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STOCHASTIC EQUILIBRIA: EXISTENCE, SPANNING NUMBER, AND THE 'NO EXPECTED FINANCIAL GAIN FROM TRADE' HYPOTHESIS

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Stochastic equilibria under uncertainty with continuous-time security trading and consumption are demonstrated in a general setting. A common question is whether the current price of a security is an unbiased predictor of the future price of the security plus intermediate dividends. This is the hypothesis of "no expected financial gains from trade." The relevance of this hypothesis in multi-good economies is called into question by the following demonstrated fact. For each set of probability assessments there exists a corresponding equilibrium, one with the original agents, original equilibrium allocations, and no expected financial gains from trade under the given probability assessments. The spanning number of the economy is defined as the fewest number of security markets required to sustain a complete markets equilibrium (in a dynamic sense made precise in the paper). The spanning number is linked directly to agent primitives, in particular the manner in which new information resolves uncertainty over time. The spanning number is shown to be invariant under bounded changes in expectations. Several examples are given in which the spanning number is finite even though the number of potential states of the world is infinite.

KEYWORDS: General equilibrium, martingales, finance, stochastic equilibrium.

1. INTRODUCTION

THIS PAPER POSES a problem for an economy whose primitives are a set of agents with preferences for, and endowments of, random streams (stochastic processes) of consumption goods: How does the manner in which agents receive new information determine the nature and number of financial securities permitting dynamically complete markets equilibria?

The receipt of new information is given by a *filtration*, basically a specification of the times at which events are revealed to be true or false. In a stochastic economy each agent, given stochastic price processes, formulates a plan for purchases at each point in time, based on information available at that time. In equilibrium, if one exists, the agents' plans must be preference-maximal subject to budget constraints and clear markets.

The results are as follows. Regularity conditions are given for the existence of a stochastic equilibrium. More importantly, however, the equilibrium shown has the property that a small number of financial securities is sufficient to dynamically span the high dimensional space of all contingent claims. Although markets are not complete at any one time, they are dynamically complete in the sense that any consumption process can be financed by trading the given set of financial securities, adjusting portfolios through time as uncertainty is resolved bit by bit. The discrete time case is effectively subsumed by the continuous time setting. In discrete time, a large finite-dimensional consumption space can be dynamically spanned by a smaller number of financial securities. The discrete time case was

[1] I am grateful for the comments of David Luenberger, David Kreps, Kenneth Arrow, Chi-fu Huang, Ruth Williams, Kai-Lai Chung, and a referee.

studied by Kreps (1982) using a different approach. In continuous time, an infinite-dimensional consumption space can be dynamically spanned by a finite number of securities, provided the information filtration has finite martingale multiplicity, a key concept outlined later in this introduction and defined precisely in the body of the paper. Several examples are given in Section 6.

The "no expected gains from trade" issue is addressed, that is, whether the current price of a security is an unbiased predictor of its value at a future date plus any intermediate dividends. The importance of fixing a relevant unit of account and set of expectations before testing this hypothesis is brought out by the following result. Having demonstrated, with a given set of expectations and numeraire, a stochastic equilibrium in which there are no expected gains from trade, a new set of expectations is specified, arbitrary except that the class of random variables with finite variance must be preserved. A new regime of financial securities and spot prices is constructed such that there exists an equilibrium with the original equilibrium allocations and with no expected gains from trade, *under the new set of expectations.*

The *spanning number* of a stochastic economy is characterized as the smallest number of financial securities having the dynamic spanning property stated. The spanning number is characterized directly in terms of the exogeneous information filtration and agents' probability assessments as *one plus the martingale multiplicity.* This is the case in both discrete and continuous time settings.

This work draws directly and significantly from a number of key sources. First, as mentioned, David Kreps (1982) is mainly responsible for the notion of dynamic spanning, following up on a long line of literature instigated by the Black-Scholes option pricing formula. The methods of Kreps (1982) do not carry over to this general setting however. Michael Harrison and David Kreps (1979) showed the key relationship between security price processes and martingales. A *martingale,* defined more precisely later, is a stochastic process whose expected future value, given current information, is merely its current value, for all future and current times. Harrison and Kreps demonstrated that if a stochastic equilibrium exists, in fact under even weaker conditions, security price processes *must* be martingales under some probability assessments, at least under a convenient choice of numeraire. That work was the central clue in the detective work leading to the present results, although it was not directly applied. The mathematics of continuous-time security trading, first applied by Merton (1971), was formalized by Harrison and Kreps (1979) and extended by Harrison and Pliska (1981), followed by Duffie and Huang (1985), to the point where it is again applied and extended here.

That brings to three the count of key ideas flowing into the present work. The fourth is the approach of showing the existence of stochastic equilibria by a dynamic implementation of an Arrow-Debreu (1954) equilibrium, opening up just enough markets to obtain dynamically complete markets. This idea appears in Kreps (1982), and was carried out in generality in Duffie and Huang (1985). The fifth line of work drawn on is a theory giving conditions for an Arrow-Debreu equilibrium in a sufficiently general setting. This breakthrough was made by Mas-Colell (1986).

How martingale multiplicity theory, a recent mathematical advance, can be exploited for dynamic security market spanning is reported in Duffie and Huang (1985). In Duffie and Huang (1985) an Arrow-Debreu economy with consumption at two points in time, 0 and T, was placed in a Radner setting; agents may learn information and trade securities during $[0, T]$. Conditions were stated under which a given Arrow-Debreu equilibrium can be implemented by continuous trading of a basis set of financial securities. This previous work did not prove that a continuous trading stochastic equilibrium actually exists, treat economies in which consumption occurs over time, characterize the spanning number directly in terms of agent primitives, nor show its variance under changes in expectations.

Given this long list of credits, the reader should have some notion of how the work proceeds. After setting up the economy and defining a stochastic equilibrium, a sizeable chore undertaken in Section 2, an Arrow-Debreu equilibrium is demonstrated for a complete markets static "scratchpad" version of the economy. The Arrow-Debreu equilibrium price functional $\Psi(\cdot)$ is associated with a candidate spot market price process ψ for the stochastic economy, such that the Arrow-Debreu market value $\Psi(c)$ of any consumption process c is its total expected future spot market cost, $E[\int_0^T \psi(t)c(t)\,dt]$, where T is the terminal date of the economy. Given a collection of financial securities, a particular consumption process c is *marketed* if there is a strategy for trading the financial securities through time such that the stream of spot market values required to finance the consumption process is precisely that generated by dividends and net sales of financial securities. When the martingale multiplicity of the given information filtration is N, a set of $N+1$ securities can be constructed such that every consumption process is marketed, or *dynamically complete markets.* How? The *gain process* G for a financial security is defined as the sum of its price process and cumulative dividend process. If the martingale multiplicity is N, one can construct N gain processes $G_1, \ldots, G_N$ with the property that, for any martingale X under consideration, there exist "appropriate" stochastic processes $\theta = \{\theta_1, \ldots, \theta_N\}$ such that

$$X_t = X_0 + \sum_{n=1}^{N} \int_0^t \theta_n(\tau)\, dG_n(\tau),$$

for all $t \geq 0$. This is basically the definition of martingale multiplicity: the smallest number of martingales with this "spanning" property. Several examples of information filtrations whose martingale multiplicities have been characterized are given in Section 3. These include event trees, and diffusion or Poisson "state variable" information. The martingale X in the above definition could represent the current conditional expected total spot market cost of an arbitrary consumption process c, or $X_t = E_t[\int_0^T c(s)\psi(s)\,ds]$. Each θ_n is a stochastic process describing the number of units of the nth security held in a portfoliio whose value replicates X through time. The integral $\int_0^t \theta_n\, dG_n$ represents the "gains" (or losses) from trading the nth security up until time t. A security whose price process is identically one is also introduced to ensure that agents are able to meet their intermediate budget constraints when following the prescribed trading strategy

θ, by borrowing or lending risklessly in numeraire terms. The Arrow-Debreu market value $\Psi(c)$, is identical with the required initial portfolio investment of $X_0 = \Psi(c)$.

Now each agent can be allocated a security trading strategy that precisely finances the stream of spot market payments required to purchase that agent's Arrow-Debreu equilibrium allocation of goods. It can also be shown that no other budget feasible trading strategy yields a strictly preferred stream of consumption. Market clearing is obtained in the spot markets by Arrow-Debreu market clearing, and in the security markets by a simple argument. In short, a dynamically complete markets stochastic equilibrium that implements the previously demonstrated Arrow-Debreu equilibrium allocation is easily constructed. All of this happens in Section 3.

In Section 4, a new set of expectations is fixed, one given by an arbitrary probability measure $\hat{P}$ that preserves the class of finite variance random variables. A stochastic equilibrium is constructed in which the "spanning" $N+1$ securities have gain processes that are martingales under $\hat{P}$. A family of such equilibria exist, all with the same agents and consumption allocations. An obvious by-product is the above mentioned caveat: when empirically or theoretically testing the hypothesis of "no expected financial gains from trade," one must carefully specify in advance the "ambient" unit of account and expectations. In general, with more than a single consumption good, there is no special numeraire that is canonical, in the sense that the "no expected gain from trade" hypothesis has an unambiguous economic relevance under the given numeraire.

In Section 5, we show that not only is the martingale multiplicity plus one a sufficient number of securities for dynamically complete markets, it is also necessary, and is thus characterized as the *spanning number.* This number is shown to be invariant under changes in expectations preserving the class of finite variance random variables. Concluding remarks are found in Section 6.

In the interests of simplicity this paper leaves out several embellishments found in the original working paper (Duffie, 1984a). For example, that paper expands the choice space to include preferences for terminal wealth. By virtue of a different approach (Duffie, 1986) to the existence of Arrow-Debreu equilibria, production is also encompassed in Duffie (1984a). Some comments on the addition of production to the model are included in the concluding section. One particularly important difference between Duffie (1984a) and the present paper lies in the formulation of security markets. In Duffie (1984a), a "security" is taken to be a claim to a specified stream of consumption goods, in the tradition of Radner's original model (1972). Here, instead, we find a considerable simplification is allowed by treating a security as a claim to a stream of financial payments, or "dividends", which are exchangeable on spot markets for consumption goods. This is in the tradition, and in some sense is a direct extension, of the fundamental work of Arrow (1953). In marrying Radner's "equilibrium of plans, prices, and price expectations" in a "sequence of markets" with Arrow's model of "the role of financial securities in the optimal allocation of risk bearing," one might describe the result as an *Arrow-Radner equilibrium* concept.

2. THE ECONOMY

This section describes the primitives for a stochastic economy: a model for uncertainty and revelation of information over time, a consumption space, endowments, and preferences. Some fundamental nonprimitive properties of a stochastic economy are also defined: the available financial securities, their price processes, and the admissible trading strategies.

Finally, the definition of a stochastic equilibrium for this economy is given.

2.1 *Uncertainty and Information Revelation*

This subsection outlines a general model for uncertainty and revelation of information for an economy comprising a finite number of agents indexed by $i \in \mathscr{I} = \{1, \ldots, I\}$.

Let Ω be the set of all possible states of the world which agents commonly believe could occur in a given economy. A "state of the world" $\omega \in \Omega$ is an exogenous train of circumstances occurring from time 0 to time T which determines the realization of every exogenous random variable relevant to the economy. The tribe[2] $\mathscr{F}$ describes the set of events, or subsets of Ω, to which agents are commonly able to assign probabilities, that is, measurable subsets of Ω. Let P_i denote agent i's subjective probability measure on $(\Omega, \mathscr{F})$, for $i \in \mathscr{I}$. We make the assumption that there are bounds on the heterogeneity of probability assessments. Specifically, there exist constants $\underline{K}$ and $\bar{K}$ such that, for any event $B \in \mathscr{F}$ and any agent i,

$$(2.1) \qquad \underline{K} P_i(B) \leq P_j(B) \leq \bar{K} P_i(B), \quad \forall j \in \mathscr{I}.$$

In other words, agents' subjective probability measures are assumed to be *uniformly absolutely continuous* with respect to one another (Halmos, 1950, p.100). This restriction makes the subsequent analysis tractable since the class of finite variance random variables is preserved if and only if the change of probability measure is of this sort. When two probability measures P and Q are uniformly absolutely continuous, we will write $P \approx Q$. Equivalent conditions for the uniform absolute continuity of two measures are given by Halmos (1950) and Allen (1983).

There is no loss in generality for the purposes of this paper, however, in proceeding as though agents have common probability assessments given by any probability measure P which is uniformly absolutely continuous with respect to the agents' probability measures (for instance, take $P = P_1$), and we shall do so. This follows from the fact that all topological properties of the consumption space described in the next section are invariant under changes of probability measure of this sort.

[2] *Tribe* is merely a simple term for *σ-algebra*.

Without loss of generality, $\mathscr{F}$ is assumed to be complete for P. Thus uncertainty for our economy can be described by the complete probability space $(\Omega, \mathscr{F}, P)$. Since all probability measures to appear are equivalent,[3] there is no ambiguity in using the symbols "a.s." for almost surely, or P-almost everywhere.

Uncertainty is resolved over time according to some *filtration* $\boldsymbol{F} = \{\mathscr{F}_t, t \in [0, T]\}$, a right-continuous increasing[4] family of subtribes of $\mathscr{F}$, with $\mathscr{F}_T = \mathscr{F}$, and $\mathscr{F}_0$ almost trivial[5] (meaning Ω is the only event of non-zero probability in $\mathscr{F}_0$). The tribe $\mathscr{F}_t$ may be interpreted as the set of events which could occur at or before time t. The descriptions of $\mathscr{F}_0$ and $\mathscr{F}_T$ imply that agents have no information about the state of the world at time 0, and that all information to be revealed is available by or at time T. The filtration is basically a specification of the order in which uncertain events are revealed to be true or false over time. For example, $\boldsymbol{F}$ might represent the information revealed by an event tree or by observing a collection of "state variable" stochastic processes.

In summary, our model for uncertainty and revelation of information over time is the filtered probability space $(\Omega, \boldsymbol{F}, P)$.

2.2 *The Consumption Space*

There are alternatives to the following setup which achieve roughly the same results. We have simply chosen one which is relatively easy to work with from among those which are reasonably general. The basic model is a choice space for agents who have preferences for consumption over time in the form of multi-dimensional consumption processes.

A stochastic process[6] X is *adapted* to the filtration $\boldsymbol{F}$ if X_t is measurable with respect to the tribe $\mathscr{F}_t$, for all $t \in [0, T]$. One could say that X is adapted to $\boldsymbol{F}$ if the value of X at any time depends only on information revealed by $\boldsymbol{F}$ up to and including that time. A process is *optional* if measurable with respect to the tribe $\mathscr{O}$ generated by right-continuous left-limits (RCLL)[7] processes. In effect, an optional process is one whose values depend on the information generated by observing only right-continuous adapted processes, rather than arbitrary adapted processes. For technical convenience we limit agents to optional consumption processes $c = \{c_t;\ t \in [0, T]\}$ satisfying $E[\int_0^T c_t^2\, dt] < \infty$. This can be relaxed slightly (Chung and Williams, 1983, pp. 63-64), but can hardly be considered restrictive. For example, any process that depends measurably on RCLL "state variable processes", such as diffusion or Poisson processes, is optional. This incorporates the continuous-time pricing models of finance, such as those of Merton (1973),

[3] Two probability measures P and Q on $\mathscr{F}$ are *equivalent* provided $P(B) = 0$ if and only if $Q(B) = 0$, for all $B \in \mathscr{F}$; that is, P and Q assign zero probability to the same events.

[4] The filtration $\boldsymbol{F}$ is *right-continuous* provided $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$ for all t. This basically means that any event known at all times after t is also known at time t. The filtration $\boldsymbol{F}$ is *increasing* provided $\mathscr{F}_t \subset \mathscr{F}_s$ whenever $s \geq t$, meaning roughly that nothing is forgotten.

[5] A subtribe is *almost trivial* if it is the tribe generated by Ω and all zero probability events in $\mathscr{F}$.

[6] For our purposes a *stochastic process* is any function $X : \Omega \times [0, T] \to R$.

[7] Call X a *right-continuous* (RC) process if, for all $(\omega, t) \in \Omega \times [0, T]$, $\lim_{s \downarrow t} X(\omega, s) = X(\omega, t)$. The analogous definitions for *left continuous* and *left limits* (LL) (meaning the left limit exists) are taken. Some authors take "RCLL" to mean RCLL almost surely. See Chung and Williams (1983) for details.

Breedon (1979), and Cox, Ingersoll, and Ross (1985). Of course, any RCLL process is itself optional. An integer number, say M, of different goods are available for consumption at any time $t \in [0, T]$. The overall consumption space is thus the vector space V of M-dimensional square-integrable optional processes $c = (c_1, \ldots, c_M)$. As usual, we identify two processes that are the same almost everywhere. We then choose the norm[8]

$$\|c\| = E\left[\int_0^T \sum_{m=1}^{M} |c_m(t)|\, dt\right], \quad c \in V,$$

in order to define convenient continuity conditions on preferences. Let V_+ denote the positive cone of V, the subset of positive consumption processes. We will write "$c \geqslant 0$" if c is in V_+ and "$c > 0$" if c is in V_+ and $c \neq 0$.

Let Φ denote the vector space of essentially bounded M-dimensional optional processes $\psi = (\psi_1, \ldots, \psi_M)$. We will take Φ as the space of *spot price processes*, leaving the terms of trade for the M goods at time t given by a (random) vector $\psi(t) = (\psi(t)_1, \ldots, \psi(t)_M)$. In other words, with spot price process $\psi \in \Phi$, $\psi_m(\omega, t)$ is the unit price of the mth good in state $\omega \in \Omega$ at time $t \in [0, T]$.

2.3. *Agents*

Each agent $i \in \mathscr{I}$ is characterized by a consumption endowment $\hat{c}_i \in V$, and a complete transitive binary preference order $\succeq_i$ on V_+. As usual, $\succ_i$ denotes the strict preference relation induced by $\succeq_i$. A preference relation $\succeq$ on a subset X of V is uniformly *proper* if there exists some scalar $\varepsilon > 0$ and vector $v \in V$ such that, for all $x \in X$ the relation $x - \alpha v + z \succeq x$, for $z \in V$ and $\alpha \in R_+$, implies that $\|z\| \geqslant \alpha\varepsilon$. In other words, the consumption choice v is so desirable that z can only compensate for some loss of v if z is sufficiently large in norm. This concept, which can be viewed as a smoothness condition on preferences, is due to Mas-Colell (1986). Richard (1985) has shown conditions under which preferences are uniformly proper. The choice v in the definition is said to be *extremely desirable* for $\succeq$, in the terminology of Yannelis and Zame (1984). We record the following assumptions for each agent $i \in \mathscr{I}$:

ASSUMPTION (A1): $c \in V_+$, $k \in V$, *and* $k > 0$ *imply that* $c + k \succ_i c$.

ASSUMPTION (A2): *The graph of* $\succ_i$ *is relatively open.*

ASSUMPTION (A3): $\sum_{j=1}^{I} \hat{c}_j$ *is extremely desirable for* $\succeq_i$.

ASSUMPTION (A4): $\forall c \in V_+, \{z \in V_+ : z \succeq_i c\}$ *is convex.*

ASSUMPTION (A5): $\hat{c}_i > 0$.

The agent assumptions may be interpreted as: (A1) strictly monotonic preferences, (A2) continuous preferences, (A3) extremely desirable aggregate endowments, (A4) convex preferences, and (A5) positive nonzero endowments. These

[8] This norm generates the same topology as the product $L^1(\Omega \times [0, T], \mathcal{O}, P \times \lambda)$-norm topology, where λ denotes Lebesgue measure. The fact that a square-integrable consumption process has finite L^1 norm follows from the Cauchy-Schwarz inequality and the fact that the underlying measure is finite.

assumptions can be weakened somewhat given the recent work of Zame (1985) and Yannelis and Zame (1984). In particular, the completeness and transivity assumptions on preferences can be eliminated with some additional work. If the preference relations are represented by utility functionals of the form $E[\int_0^T u_i(c_t)\,dt]$, sufficient conditions for Assumptions (A1)-(A4) are that u_i be concave, strictly increasing, with a finite right derivative at zero, and that $\sum_i \hat{c}_i$ be bounded away from zero. These conditions, however, are far more restrictive than Assumptions (A1)-(A4).

2.4. *Financial Securities, Gain Processes, and Trading Strategies*

We will formulate a model of financial securities in the tradition of Arrow (1953). A *security* is taken to be a claim to financial dividends that are convertible on spot markets for goods at current spot market prices. For illustration, a security paying "one dollar" in dividends at a particular time and in a particular state of the world entitles its owner to any bundle of consumption goods at that time and state with a total spot market value of one dollar. This is formalized as follows.

A *dividend process* is an adapted RCLL process D whose value $D(t)$ at any time t represents cumulative dividends paid by the underlying security up to and including time t. The price process of the corresponding security is defined by an adapted process S whose value $S(t)$ at any time t represents the market value at that time of a claim to all future dividends paid by the security, as they are paid. Our convention is that security values are *ex dividend*, meaning that $S(t)$ is the market value of the security at time t after dividends at time t have been paid, and that trades occur *ex dividend*. For illustration, if an agent buys one unit of the security at time t and sells it at a later time s, then the agent receives a total amount of dividends $D(s)-D(t)$ in the interim, and realizes a further gain (or loss) from the two transactions of $S(s)-S(t)$. Dividends can be paid in lump sums or rates; our model of D is quite general. For example the jump $\Delta D(t)\equiv D(t)-\lim_{s\uparrow t} D(s)$ in the dividend process D at time t represents a lump sum payment of $\Delta D(t)$ to each share of the security at time t. If, on the other hand, the security pays at rates of time given by a stochastic process δ, then a holder of one share receives $\int_s^t \delta(\tau)\,d\tau$ in dividends between times s and t. More generally, an agent will vary the holdings of a security according to a stochastic process θ, where $\theta(\omega, t)$ represents the number of units held at time $t\in[0, T]$ in state $\omega\in\Omega$. For a technically sound framework, as explained in earlier research on continuous-time trading (Duffie and Huang, 1985; Harrison and Pliska, 1981), a trading process θ must be *predictable*[9] meaning roughly that $\theta(t)$ must be chosen on the basis of information received up to, but not including, time t. This effectively precludes arbitrage opportunities which would otherwise be present when a price process jumps.

The obvious generalization of the above illustration of gains from trade is as follows. The financial gain up to any time $t\geq 0$ realized by holding a security

[9] A predictable process X is one that is measurable with respect to the tribe P on $\Omega\times[0, T]$ generated by adapted left-continuous processes.

with price process S and dividend process D in amounts specified by a trading process θ is the sum of $\int_0^t \theta(s)\, dD(s)$ and $\int_0^t \theta(s)\, dS(s)$, presuming the notation represents some meaningful integral which is well defined. In order to define integration in the general sense of a stochastic integral, we will define a *price process* for any security with dividend process D to be an adapted process S such that $\{G(t)=S(t)-S(0)+D(t);\, t\in[0, T]\}$ defines a semimartingale[10] G. This semimartingale G, termed the *gain process* for (D, S), describes the gain realized by holding one unit of the security. A *trading process* for a security with a nonzero gain process G is then defined to be a predictable process θ such that the stochastic integral $\int \theta\, dG$ is well-defined and such that

$$\text{(A)} \qquad E\left(\int_0^T \theta(t)^2 d[G]_t\right) < \infty,$$

where $[G]$ denotes the quadratic variation process[11] of G. The class of predictable processes θ satisfying condition (A) is commonly denoted $L^2[G]$. Conditions on θ ensuring existence of the stochastic integral $\int \theta\, dG$ may be found in Dellacherie and Meyer (1982) or Jacod (1979). For gain processes to be found in this paper, however, $\theta \in L^2[G]$ is itself a sufficient condition. For a security whose gain process G is identically zero—we have a numeraire in mind—any adapted process θ is a valid trading process since the gain from trade $\int \theta\, dG$ is trivially defined to be zero. The stochastic integral $\int \theta\, dG$ represents the total gain from trade, circumventing separate definitions of "dividend gain" $\int \theta\, dD$ and "capital gain" $\int \theta\, dS$.

Security markets will generally be characterized by a collection of $N \geqslant 1$ securities with a vector dividend process $D=(D_1, \ldots, D_N)$ and a corresponding vector price process $S=(S_1, \ldots, S_N)$. The associated vector gain process is denoted $G=(G_1, \ldots, G_N)$. A vector process $\theta=(\theta_1, \ldots, \theta_N)$ is a *trading strategy* provided θ_n is a trading process for security n, for all $n \in \{1, \ldots, N\}$. The space of trading strategies is denoted $\Theta(G)$. By the Kunita-Watanabe inequality (Jacod, 1979), $\Theta(G)$ is a vector space. In other words, a linear combination of any two trading strategies is also a trading strategy. The stochastic integral $\int \theta\, dG$, for $\theta \in \Theta(G)$, is merely the sum $\sum_{n=1}^N \int \theta_n\, dG_n$, defining total financial gains from trade for all N securities.

It is a key fact that whenever X is a martingale and $\theta \in L^2[X]$, then $\int \theta\, dX$ is also a martingale (Chung and Williams, 1983). As an important illustration, if G is a square integrable martingale gain process and θ is a trading process with respect to G, then there are no expected financial gains from trade since $\int \theta\, dG$ is a martingale. In general, if G is a vector gain process for the economy, we say there is *no expected gain from trade* if $\int \theta\, dG$ is a martingale for all $\theta \in \Theta(G)$.

[10] A *semimartingale* is a process that is the sum of a local martingale, an adapted increasing process, and an adapted decreasing process. For the definition of local martingales, which include martingales, one can refer for instance to (Dellacherie and Meyer, 1982) or (Jacod, 1977).

[11] One may refer to Dellacherie and Meyer (1982) or Jacod (1977) for a precise definition of stochastic integration and quadratic variation.

Limiting the model to gain processes that are semimartingales is not actually restrictive. Any known model of gains or losses from trading securities, whether in discrete or continuous time, is equivalent to a stochastic integral with respect to a gain process. Jacod (1979, pp. 278–279) points out that the only integrator (gain) processes that achieve the required sense of stochastic integration are semimartingales.

2.5 *Definition of Stochastic Equilibria*

A *stochastic economy* is now defined as a collection of the previously defined primitives: $\mathscr{E}_s = (\mathscr{E}, F, D)$, where $\mathscr{E}$ is the underlying Arrow–Debreu economy $(V_+, \hat{c}_i, \succeq_i; i \in \mathscr{I})$, $\boldsymbol{F}$ is the information filtration, and D is the vector of dividend processes defining available securities.

A *price system* for $\mathscr{E}_s$ is a pair (ψ, S) consisting of a spot price process $\psi \in \Phi$ and a vector price process S for the available securities. Let G denote the corresponding vector gain process. A pair $(c, \theta) \in V_+ \times \Theta(G)$ is a *budget feasible plan* for agent i, given a price system (ψ, S), if the consumption process c and trading strategy θ satisfy the budget constraint:

$$(2.2) \qquad \theta(t) \cdot [S(t) + \Delta D(t)] = \theta(0) \cdot S(0) + \int_0^t \psi(s) \cdot [\hat{c}_i(s) - c(s)]\, ds + \int_0^t \theta(s)\, dG(s)$$

for all t in $[0, T]$, and

$$(2.3) \qquad \theta(T) \cdot [S(T) + \Delta D(T)] \geqslant 0 \quad \text{a.s.}$$

Since there is no endowment of securities, the term $\theta(0) \cdot S(0)$ is necessarily zero and can be ignored. Relation (2.2) states that, at any time, the current market value of the securities portfolio is the cumulative to date spot market value of consumption endowments net of consumption purchases, plus gains (or losses) from securities trading. Relation (2.3) rules out terminal debt. A budget feasible plan (c, θ) is *optimal* for i provided there is no budget feasible plan (b, ϕ) such that $b \succ_i c$. A *stochastic equilibrium* for $\mathscr{E}_s$ is a collection $(\psi, S, (c_i, \theta^i); i \in \mathscr{I})$, where (ψ, S) is a price system for $\mathscr{E}_s$ and (c_i, θ^i) is an optimal plan for each agent $i \in \mathscr{I}$ given (ψ, S) such that security and spot markets clear, or $\sum_{i=1}^I \theta^i = 0$ and $\sum_{i=1}^I (c_i - \hat{c}_i) = 0$.

3. EXISTENCE OF STOCHASTIC EQUILIBRIA

In this section we apply Mas-Colell's (1986) proof of existence of quasi-equilibria for Arrow–Debreu economies, along with the machinery for implementing Arrow-Debreu equilibria in a Radner setting developed in Duffie and Huang (1985), to demonstrate equilibria for stochastic economies.

3.1 *The Scratchpad Economy*

The first step on the road to a stochastic equilibrium is the demonstration of an Arrow–Debreu equilibrium for the Arrow–Debreu economy $\mathscr{E} = (V_+, \hat{c}_i, \succeq_i; i \in \mathscr{I})$. For an Arrow–Debreu equilibrium, of course, every consumption process is assumed to be available for trade at time zero, leaving no incentive for markets to remain open after time zero. An *Arrow-Debreu equilibrium* for $\mathscr{E}$ is defined as a nonzero linear (price) functional Ψ on V and allocations $c_i \in V_+$ for each $i \in \mathscr{I}$ satisfying:

$$\text{(3.1a)} \qquad \Psi(c_i) \leqslant \Psi(\hat{c}_i),$$

$$\text{(3.1b)} \qquad z >_i c_i \quad \Rightarrow \quad \Psi(z) > \Psi(c_i) \quad \forall z \in V_+,$$

and

$$\text{(3.1c)} \qquad \sum_{i=1}^{I} c_i = \sum_{i=1}^{I} \hat{c}_i.$$

PROPOSITION 3.1: *Given Assumptions (A1) through (A5), $\mathscr{E}$ has an Arrow-Debreu equilibrium whose price functional Ψ is of the form, for some unique positive spot price process $\psi \in \Phi$,*

$$\text{(3.2)} \qquad \Psi(c) = E\left(\int_0^T \psi(t) \cdot c(t)\, dt\right) \quad \forall c \in V.$$

The proof of this proposition might be overlooked by those readers not interested in the technical details, which are somewhat unrelated to the main purpose at hand.

PROOF: First we will verify that $\mathscr{E}$ has a *quasi-equilibrium*, defined as a collection $(\Psi, c_i;\, i \in \mathscr{I})$ satisfying (3.1a), (3.1c), and the following substitute for (3.1b):

$$\text{(3.1d)} \qquad z \succeq_i c_i \quad \Rightarrow \quad \Psi(z) \geqslant \Psi(c_i) \quad \forall z \in V_+.$$

The consumption space V is a normed vector lattice. Although Mas-Colell's theorem of quasi-equilibrium existence (1986) assumes V is complete under its norm $\|\cdot\|$, that fact is not actually required. (See Duffie (1986, Theorem 4.2).) The only condition for Mas-Colell's theorem that is not obviously met is his "Closedness Hypothesis." For this, we will take advantage of the fact that $\|\cdot\|$-continuous preferences are automatically $\|\cdot\|_2$-continuous by virtue of the Cauchy-Schwarz inequality, where $\|\cdot\|_2$ denotes the product L^2 norm on V. For each i and each $z \in V_+$, the preferred feasible allocation set $\{c \in V_+ \colon c \leqslant \sum_{i=1}^{I} \hat{c}_i; c \succeq_i z\}$ is $\|\cdot\|_2$-weak compact by Alaoglu's Theorem. This follows from convexity of preferences (A3), the $\|\cdot\|_2$-continuity of preferences (A2), and the fact that the closure of a convex set is invariant under a duality preserving change of topology (Schaefer, 1971). Thus $\mathscr{E}$ has a quasi-equilibrium $(\Psi, c_i;\, i \in \mathscr{I})$. By construction, Ψ is positive, linear, and continuous (in either norm's topology). Thus, by an extension theorem due to Namioka reported in Schaefer (1971, p. 227), Ψ can be extended to a continuous positive linear form on the underlying space $L^1(\Omega \times [0, T], \mathcal{O}, \mu)$. Since Φ is isomorphic in the usual form of the Riesz representation theorem to the dual of this space, Ψ has a unique representation as in (3.2).

The second step is to verify that (3.1b) obtains for the given quasi-equilibrium, or that each agent has an allocation that is optimal. It is a consequence of Mas-Colell's proof of quasi-equilibrium and the extreme desirability of aggregate endowments (A3) that $\Psi(\sum_{i=1}^{I} \hat{c}_i) > 0$. Then, for some agent j, $\Psi(\hat{c}_j) > 0$. First suppose, for some nonzero $\omega \in V_+$, that $\Psi(\omega)=0$. By strict monotonicity of preferences $z=\omega+c_j >_j c_j$ and $\Psi(z)=\Psi(c_j)$. Then, for some scalar $\alpha \in (0,1)$, by continuity of preferences (A2), we would have $\alpha z \succeq_j c_j$ and $\Psi(\alpha z) < \Psi(c_j)$. But this contradicts the definition of a quasi-equilibrium. Thus $\Psi(\omega)>0$, whereupon Ψ is strictly positive, and $\Psi(\hat{c}_i)>0$ for each agent i. Then, by a similar argument, each agent i satisfies (3.1b). *Q.E.D.*

3.2. *Martingale Multiplicity*

A *square-integrable martingale* on the filtered probability space (Ω, F, P) is an adapted process X satisfying $E[X(T)^2]<\infty$ and $E[X(t)|\mathcal{F}_s]=X(s)$ for all $t \geq s$. The space of square-integrable martingales X such that $X(0)=0$ is denoted $\mathcal{M}_P^2$. A given vector of N square-integrable martingales $m=(m_1, \ldots, m_N)$ generates $\mathcal{M}_P^2$ if it has the following representation property. For any square-integrable martingale X there exist $\theta_n \in L^2[m_n]$, for $1 \leq n \leq N$, such that

$$X_t = X_0 + \sum_{n=1}^{N} \int_0^t \theta_n(s)\, dm_n(s) \quad \text{a.s.} \quad \forall t \geq 0.$$

In other words, the vector m of martingales generates the space of all square-integrable martingales provided any martingale X in $\mathcal{M}_P^2$ can be represented as the sum of stochastic integrals with respect to the basis set m. Such a vector m of martingales is known as a *martingale generator* (Jacod, 1979). For intuition we could think of X as the gain from trade (price plus dividend process) for some candidate extra security and m as the vector of gains G of the available N securities. If G happens to generate $\mathcal{M}_P^2$ then the candidate security would be redundant, for the gains achieved by holding one share could be replicated by some trading strategy θ as $X=X_0+\int \theta\, dG$.

The *multiplicity* of $\mathcal{M}_P^2$ is the minimum number of martingales required to generate $\mathcal{M}_P^2$. We could therefore think of the multiplicity as, in some sense, the dynamic analogue to the dimension of a vector space. Instead of spanning in the sense of vector addition, the multiplicity states the minimum number of martingales required to span in the sense of stochastic integration. If the martingale multiplicity is equal to N and $m=(m_1, \ldots, m_N)$ is a martingale generator, we refer to m as a *martingale basis* for $\mathcal{M}_P^2$. The zero martingale multiplicity case corresponds to a degenerate probability space, in other words, a deterministic setting. Although formally subsumed in our framework, we will ignore this case.

The following examples may provide some intuition and concreteness for martingale multiplicity, a central concept in this paper. For the results to have any interesting content we must demonstrate some finite multiplicity examples, in particular some examples in which the martingale multiplicity is significantly smaller than the "number of states of the world", which can be interpreted as the dimension of $L^2(\Omega, \mathcal{F}, P)$. Here are a few such examples.

EXAMPLE 1. *Event Trees*: Suppose the information structure is an event tree, or "finite filtration." This is the natural setting for the popular "state preference" models. In this case $L^2(\Omega, \mathcal{F}, P)$ has as its dimension the number of terminal nodes in the event tree. This is generally much larger than the martingale multiplicity, the maximum number of branches leaving any node in the tree, minus one. (See Duffie and Huang (1985) for a proof and an algorithm for constructing a martingale basis.) For instance, if the economy lives on a Markov chain, the martingale multiplicity is one less than the cardinality of the Markov state space, barring degeneracy. In this case a linear system of equations for designing a market completing set of financial securities is easy to set up and solve. This example also illustrates how discrete-time economies with a suitable information structure are subsumed within the general model.

In the following examples agents learn information by observing the evolution of a set of "state variable" processes. That is, $\boldsymbol{F}$ is the filtration generated by a family of processes that may be interpreted as descriptions of the uncertain state of the world.

EXAMPLE 2. *Brownian Motion*: The information structure is the filtration generated by an N-dimensional Wiener process (a vector of N independent Standard Brownian Motions). The dimension of $L^2(\Omega, \mathcal{F}, P)$ is of course infinite, while the martingale multiplicity is N, as shown originally by Kunita and Watanabe (1967).

EXAMPLE 3. *Diffusion State Variable Information*: Generalizing from Example 2, suppose information is generated by an N-vector of diffusion "state variable" processes. Under suitable conditions on the diffusion coefficients the martingale multiplicity is N. The details for this case are developed extensively in Huang (1986). This has been a popular model in financial economics because of the stochastic control methodology which is available for diffusions. Further extensions of this case include "generalized diffusions" (or "Ito Processes", which need not be Markov) and reflected diffusions. The martingale multiplicity is again N. See Jacod (1977) for details. In these cases any N-dimensional vector diffusion process with zero drift vector and positive definite diffusion matrix satisfying Lipschitz and growth conditions forms a martingale basis.

EXAMPLE 4. *Processes with Jumps*: Jacod (1977) provides several examples of filtrations generated by processes with jumps whose multiplicities can be characterized. If the filtration is generated by a Poisson process $\mathcal{N}$ for example, then $\{\mathcal{N}(t)-t;\ t\geq 0\}$ describes a martingale basis (Dellacherie, 1973 and 1975).

3.3. *Spanning Security Markets*

A pair (D, S) consisting of a vector dividend process D and corresponding vector price process S is *market completing* provided, for any spot process $\psi \in \Phi$ and consumption process $c \in V$, there exists a trading strategy θ such that

$$\theta(t)\cdot[S(t)+\Delta D(t)]=\theta(0)\cdot S(0)+\int_0^t \theta(s)\, dG(s)-\int_0^t \psi(s)\cdot c(s) \qquad \forall t\in[0, T] \tag{3.3}$$

and

$$\theta(T)\cdot[S(T)+\Delta D(T)]=0 \quad \text{a.s.,} \tag{3.4}$$

where G denotes the vector gain process for (D, S). If (D, S) defines the available security markets and is market completing, we say markets are *dynamically complete*, which is verbally interpreted as follows. For any given consumption plan c there is some trading strategy θ, requiring an initial investment of $\theta(0)\cdot S(0)$, that finances the stream of spot market payments required over time to purchase the consumption plan c, leaving no terminal financial obligation or surplus. In this case, that is when $(c, \theta)\in V\times\Theta(G)$ satisfies (3.3) and (3.4), we say θ *finances* c, with *initial investment* $\theta(0)\cdot S(0)$.

Let Π denote the function mapping the space of integrable dividend processes to the space of security price processes defined by

$$S(t)=\Pi(D)_t\equiv E[D_T-D_t|\mathcal{F}_t], \quad t\in[0, T]. \tag{3.5}$$

In other words, Π assigns the current market value of a security to be the conditional expectation of the remaining dividends to be paid by the security. It then follows immediately that the gain process G for $(D, \Pi(D))$ is a martingale. The notion of assigning price processes in this manner is suggested by the work of Harrison and Kreps (1979), who showed that equilibrium price processes will always be of this form, at least after selecting a numeraire and adjusting probability assessments. Here we have fixed probability assessments in advance, and will demonstrate an equilibrium by allowing spot price processes to adjust to clear both spot and security markets. Harrison and Kreps (1979), of course, dealt with consumption only at the terminal date T and with securities paying only terminal dividends, but the extension of their work to a setting such as this is straightforward, at least conceptually. Independently of this paper, Huang (1984) has extended the Harrison-Kreps results to economies with intermediate consumption and dividend payout. Here, however, the goal is not to show the martingale property as a necessary property for a given equilibrium, but to demonstrate an equilibrium with this property.

A *fundamental dividend process* is a vector dividend process $\mathcal{D}$ with $N+1$ elements having the following two properties. First, there exists a martingale generator $m=(m_1,\dots,m_N)$ such that $\mathcal{D}_n(T)=m_n(T)$ for $n=1,\dots,N$. For example, we could let $\mathcal{D}_n=m_n$. A polar case is to let $\mathcal{D}_n(t)=0$ for $0\leq t<T$ and $\mathcal{D}_n(T)=m_n(T)$. With the price process $S_n=\Pi(D_n)$, the former case leaves $S_n\equiv 0$; all gains from trade are in the form of dividends, much in the manner of the "market-to-market" nature of modern futures contracts. The latter polar case leaves all gains from trade, except at the terminal time, in the form of capital gains, such as with a forward contract. The second defining property of a fundamental dividend process is that it includes a *numeraire security*, say the zeroth security, with dividend process $\mathcal{D}_0$ defined by $\mathcal{D}_0(t)=0$ for $0\leq t<T$ and $\mathcal{D}_0(T)=1$.

PROPOSITION 3.2: *Suppose $\mathcal{D}$ is a fundamental dividend process. Then, with associated vector price process $S=\Pi(\mathcal{D})$, the dividend-price process pair $(\mathcal{D}, S)$ is* *market completing. Furthermore, given any spot price process $\psi \in \Phi$ and any $c \in V$, every trading strategy θ that finances c has initial investment*

$$(3.6)\qquad \theta(0)\cdot S(0)=E\left[\int_0^T \psi(t)\cdot c(t)\,dt\right].$$

PROOF: Let $\psi \in \Phi$ and $c \in V$ be arbitrary. By Jensen's inequality and the essential bounds on the elements of ψ, we know $x=\int_0^T \psi(t)\cdot c(t)\,dt$ is an element of $L^2(\Omega, \mathscr{F}, P)$. Then $\{X_t \equiv E(x\,|\,\mathscr{F}_t),\ t\in[0,T]\}$ defines a square-integrable martingale X (once a version of the conditional expectation process is fixed). We can therefore apply the definition of a martingale generator to get the representation:

$$(3.7)\qquad X_t = X_0 + \sum_{n=1}^{N}\int_0^t \theta_n(s)\,dG_n(s)\quad \forall t\in[0,T]\quad \text{a.s.},$$

where $\theta_n \in L^2[G_n]$, $1\leq n\leq N$. Let the trading process θ_0 be defined by the adapted process

$$(3.8)\qquad \theta_0(t)=X_t-\int_0^t \psi(s)\cdot c(s)\,ds-\sum_{n=1}^{N}\theta_n(t)S_n(t),\quad t\in[0,T].$$

Since $G_0\equiv 0$, we have $\int \theta_0\,dG_0 \equiv 0$ trivially, and thus $\theta=(\theta_0,\ldots,\theta_N)\in\Theta(G)$.

Relation (3.6) follows immediately and relation (3.3) can be verified by substituting (3.7) into (3.8). Relation (3.4) follows by evaluating the result at time T, using the definitions of X and x. Thus $(\mathscr{D}, \Pi(\mathscr{D}))$ is market completing. *Q.E.D.*

3.4. *The Main Theorem*

THEOREM 3.1: *Suppose the Arrow-Debreu economy $\mathscr{E}=(V_+, \hat{c}_i, \succeq_i;\ i\in\mathscr{I})$ satisfies assumptions (A1) through (A5). Fix a fundamental dividend process $\mathscr{D}$. Then the stochastic economy $\mathscr{E}_s=(\mathscr{E}, \mathbf{F}, \mathscr{D})$ has an equilibrium with a Pareto optimal allocation, dynamically complete markets, and no expected financial gains from trade.*

PROOF: Let $(\Psi, (c_i), i\in\mathscr{I})$ be an Arrow-Debreu equilibrium for the Arrow-Debreu economy $\mathscr{E}$, where the equilibrium price functional Ψ is represented uniquely by the spot process $\psi\in\Phi$ (Proposition 3.1). Let the securities claiming the vector dividend process $\mathscr{D}$ be assigned the vector price process $S=\Pi(\mathscr{D})$. By Proposition 3.2, $(\mathscr{D}, S)$ is market completing. By the definition of an Arrow-Debreu equilibrium, for each agent $i\in\mathscr{I}$, we know $\Psi(\hat{c}_i-c_i)=0$, or $E[\int_0^T \psi(t)\cdot[\hat{c}_i(t)-c_i(t)]\,dt]=0$. Thus, by Proposition 3.2, for each agent $i\in\{1,\ldots,I-1\}$, there exists a trading strategy θ^i such that

$$(3.9)\qquad \theta^i(t)\cdot[S(t)+\Delta\mathscr{D}(t)]=\int_0^t \psi(s)\cdot[\hat{c}_i(s)-c_i(s)]\,ds+\int_0^t \theta^i(s)\,dG(s)\qquad \forall t\in[0,T]\quad \text{a.s.},$$

where G is the vector gain process defined by $(\mathscr{D}, S)$, and

$$(3.10)\qquad \theta^i(T)\cdot[S(T)+\Delta\mathscr{D}(T)]=0\quad \text{a.s.}$$

Equations (3.9) and (3.10) show that the chosen trading strategies for agents $1, \ldots, I-1$ allow them to meet their budget constraints (2.2) and (2.3) with the plans (c_i, θ^i). Let $\theta^I = -\sum_{i=1}^{I-1} \theta^i$. Since $\Theta(G)$ is a linear space, $\theta^I \in \Theta(G)$. From the fact that $c_I = \sum_{i=1}^{I} \hat{c}_i - \sum_{i=1}^{I-1} c_i$ and the linearity of stochastic integration, relations (3.9) and (3.10) also hold for agent I.

We claim that $(\psi, S, (c_i, \theta_i); i \in \mathscr{I})$ *is a stochastic equilibrium for* $\mathscr{E}_s$. Every agent's plan is budget feasible and markets clear by construction. Suppose, for some agent i, there is a budget feasible plan $(\bar{c}, \bar{\theta})$ such that $\bar{c} \succ_i c_i$. Then the Arrow-Debreu equilibrium market value of $\bar{c}$ must be strictly higher, or $E(\int_0^T \psi(t) \cdot [\bar{c}(t) - c_i(t)]\, dt) > 0$. Since $(\bar{c}, \bar{\theta})$ is budget feasible, we can substitute from (2.2) and (2.3) to obtain

$$E\left[\int_0^T \bar{\theta}(t)\, dG(t) + \int_0^T \psi(t) \cdot [\hat{c}_i(t) - c_i(t)]\, dt\right] > 0. \tag{3.11}$$

Since $\bar{\theta} \in \Theta(G)$ and G is a martingale, $\int \bar{\theta}\, dG$ is a martingale, which has zero initial value, being a stochastic integral. The first term of (3.11) can thus be eliminated, leaving $\Psi(\hat{c}_i - c_i) > 0$, which contradicts the fact that c_i is an Arrow-Debreu equilibrium allocation for this agent. Thus each agent's allocated plan is indeed optimal.

Of course the given stochastic equilibrium is Pareto efficient since it achieves the same allocations as the corresponding Arrow-Debreu equilibrium, and the usual convexity and continuity conditions ensuring Pareto optimality for a Walrasian allocation have been assumed. (See, for example, Duffie (1986).)

By construction, markets are dynamically complete and there are no expected financial gains from trade. *Q.E.D.*

4. EQUILIBRIUM PRICES CAN BE MARTINGALES UNDER VARIOUS EXPECTATIONS

For this entire section let $(\psi, S, (c_i, \theta^i); i \in \mathscr{I})$ denote the equilibrium demonstrated in Theorem 3.1. For this equilibrium there are no expected financial gains (or losses) from trade under expectations given by the probability measure P. Of course, P was chosen arbitrarily from the set of probability measures uniformly absolutely continuous with respect to agents' probability measures, those preserving the class of finite-variance random variables. Thus we face little difficulty, at this point, in demonstrating an equilibrium with no expected gain from trade under an arbitrary new probability measure $\hat{P}$ uniformly absolutely with respect to P, denoted $\hat{P} \approx P$.

Let $\hat{\Pi}$ denote the mapping that takes any integrable dividend process D to the price process S defined by

$$S(t) = E^{\hat{P}}[D_T - D_t \mid \mathscr{F}_t], \quad t \in [0, T],$$

where $E^{\hat{P}}$ denotes expectation under $\hat{P}$. In other words, $\hat{\Pi}$ assigns the current market value to be the conditional expectation of total future dividends of the security, with expectation taken according to $\hat{P}$. The corresponding gain process G is easily verified to be a $\hat{P}$-martingale. It follows that $S = \hat{\Pi}(D)$ is in fact a price process, in the technical sense that G is a semimartingale, since a martingale is of course a semimartingale and the space of semimartingales is invariant under an equivalent change of probability measure (Jacod, 1979).

Rather than starting from scratch, we will see how the original equilibrium spot price process ψ of the last theorem may be transformed so as to preserve agents' budget feasible consumption sets when securities are assigned market values according to $\hat{\Pi}$ rather than Π. Let z denote an RCLL version[12] of the martingale $\{E(d\hat{P}/dP \mid \mathcal{F}_t), t \in [0, T]\}$, where $d\hat{P}/dP$ is the Radon–Nikodym derivative of $\hat{P}$ with respect to P. In the terminology of martingale theory, z is the *density process*. Let $\hat{\psi} \in \Phi$ denote the spot price process defined by $\hat{\psi}(t) = z(t)^{-1}\psi(t)$, $t \in [0, T]$.

LEMMA 4.1: $E(\int_0^T \psi(t) \cdot c(t)\, dt) = E^{\hat{P}}(\int_0^T \hat{\psi}(t) \cdot c(t)\, dt)\ \forall c \in V$.

PROOF: The assertion is a consequence of the following sequence of equalities. Fubini's theorem will be used twice to reverse the order of integration, relying on the joint measurability of optional processes as well as the upper and lower essential bounds on the Radon–Nikodym derivative $d\hat{P}/dP = z(T)$ a.s. for integrability. The sixth equality holds for any RCLL version of the conditional expectation process of the previous line, since all such versions are indistinguishable.[13]

$$\begin{aligned}
E^{\hat{P}}\left(\int_0^T \frac{\psi(t)}{z(t)} \cdot c(t)\, dt\right) &= E\left(\frac{d\hat{P}}{dP}\left[\int_0^T \frac{\psi(t)}{z(t)} \cdot c(t)\, dt\right]\right) \\
&= E\left(z(T)\int_0^T \frac{\psi(t)}{z(t)} \cdot c(t)\, dt\right) \\
&= E\left(\int_0^T \frac{z(T)\psi(t)}{z(t)} \cdot c(t)\, dt\right) \\
&= \int_0^T E\left(\frac{z(T)\psi(t)}{z(t)} \cdot c(t)\right) dt \\
&= \int_0^T E\left(E\left[\frac{z(T)\psi(t)}{z(t)} \cdot c(t) \mid \mathcal{F}_t\right]\right) dt \\
&= \int_0^T E[\psi(t) \cdot c(t)]\, dt \\
&= E\left(\int_0^T \psi(t) \cdot c(t)\, dt\right).
\end{aligned}$$

This completes the proof. *Q.E.D.*

[12] A process X is a version of a process Y provided, for each time t, $X(t) = Y(t)$ almost surely.

[13] Two stochastic processes X and Y are *indistinguishable* if $X(t) = Y(t)$ for all t almost surely, or in other words if they have identical sample paths with probability one.

Suppose the martingale multiplicity of $\mathcal{M}^2_{\hat{P}}$ is some integer N. (In the following section we will see that the martingale multiplicities of $\mathcal{M}^2_{\hat{P}}$ and $\mathcal{M}^2_P$ are in fact the same!) Let $\hat{\mathscr{D}}$ be a fundamental dividend process for $(\Omega, \mathbf{F}, \hat{P})$, and let $\hat{S} = \hat{\Pi}(\hat{\mathscr{D}})$ define the associated vector price process.

LEMMA 4.2: *$(\hat{\mathscr{D}}, \hat{S})$ is market completing, and for any spot price process $\phi \in \Phi$ and any consumption process $c \in V$, any trading strategy financing c has initial investment equal to $E^{\hat{P}}[\int_0^T \phi(t) \cdot c(t)\, dt]$.*

PROOF: Working on $(\Omega, \mathbf{F}, \hat{P})$ rather than $(\Omega, \mathbf{F}, P)$, the proof is a repeat of that of Proposition 3.2, once one notes that $L^2(\Omega, \mathscr{F}, P)$ and $L^2(\Omega, \mathscr{F}, \hat{P})$ have the same elements. *Q.E.D.*

The following statement is easily verified from the proof of Theorem 3.1, taking into account Lemmas 4.1 and 4.2.

PROPOSITION 4.1: *There exist trading strategies $\hat{\theta}^i$ for each $i \in \mathscr{I}$ such that $(\hat{\psi}, \hat{S}, (c_i, \hat{\theta}^i);\ i \in \mathscr{I})$ is an equilibrium for $(\mathscr{E}, \mathbf{F}, \hat{\mathscr{D}})$.*

To re-emphasize the result, for any new probability measure $\hat{P}$ preserving the set of finite variance random variables, there exists a corresponding stochastic equilibrium (under the regularity conditions stated in Theorem 3.1) with no expected financial gain from trade under $\hat{P}$. Even if the underlying Arrow-Debreu equilibrium allocation is unique, there is an entire family of stochastic equilibria with fundamentally different price behavior and identical consumption allocations.

5. THE SPANNING NUMBER OF STOCHASTIC EQUILIBRIA

Under the regularity conditions of Theorem 3.1, we have seen in Propositions 3.4 and 4.1 that, for any probability measure $\hat{P}$ uniformly absolutely continuous with respect to P, there exists a dynamically complete markets equilibrium with as few securities as the martingale multiplicity of $\mathcal{M}^2_{\hat{P}}$ plus one. Is this the smallest number? Does this number depend on the chosen price system, that is, the probability measure $\hat{P}$? An answer to the first question in a special case was proved in Duffie and Huang (1985). We will see a more general result here and provide the answer to the second question.

For the following definition we limit ourselves to economies whose gain processes have a finite variance, or $E(G(T)^2) < \infty$. For any probability measure $\hat{P}$ let the *spanning number under $\hat{P}$*, denoted $S^{\#}(\hat{P})$, be the smallest integer number of securities permitting dynamically complete markets with no expected gain from trade under $\hat{P}$. If no such integer exists, the spanning number is defined to be infinite.

PROPOSITION 5.1: *For any probability measure $\hat{P}$ uniformly absolutely continuous with respect to P, the spanning number $S^{\#}(\hat{P})$ is equal to the martingale multiplicity of $\mathcal{M}^2_{\hat{P}}$ plus one.*

PROOF: That this is a sufficient number is given by Proposition 4.1. That no fewer will suffice is given by the proof of Proposition 5.2 of Duffie and Huang (1985). Although that proposition applies to economies with consumption at times 0 and T only, the same proof also serves in this setting (with minor notational changes) and need not be repeated. *Q.E.D.*

This result states that the smallest number of security markets supporting a complete markets equilibrium when there are no expected financial gains from trade under expectations given by $\hat{P}$ is the martingale multiplicity under $\hat{P}$, plus one. Our terminology and notation leave open the possibility that the spanning number may depend on $\hat{P}$, that is, on the price system $\hat{\Pi}$. This is not the case, as one's intuition almost demands, and as will be proved shortly.

It is without loss of generality that we characterize the spanning number of economies with no expected gain from trade under some probability measure. As proved by Harrison and Kreps (1979), and in a setting more like the present one by Huang (1984), this is always the case under regularity conditions, provided no arbitrage exists. The same proof yields this result in the present setting. Whether or not it is restrictive to limit ourselves to comparisons among uniformly absolutely continuous probabilities, rather than merely equivalent probabilities, is an open question. Two probability measures that are not equivalent certainly need not have the martingale multiplicity, as is easily shown by event tree examples.

PROPOSITION 5.2 *For any probability measures P_1 and P_2 uniformly absolutely continuous with respect to P, $S^{\#}(P_1)=S^{\#}(P_2)$.*

Roughly speaking, the spanning number is independent of the probability assessments under which expected gain from trade is zero.

PROOF: The assumptions $P_1 \approx P$ and $P_2 \approx P$ imply $P_1 \approx P_2$. It has been shown (Duffie, 1986) that the martingale multiplicity is invariant among uniformly absolutely continuous probability measures. Then the result follows the Proposition 5.1. *Q.E.D.*

There exists a specific formula (derived in Duffie, 1986) for the transformation of a martingale basis under a given probability measure P to a martingale basis under a different probability measure $\hat{P} \approx P$. This formula may thus be used to design a market completing set of securities for a given "risk-neutral" probability measure.

We can now characterize the spanning number of a stochastic equilibrium directly in terms of agent primitives since, under the bounds on heterogenous expectations expressed in (2.1), every agent's probability measure is uniformly absolutely continuous with respect to P. That is, the spanning number is invariantly the martingale multiplicity plus one.

6. CONCLUDING REMARKS

The model developed in this paper pushes the "rational agent" assumption to its extreme limit in a Walrasian setting. The central concept of a stochastic equilibrium of *plans, prices, and price expectations* is that agents take the entire stochastic processes characterizing terms of trade for assets and spot consumption as given, and determine in advance their optimal consumption rates and portfolios at each point of time and in each possible state of information. This implies preposterous computing and memory ability in all but the simplest schematics of an economy. There can be mitigating factors. For instance, Bellman's principle of optimality is operative: at any time-state pair the optimal consumption-portfolio strategy is merely a "sub-strategy" of the original problem. In that case the problem can theoretically be solved by backward recursion. If, furthermore, information is Markov in nature (e.g. the filtration $\boldsymbol{F}$ is that generated by a Markov process), the existing body of stochastic control theory might be brought to bear, with two caveats. First, stochastic control theory is currently extremely limited in the range of problems which can actually be solved. Frankly, the machinery, although conceptually simple, operates on a fragile foundation of regularity conditions and often depends on the solution of obstinate partial differential equations. Merton's (1971) solution for consumption-portfolio decisions is an exceptional achievement in this regard. Second, and more important, Markov stochastic control is particularly unsuited for determining equilibrium prices in the first place (except in single agent economies). The concept of adding up agents' Bellman equations to derive aggregate demand for capital assets and consumption as functions of prices, and then inverting to get prices that clear markets, is a natural one. What is not at all clear, however, is how to formulate each agent's stochastic control problem in order to achieve this goal. In particular, what is the relevant state description? Are an agent's current portfolio holdings, current asset prices, and the current state of the exogenous environment sufficient statistics for the control problem? Do these variables together form a workable Markov process? These questions are dealt with *in extenso* in Huang (1986). Positive results depend on severely restrictive conditions. Even under ideal conditions it has yet to be demonstrated directly using the stochastic control approach that multi-agent continuous trading equilibria actually exist, despite extensive work on this problem (e.g. Brock and Magill (1979), Merton (1973), Breedon (1979), Cox, Ingersoll, and Ross (1985)). Here the existence of equilibria is based on the usual abstract topological machinery of general equilibrium theory. Although the existence result is greatly simplified by the assumed "dynamic spanning" property of the given security dividends, recent discrete-time work (Duffie, 1985) indicates that this is not a prerequisite for the existence of equilibria in a general setting. Although Hart (1975) showed that the existence of equilibria in incomplete (or dynamically incomplete) markets is a delicate issue, the assumption of purely financial rather than real security dividends simplifies matters considerably.

It is nothing new to report that Walrasian equilibria are a rather magical phenomenon in complicated economies; the problem is simply more acute here. Even if the "right" prices were taken as given by all agents, they could not plausibly be supported by isolated rational behavior if the corresponding optimization problems are intractable. This model, then, is not intended as a description of how decentralized agent optimizing behavior brings about a competitive and efficient equilibrium, although it is consistent with that paradigm. Rather, it is a study of the role of security markets in a stochastic economy under uncertainty. A full regime of time-state contingent claims is *not* a prerequisite for what is effectively a complete markets equilibrium, as made clear early on by Arrow (1953). Relatively few well chosen security markets that are always open can often serve the same purpose. The nature and minimum number of these securities depend explicitly on the manner in which agents receive information resolving uncertainty over time. This is true in both discrete and continuous time in a general probabilistic setting.

Production can be added to the model without changing the basic conclusions (Duffie, 1984a). One first demonstrates an Arrow-Debreu equilibrium for the underlying static production-exchange economy. The sufficient conditions applied in Duffie (1984a) from Duffie (1986) are restrictive. New work by Zame (1985), however, includes less restrictive sufficient conditions for this setting. Again, one assumes that the exogeneously given dividend process for zero-net supply securities have the fundamental dynamic spanning property applied in this paper. The spanning role of the firms' shares in a stock market setting is then superfluous. In principle, each firm chooses a production process $y \in V$ that maximizes the market value of the firm's share at each time and in each state of the world. From the additive nature of the function Π defined in relation (3.5) mapping the firm's dividend process $\{D_t = \int_0^t \psi_s \cdot y_s \, ds, t \in [0, T]\}$ to the firm's share price process, the Arrow-Debreu value maximizing production plan y also serves to maximize the firm's stock market value at all times and in all states. By the usual Modigliani-Miller style argument, since markets are dynamically complete, there is no role for financial decisions by the firm because shareholders can themselves make compensating financial adjustments in equilibrium.

The original working paper (Duffie, 1984a) also allows for the possibility of infinite-dimensional spot markets, extending from the M-dimensional spot market setting of this paper. An infinite-horizon setting poses no additional difficulties. Rather than a terminal budget constraint, however, one must require that each agent select a net-trade spot consumption process whose implicit initial market value in terms of securities is zero.

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