $X_t = X_0 + \int_0^t \varphi_s \, dM_s$ almost surely.

Assumption (*dynamic spanning*). There exists a probability measure Q on $(\Omega, \mathscr{F})$, uniformly equivalent[5] to P, such that the martingales $M_t^n = E^Q(D_T^n | \mathscr{F}_t)$, $t \in [0, T]$, $n \in \{1, \ldots, N\}$, form a martingale generator for $(\Omega, \mathscr{F}, \mathbb{F}, Q)$.

The dynamic spanning assumption is discussed in the setting of Brownian Motion in the next subsection. The semimartingale property and the definition of $\int \theta \, dS$ are invariant under the substitution of an equivalent probability measure. The definition of $L^1[G]$ is also invariant under the substitution of a uniformly equivalent measure Q for P, and vice versa. Likewise, the definition and topology of the consumption space L is invariant under substitution of Q for P, and vice versa. Consider the gain operator Π^Q defined by $\Pi^Q(D)_t = E^Q(D_T | \mathscr{F}_t)$.

Lemma (spanning). *Suppose D satisfies the dynamic spanning condition under the probability measure Q. Given the gain operator Π^Q and a spot price process p, any consumption process c is financed at the (unique) initial cost $\psi_p^Q(c) = E^Q (\int_0^T p_t \cdot c_t \, dt)$.*

Proof. Let $(p, c) \in L \times L$ be arbitrary. Under the dynamic spanning condition, the Q-martingales $M = (G^1, \ldots, G^N)$ defined by $G^n = \Pi^Q(D^n)$ form a martingale generator for $(\Omega, \mathscr{F}, \mathbb{F}, Q)$. Let

$$X_t = E^Q\left(\int_0^T p_s \cdot c_s \, ds \,\bigg|\, \mathscr{F}_t\right), \quad t \in [0, T].$$

Since X is a Q-martingale, by dynamic spanning there exists $\varphi = (\varphi^1, \ldots, \varphi^N) \in L^1[M]$ such that $X_t = X_0 + \int_0^t \varphi_s \, dM_s$ almost surely, $t \in [0, T]$. Let $\theta^n = \varphi^n$, $1 \leqslant n \leqslant N$, and let θ^0 be defined by

$$\theta_t^0 = X_t - \int_0^t p_s \cdot c_s \, ds - \sum_{n=1}^N \theta_t^n (S_t^n + \Delta D_t^n), \quad t \in [0, T]. \tag{11}$$

---

[5] A probability measure Q is uniformly equivalent to P if the Radon–Nikodym derivatives dQ/dP and dP/dQ are essentially bounded.

The predictability of D implies, by an argument left to the reader, that θ^0 is predictable. Since $G^0 = \Pi^Q(D^0)$ is identically equal to 1, we know that $\int \theta^0 \, dG^0 \equiv 0$. By construction, conditions (ii) and (iii) for θ to finance c are satisfied, and $\theta_0 \cdot S_0 = X_0 = \psi_p^Q(c)$. The uniqueness of $\theta_0 \cdot S_0$ (over all θ financing c) follows immediately.

Theorem 2. *Suppose that* (U_i, e^i), $i \in \{1, \ldots, m\}$, *has a (static) complete contingent-commodity market equilibrium* $(\psi, (c^i))$. *(For this, it suffices that* U_i *satisfies the regularity conditions of Theorem 1.) If the dividend process D satisfies the dynamic spanning condition, then* $((\Psi, \mathcal{F}, \mathbb{F}, P), (U_i, c^i), D)$ *has a security-spot market equilibrium with the same consumption allocation* (c^i).

Proof. Let Q be uniformly equivalent to P such that $G = \Pi^Q(D)$ is a martingale generator. Since L is a Hilbert space, the given contingent-commodity market equilibrium price function ψ has a representation of the form

$$\psi(c) = \psi_p^Q(c) = E^Q \left(\int_0^T p_t \cdot c_t \, dt \right), \quad c \in L, \tag{12}$$

for a unique spot price process $p \in L_+$. Since D satisfies the dynamic spanning condition, by the previous lemma the consumption process $c^i - e^i$ is financed by some trading strategy θ^i at the unique cost $\psi_p^Q(c^i - e^i)$. Since $(\psi_p^Q, (c^i))$ is a contingent-commodity market equilibrium, however, $\psi_p^Q(c^i - e^i) = 0$. Thus (θ^i, c^i) is a budget-feasible plan for i. We can choose θ^i in this fashion for $i < m$. Since $c^m - e^m = -\sum_{i=1}^{m-1} c^i - e^i$, and by linearity throughout, the trading strategy $\theta^m = -\sum_{i=1}^{m-1} \theta^i$ finances $c^m - e^m$ at an initial cost of zero, so (θ^m, c^m) is a budget-feasible plan for agent m. The plans (θ^i, c^i), $i \in \{1, \ldots, m\}$, are market clearing. It remains to show optimality: that there is no budget-feasible plan $(\hat{\theta}^i, \hat{c}^i)$ for some agent i such that $U_i(\hat{c}^i) > U_i(c^i)$.

We will show a contradiction, assuming that such a superior plan $(\hat{\theta}^i, \hat{c}^i)$ exists. Since $U_i(\hat{c}^i) > U_i(c^i)$ and $(\psi_p^Q, (c^i))$ is a complete contingent-commodity market equilibrium, $\psi_p^Q(\hat{c}^i) > \psi_p^Q(c^i)$. If $\hat{\theta}^i$ finances $\hat{c}^i - e^i$, however, it does so at the unique cost $\psi_p^Q(\hat{c}^i - e^i) > \psi_p^Q(c^i - e^i) = 0$, which contradicts the assumption that $(\hat{\theta}^i, \hat{c}^i)$ is budget-feasible. This proves optimality.

4.2. The dynamic spanning condition and Girsanov's Theorem

This subsection discusses sufficient conditions for a dividend process to satisfy the dynamic spanning condition.

As explained in the appendix, an integrable semimartingale X is characterized by the fact that it can be written as the sum $M + A$ of an integrable process A of finite variation and a martingale M. If D is an $\mathbb{R}^N$-valued semimartingale of the form $M + A$, where M is a martingale generator, there is no guarantee that the $\mathbb{R}^N$-valued process X defined by $X_t = E(M_T + A_T | \mathcal{F}_t)$, $t \in [0, T]$, defines a martingale generator. On the other hand, under technical regularity conditions, one can apply the Girsanov–Lenglart Theorem for the existence of a new measure Q under which D is a martingale and inherits the martingale generator property of M. Further discussion of this appears in Section 5.9.

For a concrete example, suppose that $\mathbb{F}$ is the standard filtration of a Standard Brownian Motion B in $\mathbb{R}^d$, for some dimension d. Then B is itself a martingale generator, as is any martingale in $\mathbb{R}^N$ of the form $X_t = \int_0^t \varphi_s \, dB_s$, $t \in [0, T]$, if and only if $\{\varphi_s\}$ is a $(N \times d)$-matrix-valued process of essential[6] rank d. Now, suppose that $dD_t = \mu_t \, dt + \sigma_t \, dB_t$, where $\int \sigma_t \, dB_t$ has the martingale generator property (that is, σ has essential rank d.) Under technical regularity conditions on σ and μ, there exists an equivalent probability measure Q and a Brownian Motion $\hat{B}$ in $\mathbb{R}^d$ under Q such that $dD_t = \sigma_t \, d\hat{B}_t$, which implies that D is itself a martingale generator for $(\Omega, \mathcal{F}, \mathbb{F}, Q)$. With $d = N$ for instance, it is enough that μ and σ are bounded and that σ_t has a uniformly bounded inverse. In that case, Q is defined by

$$\frac{dQ}{dP} = \exp\left[\int_0^T \varphi_t \, dB_t - \frac{1}{2} \int_0^T \varphi_t \cdot \varphi_t \, dt \right], \tag{13}$$

where $\varphi_t = \sigma_t^{-1} \mu_t$. Moroever, $\hat{B}$ is defined by $\hat{B}_t = B_t - \int_0^t \varphi_s \, ds$. Indeed this construction of $\hat{B}$ and Q succeeds under the weaker regularity conditions of the following theorem.

Theorem 3 (Girsanov). *Suppose φ is an $\mathbb{R}^d$-valued predictable process for $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F}$ is the standard filtration of a Standard Brownian Motion B in $\mathbb{R}^d$. Provided $E[\exp(\frac{1}{2} \int_0^T \varphi_t \cdot \varphi_t \, dt)] < \infty$, the Radon–Nikodym derivative given by* (13) *defines a probability measure Q such that*

$$\hat{B}_t = B_t - \int_0^t \varphi_s \, ds \,, \quad t \in [0, T] \,,$$

is a Standard Brownian Motion on $(\Omega, \mathcal{F}, \mathbb{F}, Q)$.

---

[6] The essential rank of φ is d if $\text{rank}[\varphi(\omega, t)] = d$ almost everywhere on $\Omega \times [0, T]$.

As pointed out by Harrison and Kreps (1979), and further illustrated in Section 5, Girsanov's Theorem can sometimes lead to an explicit calculation of the arbitrage price of securities.

Aside from the case of Brownian filtrations, well known examples of filtrations with an identifiable martingale generator include the standard filtrations of event trees (including finite-state Markov chains), point processes (such as a Poisson process) and Azema's martingale.

4.3. *The representative-agent asset pricing formula*

Here, we specialize to a setting that produces a continuous-time multi-agent analogue to the multi-period representative-agent formula of Section 3. This subsection and the next are based on Duffie and Zame (1989). We take our original definition of a continuous-time security-spot market model $((\Omega, \mathscr{F}, \mathbb{F}, P), (U_i, e^i), D)$, $i \in \{1, \ldots, m\}$, but adopt the assumption that there is only $l = 1$ commodity, and that for all i, U_i has a utility representation of the form

$$U_i(c) = E\left[\int_0^T u_i(c_t, t) \, dt \right], \quad c \in L_+ ,$$

where $u_i : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ is *regular*, in the sense that u_i is smooth (say C^4) restricted to (ϵ, ∞) for any $\epsilon > 0$, and, for all t, $u_i(\cdot, t) : \mathbb{R}_+ \to \mathbb{R}$ is increasing and strictly concave with unbounded derivative $u_{ic}(\cdot, t)$. Under all of these conditions, we say that U_i is additively separable and regular (u_i). As mentioned previously, the Inada condition of "infinite marginal utility at zero" implies that Pareto optimal consumption levels must be strictly positive almost everywhere, which is useful for our purposes. Unfortunately, the unboundedness of u_{ic} is also inconsistent with the properness condition used in Theorem 1. Nevertheless, we can exploit the additively separable restriction on utility for the following result, which was independently shown by Araujo and Monteiro (1989) and Duffie and Zame (1989). This type of result was later given new and successively simpler proofs by Karatzas, Lakner, Lehoczky and Shreve (1988) as well as Dana and Pontier (1989).

Proposition 4. *Suppose, for all i, that U_i is additively separable and regular. If the total endowment $e = \sum_{i=1}^m e^i$ is bounded away from zero, then the economy (U_i, e^i), $i \in \{1, \ldots, m\}$, has a complete contingent-commodity market equilibrium $(\psi, (c^i))$, $i \in \{1, \ldots, m\}$, with c^i bounded away from zero for any agent i having $e^i \neq 0$.*

Uniqueness of equilibria is discussed by Karatzas, Lakner, Lehoczky and Shreve (1988). Araujo and Monteiro (1987) have pointed out the restrictiveness of assuming that e is bounded away from zero. One may relax this assumption in a production economy.

Given an equilibrium $(\psi, (c^i))$ for the complete contingent-commodity market economy (U, e^i), a *representative agent* is a utility function $U_\lambda : L_+ \to \mathbb{R}$ of the form, for some $\lambda \in \mathbb{R}_+^m$,

$$U_\lambda(x) = \max_{x^i \in L_+, \, i \in \{1, \ldots, m\}} \sum_{i=1}^m \lambda_i U_i(x^i) \quad \text{subject to } \sum_{i=1}^m x^i \leq x, \tag{14}$$

such that (ψ, e) is the (no-trade) equilibrium for the single-agent economy (U_λ, e). Equivalently, a representative agent for $(\psi, (c^i))$ is defined by agent weights $\lambda \in \mathbb{R}_+^m$ such that $e \in \arg\max_c U_\lambda(c)$ subject to $\psi(c) \leq \psi(e)$.

Proposition 5. *Suppose, for all i, that U_i is additively separable and regular (u_i), and that $e = \Sigma_i e^i$ is bounded away from zero. There is a complete-contingent commodity market equilibrium $(\psi, (c^i))$ with a representative agent U_λ for some $\lambda \in \mathbb{R}_+^m$. Let $u_\lambda : \mathbb{R}_+ \times [0, T] \to \mathbb{R}$ be defined by*

$$u_\lambda(\alpha, t) = \max_{\alpha \in \mathbb{R}_+^m} \sum_{i=1}^m \lambda_i u_i(a_i, t) \quad \text{subject to } \sum_{i=1}^m a_i \leq \alpha. \tag{15}$$

Then, U_λ is additively separable and regular (u_λ), and λ can be chosen so that, for any $c \in L$, $\psi(c) = [\int_0^T u_{\lambda c}(e_t, t)c_t \, \mathrm{d}t]$.

The representative-agent part of the proof, due to Huang (1987), is an extension of the representative-agent construction of Section 3.2 to this infinite-dimensional setting.

Combining Proposition 5 with Theorem 2 of Section 4.2, we have the existence of a security-spot market equilibrium $(\Pi, p, (\theta^i, c^i))$, $i \in \{1, \ldots, m\}$, provided the dividend process D satisfies the dynamic spanning condition.

Given an equilibrium $(\Pi, p, (\theta^i, c^i))$, $i \in \{1, \ldots, m\}$, we now study the "real" security price process $\hat{S}$ defined by $\hat{S}_t = S_t/p_t$, $t \in [0, T]$. By "real," we mean the price relative to the numeraire defined at each time t by the consumption commodity. If the integral $\hat{D}_t = \int_0^t (1/p_s) \, \mathrm{d}D_s$ is well-defined, then $\hat{D}$ is the associated *real dividend process*. We can also define a *real security* to be a finite variation dividend process Y representing a cumulative claim to Y_t units of the consumption commodity through time t. If the integral $D_t^Y = \int_0^t p_s \, \mathrm{d}Y_s$ is a well-defined[7] (nominal) dividend process, we say that Y is

---

[7] If Y is an integrable semimartingale, then, under the conditions of Proposition 4 $\int p_t \, \mathrm{d}Y_t$ is automatically well-defined since the spot-price process p is predictable and bounded.

admissible. Any consumption process $c \in L$, for example, generates an admissible real dividend process Y defined by $Y_t = \int_0^t c_s \, ds$, which has the corresponding nominal dividend process D^Y defined by $D_t^Y = \int_0^t p_s c_s \, ds$. The introduction of any admissible real security Y has no effect on the equilibrium shown in the proof of Theorem 2.

Proposition 6. *Suppose* $((\Omega, \mathscr{F}, \mathbb{F}, P), (U_i, e^i), D)$ *is a security-spot market economy such that*:

  (i) *for all i, U_i is additively separable and regular (u_i),*
  (ii) *the aggregate endowment $e = \sum_{i=1}^m e^i$ is bounded away from zero,*
  (iii) *the security dividend process D satisfies the Dynamic Spanning condition.*

Then there is a security-spot market equilibrium $(\Pi, p, (\theta^i, c^i))$, $i \in \{1, \ldots, m\}$, with a representative agent U_λ that is additively separable and regular (u_λ), and for which the real price process S^Y of any admissible real dividend process Y satisfies

$$S_t^Y = \frac{1}{u_{\lambda c}(e_t, t)} \, E\left[\int_t^T u_{\lambda c}(e_s, s) \, dY_s \, \bigg| \, \mathscr{F}_t \right] \quad \text{a.s.,} \quad t \in [0, T] \,. \tag{16}$$

Proof. The existence of an equilibrium $(\Pi, p, (\theta^i, c^i))$, $i \in \{1, \ldots, m\}$, is guaranteed by Proposition 5 and Theorem 2 of Section 4.2, with the gain operator

$$\Pi(D) = \Pi^Q(D) = E^Q(D_T | \mathscr{F}_t) \,, \quad t \in [0, T] \,,$$

for an appropriate probability measure Q. From Proposition 4, we can also take it that the underlying complete contingent-commodity market equilibrium $(\psi, (c^i))$ has a price functional of the form

$$\psi(c) = E\left[\int_0^T u_{\lambda c}(e_t, t) c_t \, dt \right], \quad c \in L \,.$$

We know that, for a unique p in L,

$$\psi(c) = \psi_p^Q(c) = E^Q\left(\int_0^T p_t c_t \, dt \right), \quad c \in L \,.$$

It follows that $p_t = u_{\lambda c}(e_t, t) / \xi_t$, $t \in [0, T]$, where $\{\xi_t\}$ is the *density process* for Q; that is, $\xi_t = E((dQ/dP) | \mathscr{F}_t)$. [One can review Duffie (1986) for the details on this last point.]

Let Y be an admissible real dividend process. The (nominal) gain process of Y is defined by

$$\Pi^Q(D^Y)_t = E^Q\left[\int_0^T p_s \, \mathrm{d}Y_s \middle| \mathcal{F}_t\right]$$

$$= D_t^Y + E^Q\left[\int_t^T \frac{u_{\lambda c}(e_s, s)}{\xi_s} \, \mathrm{d}Y_s \middle| \mathcal{F}_t\right]$$

$$= D_t^Y + \frac{1}{\xi_t} E\left[\int_t^T u_{\lambda c}(e_s, s) \, \mathrm{d}Y_s \middle| \mathcal{F}_t\right], \quad t \in [0, T].$$

The last equality relies on an application of Fubini's Theorem for conditional expectations, which can be found in Ethier and Kurtz (1986, p. 74). Since $\xi_t = u_{\lambda c}(e_t, t)/p_t$, the corresponding real price process S^Y is therefore given by

$$S_t^Y = \frac{\Pi^Q(D^Y)_t - D_t^Y}{p_t} = \frac{1}{u_{\lambda c}(e_t, t)} E\left[\int_t^T u_{\lambda c}(e_s, s) \, \mathrm{d}Y_s \middle| \mathcal{F}_t\right], \quad t \in [0, T],$$

which completes the proof.

The representative-agent real security pricing formula (16) is an obvious analogue of the discrete-time single-agent multi-period asset pricing formula of Section 3.

Example (*the term structure*).   As an application of this asset pricing model, let Y denote the cumulative dividend process representing the payoff of a zero-coupon default-free bond of unit principal maturing at time $\tau \in [0, T]$. This means that $Y_t = 0$ for $t < \tau$, while $Y_t = 1$ for $t \geq \tau$. Equation (16) then implies that the price of this bond is zero after maturity, and at any time t before maturity has the price $E[u_{\lambda c}(e_\tau, \tau)|\mathcal{F}_t]/u_{\lambda c}(e_t, t)$. Various parametric assumptions concerning the distribution of the aggregate endowment process e and the representative-agent utility u_λ are sometimes used to calculate this conditional expectation. The most famous example is the term structure model of Cox, Ingersoll and Ross (1985a).

4.4. The consumption-based CAPM

Continuing to narrow our focus, we restrict ourselves in this subsection to the standard filtration $\mathbb{F}$ of a Standard Brownian Motion B in $\mathbb{R}^d$, for some

dimension d. With additively separable and regular utility, this produces the Consumption-Based Capital Asset Pricing Model (CCAPM) of Breeden (1979). Breeden's original proof assumes the existence of an equilibrium with pointwise interior consumption choices and optimality characterized by a smooth solution to the Bellman equation for Markov dynamic programming. This subsection shows that that representative-agent pricing approach allows for primitive conditions leading directly to an equilibrium satisfying the CCAPM.

Before proceeding, we need to record the following version of Ito's Lemma. In this setting, an Ito process in $\mathbb{R}^n$ is a semimartingale of the form

$$X_t = x + \int_0^t \mu_s \, \mathrm{d}s + \int_0^t \sigma_s \, \mathrm{d}B_s \, , \quad t \in [0, T] \, ,$$

where μ is an $\mathbb{R}^n$-valued adapted process and σ is an $n \times d$ matrix-valued predictable process. The stochastic differential form for X, which is purely formal notation, is

$$\mathrm{d}X_t = \mu_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}B_t \, .$$

It is a common abuse of the meaning of this representation of X to treat μ_t as the "instantaneous conditional expectation of $\mathrm{d}X_t$," and likewise to treat $\sigma_t \sigma_t^\top$ as the "instantaneous conditional covariance matrix of $\mathrm{d}X_t$." Of course, this can be justified for square integrable X by passing to limits the mean and covariance matrix of $X_{t+\delta} - X_t$, conditional on $\mathscr{F}_t$, as $\delta \to 0$. For the following, f_x denotes the partial derivative of a function $f : \mathbb{R}^n \times [0, T] \to \mathbb{R}$ with respect to x, and likewise for f_t and f_{xx}.

Ito's Lemma. *For any $\mathbb{R}^n$-valued Ito process X with $\mathrm{d}X_t = \mu_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}B_t$ and any C^2 function $f : \mathbb{R}^n \times [0, T] \to \mathbb{R}$, the process Y defined by $Y_t = f(X_t, t)$, $t \in [0, T]$, is also an Ito process with $\mathrm{d}Y_t = \mu_f(t) \, \mathrm{d}t + f_x(X_t, t)\sigma_t \, \mathrm{d}B_t$, where*

$$\mu_f(t) = f_x(X_t, t)\mu_t + f_t(X_t, t) + \tfrac{1}{2} \, \mathrm{tr}[\sigma_t^\top f_{xx}(X_t, t)\sigma_t] \, .$$

The conditions for Ito's Lemma can be weakened in many directions.

We now fix an economy $((\Omega, \mathscr{F}, \mathbb{F}, P), (U_i, e^i), D)$, $i \in \{1, \ldots, m\}$, satisfying the conditions of Proposition 5, where $\mathbb{F}$ is the filtration generated by a Standard Brownian Motion B in $\mathbb{R}^d$. By that proposition, there exists an equilibrium satisfying the representative-agent real asset pricing formula

$$S_t^Y = \frac{1}{u_{\lambda c}(e_t, t)} \, E\left[\int_t^T u_{\lambda c}(e_s, s) \, \mathrm{d}Y_s \, \middle| \, \mathscr{F}_t \right], \quad t \in [0, T] \, ,$$

for any admissible real dividend process Y, where u_λ defines the associated representative-agent utility function.

Since the CCAPM is by nature a statement about the "instantaneous covariance of de_t," with other variables, we need something like the following condition on the aggregate endowment.

Ito Endowments. The aggregate endowment e is an Ito process.

It is in fact enough for most of the following that e is a semimartingale. We henceforth write $de_t = \mu_e(t)\,dt + \sigma_e(t)\,dB_t$. Using Ito's Lemma and the fact (verified with the Implicit Function Theorem) that $u_{\lambda c}$ is a C^2 function, the process $\pi_t = u_{\lambda c}(e_t, t)$ has the stochastic differential representation

$$d\pi_t = [u_{\lambda cc}(e_t, t)\mu_e(t) + u_{\lambda ct}(e_t, t) + \tfrac{1}{2}u_{\lambda ccc}(e_t, t)\sigma_e(t) \cdot \sigma_e(t)]\,dt$$
$$+ u_{\lambda cc}(e_t, t)\sigma_e(t)\,dB_t\ .$$

For any admissible real dividend process Y, with real equilibrium price process $V = S^Y$, the process Z defined by $Z_t = \int_0^t \pi_s\,dY_s + \pi_t V_t$ is a martingale since, for any interval $[t, s]$,

$$E(Z_s|\mathcal{F}_t) = \int_0^t \pi_\tau\,dY_\tau + E\left[\int_t^s \pi_\tau\,dY_\tau + \pi_s\,\frac{1}{\pi_s}\,E\left(\int_s^T \pi_\tau\,dY_\tau\Big|\mathcal{F}_s\right)\Big|\mathcal{F}_t\right]$$

$$= \int_0^t \pi_\tau\,dY_\tau + \pi_t V_t = Z_t\ .$$

Suppose Y is an Ito process of the form $dY_t = \mu_Y(t)\,dt + \sigma_Y(t)\,dB_t$. This implies (by Ito's Lemma) that the real price process V is also an Ito process, with a representation of the form $dV_t = \mu_V(t)\,dt + \sigma_V(t)\,dB_t$. Again applying Ito's Lemma,

$$dZ_t = [\pi_t\mu_Y(t) + \pi_t\mu_V(t) + V_t\mu_\pi(t) + u_{\lambda cc}(e_t, t)\sigma_e(t) \cdot \sigma_V(t)]\,dt$$
$$+ \sigma_Z(t)\,dB_t\ ,$$

for some $\sigma_Z(t)$ that we need not calculate here. An Ito process $dX_t = \mu_X(t)\,dt + \sigma_X(t)\,dB_t$ is a martingale if and only if $\mu_X(t) = 0$ almost everywhere. Since Z is a martingale, we therefore have, almost everywhere,

$$\pi_t[\mu_Y(t) + \mu_V(t)] + V_t\mu_\pi(t) + \mu_{\lambda cc}(e_t, t)\sigma_e(t) \cdot \sigma_V(t) = 0\ .$$

Assuming that $V_t \neq 0$, we can divide through by $\pi_t V_t$ and rearrange to obtain

$$\frac{\mu_V(t) + \mu_Y(t)}{V_t} - r_t = \frac{-u_{\lambda cc}(e_t, t)}{u_{\lambda c}(e_t, t)} \frac{\sigma_V(t)}{V_t} \cdot \sigma_e(t) , \tag{17}$$

where $r_t = -\mu_\pi(t)/\pi_t$. Formally speaking, if we treat $(dV_t + dY_t)/V_t$ as the "instantaneous real return" on the security, it is natural to treat

$$\bar{R}_t = \frac{\mu_V(t) + \mu_Y(t)}{V_t}$$

as the "instantaneous mean rate of return" and $(\sigma_V(t)/V_i) \cdot \sigma_e(t)$ as the "instantaneous covariance between the return dV_i/V_i and aggregate consumption increment de_t," following the heuristic conventions outlined earlier. If the return is "riskless," that is, if $\sigma_V(t) = 0$, then we have $\bar{R}_t = r_t$, so we call r_t the riskless rate of return. Since $\pi_t = u_{\lambda c}(e_t, t)$ is the "representative-agent marginal utility," we can therefore view the riskless rate $r_t = -\mu_\pi(t)/\pi_t$ as the exponential rate of decline of the representative-agent marginal utility, a characterization uncovered (in a more narrow single-agent Markov setting) by Cox, Ingersoll and Ross (1985b). The difference $\bar{R}_t - r_t$ is known as the excess mean rate of return of the asset, and based on (17) satisfies the proportionality restriction

$$\bar{R}_t - r_t = \frac{-u_{\lambda cc}(e_t, t)}{u_{\lambda c}(e_t, t)} \sigma_R(t) \cdot \sigma_e(t) , \tag{18}$$

where $\sigma_R(t) = \sigma_V(t)/V_t$. In words, the mean excess rate of return on a security is proportional to "instantaneous covariance" with aggregate consumption increments. The constant of proportionality is the risk aversion coefficient of the representative agent. This is a form of the CCAPM. We summarize as follows.

Proposition 7 (CCAPM). *Suppose the conditions of Proposition 5 are satisfied, that $\mathbb{F}$ is the standard filtration of a Standard Brownian Motion B in $\mathbb{R}^d$, and that the aggregate endowment process e is an Ito process. Then there exists a security-spot market equilibrium in which, at any time t, the return of any security (with non-zero price) satisfies (18).*

We can also view the CCAPM in a traditional "beta" form. Because of the dynamic spanning condition, one can assume without loss of generality that there is some security whose real price process, say V^*, has a diffusion process σ^* with $\sigma^* = k_t \sigma_e$ for some positive predictable k, characterizing the security

as one whose return is "instantaneously perfectly correlated" with aggregate consumption increments. For such a security, the instantaneous mean rate of return, denoted $\bar{R}_t^*$, satisfies the CCAPM, implying that

$$\bar{R}_t^* - r_t = \frac{u_{\lambda cc}(e_t, t)}{u_{\lambda c}(e_t, t)} \; \sigma_{R^*}(t) \cdot \sigma_e(t) , \tag{19}$$

where $\sigma_{R^*}(t) = \sigma^*(t)/V_t^*$. One defines for any given security the "instantaneous regression coefficient"

$$\beta_R(t) = \frac{\sigma_R(t) \cdot \sigma_{R^*}(t)}{\sigma_{R^*}(t) \cdot \sigma_{R^*}(t)}$$

(assuming that $\sigma_{R^*}(t) \neq 0$), as the beta of that security relative to aggregate consumption. Combining this expression with the originally stated form (18) of the CCAPM, we have the traditional "beta form"

$$\bar{R}_t - r_t = \beta_R(t)(\bar{R}_t^* - r_t) , \tag{20}$$

satisfied by all securities (with non-zero market values). The beta form (20) is implied by, but does not imply, the representative-agent form (18) of the CCAPM since (20) applies even if the representative-agent risk aversion coefficient defined by $-u_{\lambda cc}(e_t, t)/u_{\lambda c}(e_t, t)$ is replaced in (18) by any other coefficient. For example (under strong conditions on an equilibrium), a version of the beta form of the CCAPM is satisfied even without dynamic spanning. The supporting arguments may be found in Breeden (1979). At this writing, however, primitive conditions for multi-agent equilibrium that do not require dynamic spanning remain to be shown.

Of course, Sections 4.3 and 4.4 are based on the strong assumption of additively separable utility; for extensions, see Section 6.6.

5. Continuous-time derivative asset pricing

5.1. Prologue

This section characterizes, with the aid of martingale theory, the arbitrage-free pricing of derivative assets, those whose dividends can be financed by trading other "primitive" securities. Under the assumption of no arbitrage opportunities, the price of a derivative asset is the initial investment cost in primitive securities required to replicate the dividends of the derivative asset. If this were not the case, a position in the derivative asset combined with an offsetting

position in the replicating trading strategy would produce an arbitrage. This obviously ignores transactions costs.

Of course, the primitive securities must themselves be priced, say by an equilibrium asset pricing model or even by actual financial markets, but it is nevertheless useful to have a model that prices some (derivative) securities relative to other (primitive) securities. The most famous example of this is the Black–Scholes (1973) formula for the price of a European call option on a security whose price process is a geometric Brownian motion. Arbitrage pricing is perhaps the most actively used asset pricing technique in practical applica- tions.

A large part of this section follows the lines of Harrison and Kreps (1979), establishing, in the manner of Section 3.1, that the absence of arbitrage implies the existence of an equivalent martingale measure. From this, any derivative security price can be calculated as the expected discounted present value of the security's dividend stream, substituting the equivalent martingale measure for the originally given probability measure in calculating the expectation.

5.2. The setup

A basic primitive is a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F}, P)$, where $\mathbb{F} = \{\mathscr{F}_t : t \in [0, T]\}$ is an augmented filtration of σ-algebras satisfying the usual conditions, as explained in the Appendix. The σ-algebra $\mathscr{F}_t$ is the set of events characterizing information held by investors at time t. For simplicity, we take it that $\mathscr{F}_0$ is almost trivial, in that it includes no events with probability in $(0, 1)$, and without loss of generality take $\mathscr{F} = \mathscr{F}_T$.

The *short-term rate*, if it exists, is an adapted process r satisfying $\int_0^T |r_t| \, dt < \infty$ almost surely, with r_t interpreted as the dividend rate demanded at time t on a security whose price is always equal to 1. That is, r_t is the continuously compounding interest rate on riskless deposits at time t. The existence of the short-term rate is itself an assumption that can be avoided for the following, at some cost in concreteness. We actually assume, henceforth, that the short-term rate exists and is bounded.

By initially investing one unit of account at the short-term rate and continu- ally reinvesting the original deposit and accumulated interest dividends at the short rate, the total balance Z_t held at time t is determined by the ordinary differential equation

$$\frac{dZ_t}{dt} = r_t Z_t \,, \qquad Z_0 = 1 \,.$$

The solution is of course $Z_t = \exp(\int_0^t r_s \, ds)$.

Likewise, investing one unit of account at any time t in the same short-rate investment strategy yields by time τ

$$f_{t,\tau} = \exp\left(\int_t^\tau r_s \, \mathrm{d}s\right).$$

Also given are N securities with cumulative dividend processes $D^1, \ldots, D^N$ and price processes $S^1, \ldots, S^N$. By definition, these are integrable semimartingales, with D predictable. We let $\{D_t^0 = \int_0^t r_s \, \mathrm{d}s : 0 \leq t \leq T\}$ denote the cumulative interest dividends on the short-term deposit security, with associated price process S^0 identically equal to 1. This makes for the vector dividend process $D = (D^0, \ldots, D^N)$ and price process $S = (S^0, \ldots, S^n)$. The associated gain process is $G = D + S$.

Unless otherwise stated, we continue to use the convention that the price processes are *ex dividend*, meaning that the *cum dividend* market value of a unit of security n at time t is $S_t^n + \Delta D_t^n$, the price plus any lump sum dividend paid at that time t.

As in Section 4, a trading strategy is an $\mathbb{R}^{N+1}$-valued process $\theta \in L^1[G]$. Aside from the natural informational constraint, the restriction that $\theta \in L^1[G]$ is technical, mildly limiting the speed and sizes of trades, and is automatically satisfied in a finite-dimensional setting. Several alternative sets of technical assumptions will lead to the basic conclusions of this section, as shown for example by Dybvig and Huang (1988).

5.3. Arbitrage and self-financing strategies

A dividend process C is *financed* by a trading strategy θ if

$$\theta_t \cdot S_t = \theta_0 \cdot S_0 + \int_0^t \theta_s \, \mathrm{d}G_s - C_{t-}, \quad t \in [0, T], \tag{21}$$

meaning that the current market value $\theta_t \cdot S_t$ of the strategy at time t is the initial investment value $\theta_0 \cdot S_0$, plus the trading gains $\int_0^t \theta_s \, \mathrm{d}G_s$, less the cumulative dividends C_t removed from the strategy by time t.

An *arbitrage* is a trading strategy θ with initial investment value $\theta_0 \cdot S_0 \leq 0$, financing a non-negative dividend process D^θ, and having a non-negative cum-dividend final value $\theta_T \cdot (S_T + \Delta D_T)$, with one of these three non-zero. The basic goal of this section is to characterize the prices of securities under an assumption of no arbitrages.

A trading strategy θ is *self-financing* if it finances a zero dividend process, meaning that the strategy neither generates nor requires income during $(0, T)$.

Lemma 1. *There is an arbitrage if and only if there is a self-financing arbitrage.*

Proof. A self-financing arbitrage is an arbitrage. Suppose there is an arbitrage θ. By the definition of a dividend process, the dividend process D^{θ} financed by θ is a semimartingale. Consider the trading strategy $\varphi = (\varphi^0, \ldots, \varphi^N)$ defined by $\varphi^j = 0$, $j \neq 0$ and $\varphi_t^0 = \int_0^t f_{s,t} \, dD_s^{\theta}$. In other words, φ re-invests the dividends financed by θ at the short rate. The strategy $\theta + \varphi$ is then a self-financing arbitrage. (It is clearly an arbitrage, and is self-financing by a calculation using Ito's Lemma.)

If T is the terminal date of the economy, it seems compelling that $S_T = 0$ since there are no dividends after time T. As pointed out by Ohashi (1987), this is actually an assumption that goes beyond the absence of arbitrage, since it may be impossible to carry out an arbitrage with $S_T \neq 0$ if S_T is not measurable with respect to $\mathscr{F}_{T-}$, the left limit of the filtration at T. This may be viewed as a technical limitation of the model that can be eliminated by any of several minor assumptions. For example, we could allow an extra round of $\mathscr{F}_T$-measurable trades at time T, or could extend the time horizon of the economy to $[0, \infty)$. Unless otherwise stated, we do not assume that $S_T = 0$.

5.4. The arbitrage pricing functional

Let Θ denote the space of self-financing trading strategies and $M = \{\theta_T \cdot (S_T + \Delta D_T): \theta \in \Theta\} \subset L^1(P)$, the *marketed subspace* of potential final values.

Proposition 8. *There is no arbitrage if and only if there is a unique strictly positive linear functional $\psi : M \to \mathbb{R}$ defined by $\psi[\theta_T \cdot (S_T + \Delta D_T)] = \theta_0 \cdot S_0$.*

Proof. Suppose there is no self-financing arbitrage. For two self-financing strategies θ and φ satisfying $\theta_T \cdot (S_T + \Delta D_T) = \varphi_T \cdot (S_T + \Delta D_T)$, we claim that $\theta_0 \cdot S_0 = \varphi_0 \cdot S_0$. If not, say if $\theta_0 \cdot S_0 > \varphi_0 \cdot S_0$, then $\varphi - \theta$ is a self-financing arbitrage. Thus ψ is well defined. Strict positivity of ψ follows directly from the definition of an arbitrage. Conversely, if ψ is uniquely well defined and strictly positive, there is no self-financing arbitrage. By Lemma 1, it suffices to examine self-financing arbitrages.

Our objective now is to characterize, under the assumption of no arbitrage, the *arbitrage pricing functional ψ* given by Proposition 8.

5.5. Numeraire-invariance

Before proceeding, we will put in place for later use a natural fact: changing the numeraire for prices and dividends has no real effects. A *price deflator* is a positive predictable semimartingale β that is bounded and bounded away from zero. For example, β_t could be the reciprocal of the price of a particular security (such as a foreign currency) or commodity (such as gold). The following proposition states the obvious fact that re-expressing all prices and dividends with respect to a price deflator has no impact on the ability of a trading strategy to finance a dividend process, nor on the real price at which it is financed. First, let D^β be the *deflated dividend process* defined by $D_t^\beta = \int_0^t \beta_s \, dD_s$, and S^β be the *deflated price process* defined $S_t^\beta = \beta_t S_t$.

Proposition 9 (numeraire-invariance). *Let β be any price deflator. Suppose θ finances a finite variation dividend process C, given securities defined by the dividend process D and price process S. Then, given the securities defined by the deflated dividend process D^β and deflated price process S^β, the same trading strategy θ finances the deflated dividend process C^β defined by $C_t^\beta = \int_0^t \beta_s \, dC_s$.*

The proof by Huang (1985a) is a lengthy application of Ito's Lemma for semimartingales and is not repeated here. The following corollary is immediate from the definitions of M, Θ and ψ.

Corollary. *If (D, S) admits no arbitrage, then (D^β, S^β) admits no arbitrage. The marketed subspaces M under (D, S) and M^β under (D^β, S^β) are related by $x \in M$ if and only if $x\beta_T \in M^\beta$. The respective spaces Θ and Θ^β of self-financing trading strategies are the same. In the absence of arbitrage, the respective arbitrage pricing functionals ψ and ψ^β are related by $\beta_0 \psi(x) = \psi^\beta(x\beta_T)$.*

5.6. Equivalent martingale measure

We now consider the price deflator δ defined by $\delta_t = Z_t^{-1} = \exp(-\int_0^t r_s \, ds)$, and define the deflated gain process G^δ by $G^\delta = D^\delta + S^\delta$. This is merely the gain relative to the numeraire defined by the market value Z of the short-rate re-investment strategy. An *equivalent martingale measure* is a probability measure Q, equivalent to P, such that G^δ is a Q-martingale. That is, under an equivalent martingale measure Q, for any times t and $\tau \geq t$, $E^Q(G_\tau^\delta | \mathcal{F}_t) = G_t^\delta$, where E^Q denotes expectation with respect to Q, implying that

$$S_t = \frac{1}{\delta_t} E^Q\left(\int_t^\tau \delta_s \, dD_s + \delta_\tau S_\tau \, \Big| \, \mathcal{F}_t \right), \tag{22}$$

a useful formula.

In spirit, based on the same arguments used in Section 3.1, the existence of an equivalent martingale measure is equivalent to the absence of arbitrage. Unfortunately, in an infinite-dimensional setting, this equivalence can be upset by various technical problems, as explained for example by Back and Pliska (1989). The principal difficulty is that there is no general result guaranteeing that the arbitrage pricing functional ψ can be extended to a strictly positive linear functional on $L^1(P)$. If $\mathcal{F}$ is finite, the extension follows immediately from the finite-dimensional lemma of section 3.1. For now, we will merely take $\mathcal{F}$ to be finite, and later return to provide other sufficient conditions for a strictly positive linear extension. The following theorem is conceptually the same as the main result of Harrison and Kreps (1979).

Theorem 4 (Harrison-Kreps). *Suppose $\mathcal{F}$ is finite. Then there is no arbitrage if and only if there is an equivalent martingale measure.*

The following proof is written as though $\mathcal{F}$ is general, since the arguments are general, with the exception of the extension result, and can be used again later.

Proof. (Only if): Suppose there is no arbitrage. Let ψ be defined by Proposition 8. By the extension lemma of Section 3.1, ψ has a strictly positive linear extension $\Psi : L^1(P) \to \mathbb{R}$. By a result sometimes known as Choquet's Theorem, any non-negative linear functional on $L^1(P)$ is continuous, so that Ψ is continuous. [See, for example, Schaefer (1974).] By the Riesz Representation Theorem for $L^1(P)$, there is a unique bounded strictly positive random variable π such that

$$\Psi(x) = E(\pi x), \quad x \in L^1(P).$$

We define a measure Q on $(\Omega, \mathcal{F})$ by

$$Q(A) = E(1_A Z_T \pi), \quad A \in \mathcal{F}.$$

We have $Q(\Omega) = E(Z_T \pi) = \psi(Z_T) = 1$, since Z_T is by definition the final payoff of a strategy requiring an initial investment of 1 unit of account. Thus Q is a probability measure, and is equivalent to P since $Z_T \pi$ is strictly positive almost surely. For any random variable x integrable with respect to P, the expectation of x with respect to Q is well defined since the Radon–Nikodym derivative $\mathrm{d}Q/\mathrm{d}P = Z_T \pi$ is bounded.

For any security j, we will show that $\{\int_0^t \delta_s \, \mathrm{d}D_s^j + \delta_t S_t^j : t \in [0, T]\}$ is a martingale under Q, completing the "only if" portion of the proof. This is trivial for $j = 0$. For any $j \geqslant 1$, it is enough to show, for any times t and $s > t$ and any event $A \in \mathcal{F}_t$, that

$$E^Q(1_A \delta_t S_t^j) = E^Q \left[1_A \left(\int_t^s \delta_v \, dD_v^j + \delta_s S_s^j \right) \right].$$ (23)

To this end, consider the trading strategy θ defined by:
  (1) at any time $\tau \leq t$, let $\theta_\tau = 0$;
  (2) at any time $\tau \in (t, s]$, let
      (a) $\theta_\tau^j = 1_A$,
      (b) $\theta_\tau^k = 0$, $k \notin \{0, j\}$,
      (c) $\theta_\tau^0 = 1_A(-S_t^j f_{t,\tau} + \int_t^\tau f_{v,\tau} \, dD_v^j)$;
  (3) at any time $\tau \in (s, T]$,
      (a) $\theta_\tau^k = 0$, $k \neq 0$,
      (b) $\theta_\tau^0 = V_s f_{s,\tau}$, where $V_s = 1_A(S_s^j - f_{t,s} S_t^j + \int_t^s f_{v,s} \, dD_v^j)$.
The strategy θ merely holds one share of security j between times t and s in event A, financing the cost S_t^j by borrowing at the short rate, and continually re-investing the dividends at the short rate. At time s, the unit of security j is sold, and the entire resulting balance V_s is re-invested at the short rate until time T.

The final market value of the strategy is θ_T^0. Since the initial investment is $\theta_0 \cdot S_0 = 0$, we have, by definition of ψ,

$$0 = \psi(\theta_T^0) = E(\pi \theta_T^0) = E^Q(\delta_T \theta_T^0).$$ (24)

The definition of θ_T^0 in 3(b), however, implies that (23) and (24) are equivalent, proving the "only if" part of the result.

(If): Suppose Q is an equivalent martingale measure. Let θ be a self-financing trading strategy. The numeraire-invariance Proposition 9 implies that

$$\delta_T[\theta_T \cdot (S_T + \Delta D_T)] = \theta_0 \cdot S_0 + \int_0^T \theta_t \, dG_t^\delta.$$ (25)

Since dQ/dP is bounded (because $\mathcal{F}$ is finite) and δ is bounded, θ is in $L_Q^1[G^\delta]$, where the notation indicates expectation relative to Q. That is, $\int \theta \, dG^\delta$ is a martingale with respect to Q. Taking expectations with respect to Q on each side of (25) leaves

$$E^Q[\delta_T \theta_T \cdot (S_T + \Delta D_T)] = \theta_0 \cdot S_0.$$

This defines the pricing functional ψ of Proposition 8 by $\psi(x) = E^Q(\delta_T x)$, $x \in M$. As such, ψ is linear and strictly positive, implying the absence of arbitrage.

5.7. Alternate sufficient conditions for equivalent martingale measures

Theorem 4 is the main result of this section, but relies on the assumption that there is only a finite number of distinct events. There are at least two other sufficient conditions that have been studied in the literature:

(1) the existence of an optimal policy for some agent whose preferences satisfy regularity conditions;

(2) the absence of a *free lunch*, an approximate notion of arbitrage due to Kreps (1981).

We will state the sufficiency of these conditions in turn. We omit proofs, since these alternative sufficient conditions for equivalent martingale measures merely shore up the natural intuition of Theorem 4 with the technical qualifications required in an infinite-dimensional setting for a strictly positive linear extension of the arbitrage pricing functional.

Consider some agent with a utility function $U : L^1(P) \rightarrow \mathbb{R}$ facing the problem

$$\max_{\theta \in \Theta} U[\theta_T \cdot (S_T + \Delta D_T)] . \tag{26}$$

Proposition 10. *Suppose U is quasi-concave, continuous and strictly increasing. Then there exists an equivalent martingale measure if problem* (26) *has a solution.*

Harrison and Kreps (1979) call the existence of a solution to (26) *viability*. Their proof of a result essentially the same as Proposition 10 will also suffice here. Naturally, the proof first uses the fact that viability implies lack of arbitrage. The arbitrage pricing functional ψ has a strictly positive linear extension given by a re-scaling of the linear functional defining a separating hyperplane between:

(1) the upper contour set $\{x \in L^1(P): U(x) \geq U(x^*)\}$ at $x^* = \theta_T^* \cdot (S_T + \Delta D_T)$, the final wealth financed by a solution θ^* to (26), and

(2) the budget feasible set $\{x \in M: \psi(x) \leq \psi(x^*)\}$.

Given this extension of ψ, the proof of an equivalent martingale measure follows the "only if" part of the proof of Theorem 4. The basic idea of the result extends to a model with preferences over multiple commodities and over consumption processes on $[0, T]$. Essentially, the desired extension of ψ is a shadow price or Lagrange multiplier for the final wealth budget constraint.

Now we record the fact that the absence of a free lunch, a construction due to Kreps (1981), is also a sufficient condition for the existence of an equivalent martingale measure. In the context of securities with no arbitrage, with associated marketed subspace M and arbitrage price functional ψ, a *free lunch* is a sequence $\{(m_n, x_n)\}$ in $M \times L^1(P)$ satisfying:

(1) $m_n \geq x_n$,

(2) x_n converges in $L^1(P)$ to some non-zero $k \in L^1(P)_+$, and

(3) $\liminf \psi(m_n) \leq 0$.

The three conditions suggest the possibility, in a limiting sense, of obtaining a payoff k "better than zero" at a zero or negative price. The absence of free lunches implies the absence of arbitrages, and a bit more.

Proposition 11. *Suppose $(\Omega, \mathcal{F}, P)$ is separable. If there is no free lunch, then there exists an equivalent martingale measure.*

The separability of $(\Omega, \mathcal{F}, P)$ is a mild regularity condition that is satisfied, for example, if $\mathcal{F}$ is the σ-algebra generated by a Standard Brownian Motion in some Euclidean space. The proof by Duffie and Huang (1985) shows that the absence of free lunches implies that the arbitrage pricing functional ψ has a strictly positive linear extension.

In general, we can draw on the following result for other possible sufficient conditions.

Proposition 12. *Suppose there is no arbitrage and the arbitrage pricing functional ψ has a strictly positive linear extension to $L^1(P)$. Then there is an equivalent martingale measure.*

Again, the proof is the "only if" portion of the proof of Theorem 4.

5.8. Equivalent martingale measure and the state price process

Given the setup $((\Omega, \mathcal{F}, \mathbb{F}, P), (D, S))$ of Section 5.2, suppose the hypotheses of Proposition 12 are satisfied. Then there is an equivalent martingale measure Q, where the Radon–Nikodym derivative dQ/dP is bounded. Recall from Section 4 that the density process ξ for Q is defined by $\xi_t = E((dQ/dP)/\mathcal{F}_t)$.

We now pick a particular security of the $N + 1$, with price process, say V, and dividend process, say C. For the next result, we will use the assumption that C is of finite variation. For example, C is of finite variation if defined by $C_t = \int_0^t p_s \cdot c_s \, ds$ for some multi-commodity consumption process c and spot price process p whose product is integrable. More generally, as described in the Appendix, a finite variation process is a semimartingale that can be written as the sum of an increasing and a decreasing process.

From (22), for any times t and τ with $\tau \geq t$,

$$V_t = \frac{1}{\delta_t} E^Q \left[\int_t^\tau \delta_s \, dC_s + \delta_\tau V_\tau \,\middle|\, \mathcal{F}_t \right]. \tag{27}$$

The following result[8] makes a connection between the equivalent martingale measure and the state price process.

Proposition 13. *Let π be the process defined by $\pi_t = \delta_t \xi_t$. If C is of finite variation, then for any times t and $\tau \geq t$,*

$$V_t = \frac{1}{\pi_t} E\left[\int_t^\tau \pi_s \, dC_s + \pi_\tau V_\tau \,\Big|\, \mathscr{F}_t\right].$$

The proof follows the lines of the proof of Proposition 6, where the *state price process* π is identified as the marginal utility process $\{u_{\lambda c}(e_t, t)\}$ for a particular representative-agent equilibrium with additive separable utilities.

5.9. Arbitrage pricing of redundant securities

Once again, consider the setup described in Section 5.2, with the no-arbitrage hypotheses of Proposition 12. An equivalent martingale measure Q for (D, S) is fixed for this subsection. In addition to the given set (D, S) of securities, consider a new security with a dividend process C and price process V. We are interested in knowing whether the same equivalent martingale measure Q will also serve to price C, that is, whether (27) applies for all t and $\tau \geq t$. In that case, we say that C *is priced by* Q. An obvious sufficient condition is the existence of a trading strategy θ that finances C.

Lemma 2. *Suppose $((D, C), (S, V))$ admits no arbitrage. If $(S_T, V_T) = 0$ and there exists a trading strategy θ that finances C, then C is priced by Q.*

Proof. Suppose that θ finances C. Pick any time t. By writing down the financing condition under the deflator δ, taking expectation at t given $\mathscr{F}_t$, and using $S_T = 0$, we have

$$\theta_t \cdot S_t = \frac{1}{\delta_t} E^Q\left(\int_t^T \delta_s \, dC_s \,\Big|\, \mathscr{F}_t\right).$$

It remains to confirm (27) by showing that $V_t = \theta_t \cdot S_t$. If this is not the case, say if $V_t > \theta_t \cdot S_t$ on some event A in $\mathscr{F}_t$ with $P(A) > 0$, consider the following trading strategy. Let φ be the trading strategy that invests at the short rate, at any time s, $f_{t,s}(V_t - \theta_t \cdot S_t) 1_{(t,T) \times A}(s)$, plus the $\mathbb{R}^{N+2}$-valued trading strategy $1_{A \times (t,T]}(\theta, -1)$, holding -1 units of C and adopting the strategy θ from time t.

---

[8]At the author's request, Steven Shreve constructed a counter-example for the case of a dividend process C that is not of finite variation.

Then φ is a self-financing arbitrage, with zero initial investment and final value equal to $f_{t,T}(V_t - \theta_t \cdot S_t)1_A > 0$. This contradicts the assumed absence of arbitrage, proving the result.

The lemma suggests that we can price C according to the same equivalent martingale measure Q if C is redundant, in the sense that it can be financed by trading the original securities. (The qualification that $(S_T, V_T) = 0$ is to be expected given the discussion at the end of Section 5.3.) It is natural to expect redundancy of any dividend process C given (D, S) if D satisfies a dynamic spanning condition like that described in Section 4. The next result develops an alternative spanning condition directly on the martingale component of the gain process $G = D + S$. As an integrable semimartingale, each gain process G^j can be written as the sum of a martingale M^j and a bounded variation predictable process A^j with $A_0^j = 0$. A *special semimartingale* is a semimartingale (or vector of semimartingales) with a unique such decomposition. For example, any semimartingale with bounded jumps (in particular, any continuous semimartingale) is special.

Proposition 14. *Suppose the gain process G is special and dP/dQ is bounded or every Q-martingale has bounded jumps. If the martingale component of G is a martingale generator under P, then G^δ is a martingale generator under Q and any dividend process can be financed by some trading strategy.*

Proof. Let M be the martingale component of G. It is immediate that Y is a martingale generator when defined by $Y_t = \int_0^t \delta_s \, dM_s$. By Lemma 3.2 in Duffie (1985), Y is special under Q. By the uniqueness of the decomposition of Y under Q, it follows that G^δ is the martingale component under Q of Y, which by Theorem 3.2 in Duffie (1985) implies that G^δ is a martingale generator under Q. The remainder of the proof is an obvious extension of the proof of the spanning lemma of Section 4.1.

Corollary. *Under the assumptions of Proposition 14, there is a unique equivalent martingale measure.*

Proof. The fact that G^δ is a martingale generator under an equivalent martingale measure Q implies that $M = L^1(P)$. This requires that, for any event $A \in \mathcal{F}$, $Q(A) = \psi(Z_T 1_A)$, which fixes Q.

5.10. The Brownian case: spanning and Girsanov's Theorem

This subsection explores the implications of the last in a setting of Brownian information. We continue under the hypotheses of Proposition 12 and fix an

equivalent martingale measure Q. The hypotheses of Proposition 14 are easily checked if $\mathbb{F}$ is generated by some Standard Brownian Motion B, say in $\mathbb{R}^d$. In that case, let M denote the martingale part of G. As stated in Section 4.2, B is itself a martingale generator, so we can always write, for some predictable process σ that is $K \times d$ matrix-valued, $M_t = \int_0^t \sigma_s \, dB_s$, $0 \le t \le T$. Provided the rank of σ is d almost everywhere, M is a martingale generator, and the conditions of Proposition 14 are satisfied since every martingale on a Brownian filtration has continuous sample paths (almost surely).

The Brownian case is also particularly nice because it allows us to calculate Q. Suppose, for example, that G is an Ito process. In that case, we can always write

$$dG_t^\delta = \mu_t \, dt + \delta_t \sigma_t \, dB_t \,,$$

for some adapted (vector) drift process μ, with σ as described in the last paragraph. Assuming that σ has rank d almost everywhere, we can define an $\mathbb{R}^d$-valued adapted process φ with $\sigma_t \varphi_t = -\mu_t/\delta_t$ almost everywhere. Ignoring integrability for the moment, let $\hat{B}_t = B_t - \int_0^t \varphi_s \, ds$. Then[9]

$$\begin{aligned}
dG_t^\delta &= \mu_t \, dt + \delta_t \sigma_7 \, dB_t \\
&= \mu_t \, dt + \delta_t \sigma_t (d\hat{B}_t + \varphi_t \, dt) \\
&= \mu_t \, dt + \delta_t \sigma_t \, d\hat{B}_t - \mu_t \, dt \\
&= \delta_t \sigma_t \, d\hat{B}_t \,.
\end{aligned}$$

Since Q is uniquely defined, according to the corollary to Proposition 14, Q must be that measure obtained by an application of Girsanov's Theorem. That is, it must be the case that dQ/dP is defined by (13) and that $\hat{B}$ is a Standard Brownian Motion under Q.

In short, this provides us with a direct calculation of dQ/dP, which can then be used to calculate the price of any security, say by (27). The qualification in Lemma 2 that "$V_T = 0$" is automatically satisfied in this setting, since every semimartingale on a Brownian filtration is predictable.[10]

For example, consider an additional security with dividend process C defined by $C_t = 0$, $t \le \tau$, and $C_t = H$, $t \ge \tau$, where τ is a stopping time and H is

---

[9]For disciples of semimartingale theory, a more direct way to see this representation of G^δ under Q is to check that the matrix-valued "sharp brackets" process $\langle G^\delta, G^\delta \rangle$ is preserved under a change of equivalent measure. Since this process is differentiable with respect to time and G^δ is a martingale under Q, there exists a Brownian motion $\hat{B}$ under Q such that $dG_t^\delta = \delta_t \sigma_t \, d\hat{B}_t$. For details, see, for example, Jacod (1979).

[10]This predictability is proved in a written communication from Kai Lai Chung and Ruth Williams. See also Proposition 4 of Ohashi (1987), which is relevant since the Brownian filtration is left-continuous.

$\mathscr{F}_\tau$-measurable. In other words, C pays a lump sum dividend of H at the stopping time τ. Assuming σ has rank d almost everywhere, (27) implies that the unique arbitrage free-price process V of the additional security satisfies

$$V_t = \frac{1}{\delta_t} E^Q(\delta_\tau H | \mathscr{F}_t), \quad t < \tau, \tag{28}$$

where

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \exp\left(\int_0^T \varphi_t \, \mathrm{d}B_t - \frac{1}{2} \int_0^T \varphi_t \cdot \varphi_t \, \mathrm{d}t\right),$$

and where φ is defined by $\sigma_t \varphi_t = -\mu_t/\delta_t$.

In many applications, $D = 0$ and $H = g(S_\tau, \tau)$ for some $g : \mathbb{R}^{N+1} \times [0, T] \to \mathbb{R}$. We can take it that $\mathrm{d}S_t = \nu_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}B_t$ for some drift process ν. We know that $S^\delta = G^\delta$ is a martingale under Q, so an application of Ito's Lemma implies that

$$\mathrm{d}S_t = r_t S_t \, \mathrm{d}t + \sigma_t \, \mathrm{d}\hat{B}_t, \tag{29}$$

where $\hat{B}$ is the Standard Brownian Motion under Q constructed above. Rather than using $\mathrm{d}Q/\mathrm{d}P$ explicitly as in (28), we can instead use the expression (29) for $\mathrm{d}S_t$ under Q to represent the arbitrage-free price of the additional security in the form

$$V_t = \frac{1}{\delta_t} E^Q[\delta_\tau g(S_\tau, \tau) | \mathscr{F}_t], \quad t < \tau.$$

As a special case, suppose $N = d = 1$, $r_t = \bar{R}$ for some constant $\bar{R}$ and $\sigma_t = \bar{\sigma} S_t$ for some constant $\bar{\sigma}$. By Ito's Lemma, $S_\tau = S_0 \exp[(\bar{R} - \bar{\sigma}^2/2)\tau + \hat{B}_\tau]$, yielding the arbitrage free initial price

$$V_0 = E^Q[\mathrm{e}^{-\bar{R}\tau} g(S_\tau, \tau)].$$

The best known example is the Black–Scholes (1973) option pricing formula, for which $\tau = T$ and $g(x, T) = (x - \bar{x})^+$ is the expiry value[11] of a European call option with exercise price $\bar{x}$. In that case, we have the explicit calculation,

---

[11] The option gives its owner the right, but not the obligation, to purchase the underlying asset at the exercise price $\bar{x}$ fixed in advance. If the underlying price X_T at the expiry date T of the option exceeds the exercise price, the option holder will exercise the option for a net payoff of $X_T - \bar{x}$. Otherwise, the option expires with no value. Thus $g(x, T) = (x - \bar{x})^+ \equiv \max(x - \bar{x}, 0)$.

known as the Black–Scholes option pricing formula,

$$V_0 = S_0 \Phi(d_1) - \mathrm{e}^{-\bar{R}T} \bar{x} \Phi(d_1 - \bar{\sigma}\sqrt{T}) \,,$$

where Φ is the standard normal cumulative distribution function and

$$d_1 = \frac{1}{\bar{\sigma}\sqrt{T}} \log\left(\frac{S_0 \, \mathrm{e}^{\bar{R}T}}{K}\right) + \frac{\bar{\sigma}\sqrt{T}}{2} \,.$$

The Black–Scholes formula was originally computed more tediously by a direct solution of a partial differential equation studied in the next subsection.

If the dividend $g(S_\tau, \tau)$ is paid at a stopping time τ chosen by the owner of the security, the absence of arbitrage implies that τ is rationally chosen so as to maximize the market value of the security. That is,

$$V_0 = \sup_\tau E^Q[\delta_\tau g(S_\tau, \tau)] \,.$$

For instance, an American put option has payoff $g(S_\tau, \tau) = (\bar{x} - S_\tau)^+$ at an exercise date τ chosen by the holder of the option. Progress on this problem has been made in sources cited in Section 6.10.

5.11. The Markov case: the PDE for derivative asset prices

This subsection characterizes arbitrage-free derivative asset prices in a Markov state space setting. We will derive a partial differential equation (PDE) for the derivative asset price, and then provide sufficient conditions for the existence of a smooth solution. Of course, the solution is exactly that defined by the conditional expectation (27), but the equivalent martingale measure Q is implicit, rather than explicit, in the PDE. Finally, we mention several techniques that are commonly used for solving the PDE, at least numerically. In practice, a Markov setting is the most commonly found in application because of its computational and econometric advantages.

The state of the market model is defined by an $\mathbb{R}^K$-valued process $\{X_t\}$ satisfying the stochastic differential equation

$$\mathrm{d}X_t = \nu(X_t, t) \, \mathrm{d}t + \eta(X_t, t) \, \mathrm{d}B_t \,, \quad X_0 = x \,,$$

where B is a Standard Brownian Motion in $\mathbb{R}^d$ and $\nu : \mathbb{R}^K \times [0, T] \to \mathbb{R}^K$ and $\eta : \mathbb{R}^K \times [0, T] \to \mathbb{R}^{K \times d}$ satisfy regularity conditions ensuring existence and strong uniqueness of solutions. Details can be found, for example, in Chung and Williams (1989); it is enough that both ν and η are Borel measurable and

satisfy a Lipschitz[12] condition as well as a growth[13] condition, both with respect to their first (state) argument.

The "primitive" securities are defined by functions $(\mathcal{S}, \delta, R)$ on $\mathbb{R}^K \times [0, T]$ that satisfy regularity conditions to be added later. Specifically, the $\mathbb{R}^N$-valued function $\mathcal{S}$ defines the N "risky" security prices by $\mathcal{S}(X_t, t) = (S_t^1, \ldots, S_t^N)$. The corresponding N dividend processes are defined by $D_t^j = \int_0^t \delta_j(X_s, s) \, ds$, $j \geq 1$. As usual, security number zero has price identically equal to 1 and a dividend rate equal to the short rate process r, in this case given by $r_t = R(X_t, t)$. For convenience, we depart from our usual convention and take the *cum dividend* security pricing convention.

For a full general equilibrium setting with this form of price behavior, consider the equilibrium described by Proposition 6. Suppose the exogenous Markov process X determines the aggregate endowment process e for that economy by $e_t = h(X_t, t)$ for some smooth function h, and suppose each primitive security $j \geq 1$ has a real dividend process of the form $D_t^j = \int_0^t f_j(X_s, s) \, ds$, $t < T$ and $D_T^j = \int_0^T f_j(X_s, s) \, ds + g_j(X_T, T)$, for measurable f_j and g_j. Relation (16) and the calculation $r_t = -\mu_\pi(t)/\pi(t)$ of the short rate imply that $S_t = (1, \mathcal{S}(X_t, t))$ and that $r_t = R(X, t)$ for measurable functions $\mathcal{S}$ and R. See Huang (1987) for extensive analysis of such a Markovian equilibrium.

An additional security, to be priced, has a dividend process C defined by $C_t = \int_0^t f(X_s, s) \, ds$, $t < T$ and $C_T = \int_0^T f(X_s, s) \, ds + g(X_T, T)$, where f and g are real-valued functions on $\mathbb{R}^K \times [0, T]$ with properties to be specified. In many applications, such as the original Black–Scholes model, the state process X is actually the security price process S itself. In that case, the additional security to be priced is called *derivative* because its dividends are functions of the underlying asset price process. For example, in the Black–Scholes call option pricing model, X is a geometric Brownian Motion describing the price of a given security (that has no dividends), and the derivative dividend process is defined by $f = 0$ and $g(x, T) = (x - \bar{x})^+$, where $\bar{x}$ is the option's exercise price, as explained in Section 5.10.

We presume that the dividend process C defined by f and g can be financed given (D, S), and later return to provide sufficient conditions for this assumption, as well as several other assumptions made (rather loosely) along the way to a conjectured solution for the price process V. At the final stage, we can state a formal theorem.

The absence of arbitrage implies restrictions on the price process V for C. Rather than pursuing the existence of an equivalent martingale measure, however, we will use the redundancy of C and the absence of arbitrage to

---

[12]There exists a constant k such that $\|\eta(x, t) - \eta(y, t)\| \leq k\|x - y\|$ for all x and y and all t.
[13]There exists a constant k such that $\|\eta(x, t)\| \leq k(1 + \|x\|)$ for all x and all t.

derive a PDE in $\mathbb{R}^K \times [0, T]$ whose solution J, if sufficiently well behaved, evaluates V as $V_t = J(X_t, t)$.

Assuming that J is sufficiently smooth for an application of Ito's Lemma, $V_t = J(X_t, t)$ implies that

$$dV_t = \mathscr{D}J(X_t, t)\,dt + J_x(X_t, t)\eta(X_t, t)\,dB_t\,, \tag{30}$$

where, for any smooth function H,

$$\mathscr{D}H(x, t) = H_x(x, t)\nu(x, t) + H_t(x, t) + \tfrac{1}{2}\,\mathrm{tr}[\eta(x, t)^{\top}H_{xx}(x, t)\eta(x, t)]\,,$$

with subscripts indicating the obvious partial derivatives.

By assumption, a trading strategy θ finances the dividend process C. Barring arbitrage, this means that, for all t,

$$\theta_t \cdot S_t = V_t \tag{31}$$

and

$$\theta_t \cdot S_t = \theta_0 \cdot S_0 + \int_0^t \theta_s\,dG_s - C_t\,. \tag{32}$$

Substituting the various functions applied above, and denoting $\theta_t^0 = b_t$ and $(\theta_t^1, \ldots, \theta_t^N) = a_t$, relation (31) implies that

$$a_t \cdot \mathscr{S}(X_t, t) + b_t = J(X_t, t)\,, \quad t \in [0, T]\,. \tag{33}$$

From (31) and (32), with the obvious notational shorthand,

$$dV_t + f(X_t, t)\,dt = a_t \cdot [\delta(X_t, t)\,dt + \mathscr{D}\mathscr{S}(X_t, t)\,dt + \mathscr{S}_x(X_t, t)\eta(X_t, t)\,dB_t]$$
$$+ b_t R(X_t, t)\,dt\,. \tag{34}$$

Ito processes are special semimartingales and can therefore be uniquely decomposed as the sum of a constant, a stochastic integral with respect to the Brownian motion B and an ordinary Lebesgue integral with respect to "time" t. This means that we can equate the coefficients of dB_t and dt separately in (34), using (30), to derive several necessary conditions for no arbitrage and the fact that θ finances C. First, equating coefficients in dB_t from (34) leaves (almost everywhere)

$$a_t \mathscr{S}_x(X_t, t)\eta(X_t, t) = J_x(X_t, t)\eta(X_t, t)\,, \quad t \in [0, T]\,. \tag{35}$$

In order to find a_t satisfying (35), it is enough (and close to necessary) that $\mathscr{S}_x$ is everywhere of rank K, in which case

$$a_t = J_x(X_t, t)\mathscr{S}_x(X_t, t)^{\top}[\mathscr{S}_x(X_t, t)\mathscr{S}_x(X_t, t)^{\top}]^{-1} . \tag{36}$$

Next, (33) and (36) imply that

$$b_t = J(X_t, t) - J_x(X_t, t)\mathscr{S}_x(X_t, t)[\mathscr{S}_x(X_t, t)\mathscr{S}_x(X_t, t)^{\top}]^{-1}\mathscr{S}(X_t, t) . \tag{37}$$

Finally, equating the coefficients of dt in (34), using (36) and (37), leaves

$$\begin{aligned} R(X_t, t)J(X_t, t) &= J_x(X_t, t)\mu(X_t, t) + J_t(X_t, t) \\ &\quad + \tfrac{1}{2}\operatorname{tr}[\eta(X_t, t)^{\top}J_{xx}(X_t, t)\eta(X_t, t)] + f(X_t, t) , \end{aligned} \tag{38}$$

where $\mu : \mathbb{R}^K \times [0, T] \to \mathbb{R}^K$ is defined by

$$\begin{aligned} \mu(x, t) &= S_x(x, t)^{\top}[\mathscr{S}_x(x, t)\mathscr{S}_x(x, t)^{\top}]^{-1}[R(x, t)\mathscr{S}(x, t) - \delta(x, t) - \mathscr{S}_t(x, t) \\ &\quad - \tfrac{1}{2}q(x, t)] , \end{aligned} \tag{39}$$

and where $q_j(x, t) = \operatorname{tr}[\eta(x, t)^{\top}\mathscr{S}^j_{xx}(x, t)\eta(x, t)]$.

Of course, (38) is automatically satisfied if J solves the parabolic PDE in $\mathbb{R}^K \times [0, T]$ given by

$$\begin{aligned} R(x, t)J(x, t) &= J_x(x, t)\mu(x, t) + J_t(x, t) \\ &\quad + \tfrac{1}{2}\operatorname{tr}[\eta(x, t)^{\top}J_{xx}(x, t)\eta(x, t)] + f(x, t) . \end{aligned} \tag{40}$$

The boundary condition imposed on (40) by equating the cum dividend final market value $J(X_T, T)$ with the final payoff $g(X_T, T)$ is

$$J(x, T) = g(x, T) , \quad x \in \mathbb{R}^K . \tag{41}$$

We can immediately conjecture a solution to (40), (41) by applying Ito's Lemma. For each $(x, t) \in \mathbb{R}^K \times [0, T]$, assuming the expectation is well defined, let

$$J(x, t) = E\left[\int_t^T e^{-\varphi(s)}f(Y_s^{x, t}, s)\, ds + e^{-\varphi(T)}g(Y_T^{x, t}, T)\right] , \tag{42}$$

where $\{Y_s^{x, t}: t \le s \le T\}$ solves

$$Y_s^{x,t} = x + \int_t^s \mu(Y_\tau^{x,t}, \tau) \, \mathrm{d}\tau + \int_t^s \eta(Y_\tau^{x,t}, \tau) \, \mathrm{d}B_\tau \,, \quad s \geq t \,, \tag{43}$$

and where

$$\varphi(s) = \int_t^s R(Y_\tau^{x,t}, \tau) \, \mathrm{d}\tau \,.$$

A unique solution to (43) exists under the usual conditions on μ and η mentioned above. If J is indeed well defined by (42) and smooth enough for an application of Ito's Lemma, it follows immediately from Ito's Lemma that J solves the PDE (40) with boundary condition (41). This is often called the *Feynman–Kac* solution of the PDE.

All of the above calculations can be justified with known conditions on the functions (μ, η, δ, R, f, g) under which (42) is well defined and generates the unique solution J to (40), (41) satisfying a growth condition in the state variable. Typical alternative sets of conditions are due to Dynkin (1965), Freidlin (1985) and Krylov (1980). The following result is representative.

Krylov's Theorem. *Suppose μ and η satisfy a Lipschitz condition in the state variable, and that all of the functions (μ, η, δ, R, f, g) are Borel measurable, have two continuous derivatives with respect to the state variable, and that the functions and their first and second derivatives with respect to the state variable satisfy a growth condition with respect to the state variable. Then (42) defines a solution J to the PDE (40), (41), the unique solution satisfying a growth condition with respect to the state variable.*

If $\eta\eta^\top$ has eigenvalues bounded away from zero (or "uniform ellipticity"), Krylov's smoothness conditions can be weakened significantly. Our prior analysis now justifies the following claim.

Corollary. *Suppose $((D, C), (S, V))$ admits no arbitrage, rank($\mathcal{S}_x$) = K everywhere, and (μ, η, δ, R, f, g) satisfies Krylov's conditions. Then J is well defined by (42), C is financed by the trading strategy (b, a) defined by (36) and (37), and the price process V of C is given by $V_t = J(X_t, t)$.*

As an example, we can take the case $\delta = 0$ and $\mathcal{S}(x, t) = x$, in which case $\mu(x, t) = R(x, t)x$. Then

$$V_0 = E\left[\int_0^T e^{-\varphi(t)} f(Y_t, t) \, \mathrm{d}t + e^{-\varphi(T)} g(Y_T, T) \right], \tag{44}$$

where Y is the "pseudo-price process" defined by the stochastic differential equation

$$\mathrm{d}Y_t = r(Y_t, t)Y_t \, \mathrm{d}t + \sigma(Y_t, t) \, \mathrm{d}B_t \,, \quad Y_0 = X_0 \,, \tag{45}$$

and where $\varphi(t) = \int_0^t r(Y_s, s) \, \mathrm{d}s$. Of course, the distribution of the pseudo-price process Y under P is the same as that of the price process S itself under the equivalent martingale measure Q, as shown by comparing (29) and (45), and the solution given here for V_0 is exactly that obtained in Section 5.10.

In particular, we can easily recover the Black–Scholes formula in the case $K = 1$, $R(x, t) = \bar{R}$, $f = 0$, $g(x, t) = (x - \bar{x})^+$ and $\sigma(x, t) = \bar{\sigma}x$, where $\bar{R}$, $\bar{x}$ and $\bar{\sigma}$ are positive constants. It follows from (44) that

$$V_0 = E[\mathrm{e}^{-\bar{R}T}(Y_T - \bar{x})^+] \,, \tag{46}$$

where $Y_T = X_0 \exp[(\bar{R} - \bar{\sigma}^2/2)T + \bar{\sigma}B_T]$. Relation (46) defines the Black–Scholes option pricing formula, as stated in the Section 5.10. Of course, the payoff function $(x, t) \mapsto (x - \bar{x})^+$ is not as smooth as required by Krylov's conditions, being non-differentiable at $\bar{x}$, but those conditions can be extended to incorporate a function g that is continuous with finitely many pieces that are smooth in Krylov's sense, yielding a solution J that is smooth in $\mathbb{R}^K \times [0, T)$, but not of course at T.

5.12. Approximate solution of the arbitrage PDE

The Black–Scholes option pricing formula is one of several closed-form solutions available for arbitrage pricing of particular derivative securities in this setting. (Some of the other examples are cited in Section 6.10.) As a practical technique for pricing many different forms of derivative securities, however, one typically relies on approximate solutions, usually obtained with the aid of a computer. Commonly used algorithms involve Monte-Carlo simulation of the expectation in (43) or direct numerical solution of the PDE (40), say by finite-difference or finite-element algorithms (see Section 6.10 for references). For simple problems based on a geometric Brownian price process, solutions are also frequently estimated by approximating the "pseudo-price" process Y with a binomial process, calculating the discrete analogue to (42) by a backward recursion, and then improving the approximation error by reducing the length of a trading period. The latter approach was popularized by Cox, Ross and Rubinstein (1979), who showed by an explicit calculation (involving the central limit theorem) that a natural binomial approximation of the price

process X leads to a sequence of option prices converging, with the number of trading periods per unit of time, to the Black–Scholes formula.

5.13. Extensions of the PDE method

The same PDE approach can be extended so as to allow T to be replaced by a stopping time τ defined as the hitting time of (X_t, t) on some regular set, or "liquidation boundary." Dynkin (1965), for example, shows sufficient conditions on the coefficient functions and the liquidation boundary for an analogue to Krylov's Theorem.

In principle, although there are few available results, the PDE (40) also extends to the pricing of securities that the holder may exchange at any time for a pre-arranged liquidation value, the classic example being an American put option, whose liquidation value is the excess (if any) of the option's exercise price over the current price of the underlying security. Recent literature on the American put is cited in Section 6.10. Although it is unrealistic to expect a closed-form solution for the American put, there has been much progress in defining the optimal liquidation boundary in the Black–Scholes setting. The optimal liquidation boundary is that yielding the supremum arbitrage-free value for the derivative security. The PDE (40), with the associated *free boundary*, is often termed a Stefan problem.

The PDE approach can also be extended in like generality to the pricing of continuously re-settled securities, such as futures and futures options, as shown by Black (1976), Cox, Ingersoll and Ross (1981b) and Duffie and Stanton (1988).

6. Further reading

This section points to additional sources of reading on the topics presented in this chapter, as well as a range of literature on related topics that have not been reviewed.

6.1. General references

There are a number of monographs presenting the topics of this chapter at various levels, including Fama and Miller (1972), Mossin (1973), Fama (1976), Ingersoll (1987), Jarrow (1988), Huang and Litzenberger (1988) and Duffie (1988). Survey articles include those of Kreps (1979), Merton (1987), Rothschild (1986), Marimon (1987), Radner (1988), DeMarzo and Van Nuys (1988) and Constantinides (1989).

6.2. Finite-dimensional general equilibrium in security markets

Chapter 30 reviews the literature on existence, optimality, and multiplicity of finite-dimensional security-market equilibria. We will therefore limit ourselves here to mentioning, in addition to Arrow (1953), the key contributions of Debreu (1953), Radner (1967, 1972) and Hart (1975) in formulating the central issues. For existence of equilibria with references defined directly on a linear space of portfolio choices, see Hart (1974), Nielson (1986b) and Werner (1987).

6.3. Spanning and the behavior of the firm

The Modigliani and Miller (1958) results on the irrelevance of financial policy are given a general finite-dimensional treatment in incomplete markets by Duffie and Shafer (1986) and DeMarzo (1988). Financial policy is relevant under almost any departure from the neo-classical assumptions, such as taxes [Miller (1977)], asymmetric information [Jensen and Meckling (1976), Ross (1977), Myers and Majluf (1984), Duffie and DeMarzo (1988)], introduction of options on the firm [Detemple, Gottardi and Polemarchakis (1989)] or bankruptcy [Hellwig (1981)].

There is not yet a generally accepted paradigm for the production decisions of the firm without some sort of spanning condition. Arrow and Debreu (1954) merely took it as an axiom of competitive behavior that firms maximize their market value. With complete spanning, of course, shareholders unanimously support this objective, since it generates maximal budget-feasible choice sets for shareholders. This unanimity result was extended by Diamond (1967) and Ekern and Wilson (1974) to the case of security markets that span the set of feasible dividends of the firm. Makowski (1983) pointed out that this spanning condition is automatically satisfied if shareholders act as though the span of security markets is fixed, independently of the firm's choice. Duffie and Shafer (1986) showed that, if shareholder's do not treat the span of markets as fixed, then, generically, all but at most one shareholder objects to maximizing market value. Of course, the very objective of value maximization is not well defined unless firms have conjectures concerning the value of securities outside of the current span of markets. Duffie and Shafer (1986) show generic existence of equilibria when conjectures are defined by state-prices, in the sense of Section 3.1. Drèze (1974), instead, takes it as an axiom that firms maximize according to state prices defined by a weighted sum of agents' marginal rates of substitution (given by the vector $\nabla U_i(c^i)$, in the notation of Section 3.2), with weights proportional to shareholdings. This objective generates constrained Pareto optimal allocations with a single spot consumption commodity, although

Geanakoplos, Magill, Quinzii and Drèze (1987) overturn this optimality property with multiple spot commodities.

Further characterization of the theory of the firm in incomplete security markets is provided by Aghion and Bolton (1986), Allen and Gale (1987), Bester (1982), Brealey and Myers (1984), Burke (1986), DeMarzo (1987), DeMarzo and Duffie (1988), Fama (1978), Gevers (1974), Grossman and Hart (1979, 1987), Grossman and Stiglitz (1976), Harris and Raviv (1987), Hart (1977, 1979, 1987), Jensen and Long (1972), Kreps (1979), Leland (1973, 1974, 1978), Marimon (1987), Merton and Subrahmanyam (1974), Myers (1984), Radner (1974), Satterthwaite (1981) and Stiglitz (1972, 1974, 1982).

6.4. Mutual funds and factors in asset prices

The CAPM is a single-factor pricing model; the factor is the market portfolio. The CCAPM is also a single-factor model; the factor at each point in time is the growth rate of consumption over the next "instant." A general multi-period single-factor model always applies under mild regularity conditions, as shown, for example, by Hansen and Richard (1987); the general problem is econometric identification of the factor. The CAPM is based on the sufficiency of two mutual funds for Pareto optimality; further sufficient conditions are given by Cass and Stiglitz (1970) and Ross (1978b); additional recommended readings are the papers by Rubinstein (1974), Nielson (1986a) and Stiglitz (1989).

Ross (1976a) described a multi-factor asset pricing model called the APT; sufficient conditions are provided by Huberman (1982) and Connor (1984). Approximate multi-factor models are characterized by Chamberlain (1983b) and Chamberlain and Rothschild (1983).

6.5. Asymmetric information

Most of the available asset pricing results with asymmetric information are based on strong parametric assumptions, as in Admati (1985), Bray (1981), Carino (1987), Dothan and Feldman (1986), Gennotte (1984) and Grossman (1976). Hindy (1989) presents a non-parametric but "bounded rationality" model.

Examples of the literature on asset valuation with a specialist market maker and asymmetric information include the work of Admati and Pfleiderer (1988), Glosten and Milgrom (1985) and Haggerty (1985). This is a very small sample; Bhattacharya and Constantinides (1989) have edited a selection of readings on the role of information in financial economics.

6.6. Equilibrium asset pricing models

Further examples of asset pricing models under the additive separable preference assumptions of Sections 3.2, 3.3 and 4.3 include the papers of Back (1988), Breeden (1986), Breeden and Litzenberger (1978), Grauer and Litzenberger (1979), Kraus and Litzenberger (1975), and Merton (1973a).

By relaxing the additively separable model of preferences described in Sections 3.3 and 4.3, a range of alternative asset pricing formulas can be achieved. Asset pricing models based on alternative preference specifications have been described by Bergman (1985), Constantinides (1988), Duffie and Epstein (1989), Epstein and Zin (1989a) and Sundaresan (1989). Hindy and Huang (1989) formalize the notion of intertemporal substitution of consumption, relaxing the continuous-time assumption of consumption at rates.

6.7. Extended notions of spanning

Models of general equilibrium based on multi-period notions of spanning have been developed by Friesen (1974), Kreps (1982) and Duffie and Huang (1985), which introduces the dynamic spanning condition of Section 4. For technical results on the closely associated problem of "martingale multiplicity," the reader is referred to Clark (1970), Davis and Varaiya (1974), Kunita and Watanabe (1967) and Jacod (1977).

Static notions of spanning based on the formation of options and compound options are due to Ross (1976b), Breeden and Litzenberger (1978), Brown and Ross (1988), Jarrow and Green (1985) and Nachman (1988).

6.8. Asset pricing with "frictions"

The majority of this chapter is based on the costless formation of unrestricted portfolios. For asset-pricing models incorporating one or more of transactions costs, taxes, cash-in-advance constraints, credit constraints, and so on, please see Back (1986), Back and Pliska (1986), Constantinides (1986), Dybvig and Huang (1988), Grossman and Laroque (1987), Lucas and Stokey (1987), Leland (1985), Prisman (1985), Ross (1987), Svensson (1988), Scheinkman and Weiss (1986) and Townsend (1984). Svensson (1988) and Scheinkman (1989) discuss the limited results available on asset pricing results that explicitly incorporate the incomplete markets constraint.

6.9. Technical references on continuous-time models

Brownian motion was actually formulated by Bachelier (1900) as a model of stock prices. On diffusion processes and stochastic differential equations, standard references include Arnold (1974), Freedman (1983), Friedman (1975), Gihman and Skorohod (1972), Ikeda and Watanabe (1981), Karatzas and Shreve (1988), Krylov (1980) and Lipster and Shiryayev (1977). More abstract treatments of stochastic integration are provided by Chung and Williams (1983), Dellacherie and Meyer (1978, 1982), Durrett (1984), Ethier and Kurtz (1986), Jacod (1979), Kopp (1984) and Protter (1989). The central results on the behavior of semimartingales under a change of probability measure are given by Girsanov (1960), Lenglart (1977) and Memin (1980).

The technical foundations of continuous-time security prices and trading in an abstract setting has been developed in a series of papers by Harrison and Kreps (1979), Harrison and Pliska (1981, 1983), Huang (1985a,b) and Pliska (1982).

6.10. Derivative asset pricing

Varian (1987) is a useful introductory exposition of the arbitrage notion of derivative asset pricing. The literature that applies the arbitrage approach to the pricing of derivative securities (that is, securities whose dividends are functions of the prices or dividends of other securities) is far too voluminous to even list here. The most famous example is the option pricing formula of Black and Scholes (1973), which is given a detailed treatment in the book by Cox and Rubinstein (1985). Other prominent examples are the papers by Breeden and Litzenberger (1978), Brennan and Schwartz (1977, 1979), Carr (1987), Cheng (1987), Cox, Ingersoll and Ross (1981a,b), Cox and Ross (1976), Cox, Ross and Rubinstein (1979), Duffie and Stanton (1989), Geske (1979), Goldman, Sosin and Gatto (1979), Heath, Jarrow and Morton (1987), Hemler (1987), Ho and Lee (1986), Hull (1989), Ingersoll (1977), Johnson (1987), Margrabe (1978), Merton (1973b, 1974, 1976, 1977), Müller (1985) and Richard (1978). Numerical solution methods are described by Boyle (1977), Jones and Jacobs (1986), Miltersen and Nielsen (1989) and Geske and Shastri (1985). Progress on the pricing of American options can be found in Geske and Johnson (1984), Parkinson (1977), Jamshidian (1989), Carr and Myneni (1989), Carr, Jarrow and Myneni (1989), Barone-Adesi and Elliott (1989) and Kim (1989).

6.11. Infinite horizon recursive models

The model in Section 3.3 is usually presented in a Markov setting, as in Lucas (1978) and Prescott and Mehra (1980). The determination of an equilibrium is more interesting with production, as shown by Brock (1979, 1982). The monograph by Stokey and Lucas (with Prescott) (1989) is a good source for details.

6.12. Estimation

The many papers on econometric estimation of the asset pricing models presented in this chapter include those of Breeden, Gibbons and Litzenberger (1986), Brown and Gibbons (1985), Epstein and Zin (1989), Hansen (1989) and Hansen and Singleton (1982, 1983). Singleton (1987) surveys some of the econometric work on asset pricing models.

Appendix: Stochastic integration

This appendix is provided for the convenience of those readers interested in the definition of stochastic integration and the underlying technical details. Before beginning, however, we assure the reader that, limiting attention to a large subclass of integrands (θ) and integrators (G), the stochastic integral $\int \theta \, dG$ is nothing more than the limit in probability of the obvious sum,

$$\sum_{j=0}^{k-1} \theta_{t_j}(G_{t_{j+1}} - G_{t_j}) \,,$$

as the maximum length of a time interval $t_{j+1} - t_j$ converges to zero. For this limited but easy definition of the stochastic integral, see Protter (1989).

As primitives, we have a probability $(\Omega, \mathcal{F}, P)$, a time interval $\mathcal{T} = [0, T]$ or $\mathcal{T} = [0, \infty)$, and a family $\mathbb{F} = \{\mathcal{F}_t : t \in \mathcal{T}\}$ of sub-σ-algebras of $\mathcal{F}$ satisfying the usual conditions:

(1) $\mathcal{F}_t \subset \mathcal{F}_s$ whenever $s \geq t$ (increasing);
(2) $\mathcal{F}_0$ includes all subsets of zero-probability events in $\mathcal{F}$ (augmentation);
(3) for all $t \in \mathcal{T}$, $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s$ (right-continuity).

A stochastic process is a family $X = \{X_t : t \in \mathcal{T}\}$ of random variables. Unless otherwise stated, we take a stochastic process to be real-valued. A process X is *adapted* if X_t is $\mathcal{F}_t$-measurable for all t. An adapted process X is *integrable* if $E(|X_t|)$ is finite for all t. A martingale is an adapted integrable stochastic process X with the property:

$$E(X_t | \mathscr{F}_s) = X_s, \quad \text{a.s. whenever } s \leq t,$$

where $E(\cdot | \mathscr{F}_s)$ denotes conditional expectation.

Example (*Brownian motion*). A stochastic process B on some probability space is a Standard Brownian Motion if:

(a) for any $0 \leq s < t < \infty$, $B_t - B_s$ is normally distributed with zero expectation and variance equal to $t - s$;

(b) for any $0 \leq t_0 < t_1 < \cdots < t_l < \infty$, the random variables $\{B(t_0), B(t_k) - B(t_{k-1}): 1 \leq k \leq l\}$, are independent; and

(c) $P(B_0 = 0) = 1$.

For $d \in \mathbb{N}$, a Standard Brownian Motion in $\mathbb{R}^d$ is an $\mathbb{R}^d$-valued process $(B^{(1)}, \ldots, B^{(d)})$ made up of d independent Standard Brownian Motions.

It is normal to use a filtration $\mathbb{F}$ with respect to which the Standard Brownian Motion B is a martingale. For example, we could take $\mathscr{F}_t$ to be the σ-algebra generated by $\{B_s: 0 \leq s \leq t\}$ as well as the subsets of zero-probability events in $\mathscr{F}$. The resulting filtration $\mathbb{F} = \{\mathscr{F}_t: t \in \mathscr{T}\}$ is called the *standard filtration* of B. This ends the example.

A process X is left-continuous if $\lim_{t \uparrow s} X_t = X_s$ for all s almost surely. The predictable σ-algebra on $\Omega \times \mathscr{T}$ is that generated by the left-continuous adapted processes. A stochastic process θ is predictable if $\theta : \Omega \times \mathscr{T} \to \mathbb{R}$ is measurable with respect to the predictable σ-algebra. In continuous-time settings, it is natural to restrict agents to predictable strategies.

A martingale X is square-integrable if $\{X_t^2: t \in \mathscr{T}\}$ is an integrable process. The *quadratic variation* of a square-integrable martingale S is the unique increasing process denoted $[S]$ such that, for each $t \in \mathscr{T}$,

$$[S_t] = \lim_{n \to \infty} \sum_{i=0}^{2^n - 1} [S(t_{i+1}^n) - S(t_i^n)]^2,$$

where $t_i^n = i2^{-n}t$ for $0 \leq i \leq 2^n$. [The limit is in the space $L_1(P)$.] Roughly speaking, $[S_t]$ is the limit of squared changes of S during $[0, t]$, where the length of time intervals over which the changes are measured shrinks to zero. For a Standard Brownian Motion B, $[B_t] = t$ almost surely for all t.

Let $\mathcal{M}^2$ denote the space of square-integrable martingales. For each $S \in \mathcal{M}^2$, let $L^2[S]$ denote the space consisting of any predictable process θ with

$$E\left(\int_0^t \theta_\tau^2 \, \mathrm{d}[S]_\tau\right) < \infty \quad \text{for all } t \text{ in } \mathscr{T}.$$

(Since $[S]$ is increasing and θ_τ^2 is positive, the integral $\int_0^t \theta_\tau^2 \, d[S]_\tau$ is always well defined, although possibly $+\infty$, for each t and each ω in Ω as a Stieltjes integral.) We will next define a stochastic integral $\int \theta \, dS$ for $S \in \mathcal{M}^2$ and $\theta \in L^2[S]$.

We first take the case $\mathcal{T} = [0, T]$. A stochastic process θ is *elementary* if in each state $\omega \in \Omega$ there is partition $\{(0, t_1], (t_1, t_2], \ldots, (t_k, T]\}$ such that θ is constant over each set in the partition. That is, an elementary process is piecewise constant and left-continuous. The stochastic integral $\int_0^T \theta_t \, dS_t$ is easily and intuitively defined for any elementary process θ as a sum of the form

$$\int_0^T \theta_t \, dS_t = \sum_{\{k : t_k \le T\}} \theta(t_{k-1})[S(t_k) - S(t_{k-1})] .$$

This defines a process $\int \theta \, dS = \{\int_0^t \theta_\tau \, dS_\tau : t \in \mathcal{T}\}$. Let $L^2[S]_\mathscr{E} = \{\theta \in L^2[S]: \theta$ is elementary$\}$. The following lemma can be proved as an exercise.

Lemma. *If $S \in \mathcal{M}^2$ and $\theta \in L^2[S]_\mathscr{E}$, then $\int \theta \, dS \in \mathcal{M}^2$.*

We next define a norm $\|\cdot\|_{\mathcal{M}^2}$ on $\mathcal{M}^2$ (that gives $\mathcal{M}^2$ the structure of a Hilbert space) by

$$\|S\|_{\mathcal{M}^2} = \sqrt{\mathrm{var}(S_T)} , \quad S \in \mathcal{M}^2 .$$

Likewise, for each $S \in \mathcal{M}^2$, a semi-norm $\|\cdot\|_S$ is defined on $L^2[S]$ by

$$\|\theta\|_S = \left[E\left(\int_0^T \theta_t^2 \, d[S]_t \right) \right]^{1/2} , \quad \theta \in L^2[S] .$$

It turns out that

$$\|\theta\|_S = \left\| \int \theta \, dS \right\|_{\mathcal{M}^2} , \quad \theta \in L^2[S]_\mathscr{E}, \ S \in \mathcal{M}^2 ,$$

which defines an isometry that can be extended to $L^2[S]$, allowing us to define the stochastic integral $\int \theta \, dS$ for any $\theta \in L^2[S]$ as follows.

Theorem (definition of stochastic integration). *For any $S \in \mathcal{M}^2$ and any $\theta \in L^2[S]$, there exists a sequence $\{\theta_n\}$ in $L^2[S]_\mathscr{E}$ such that $\|\theta_n - \theta\|_S \to 0$. There is a unique martingale in $\mathcal{M}^2$, denoted $\int \theta \, dS$, such that for any such sequence $\{\theta_n\}$, $\{\int \theta_n \, dS\}$ converges in $\|\cdot\|_{\mathcal{M}^2}$ to $\int \theta \, dS$.*

This definition of the stochastic integral is extended from $\mathcal{T} = [0, T]$ to $\mathcal{T} = [0, \infty)$ by defining $\int \theta \, dS$ on $[0, \infty)$ via its restriction to $[0, T]$ for each T. While the above definition is perfectly satisfactory for many applications, it must be extended to handle more general processes θ and S. In order to do this, we next define the most general class of such S for which a stochastic integral can be defined with reasonable properties; this is the class of *semi-martingales*. We first need a few additional definitions.

A $\mathcal{T}$-valued random variable τ is a stopping time if the event $\{\omega \in \Omega : \tau(\omega) \leqslant t\}$ is in $\mathcal{F}_t$ for all t in $\mathcal{T}$. For an adapted process X and stopping time τ, the *stopped process* X^τ is defined by $X^\tau(t) = X(t)$, $t \leqslant \tau$, and $X^\tau(t) = X(\tau)$, $t > \tau$. An adapted process X is a *local martingale* if there is a sequence $\{\tau_n\}$ of stopping times with $\tau_n \leqslant \tau_{n+1}$ and $\lim_{n \to \infty} \tau_n = \infty$ almost surely such that X^{τ_n} is a martingale for all n. (In particular, a martingale is a local martingale.)

A stochastic process X is a *finite variation* process if $X = A - B$, where A and B are adapted processes that are increasing (almost surely). A stochastic process S is a *semimartingale* if $S = M + A$ for some local martingale M and finite variation process A.

A stochastic integral $\int \theta \, dS$ is defined, for predictable θ and semimartingale S, if there is a decomposition $S = M + A$ of S as the sum of a local martingale M and a finite variation process A such that $\int \theta \, dM$ and $\int \theta \, dA$ are well defined. In that case, $\int \theta \, dS = \int \theta \, dM + \int \theta \, dA$ does not depend on the decomposition. While we do not define $\int \theta \, dM$ and $\int \theta \, dA$ explicitly, the former is a natural extension of the integral $\int \theta \, dM$ for $M \in \mathcal{M}^2$ and $\theta \in L^2[M]$, while $\int_0^T \theta_t \, dA_t$ is the classical Stieltjes integral for each $\omega \in \Omega$.

For any semimartingale S, we let $L^1[S]$ denote the set of predictable θ such that the stochastic integral $\int \theta \, dS$ is a well-defined and integrable process. Given an $\mathbb{R}^N$-valued process $S = (S^1, \ldots, S^N)$ for which S^n is a semimartingale, $n \in \{1, \ldots, N\}$, we can define $\theta = (\theta^1, \ldots, \theta^N) \in L^1[S]$ and $\int \theta \, dS$ by a natural extension of the one-dimensional case. For a precise definition, see Jacod (1979). One should think of $\int \theta \, dS$ as $\sum_{n=1}^N \int \theta^n \, dS^n$, although, in pathological cases, this is only true in a limiting sense. (If $\mathbb{F}$ is the standard filtration of a Brownian motion, $\int \theta \, dS = \sum_{n=1}^N \int \theta^n \, dS^n$.) Protter (1989) is an excellent introduction to stochastic integration.

References

Admati, A. (1985) 'A noisy rational expectations equilibrium for multi-asset securities markets', *Econometrica*, 53: 629–657.

Admati, A. and P. Pfleiderer (1988) 'A theory of intraday patterns: volume and price variability', *Review of Financial Studies*, 1: 3–40.

Aghion, P. and P. Bolton (1986) 'An "incomplete contracts" approach to bankruptcy and the optimal financial structure of the firm', Unpublished, Department of Economics, Harvard University.

Allen, F. and D. Gale (1987) 'Optimal security design', Unpublished, Department of Economics, University of Pittsburgh.

Araujo, A. and P. Monteiro (1987) 'Generic non-existence of equilibria in finance models', Unpublished, Instituto de Matemática Pura e Aplicada, Rio de Janeiro.

Araujo A. and P. Monteiro (1989) 'Existence without uniform conditions', *Journal of Economic Theory*, 48: 416–427.

Arnold, L. (1974) *Stochastic differential equations: theory and applications*. New York: Wiley.

Arrow, K. (1951) 'An extension of the basic theorems of classical welfare economics', In: J. Neyman, ed., *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*. Berkeley: University of California Press, pp. 507–532.

Arrow, K. (1953) 'Le rôle des valeurs boursières pour la repartition la meillure des risques', Colloq. Internat. Centre National de la Recherche Scientifique (Paris, 1952), *Econometrie*, 40: 41–47; discussion, pp. 47–48, CNRS Paris, 1953. English translation: (1964) *Review of Economic Studies* 31: 91–96.

Arrow, K. and G. Debreu (1954) 'Existence of an equilibrium for a competitive economy', *Econometrica*, 22: 265–290.

Bachelier, L. (1900) 'Théorie de la speculation', *Annales Scientifiques de l'Ecole Normale Supérieure*, troisième serie, 17: 21–88. Translation: P. Cootner, ed., *The random character of stock market prices*: Cambridge, MA: MIT Press.

Back, K. (1986) 'Securities market equilibrium without bankruptcy: contingent claim valuation and the Martingale property', Research Paper 683, Center For Mathematical Studies in Economics and Management Science, Northwestern University.

Back, K. (1988) 'Correlation of consumption, state prices, and risk premia for general asset price processes', Unpublished, Department of Finance, Indiana University.

Back, K. and S. Pliska (1986) 'Discrete versus continuous trading in securities markets with net worth constraints', Working Paper 700, Center for Mathematical Studies in Economics and Management Science, Northwestern University.

Back, K. and S. Pliska (1989) 'On the fundamental theorem of asset pricing with an infinite state space', Unpublished, Department of Finance, Indiana University.

Barone-Adesi, G. and R. Elliott (1989) 'The valuation of American options', Unpublished, University of Alberta.

Bergman, Y. (1985) 'Time preference and capital asset pricing models', *Journal of Financial Economics*, 14: 145–159.

Bester, H. (1982) 'On shareholder unanimity in the mean-variance model', *Economics Letters*, 10: 363–367.

Bhattacharya, S. and G. Constantinides, eds. (1989) *Financial markets with incomplete information, frontiers of modern financial theory*, Vol. 2. Totowa, NJ: Rowman and Littlefield.

Black, F. (1972) 'Capital market equilibrium with restricted borrowing', *Journal of Business*, 45: 444–454.

Black, F. (1976) 'The pricing of commodity contracts', *Journal of Financial Economics*, 3: 167–179.

Black, F. and M. Scholes (1973) 'The pricing of options and corporate liabilities', *Journal of Political Economy*, 81: 637–654.

Boyle, P. (1977) 'Options: a Monte Carlo approach', *Journal of Financial Economics*, 4: 323–338.

Bray, M. (1981) 'Futures trading, rational expectations and the efficient markets hypothesis', *Econometrica*, 49: 575–596.

Brealey, R. and S. Myers (1984) *Principles of corporate finance* (2nd edn.). New York: McGraw-Hill.

Breeden, D. (1979) 'An intertemporal asset pricing model with stochastic consumption and investment opportunities', *Journal of Financial Economics*, 7: 265–296.

Breeden, D. (1986) 'Consumption, production, inflation and interest rates', *Journal of Financial Economics*, 16: 3–39.

Breeden, D., M. Gibbons and R. Litzenberger (1986) 'Empirical tests of the consumption-oriented CAPM', Research Paper 879, Graduate School of Business, Stanford University.

Breeden, D. and R. Litzenberger (1978) 'Prices of state-contingent claims implicit in option prices', *Journal of Business*, 51: 621–651.

Brennan, M. and E. Schwartz (1977) 'The valuation of American put options', *Journal of Finance*, 32: 449–462.

Brennan, M. and E. Schwartz (1979) 'A continuous time approach to the pricing of bonds', *Journal of Banking and Finance*, 3: 133–155.

Brock, W. (1979) 'An integration of stochastic growth theory and the theory of finance, Part I: The growth model', in: J. Green and J. Scheinkman, eds., *General equilibrium, growth, and trade*. New York: Academic Press, pp. 165–192.

Brock, W. (1982) 'Asset prices in a production economy', in: J. McCall, ed., *The economics of information and uncertainty*. Chicago, IL: University of Chicago Press, pp. 1–46.

Brown, D. and M. Gibbons (1985) 'A simple econometric approach for utility-based asset pricing models', *Journal of Finance*, 40: 359–381.

Brown, D. and S. Ross (1988) "Spanning, valuation, and options', Unpublished, Cowles Foundation, Yale University.

Burke, J. (1986) Existence of equilibrium for incomplete market economies with production and stock trading', Unpublished, Department of Economics, Texas A & M University.

Carino, D. (1987) 'Multiperiod security markets with diversely informed agents', PhD Thesis, Engineering-Economic Systems Department, Stanford University.

Carr, P. (1987) 'Treasury bond futures and the quality option', Unpublished, Graduate School of Management, University of California, Los Angeles.

Carr, P. and R. Myneni (1989) 'A representation of the American equity put option', Unpublished, Cornell University.

Carr, P., R. Jarrow and R. Myneni (1989) 'Alternative characterizations of American put options', Unpublished, Cornell University.

Cass, D. and J. Stiglitz (1970) 'The structure of investor preferences and asset returns, and separability in portfolio allocation: a contribution to the pure theory of mutual funds', *Journal of Economic Theory*, 2: 122–160.

Chamberlain, G. (1983a) 'A characterization of the distributions that imply mean-variance utility functions', *Journal of Economic Theory*, 29: 185–201.

Chamberlain, G. (1983b) 'Funds, factors, and diversification in arbitrage pricing models', *Econometrica*, 51: 1305–1323.

Chamberlain, G. (1988) 'Asset pricing in multiperiod security markets', *Econometrica*, 56: 1283–1300.

Chamberlain, G. and M. Rothschild (1983) 'Arbitrage, factor structure, and mean-variance analysis on large asset markets', *Econometrica*, 51: 1281–1304.

Cheng, S. (1987) 'Pricing models for multiple-currency option bonds', Unpublished, Graduate School of Business, Stanford University.

Chew, S. (1989) 'Axiomatic utility theories with the betweenness property', *Annals of Operations Research*, 19: 273–298.

Christensen, P. (1987) 'An intuitive approach to the Harrison and Kreps concept of arbitrage pricing for continuous time diffusions', Unpublished, Department of Management, Odense University, Denmark.

Chung, K.L. and R. Williams (1983) *An introduction to stochastic integration*. Boston: Birkhäuser.

Clark, J. (1970) 'The representation of functionals of Brownian Motion by stochastic integrals', *Annals of Mathematical Statistics*, 41: 1282–1295.

Clark, S. (1990) 'The valuation problem in arbitrage price theory'. Unpublished. Department of Statistics, University of Kentucky, *Journal of Mathematical Economics*, in press.

Connor, G. (1984) 'A unified beta pricing theory', *Journal of Economic Theory*, 34: 13–31.

Constantinides, G. (1982) 'Intertemporal asset pricing with heterogenous consumers and without demand aggregation', *Journal of Business*, 55: 253–267.

Constantinides, G. (1986) 'Capital market equilibrium with transactions costs', *Journal of Political Economy*, 94: 842–862.

Constantinides, G. (1988) 'Habit formation: a resolution of the equity premium puzzle', Unpublished, Graduate School of Business, University of Chicago.

Constantinides, G. (1989) 'Theory of valuation: overview and recent developments', in: S. Bhattacharya and G. Constantinides, *Theory of valuation: Frontiers of modern financial theory*, Vol. 1. Totowa, NJ: Rowman and Littlefield.

Cox, J., J. Ingersoll and S. Ross (1981a) 'A re-examination of traditional hypotheses about the term structure of interest rates', *Journal of Finance*, 36: 769–799.

Cox, J., J. Ingersoll and S. Ross (1981b) 'The relation between forward prices and futures prices', *Journal of Financial Economics*, 9: 321–346.

Cox, J., J. Ingersoll and S. Ross (1985a) 'A theory of the term structure of interest rates', *Econometrica*, 53: 385–408.

Cox, J., J. Ingersoll and S. Ross (1985b) 'An intertemporal general equilibrium model of asset prices', *Econometrica*, 53: 363–384.

Cox, J. and S. Ross (1976) 'The valuation of options for alternative stochastic processes', *Journal of Financial Economics*, 3: 145–166.

Cox, J., S. Ross and M. Rubinstein (1979) 'Option pricing: a simplified approach', *Journal of Financial Economics*, 7: 229–263.

Cox, J. and M. Rubinstein (1985) *Options markets.* Englewood Cliffs, NJ: Prentice Hall.

Dana, R.-A. (1990) 'Existence, uniqueness and determinacy of Arrow–Debreu equilibria in finance models'. Unpublished. Laboratoire de Mathématiques Fondamentales, University of Paris VI.

Dana, R.-A. and M. Pontier (1989) 'On the existence of a stochastic equilibrium, a remark', Unpublished, Laboratoire de Mathématiques Fondamentales, Université de Paris VI.

Davis, M.H. and P. Varaiya (1974) 'The multiplicity of an increasing family of σ-fields', *The Annals of Probability*, 2: 958–963.

Debreu, G. (1951) 'The coefficient of resource utilization', *Econometrica*, 19: 273–292.

Debreu, G. (1953) 'Une economie de l'incertain', Unpublished, Electricité de France.

Debreu, G. (1959) *Theory of value.* Cowles Foundation Monograph 17, New Haven, CT: Yale University Press.

Dekel, E. (1986) 'An axiomatic characterization of preferences under uncertainty', *Journal of Economic Theory*, 40: 304–318.

Dellacherie, C. and P. Meyer (1978) *Probabilities and potential.* New York: North-Holland.

Dellacherie, C. and P. Meyer (1982) *Probabilities and potential B: Theory of Martingales.* New York: North-Holland.

DeMarzo, P. (1987) 'Majority rule and corporate control: the rule of the dominant shareholder', Unpublished, Department of Economics, Stanford University.

DeMarzo, P. (1988) 'An extension of the Modigliani–Miller Theorem to stochastic economies with incomplete markets', *Journal of Economic Theory*, 45: 353–369.

DeMarzo, P. and D. Duffie (1991) 'Corporate financial hedging with proprietary information', Unpublished, Stanford University; *Journal of Economic Theory*, 53: in press.

DeMarzo, P. and K. Van Nuys (1988) 'Lecture notes in theoretical finance', Unpublished, Institute for Mathematical Studies in the Social Sciences Lectures in Economics, based on Lectures of William A. Brock at the Graduate School of Business, Stanford University.

Detemple, J., P. Gottardi and H. Polemarchakis (1989) 'The relevance of financial policy', Working Paper, Graduate School of Business, Columbia University.

Detemple J. and F. Zapatero (1990) 'Asset prices in an exchange economy with habit formation', Unpublished. Columbia University.

Diamond, P. (1967) 'The role of a stock market in a general equilibrium model with technological uncertainty', *American Economic Review*, 57: 759–776.

Dothan, M. and D. Feldman (1986) 'Equilibrium interest rates and multiperiod bonds in a partially observable economy', Unpublished, Vanderbilt University.

Drèze, J. (1974) 'Investment under private ownership: optimality, equilibrium and stability', in: J. Drèze, ed., *Allocation under uncertainty: equilibrium and optimality.* New York: Wiley, pp. 129–165.

Duffie, D. (1985) 'Predictable representation of Martingale spaces and changes of probability measure', in: J. Azema and M. Yor, eds., *Seminaire de Probabilités.* XIX, Lecture Notes in Mathematics Number 1123. New York: Springer-Verlag, pp. 278–284.

Duffie, D. (1986) 'Stochastic equilibria: existence, spanning number, and the "no expected financial gain from trade" hypothesis', *Econometrica*, 54: 1161–1184.

Duffie, D. (1988) *Security markets: stochastic models.* Boston: Academic Press.

Duffie D. (1990) 'The nature of incomplete security markets', Unpublished. Graduate School of

Business, Stanford University, forthcoming in: J.-J. Laffont and C. Sims, eds., *Advances in Econometrics, Sixth World Congress.* Cambridge University Press.

Duffie, D. and L. Epstein (1989) 'Stochastic differential utility and asset pricing', Unpublished, Graduate School of Business, Stanford University.

Duffie, D., J. Geanakoplos, A. Mas-Colell and A. McLennan (1988) 'Stationary Markov equilibria', Unpublished, Graduate School of Business, Stanford University.

Duffie, D. and C. Huang (1985) 'Implementing Arrow-Debreu equilibria by continuous trading of few long-lived securities', *Econometrica*, 53: 1337–1356.

Duffie, D. and W. Shafer (1986) 'Equilibrium and the role of the firm in incomplete markets', Research Paper 915, Graduate School of Business, Stanford University.

Duffie, D. and C. Skiadas (1990) 'Continuous-time security pricing: a utility gradient approach', Unpublished. Graduate School of Business, Stanford University.

Duffie, D. and R. Stanton (1988) 'Pricing continuously resettled contingent claims', Unpublished, Graduate School of Business, Stanford University.

Duffie, D. and W. Zame (1959) 'The consumption-based capital asset pricing model', *Econometrica*, 57: 1279–1297.

Durrett, R. (1984) *Brownian motion and martingales in analysis.* Belmont, CA: Wadsworth.

Dybvig, P. and C.-F. Huang (1988) 'Nonnegative wealth, absence of arbitrage, and feasible consumption plans', *Review of Financial Studies*, 1: 377–402.

Dynkin, E. (1965) *Markov Processes.* Berlin, Springer-Verlag.

Ekern, S. and R. Wilson (1974) 'On the theory of the firm in an economy with incomplete markets', *Bell Journal of Economics and Management Science*, 5: 171–180.

Epstein, L. and S. Zin (1989a) 'Substitution, risk aversion and the temporal behavior of consumption and asset returns: a theoretical framework', *Econometrica*, 57: 937–969.

Epstein, L. and S. Zin (1989b) 'Substitution, risk aversion and the temporal behavior of consumption and asset returns: an empirical analysis', Unpublished, Department of Economics, University of Toronto.

Ethier, S. and T. Kurtz (1986) *Markov processes: characterization and convergence.* New York: Wiley.

Fama, E. (1976) *Foundations of finance.* New York: Basic Books.

Fama, E. (1978) 'The effects of a firm's investment and financing decisions on the welfare of its security holders', *American Economic Review*, 68: 272–284.

Fama, E. and M. Miller (1972) *The theory of finance.* New York: Holt, Rinehart and Winston.

Freedman, D. (1983) *Brownian motion and diffusion.* New York: Springer-Verlag.

Freidlin, M. (1985) *Functional Integration and Partial Differential Equations.* Princeton University Press.

Friedman, A. (1975) *Stochastic differential equations and applications*, vol. I. New York: Academic Press.

Friesen, P. (1974) 'A reinterpretation of the equilibrium theory of Arrow and Debreu in terms of financial markets', Technical Report 126, Institute for Mathematical Studies in the Social Sciences, Stanford University.

Gale, D. (1960) *The theory of linear economic models.* New York: McGraw-Hill.

Geanakoplos, J., M. Magill, M. Quinzii and J. Drèze (1987) 'Generic inefficiency of stock market equilibrium when markets are incomplete', Working Paper M8735, Department of Economics, University of Southern California.

Gennotte, G. (1984) 'Continuous-time production economies under incomplete information I: a separation theorem', Working Paper 1612-84, Sloan School, Massachusetts Institute of Technology.

Geske, R. (1979) 'The valuation of compound options', *Journal of Financial Economics*, 7: 63–81.

Geske, R. and H. Johnson (1984) 'The American put option valued analytically', *Journal of Finance*, 39: 1511–1524.

Geske, R. and K. Shastri (1985) 'Valuation by approximation: a comparison of alternative option valuation techniques', *Journal of Financial and Quantitative Analysis*, 20: 45–71.

Gevers L. (1974) 'Competitive equilibrium of the stock exchange and Pareto efficiency', in: J. Drèze, ed., *Allocation under uncertainty: equilibrium and optimality.* New York: Wiley, pp. 167–191.

Gihman, I. and A. Skorohod (1972) *Stochastic differential equations*. New York: Springer-Verlag.

Girsanov, I. (1960) 'On transforming a certain class of stochastic processes by absolutely continuous substitution of measures', *Theory of Probability and Its Applications*, 5: 285–301.

Glosten, L. and P. Milgrom (1985) 'Bid, ask, and transaction prices in a specialist market with heterogeneously informed traders', *Journal of Financial Economics*, 14: 71–100.

Goldman, B., H. Sosin and M. Gatto (1979) 'Path dependent options: "buy at the low, sell at the high"', *Journal of Finance*, 34: 1111–1127.

Grauer, F. and R. Litzenberger (1979) 'The pricing of commodity futures contracts, nominal bonds, and other risky assets under commodity price uncertainty', *Journal of Finance*, 44: 69–84.

Grossman, S. (1976) 'On the efficiency of competitive stock markets where traders have diverse information', *Journal of Finance*, 31: 573–585.

Grossman, S. and O. Hart (1979) 'A theory of competitive equilibrium in stock market economies', *Econometrica*, 47: 293–330.

Grossman, S. and O. Hart (1987) 'One share/one vote and the market for corporate control', Unpublished, Department of Economics, Princeton University.

Grossman, S. and G. Laroque (1989) 'Asset pricing and optimal portfolio choice in the presence of illiquid durable consumption goods', *Econometrica*, 58: 25–52.

Grossman, S. and J. Stiglitz (1976) 'On stockholder unanimity in making production and financial decisions', Technical Report No. 224, Institute for Mathematical Studies in the Social Sciences, Stanford University.

Haggerty, K. (1985) Equilibrium bid-ask spreads in a dealership market, PhD Thesis, Graduate School of Business, Stanford University.

Hansen, L. (1989) 'Semi-parametric efficiency bounds for linear time series models', Unpublished, University of Chicago.

Hansen, L. and S. Richard (1987) 'The role of conditioning information in deducing testable restrictions implied by dynamic asset pricing models', *Econometrica*, 55: 587–614.

Hansen, L. and K. Singleton (1982) 'Generalized instrumental variables estimation of nonlinear rational expectations models', *Econometrica*, 50: 1269–1286.

Hansen, L. and K. Singleton (1983) 'Stochastic consumption, risk aversion, and the temporal behavior of asset returns', *Journal of Political Economy*, 91: 249–265.

Harris, M. and A. Raviv (1987) 'Corporate governance: voting rights and majority rules', Working Paper 33, Kellogg School, Northwestern University.

Harrison, J.M. and D. Kreps (1979) 'Martingales and arbitrage in multiperiod securities markets', *Journal of Economic Theory*, 20: 381–408.

Harrison, J.M. and S. Pliska (1981) 'Martingales and stochastic integrals in the theory of continuous trading', *Stochastic Processes and Their Applications*, 11: 215–260.

Harrison, J.M. and S. Pliska (1983) 'A stochastic calculus model of continuous trading: complete markets', *Stochastic Processes and their Applications*, 15: 313–316.

Hart, O. (1974) 'On the existence of equilibria in a securities market model', *Journal of Economic Theory* 9: 293–311.

Hart, O. (1975) 'On the optimality of equilibrium when the market structure is incomplete', *Journal of Economic Theory*, 11: 418–443.

Hart, O. (1977) 'Takeover bids and stock market equilibrium', *Journal of Economic Theory*, 9: 53–83.

Hart, O. (1979) 'On shareholder unanimity in large stock market economies', *Econometrica*, 47: 1057–1082.

Hart, O. (1987) 'Capital structure as a control mechanism in corporations', Research Paper 441, Department of Economics, Massachusetts Institute of Technology.

Heath, D., R. Jarrow and A. Morton (1987) 'Bond pricing and the term structure of interest rates: a new methodology', Unpublished, Cornell University.

Hellwig, M. (1981) 'Bankruptcy, limited liability, and the Modigliani–Miller theorem', *American Economic Review*, 71: 155–170.

Hemler, M. (1987) 'The quality delivery option in treasury bond futures contracts', Unpublished, Graduate School of Business, University of Chicago.

Hicks, J. (1939) *Value and capital*. Oxford: Clarendon Press.

Hindy, A. (1989) 'An equilibrium model of futures markets dynamics', Unpublished, Sloan School of Management, M.I.T.

Hindy, A. and C.F. Huang (1989) 'On intertemporal preferences with a continuous-time dimension II: The case of uncertainty', Unpublished, Sloan School of Management, M.I.T.

Ho, T. and S. Lee (1986) 'Term structure movements and pricing interest rate contingent claims', *Journal of Finance*, 41: 1011–1029.

Huang, C.F. (1985a) 'Information structures and viable price systems', *Journal of Mathematical Economics*, 14: 215–240.

Huang, C.F. (1985b) 'Information structure and equilibrium asset prices', *Journal of Economic Theory*, 31: 33–71.

Huang, C.F. (1987) 'An intertemporal general equilibrium asset pricing model: the case of diffusion information', *Econometrica*, 55: 117–142.

Huang, C. and R. Litzenberger (1988) *Foundations for financial economics*. Amsterdam: North-Holland.

Huberman, G. (1982) 'A simple approach to arbitrage pricing theory', *Journal of Economic Theory*, 28: 183–191.

Hull, J. (1989) *Options, futures, and other derivative securities*. Englewood Cliffs, NJ: Prentice Hall.

Ikeda, N. and S. Watanabe (1981) *Stochastic differential equations and diffusion processes*. Amsterdam: North-Holland.

Ingersoll, J. (1977) 'A contingent-claims valuation of convertible securities', *Journal of Financial Economics*, 4: 289–322.

Ingersoll, J. (1987) *Theory of financial decision making*. Totowa, NJ: Rowman and Littlefield.

Jacod, J. (1977) 'A general theorem of representation for martingales', *Proceedings of the Symposia in Pure Mathematics*, 31: 37–53.

Jacod, J. (1979) *Calcul stochastique et problèmes de martingales*, Lecture Notes in Mathematics No. 714. Berlin: Springer-Verlag.

Jamshidian, F. (1989) 'Free-boundary formulas for American options', Unpublished, Merrill Lynch Capital Markets.

Jarrow, R. (1988) *Finance Theory*. Englewood Cliffs, NJ: Prentice Hall.

Jarrow, R. and R. Green (1985) 'Spanning and completeness in markets with contingent claims', Unpublished, Cornell University.

Jensen, M. and J. Long (1972) 'Corporate investment under uncertainty and Pareto optimality in the capital markets', *Bell Journal of Economics and Management Science*, 3: 151–174.

Jensen, M. and W. Meckling (1976) 'Theory of the firm: managerial behavior, agency costs and ownership structure', *Journal of Financial Economics*, 3: 305–360.

Johnsen T. and J. Donaldsen (1985) 'The structure of intertemporal preferences under uncertainty and time consistent plans', *Econometrica*, 53: 1451–1463.

Johnson, H. (1987) 'Options on the maximum or the minimum of several assets', *Journal of Financial and Quantitative Analysis*, 22: 277–283.

Jones, R. and R. Jacobs (1986) 'History dependent financial claims: Monte Carlo valuation', Unpublished, Simon Fraser University.

Kandori, M. (1988) 'Equivalent equilibria', *International Economic Review*, 29: 401–417.

Karatzas, I., P. Lakner, J. Lehoczky and S. Shreve (1988) 'Dynamic equilibria in a multi-agent economy: construction and uniqueness', Research Paper 88-9, Carnegie-Mellon University.

Karatzas, I. and S. Shreve (1988) *Brownian motion and stochastic calculus*. New York: Springer-Verlag.

Kim, I.J. (1989) 'The analytic valuation of American puts', Unpublished, New York University.

Kopp, P. (1984) *Martingales and stochastic integrals*. Cambridge: Cambridge University Press.

Kraus, A and R. Litzenberger (1975) 'Market equilibrium in a multiperiod state preference model with logarithmic utility', *Journal of Finance*, 30: 1213–1227.

Kreps, D. (1979) 'Three essays on capital markets', Technical Report 298, Institute for Mathematical Studies in The Social Sciences, Stanford University.

Kreps, D. (1981) 'Arbitrage and equilibrium in economies with infinitely many commodities', *Journal of Mathematical Economics*, 8: 15–35.

Kreps, D. (1982) 'Multiperiod securities and the efficient allocation of risk: a comment on the Black–Scholes option pricing model', in: J. McCall, ed., *The economics of uncertainty and information*. Chicago, IL: University of Chicago Press, pp. 203–232.

Kreps, D. and E. Porteus (1978) 'Temporal resolution of uncertainty and dynamic choice theory', *Econometrica*, 46: 185–200.

Krylov, N. (1980) *Controlled diffusion processes*. New York: Springer-Verlag.

Kunita, H. and S. Watanabe (1967) 'On square-integrable martingales', *Nagoya Mathematics Journal*, 30: 209–245.

Leland, H. (1973) 'Capital asset markets, production, and optimality: a synthesis', Technical Report 115, Institute for Mathematical Studies in the Social Sciences, Stanford University.

Leland, H. (1974) 'Production theory and the stock market', *Bell Journal of Economics and Management Science*, 5: 125–144.

Leland, H. (1978) 'Information, management choice, and stockholder unanimity', *Review of Economic Studies*, 45: 527–534.

Leland, H. (1985) 'Option pricing and replication with transactions costs', *Journal of Finance*, 40: 1283–1301.

Lenglart, E. (1977) 'Transformation des martingales locales par changement absolument continu de probabilites', *Zeitschrift für Wahrscheinlichkeitstheorie*, 39: 65–70.

LeRoy, S. (1973) 'Risk aversion and the martingale property of stock prices', *International Economic Review*, 14: 436–446.

Lintner, J. (1965) 'The valuation of risky assets and the selection of risky investment in stock portfolios and capital budgets', *Review of Economics and Statistics*, 47: 13–37.

Lipster, R. and A. Shiryayev (1977) *Statistics of random processes I: General theory*. New York: Springer-Verlag.

Lucas, R. (1978) 'Asset prices in an exchange economy', *Econometrica*, 46: 1429–1445.

Lucas, R. and N. Stokey (1987) 'Money and interest in a cash-in-advance economy', *Econometrica*, 55: 491–514.

Luenberger, D. (1969) *Optimization by vector space methods*. New York: Wiley.

Machina, M. (1982) ' "Expected utility" analysis without the independence axiom', *Econometrica*, 50: 277–323.

Magill, M. and W. Shafer (1991) 'Incomplete markets', in: W. Hildenbrand and H. Sonnenschein, eds., *Handbook of mathematical economics*, Vol. IV. Amsterdam: Elsevier (this Handbook).

Makowski, L. (1983) 'Competitive stock markets', *Review of Economic Studies*, 50: 305–330.

Margrabe, W. (1978) 'The value of an option to exchange one asset for another', *Journal of Finance*, 33: 177–186.

Marimon, R. (1987) 'Kreps' "Three Essays on Capital Markets" Almost Ten Years Later', Unpublished, Department of Economics, University of Minnesota; *Revista Española de Economia*, in press.

Mas-Colell, A. (1985) *The theory of general economic equilibrium – a differentiable approach*. Cambridge: Cambridge University Press.

Mas-Colell, A. (1986) 'The price equilibrium existence problem in topological vector lattices', *Econometrica*, 54: 1039–1054.

Mas-Colell, A. and W. Zame (1991) 'Infinite-dimensional equilibria', in: W. Hildenbrand and H. Sonnenschein, eds., *Handbook of mathematical economics*, Vol. IV. Amsterdam: Elsevier (this Handbook).

McKenzie, L. (1954) 'On equilibrium in Graham's model of world trade and other competitive systems', *Econometrica*, 22: 147–161.

Memin, J. (1980) 'Espaces de semimartingales et changement de probabilités', *Zeitschrift für Wahrscheinlichkeitstheorie*, 52: 9–39.

Merton, R. (1973a) 'An intertemporal capital asset pricing model', *Econometrica*, 41: 867–888.

Merton, R. (1973b) 'The theory of rational option pricing', *Bell Journal of Economics and Management Science*, 4: 141–183.

Merton, R. (1974) 'On the pricing of corporate debt: the risk structure of interest rates', *Journal of Finance*, 29: 449–470.

Merton, R. (1976) 'Option pricing when the underlying stock returns are discontinuous', *Journal of Financial Economics*, 5: 125–144.

Merton, R. (1977) 'On the pricing of contingent claims and the Modigliani–Miller theorem', *Journal of Financial Economics*, 5: 241–250.

Merton, R. (1987) 'Capital market theory and the pricing of financial securities', Working Paper 1818-86, Sloan School of Management, Massachusetts Institute of Technology.

Merton, R. and M. Subrahmanyam (1974) 'The optimality of a competitive stock market', *Bell Journal of Economics and Management Science*, 5: 145–170.

Miller, M. (1977) 'Debt and taxes', *Journal of Finance*, 32: 261–275.

Miltersen, K. and L. Nielsen (1989) 'The pricing of contingent claims written on bonds by simulation of bond price processes', Unpublished, Odense University.

Modigliani, F. and M. Miller (1958) 'The cost of capital, corporation finance, and the theory of investment', *American Economic Review*, 48: 261–297.

Mossin, J. (1973) *Theory of financial markets*. Englewood Cliffs, NJ: Prentice Hall.

Müller, S. (1985) *Arbitrage pricing of contingent claims*, Lecture Notes in Economics and Mathematical Systems, Vol. 254. New York: Springer-Verlag.

Myers, S. (1984) 'The capital structure puzzle', *Journal of Finance*, 39: 575–592.

Myers, S. and N. Majluf (1984) 'Corporate financing and investment decisions when firms have information that investors do not have', *Journal of Financial Economics*, 13: 187–221.

Nachman, D. (1988) 'Spanning and completeness with options', *Review of Financial Studies*, 1: 311–328.

Nielson, L.T. (1985) 'Preference structure and equilibrium in the classical capital asset pricing model', Unpublished, Department of Finance, University of Texas at Austin.

Nielson, L.T. (1986a) 'Mutual fund separation, factor structure and robustness', Unpublished, Department of Finance, University of Texas at Austin.

Nielson, L.T. (1986b) 'Asset market equilibrium with short selling', Unpublished, Department of Economics, University of Texas, Austin.

Nielson, L.T. (1987) 'Equilibrium in CAPM without a riskless asset', Working Paper 87/88-2-3, Department of Finance, University of Texas at Austin.

Nielson, L.T. (1989a) 'Existence of equilibrium in the CAPM', Unpublished, Institute of Economics, University of Copenhagen.

Nielson, L.T. (1989b) 'Existence of equilibrium in CAPM: further results', Unpublished, Institute of Economics, University of Copenhagen.

Ohashi, K. (1987) 'On security prices at the terminal date in a continuous time trading with dividends model', Unpublished, Hitosubashi University.

Parkinson, M. (1977) 'Option pricing: the American put', *Journal of Business*, 50: 21–36.

Pliska, S. (1982) 'A stochastic calculus model of continuous trading: return processes and investment plans', Research Paper 517, Center for Mathematical Studies in Economics and Management Science, Northwestern University.

Prescott, E. and R. Mehra (1980) 'Recursive competitive equilibrium: the case of homogeneous households', *Econometrica*, 48: 1365–1379.

Prisman, E. (1985) 'Valuation of risky assets in arbitrage free economies with frictions', Unpublished, University of Arizona.

Protter, P. (1989) *Stochastic integration and stochastic differential equations*. Berlin: Springer-Verlag.

Radner, R. (1967) 'Equilibre des marchés a terme et au comptant en cas d'incertitude', *Cahiers d'Econométrie*, 4: 35–52.

Radner, R. (1972) 'Existence of equilibrium of plans, prices and price expectations in a sequence of markets', *Econometrica*, 40: 289–303.

Radner, R. (1974) 'A note on unanimity of stockholders' preferences among alternative production plans: a reformulation of the Ekern–Wilson model', *The Bell Journal of Economics and Management Science*, 5: 181–186.

Radner, R. (1988) 'Intertemporal general equilibrium', Unpublished, AT&T Bell Laboratories, Murray Hill, NJ.

Richard, S. (1978) 'An arbitrage model of the term structure of interest rates', *Journal of Financial Economics*, 6: 33–57.

Ross, S.A. (1976a) 'The arbitrage theory of capital asset pricing', *Journal of Economic Theory*, 13: 341–360.

Ross, S.A. (1976b) 'Options and efficiency', *Quarterly Journal of Economics*, 90: 75–89.
Ross, S.A. (1976c) 'Risk, return, and arbitrage', in: I. Friend and J. Bickler, eds. *Risk and return in finance*. Ballinger, Cambridge, MA.
Ross, S.A. (1977) 'The determination of financial structure: the incentive-signalling approach', *Bell Journal of Economics* (Spring) 23–40.
Ross, S.A. (1978a) 'A simple approach to the valuation of risky streams', *Journal of Business*, 51: 453–475.
Ross, S.A. (1978b) 'Mutual fund separation in financial theory – the separating distributions', *Journal of Economic Theory*, 17: 254–286.
Ross, S.A. (1987) 'Arbitrage and martingales with taxation', *Journal of Political Economy*, 95: 371–393.
Rothschild, M. (1986) 'Asset pricing theories', in: W. Heller and D. Starrett, eds., *Uncertainty, information and communication – essays in honor of Kenneth J. Arrow*, Vol. III. Cambridge: Cambridge University Press, pp. 97–128.
Rubinstein, M. (1974) 'An aggregation theorem for securities markets', *Journal of Financial Economics*, 1: 225–244.
Rubinstein, M. (1976) 'The valuation of uncertain income streams and the pricing of options', *Bell Journal of Economics*, 7: 407–425.
Satterthwaite, M. (1981) 'On the scope of the stockholder unanimity theorems', *International Economic Review*, 22: 119–133.
Savage, L.J. (1954) *The foundations of statistics*. New York: Wiley.
Schaefer, H. (1974) *Banach lattices and positive operators*. Berlin: Springer-Verlag.
Scheinkman, J. (1989) 'Market incompleteness and the equilibrium valuation of assets', in: S. Bhattacharya and G. Constantinides, eds., *Theory of valuation, frontiers of modern financial theory*, Vol. 1. Totowa, NJ: Rowman and Littlefield, pp. 45–52.
Scheinkman, J. and L. Weiss (1986) 'Borrowing constraints and aggregate economic activity', *Econometrica*, 54: 23–46.
Sharpe, W. (1964) 'Capital asset prices: a theory of market equilibrium under conditions of risk', *Journal of Finance*, 19: 425–442.
Singleton, K. (1987) 'Speculation and estimation of intertemporal asset pricing models', Research Paper 964, Graduate School of Business, Stanford University.
Stiglitz, J. (1972) 'On the optimality of the stock market allocation of investment', *Quarterly Journal of Economics*, 86: 25–60.
Stiglitz, J. (1974) 'On the irrelevance of corporate financial policy', *American Economic Review*, 64: 851–866.
Stiglitz, J. (1982) 'The inefficiency of the stock market equilibrium', *Review of Economic Studies*, 49: 241–261.
Stiglitz, J. (1989) 'Mutual funds, capital structure, and economic efficiency', in: S. Bhattacharya and G. Constantinides, eds., *Theory of valuation, frontiers of modern financial theory*, Vol. 1. Totowa, NJ: Rowman and Littlefield, pp. 342–356.
Stokey, N. and R. Lucas (with E. Prescott) (1989) *Recursive methods in economic dynamics*. Cambridge, MA: Harvard University Press.
Sundaresan, S. (1985) 'Intertemporally dependent preferences and the volatility of consumption and wealth', *Review of Financial Studies*, 2: 73–90.
Svensson, L. (1988) 'Portfolio choice and asset pricing with nontraded assets', Unpublished, Institute for International Economic Studies, University of Stockholm.
Townsend, R. (1984) 'Asset prices in a monetary economy', Unpublished, Department of Economics, University of Chicago.
Varian, H. (1987) 'The arbitrage principle in financial economics', *Economic Perspectives*, 1: 55–72.
Werner, J. (1987) 'Arbitrage and the existence of competitive equilibrium', *Econometrica*, 55: 1403–1418.