Stochastic Equilibria with Incomplete Financial Markets

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ABSTRACT

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We demonstrate the existence of equilibria with incomplete financial markets for stochastic economies whose information structure is given by an event tree, restricting attention to purely financial securities, those paying in units of account (e.g. "dollars"). Financial markets may be incomplete: some consumption streams may be impossible to obtain by any trading strategy. Securities may be individually precluded from trade at arbitrary states and dates. Sufficient conditions for the existence of stochastic equilibria are: continuous, convex, strictly monotonic preferences and strictly positive aggregate endowments. These conditions are weakened. A corollary states that any regime of security prices precluding arbitrage can be embedded in an equilibrium.

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Stochastic Equilibria with Incomplete Financial Markets

This paper shows that the existence of stochastic equilibria with incomplete markets is not at all problematic, *provided securities are purely financial.* This is a restrictive setting, but the results may clarify the role of real securities, as opposed to financial securities, in Hart's [11] famous examples of non–existence of equilibria with incomplete markets. The results add weight to similar conclusions by Cass [4] and Werner [17] for economies with a single round of security trading. A by–product of the methods used in this paper is a proof that any regime of stochastic security price processes that precludes arbitrage can be embedded in an equilibrium under typical competitive conditions.

The central result is a demonstration of Walrasian equilibria with incomplete financial markets for stochastic economies whose information structure is given by a finite event tree. In Radner's model [16] of a sequence of markets with the same information structure, a security was defined to be a claim to a particular bundle of commodities at each node, or "state–date pair", in the event tree. As shown by Hart [11], Radner's model of equilibrium with these "real" securities is fraught with problems related to discontinuities in the map from spot goods price processes to the subspace of consumption streams attainable by trading strategies. Equilibria can fail to exist in Radner's model under standard conditions of continuity and convexity on agent characteristics. Subsequent work by Kreps [13], McManus [15], and Magill and Shafer [14] has shown that given a sufficient number of securities, markets are generically complete in some sense, and a demonstration of equilibria can proceed. The focus of this work, however, is incomplete markets. Provided intertemporal markets are purely financial, equilibria exist under the standard assumptions of general equilibrium theory. By "purely financial", we mean that a security is defined as a contingent stream of financial credits or debits, in units of account. This is precisely the definition of a security chosen over thirty years ago by Arrow [1]. This would preclude forward contracts for commodities, for example, but not bonds, other loan contracts, insurance contracts, financial forwards and futures, foreign exchange contracts, and the like. Although there is still a link, with financial securities, between spot price processes and the subspace of consumption streams attainable by trade, this link is sufficiently "smooth" to rule out the problems encountered with real securities.

Under general conditions for the existence of an equilibrium, we have the following corollary. Security prices preclude arbitrage, by definition, if any trading strategy generating positive non–zero dividends requires a strictly positive initial investment. *For any given security price processes precluding arbitrage there exists a stochastic equilibrium embedding those price processes.* The result comes about from the ability of spot price processes for commodities to adjust themselves to clear both spot and security markets.

This paper extends the recent contributions of Cass [4] and Werner [17] on the existence of equilibria in economies with incomplete financial markets. In these two models, as in the original complete financial markets model of Arrow [1], security trading occurs once. As Werner states, his results "can easily be generalized to the case of more dates with securities which cannot be re–traded, for example, securities which 'live' for only a single period" [17,p.2]. In this paper existence with incomplete financial markets is extended to economies whose financial securities trade at any subsets of dates and states in a finite event tree, for example, "long–lived securities" [13]. The sufficient conditions accomodate securities with arbitrary dividend schemes, including no securities at all. Werner finds a fixed point simultaneously in spot consumption markets and security markets. Here the analysis is considerably simplified by proceeding in three steps. First any arbitrage–free price processes are chosen for securities. Then, an equilibrium spot price process is demonstrated by applying a fixed point theorem in consumption markets, forcing agents to stay on the subspace of consumption processes "reachable" by some security trading strategy. Finally, we allocate market clearing security trading strategies to agents via a simple device. This follows the ideas used in Duffie and Huang [8] and Duffie [6], where financially complete markets play a major role.

A. Event Trees All agents in our economy are assumed to learn information according to an event tree Ξ, such as that depicted in Figure 1. The number of nodes in Ξ is assumed finite. The unique node in Ξ with no predecessor is the *root node*, denoted ξ_0. For any node $\xi \in \Xi$ other than the root node, ξ_- designates the unique predecessor of ξ. The number of immediate successor nodes of any ξ in Ξ is denoted $\#\xi$. A node $\xi \in \Xi$ is *terminal* if $\#\xi = 0$, and otherwise *non–terminal*. The immediate successor nodes of any non–terminal node $\xi \in \Xi$ are labeled $\xi_{+1}, \ldots, \xi_{+K}$, where $K = \#\xi$. The sub–tree with root $\xi \in \Xi$ is denoted $\Xi(\xi)$; in particular, $\Xi = \Xi(\xi_0)$. This notation is illustrated in Figure 1, which will perhaps serve in the absence of formal definitions from directed graph theory, for which Avondo–Bodino [2], for example, may be consulted. We are conveniently using the symbol Ξ to denote the event tree itself, or its structure as a directed graph, as well as the set of nodes in the tree. Static complete markets equilibria based on the same model of uncertainty are demonstrated by Debreu [6; Chapter 7].

B. The Economy With each node $\xi \in \Xi$ is associated a *spot consumption space* Z_ξ, a vector space representing the nature of "spot market choices" at node ξ. For instance, if there is an integer number $\ell \geq 1$ of different scalar–valued commodities available for consumption at node ξ, a suitable vector space might be $Z_\xi = R^\ell$. For simplicity we take each spot consumption space Z_ξ to be Euclidean and adopt the usual product norm and ordering for the product space $L = \Pi_{\xi \in \Xi} Z_\xi$, which represents consumption choices throughout the event tree. In other words L is equivalent to R^n for some integer n. With ℓ goods at each node and $\overline{K}$ nodes in all, for example, $L = R^{\overline{K}\ell}$. As usual, $x \gg 0$ means $x \in int(L_+)$, where L_+ denotes the positive cone of L. A choice vector x in L, also referred to as a *consumption process*, represents spot consumption $x(\xi) \in Z_\xi$ at a generic node $\xi \in \Xi$. A *spot price process* is a vector ψ in L such that $\psi(\xi) \neq 0$ for all $\xi \in \Xi$. That is, at node $\xi \in \Xi$ the spot consumption vector $z \in Z_\xi$ has a spot market value $\psi(\xi) \cdot z$. We assume spot markets are complete. It seems natural to require, as we have, that a spot price process ψ have a non–zero value $\psi(\xi)$ at each node ξ. This extends the definition of a price functional in a static economy to be a non–zero linear functional.

A *security* is a claim to a dividend process $d \in \mathcal{D}$, where $\mathcal{D}$ denotes the space of real valued functions on Ξ. We will speak simply of "security *d*" rather than "the security claiming dividends from the process *d*." The holder at node ξ of a unit

Figure 1. Event Tree Notation

of security d is entitled at node ξ to any spot market consumption bundle $z \in Z_\xi$ of market value $\psi(\xi) \cdot z = d(\xi)$. Some models, for example Radner's [16], treat a "security" as a claim to a consumption process $x \in L$. Under that convention, the holder at node ξ of a security claiming x is entitled to collect the consumption bundle $x(\xi) \in Z_\xi$. With complete spot markets, this is equivalent to a claim to the dividend process $\psi \,\square\, x \in \mathcal{D}$ defined by

$$[\psi \,\square\, x](\xi) = \psi(\xi) \cdot x(\xi), \quad \xi \in \Xi.$$

We take the number of securities to be some integer $N \geq 1$. Each security d_n, for $1 \leq n \leq N$, is associated with a *price process* $S_n \in \mathcal{D}$. In other words, $S_n(\xi)$ is the market value of d_n at node $\xi \in \Xi$. It will be convenient to treat $S_n(\xi)$ as the market value of d_n after the dividend $d_n(\xi)$ has been "declared," but before it has been paid. The vector $d = (d_1, \ldots, d_N)$ of dividend processes is associated with a vector $S = (S_1, \ldots, S_N)$ of price processes. Spot and security markets are commonly denominated, so a portfolio $\alpha \in R^N$ of securities can be "sold" at node ξ for spot consumption of total market value $\alpha \cdot S(\xi)$.

The triple (d, S, ψ) is a complete characterization of trading opportunities, or a *market system.*

C. Security Trading Strategies A *trading strategy* $\theta = (\theta_1, \ldots, \theta_N)$ is an R^N–valued function on Ξ. The scalar $\theta_n(\xi)$ represents the number of units of security d_n held at node ξ when strategy θ is followed. We adopt the convention that $\theta(\xi)$ represents the portfolio held *after* trading at node ξ has occurred, but *before* dividends $d(\xi) \in R^N$ are paid. That is, a spot market value $\theta(\xi) \cdot d(\xi)$ accrues to strategy θ at node ξ. Because trades occur at pre–dividend security values $S(\xi)$, our conventions are consistent. The pre–trade holdings of any strategy θ at the root node is $\theta(\xi_{0-}) = 0$ by a notational convention, since no agent will be endowed initially with securities. The *dividend process* $d^\theta \in \mathcal{D}$ *generated by a trading strategy* θ is defined by

$$d^\theta(\xi) = [\theta(\xi_-) - \theta(\xi)] \cdot S(\xi) + \theta(\xi) \cdot d(\xi), \quad \xi \in \Xi.$$

The first term is the market value of the portfolio of securities sold at node ξ; the second is the number of units of account paid in dividends at node ξ. A trading strategy θ *finances* a consumption process x if $d^\theta \geq \psi \square x$, that is, if $d^\theta - \psi \square x \in \mathcal{D}_+$.

Since it is natural for some securities to be unavaiable for trade at some nodes, we define restrictions on admissiblee trading strategies by limiting agents to some vector subspace Θ of the set of trading strategies. Precluding trade of certain securities at certain nodes is indeed a linear restriction.

D. Definition of Equilibrium We are ready to formulate equilibrium in a simple setting. A *stochastic economy* is a collection $\mathcal{E} = ((\omega_i, \succeq_i), \Xi, d, \Theta)$, $i \in$

$\mathcal{I} = \{1, \ldots, I\}$ consisting of an event tree Ξ, securities $d = (d_1, \ldots, d_N)$ on Ξ, an admissible vector space Θ of trading strategies, and agents 1 through I defined by an endowment $\omega_i \in L_+$ and a preference relation $\succeq_i$ on L_+. Taking a market system (d, S, ψ) as given, a *budget feasible plan* for agent i is a pair (θ_i, x_i) made up of a consumption process $x_i \in L_+$ and a trading strategy θ_i in Θ financing the net trade $x_i - \omega_i$. A budget feasible plan (θ_i, x_i) is *optimal* for i if there is no other budget feasible plan for i (θ_i', x_i') for agent i such that $x_i' \succ_i x_i$. The collection $((S, \psi); (\theta_i, x_i)), \quad i \in \mathcal{I}$, is an *equilibrium* for $\mathcal{E}$ provided (θ_i, x_i) is an optimal plan for each agent i given market system (d, S, ψ), and *markets clear* for all $\xi \in \Xi$, or $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} \omega_i$ and $\sum_{i=1}^{I} \theta_i = 0$.

E. Arbitrage–Free Security Markets A security price process S is *arbitrage–free* for the dividend process d provided there is no trading strategy $\theta \in \Theta$ such that $d^\theta \geq 0$ and $d^\theta \neq 0$. Given a vector of securities d it is a simple matter to construct an arbitrage–free price process S for d. For example, let

$$S(\xi) = \Lambda(d)_\xi = \sum_{\eta \in \Xi(\xi)} d(\eta), \qquad \xi \in \Xi.$$

The "current" value of a security at node ξ is assigned by the operator Λ to be the sum of its dividends in the entire sub–tree $\Xi(\xi)$ subtended by ξ. More generally, let Ω denote the set of terminal nodes of Ξ and let π denote a strictly positive real–valued function on Ω. For convenience we call π a *normalization*. For intuition, Ω may be thought of as a probability space with the power set σ–algebra 2^Ω , and $\pi(\xi)$ treated as the probability of reaching terminal node ξ. We extend π to the entire tree Ξ as follows. For each predecessor node ξ of a terminal node let $\pi(\xi) = \sum_{k=1}^{K} \pi(\xi_{+k})$, where $K = \#\xi$. In this fashion we continue extending π by backward recursion. Now assign security price processes S to the securities d by the operator Λ^π defined by

$$S(\xi) = \Lambda^\pi(d)_\xi = \frac{1}{\pi(\xi)} \sum_{\eta \in \Xi(\xi)} \pi(\eta) d(\eta), \qquad \xi \in \Xi.$$

Treating Ω as a probability space and constructing the filtration of sub–σ–algebras of 2^Ω determined by Ξ in the obvious way, Λ^π is effectively a conditional expectation operator. It easily shown by a backward induction argument that $\Lambda^\pi(d)$ is arbitrage–free for d under any normalization π. See, for example, Duffie [6] for a transformation of the spot price process, in response to a change in the normalization π, that leaves budget feasible sets unchanged.

F. Wealth Accessibility Just as in static general equilibrium theory, in order to prove the existence of equilibria it helps to ensure that each agent has a feasible choice strictly smaller in market value than that agent's wealth. In a stochastic setting, this should be the case at each node in the event tree Ξ. For example, suppose the endowment ω_i is positive and non–zero at each node, or $\omega_i(\xi) > 0$ for all $\xi \in \Xi$. Provided the spot price process ψ is strictly positive, $\psi \gg 0$, this endowment assumption ensures that the zero choice is strictly smaller in market value than the endowment at each node. This assumption is unnecessarily strong, however, for an agent can obtain wealth at a given node ξ either by selling endowments at ξ or by selling endowments at a predecessor of ξ and transferring the wealth "downstream" to ξ via security markets. The key is that the agent has access to wealth at all nodes in the tree through the combination of endowmnent sales and trading strategies. The formal condition is that there exists a trading strategy generating dividends that, when added to the market values of endowments at strictly positive spot prices, leaves strictly positive wealth at each node. This might be generically the case if true for a particular $\psi \gg 0$ and arbitrage–free S. A formal definition is offered on the basis of a given arbitrage–free security price process S.

Wealth Accesibility. *Agent i has wealth accessibility given security price process S if, for all spot price processes $\psi \gg 0$, there exists a trading strategy $\theta \in \Theta$ such that $d^\theta + \psi \,\square\, \omega_i \gg 0$.*

A sufficient condition for wealth accessibility is that $\omega_i(\xi) > 0$ for all $\xi \in \Xi$. A necessary condition is non–zero endowments at the root node ξ_0. Provided there exists a trading strategy paying strictly positive dividends at each successor of ξ_0, this would also be a sufficient condition. For example, one could sell some endowments at the root node, place the proceeds in a riskless bond (or any security that, at each node, pays postive dividends and has strictly positive price), and sell off part of the bond at each subsequent node. In the two period model of Werner, wealth accessibility is similar to conditions $(*)$ and $(**)$ [17,p.6], but weaker since Werner assumes that the dividends vector d is positive and non–zero at each terminal node.

G. The Main Theorem We are ready to state and prove the main result of the paper. There are no restrictions on the nature of securities; they may have arbitrary patterns of positive, negative, or zero dividends and be restricted individually from trade at any subsets of nodes in the event tree. A preference relation (complete transitive binary order) $\succeq$ on L_+ is *strictly monotonic* if $x + y \succ x$ whenever x and y are elements of L_+ and $y \neq 0$. The relation $\succeq$ is *convex* if $\{x \in L_+ : x \succeq y\}$ is convex for all $y \in L_+$, and *continuous* if $\{x \in L_+ : x \succeq y\}$ and $\{x \in L_+ : y \succeq x\}$ are closed for all $y \in L_+$. These are standard definitions.

THEOREM. *An economy $\mathcal{E} = ((\omega_i, \succeq_i), \Xi, d, \Theta)$ has an equilibrium under the conditions:*

(a) for all i, $\succeq_i$ is convex, continuous, and strictly monotonic,

(b) $\sum_{i \in \mathcal{I}} \omega_i \gg 0$, and

(c) there is an arbitrage–free security price process S for d such that every agent i has wealth accessibility given S.

COROLLARY I. *There exists an equilibrium for $\mathcal{E}$ under the conditions (a), (b), and*

(c') $\omega_i(\xi) \neq 0$ for all $\xi \in \Xi$, for all i in $\mathcal{I}$.

COROLLARY II. *Let $\mathcal{E}$ be an economy satisfying conditions (a), (b), and (c'), and let $\hat{S}$ be any security price process that is arbitrage–free for d. Then $\mathcal{E}$ has an equilibrium of the form $((\hat{S}, \psi); (x_i, \theta_i))$.*

Corollary II claims, under general equilibrium conditions on agents, that there exists an equilibrium embedding any given arbitrage–free security price process.

Our proof of the main theorem will proceed in the usual fashion, establishing regularity conditions for budget and demand correspondences and applying a fixed point theorem. The equilibrium demonstrated will be of the form $(S, \psi, (\theta_i, x_i))$, $i \in \mathcal{I}$, where S is the arbitrage–free security price process referred to in the wealth accessibility assumption (c), and $\psi \in$ *int*(Δ), where Δ denotes the *unit simplex* for L. To repeat, the wealth accessibility assumption (c) is a weaker assumption than (c'): positive non–zero endowments at each node, since any arbitrage–free security price process $\hat{S}$ allows wealth accessibility for every agent in this case. This establishes Corollaries I and II, once the main theorem is proved.

H. Proof of the Main Theorem Let S denote the arbitrage–free security price process assumed by condition (c). For any spot price process ψ and any agent i let $\beta_i(\psi)$ denote the budget feasible set $\{x \in L_+ : \exists \theta \in \Theta; d^\theta + \psi \,\square\, (\omega_i - x) \geq 0\}$. Since $\square$ is a (trivially continuous) linear function and L_+, $\mathcal{D}_+$, and Θ are closed and convex, β_i defines a closed convex–valued correspondence on L into the subsets of L_+.

Lemma 1. *The budget correspondences $\{\beta_i, i \in \mathcal{I}\}$ are closed and convex–valued.*

Lemma 2. *For each $i \in \mathcal{I}$, the budget correspondence β_i is compact and lower hemicontinuous at any $\psi \gg 0$.*

proof: For the compactness assertion, let $D = \{d^\theta \in \mathcal{D} : \theta \in \Theta\}$ and let Π be the linear functional on D defined by $\Pi(\delta) = -\delta(\xi_0)$, $\delta \in D$. We may think of $\Pi(\delta)$ as the initial investment required to finance the dividend stream δ. Since S is arbitrage–free for d, Π is a trivially strictly positive linear functional D and thus has a strictly positive linear extension also denoted Π to $\mathcal{D}$. For $\psi \gg 0$, it follows that $\{x \in L_+ : \Pi(\psi \,\square\, (x - \omega_i)) \leq \psi(\xi_0) \cdot \omega_i(\xi_0)\}$ is compact, and thus $\beta_i(\psi)$ is compact. The lower hemicontinuity follows from standard arguments, such as in Hildenbrand [12,p.99]. That is, let $\check{\beta}_i$ be the correspondence defined by $\check{\beta}_i(\psi) = \{x \in L_+ : \exists \theta \in \Theta; d^\theta + \psi \,\square\, (\omega_i - x) \gg 0\}$. By the wealth accessibility assumption (c), $\check{\beta}_i(\psi)$ is not empty for $\psi \gg 0$. Let $x \in \check{\beta}_i(\psi)$ and choose a sequence $\{x_n\}$ in L_+ converging to x. If $\{\psi_n\}$ is a sequence in L converging to ψ there is an integer $\overline{n}$ so large that $x_n \in \check{\beta}_i(\psi_n)$ for all $n \geq \overline{n}$, implying $\check{\beta}_i$ is lower hemicontinuous at ψ. Since the closure of a lower hemicontinuous correspondence is lower hemicontinuous, the result is proved. ■

The *demand correspondence* φ_i for agent i is defined by

$$\varphi_i(\psi) = \{x \in \beta_i(\psi) : \{x' \in \beta_i(\psi) : x' \succ_i x\} = \emptyset\}, \quad \psi \in L_+.$$

The *compactified budget correspondence* $\hat{\beta}_i$ is defined by $\hat{\beta}_i(\psi) = \{x \in \beta_i(\psi) : x \leq 2\sum_{i\in I}\omega_i\}$, and the *compactified demand correspondence* by $\tilde{\varphi}_i(\psi) = \{x \in \hat{\beta}_i(\psi) : \{x' \in \hat{\beta}_i(\psi) :\ x' \succ_i x\} = \emptyset\}$. Finally, in order to "smooth" the behavior of preferences on the boundary of the price simplex, we turn to the *quasi–demand correspondence* $\hat{\varphi}_i$ defined by

$$\hat{\varphi}_i(\psi) = \beta_i(\psi), \qquad \psi \,\square\, \omega_i = 0,$$

$$\hat{\varphi}_i(\psi) = \tilde{\varphi}_i(\psi), \qquad \psi \,\square\, \omega_i \neq 0.$$

We note that wealth accessibility assumption (c) implies that $\hat{\varphi}_i(\psi) = \tilde{\varphi}_i(\psi)$ for all $\psi \gg 0$. Furthermore, if $x_i \in \tilde{\varphi}_i(\psi)$ and $x_i \leq \sum_{i\in\mathcal{I}}\omega_i$, then $x_i \in \varphi(\psi)$, meaning there exists θ_i in Θ such that (θ_i, x_i) is an optimal plan for i given (d, S, ψ).

LEMMA 3. *The quasi–demand correspondences $\{\hat{\varphi}_i,\ i \in \mathcal{I}\}$ are closed on L_+.*

PROOF: If $\psi \,\square\, \omega_i = 0$ then $\hat{\varphi}_i(\psi) = \beta_i(\psi)$, and β_i is closed by Lemma 1. If $\psi \,\square\, \omega_i \neq 0$ let $\{\psi_n\}$ be a sequence in L_+ converging to ψ and $x_n \in \hat{\varphi}_i(\psi_n)$ with $x_n \to x$. For an integer $\overline{n}$ sufficiently large, $\hat{\varphi}_i(\psi_n) = \tilde{\varphi}_i(\psi_n)$ and $x_n \in \tilde{\varphi}_i(\psi_n)$ for all $n \geq \overline{n}$. Let $y \in \hat{\beta}_i(\psi)$ and $\alpha \in (0, 1)$. For an integer $\hat{n} \geq \overline{n}$ sufficiently large, $\alpha y \in \hat{\beta}_i(\psi_n)$ implying $x_n \succeq_i \alpha y$ for all $n \geq \hat{n}$. Thus, by continuity of $\succeq_i$, $x \succeq_i \alpha y$. Allowing α to approach 1, we have $x \succeq_i y$, again by continuity of $\succeq_i$, implying $x \in \tilde{\varphi}(\psi) = \hat{\varphi}_i(\psi)$ and $\hat{\varphi}_i$ is closed. ■

LEMMA 4. *For all $i \in \mathcal{I}$, $\hat{\varphi}_i(\psi)$ is non–empty, convex and compact–valued, and upper hemicontinuous at any $\psi \gg 0$.*

PROOF: Given the lower hemicontinuity of β_i for $\psi \gg 0$ established in Lemma 2, the assertion follows by a standard argument, used for example by Hildenbrand [12,p.100]. ■

Let $\hat{\Phi}$ denote the *quasi–excess demand correspondence* defined by

$$\hat{\Phi}(\psi) = \sum_{i\in\mathcal{I}} \hat{\varphi}_i(\psi) - \{\sum_{i\in\mathcal{I}} \omega_i\}, \quad \psi \in L_+.$$

The total correspondence $\hat{\Phi}$ inherits from the individual correspondences $\{\hat{\varphi}_i,\ i \in \mathcal{I}\}$ the property of being non–empty, convex, and compact valued and upper hemicontinuous at any $\psi \gg 0$. For each $\psi \in L_+$ let Ψ_ψ denote the positive linear functional on L given by $\Psi_\psi(x) = \Pi(\psi \,\square\, x) + \psi(\xi_0) \cdot x(\xi_0)$. At any $\psi \gg 0$ strict monotonicity implies *Walras' Law*: for any $x \in \hat{\Phi}(\psi)$, $\Psi_\psi(x) = 0$; that is, there is a trading strategy financing the excess demand x with no excess dividends. Let $\{\Delta^n\}$ denote an increasing sequence of compact convex subsets of *int* (Δ) such that $\bigcup_n \Delta^n =$ *int* (Δ).

LEMMA 5. *For any positive integer n there exists $\psi_n \in \Delta^n$ and $x_n \in \hat{\Phi}(\psi_n)$ such that $\Psi_\psi(x_n) \leq 0$ for all $\psi \in \Delta^n$.*

PROOF: Let C^n be a compact convex set containing $\hat{\Phi}(\psi)$ for all $\psi \in \Delta^n$. Let μ_n denote the correspondence from C^n to Δ^n defined by

$$\mu_n(x) = \{\psi \in \Delta^n : \ \Psi_\psi(x) = \max\{\Psi_\gamma(x) : \ \gamma \in \Delta^n\}\}\,.$$

Let Φ^n denote the restriction of $\hat{\Phi}$ to Δ^n. Now $\mu_n \times \Phi^n$ is an upper hemicontinuous correspondence from a compact convex non–empty subset $C^n \times \Delta^n$ (of a locally convex space) to itself, and, for all $(x, \psi) \in C^n \times \Delta^n$, $\mu_n \times \Phi^n(x, \psi)$ is not empty and convex. By a well known fixed point theorem [3,p.270], $\mu_n \times \Phi^n$ has a fixed point, say (x_n, ψ_n). Since Walras' Law implies $\Psi_{\psi_n}(x_n) = 0$, the result follows from the definition of μ_n. ■

For each integer n the pair $(x_n, \psi_n) \in L \times \Delta$ satisfying the properties of the previous lemma corresponds to some $(x_1^n, \ldots, x_I^n, \psi_n) \in L_+^I \times \Delta$ satisfying $\sum_{i \in \mathcal{I}} x_i^n - \omega_i = x_n$ and $x_i^n \in \hat{\varphi}_i(\psi_n)$. The sequence $\{x_1^n, \ldots, x_I^n, \psi_n\}$ is bounded and thus has a convergent subsequence with a limit $(x_1^*, \ldots, x_I^*, \psi)$ in $L_+^I \times \Delta$. Since $\psi^* \neq 0$ and $\sum_{i \in \mathcal{I}} \omega_i \gg 0$, we know that for some agent j, $\psi \,\square\, \omega_j \neq 0$. By Lemma 3, $x_j^* \in \tilde{\varphi}(\psi^*)$. Let $x^* = \sum_{i \in \mathcal{I}} x_i^* - \omega_i$. By Lemma 5, for all $\psi \gg 0$, $\Psi_\psi(x^*) \leq 0$. Since Walras' Law for each n implies $\Psi_{\psi^*}(x^*) = 0$ in the limit, we then have $x^* = 0$, zero excess demand, implying $x_j^* \leq \sum_{i \in \mathcal{I}} \omega_i$, which implies $\psi^* \gg 0$ by strict monotonicity for agent j. Then, by wealth accessibility condition (c), $x_i^* \in \varphi_i(\psi^*)$ for each i in $\mathcal{I}$. Thus spot markets clear and optimality is obtained. For security market clearing, let θ_i be a trading strategy in Θ that finances $x_i - \omega_i$ for each $i \in \{1, \ldots, I-1\}$, and let $\theta^I = -\sum_{i=1}^{I-1} \theta_i$. Then θ_I finances $x_I - \omega_I = -\sum_{i=1}^{I-1} x_i - \omega_i$ and $\sum_{i \in \mathcal{I}} \theta_i = 0$. This completes the proof.

I. Concluding Remarks The result is easily generalized to incomplete preference relations and general closed convex bounded from below consumption sets. For the given setting, we have *constrained Pareto Optimality*, in the sense that there are no feasible transfers in the subspace $M(\psi, S, d) = \{x \in L : \exists\theta \in \Theta; d^{\theta} = \psi \,\square\, x\}$, defining those choices obtainable by some trading strategy, such that the sum of the transfers is zero and the resulting allocation Pareto dominates the given equilibrium alocation. We note, however, that this is a weak sense of constrained optimality, and refer readers to Geanakoplos and Polemarchakis [10] for a version of constrained optimality which equilibrium allocations in differentiable incomplete markets economies generically fail to achieve. We also refer to Geanakoplos and Mas–Colell [9] for a study of the indeterminacy of equilibrium allocations with incomplete financial markets; characterizing the dimension of the equilibrium allocation manifold generically. Cass [5] has reached similar conclusions.

A necessary and sufficient number of securities for complete markets equilibria is the maximum of $\#\xi$ over all non–terminal ξ in the event tree Ξ, that is, the greatest number of branches leaving any node in the tree. This is the notion of Kreps [13], as extended by Duffie and Huang [8].

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List of Symbols

$\square$	special symbol, a square
$\mathcal{D}$	script upper case D
$\mathcal{E}$	script upper case E
Ξ	Greek upper case xi
ℓ	script lower case l (ell)