

# Pricing continuously resettled contingent claims\*

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This paper is a study of continuously resettled contingent claims prices in a stochastic economy. As special cases, the relationship between futures and forward prices is analyzed, and a preference-free expression is derived for these prices, as well as the price of a continuously resettled futures option, whose formula differs from Black's futures option pricing formula due to the effects of marking-to-market the changes in the futures option premium.

## 1. Introduction

This paper derives prices of continuously resettled contingent claims in a Markov diffusion setting. This extends the work of Cox, Ingersoll, and Ross (CIR) (1981) by pricing a class of assets from which the futures contract and modern futures option are special cases. The CIR formula for futures prices in a continuous-time setting depends on a risk-premium term that is given an explicit solution in this paper in terms of other parameters. CIR, moreover, characterized futures prices in a discrete-time setting, and then extended to continuous-time by asserting an analogy that is made explicit here.

### *Continuous resettlement*

By analogy with a futures contract, a general continuously resettled contingent claim is completely specified by its maturity date  $T$ , dividend rate  $\{d_t, t \leq T\}$  (zero in the case of a futures contract), and an underlying process

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$\{P_t, t \leq T\}$  (for example, the price of the underlying asset in the case of certain futures contracts), to which the claim is linked.

The continuous resettlement price is a 'price' process  $\{Q_t, t \leq T\}$ , for example, the futures price process. Changes in this price are continuously credited to the holder of the claim's margin account, together with any dividends paid by the asset. We are not aware of any currently traded, continuously resettled claims that pay dividends in addition to resettlement payments. A potential example would be a continuously resettled swap contract. The margin account earns interest at the short-term risk-free rate. The process  $Q$  is such that:

- (1) the current market value of the resettled claim is always zero, and
- (2) at maturity,  $Q_T = P_T$ .

A futures contract satisfies both of these conditions, as do certain of the modern futures options contracts. With conventional options, the option premium is paid out to purchase the option, and no further money changes hands until the option is exercised. The same arrangement applies to certain futures options. This type of option was studied by Black (1976) and by Ramaswamy and Sundaresan (1985). Certain modern futures options, however, are marked-to-market. [For an overview of the different types of margin systems, see Fitzgerald (1987).] The systems employed at LIFFE in London and (to a lesser extent) at the Chicago Mercantile Exchange (CME) correspond in principle to our definition of continuous resettlement.

The remainder of the paper is organized as follows. Section 2 lays out the primitive processes and functions describing the market. The main results, in section 3, include two useful pricing lemmas, as well as a series of results on continuously resettled prices that extend and sharpen the basic results of Cox, Ingersoll, and Ross (1981). Section 4 contains some discussion as well as several simple examples for futures and futures option prices, including a comparison of futures options of the conventional (not marked-to-market) and resettled varieties.

## 2. The primitives

We represent the state of the economy by a state vector  $X_t \in \mathbb{R}^k$  satisfying the stochastic differential equation

$$dX_t = \nu(X_t, t) dt + \eta(X_t, t) dB_t,$$

where  $B$  is a  $k$ -dimensional Standard Brownian Motion and  $\nu$  and  $\eta$  are functions satisfying regularity conditions to be described. We assume that

there are  $k$  risky securities and short-term riskless borrowing, and define:

$P_t = \mathcal{P}(X_t, t) \in \mathbb{R}$  = underlying process for continuously resettled asset,  
 $d_t = D(X_t, t) \in \mathbb{R}$  = rate of dividend payment of continuously resettled claim,  
 $S_t = \mathcal{S}(X_t, t) \in \mathbb{R}^k$  = vector of risky security prices,  
 $\delta_t = \Delta(X_t, t) \in \mathbb{R}^k$  = rate of dividend payment of risky securities, and  
 $r_t = R(X_t, t) \in \mathbb{R}_+$  = instantaneous risk-free rate,

where the functions  $\mathcal{S}$ ,  $\Delta$ , and  $R$  satisfy regularity conditions to be described. Huang (1987) shows that a continuous-time general equilibrium [of the sort demonstrated by Duffie and Zame (1989)] can have this Markovian structure. We use  $E_t(\cdot)$  to denote expectation conditional on the  $\sigma$ -algebra generated by  $\{B_s: 0 \leq s \leq t\}$ , that is, the ‘information set’ at time  $t$ . The notation  $\text{cov}_t(\cdot, \cdot)$  is defined similarly. For simplicity, we suppress the usual qualification: ‘almost surely’.

*Regularity conditions*

For simplicity, define an arbitrage to be a self-financing trading strategy with negative initial market value, which has no future cash flow [for a technical definition, including integrability or boundedness conditions on trading strategies, see, for example, Duffie (1988) or Dybvig and Huang (1988)]. All of our results are based on an assumed absence of arbitrage. Assuming that the matrix  $\mathcal{S}_x(x, t)$  is everywhere nonsingular, let  $\mu: \mathbb{R}^k \times \mathbb{R} \rightarrow \mathbb{R}$  be defined for later purposes by

$$\mu(x, t) = \mathcal{S}_x^{-1} \left( R\mathcal{S} - \Delta - \mathcal{S}_t - \frac{1}{2} \text{tr} \left[ \eta^\top \mathcal{S}_{xx} \eta \right] \right), \tag{1}$$

suppressing the  $(x, t)$  arguments whenever convenient, and where, for example,  $\mathcal{S}_x \equiv \partial \mathcal{S} / \partial x$  and  $\mathcal{S}_{xx}$  is defined coordinate-by-coordinate in the obvious way.

We assume for the remainder of the paper that the functions  $\mu$  and  $\eta$  defined above satisfy Lipschitz conditions.<sup>1</sup> The functions  $D$ ,  $H$  (to be defined later), and  $R$ , all defined on  $\mathbb{R}^k \times [0, T]$  for some fixed time  $T > 0$ , also satisfy a Lipschitz condition. In  $\mathbb{R}^k \times [0, T]$ , the functions  $\mu$ ,  $\eta$ ,  $D$ ,  $H$ ,  $R$ ,  $\mu_x$ ,  $\eta_x$ ,  $D_x$ ,  $H_x$ ,  $R_x$ ,  $\mu_{xx}$ ,  $\eta_{xx}$ ,  $D_{xx}$ ,  $H_{xx}$ , and  $R_{xx}$  exist, are continuous, and satisfy a growth condition.<sup>2</sup> These conditions, imposed for the remainder of the paper, can be weakened, as indicated in the paper’s concluding remarks.

<sup>1</sup>Define  $\|L\| \equiv [\text{tr}(LL^\top)]^{1/2}$  for  $L$  in the space of  $k \times n$  matrices,  $M^{k,n}$ . Then  $f: \mathbb{R}^k \times [0, \infty) \rightarrow M^{k,n}$  satisfies a Lipschitz condition (in  $x$ ) if there exists a scalar  $\lambda$  such that  $\|f(x, t) - f(y, t)\| \leq \lambda \|x - y\|$  for all  $x$  and  $y$  in  $\mathbb{R}^k$  and  $t \geq 0$ .

<sup>2</sup> $f$  satisfies a growth condition (in  $x$ ) if there exists a scalar  $\lambda$  such that  $\|f(x, t)\| \leq \alpha(1 + \|x\|)$  for all  $x$  and  $y$  in  $\mathbb{R}^k$  and  $t \geq 0$ .

**3. Main results**

The following pricing result, although new, is similar in spirit to others in the literature.

*Lemma 1.* *Suppose there is no arbitrage. The market value,  $g_t$ , at time  $t$  of a security that pays out dividends at rate  $d_\tau$  for all  $\tau \in [t, T]$ , and has a terminal payoff of  $h_T = H(X_T)$  at time  $T$ , is given by*

$$\begin{aligned} g_t &= G(X_t, t) \\ &= E_t \left( H(\hat{X}_T) \exp \left[ - \int_t^T R(\hat{X}_\tau, \tau) d\tau \right] \right) \\ &\quad + E_t \left( \int_t^T D(\hat{X}_s, s) \exp \left[ - \int_t^s R(\hat{X}_\tau, \tau) d\tau \right] ds \right), \end{aligned} \tag{2}$$

where the process  $\{\hat{X}_\tau: \tau \in [t, T]\}$  is defined by

$$\hat{X}_t = X_t, \tag{3}$$

$$d\hat{X}_\tau = \mu(\hat{X}_\tau, \tau) d\tau + \eta(\hat{X}_\tau, \tau) dB_\tau. \tag{4}$$

To prove this, we suppose that  $G(X_\tau, \tau)$  is indeed the price of the security at time  $\tau$ ,  $t \leq \tau \leq T$ , and that  $G: \mathbb{R}^k \times [0, T] \rightarrow \mathbb{R}$  is  $C^{2,1}$  (that is, twice continuously differentiable with respect to  $x$ , and continuously differentiable with respect to  $t$ ). We also assume that there exists some self-financing trading strategy  $(a, b)$  in the risky assets and riskless borrowing that replicates this asset. Later, we confirm these assumptions. By the definition of a self-financing strategy, for  $t \leq \tau \leq T$ ,

$$\begin{aligned} G(X_\tau, \tau) &= a_\tau S_\tau + b_\tau \\ &= G(X_t, t) + \int_t^\tau a_s dS_s + \int_t^\tau [a_s \delta_s + b_s r_s] ds - \int_t^\tau d_s ds. \end{aligned} \tag{5}$$

By Ito's Lemma, continuing to suppress arguments of functions wherever notationally convenient,

$$\mathcal{S}(X_\tau, \tau) = \mathcal{S}(X_t, t) + \int_t^\tau \mathcal{D}\mathcal{S} ds + \int_t^\tau \mathcal{S}_x \eta dB_s, \tag{6}$$

$$G(X_\tau, \tau) = G(X_t, t) + \int_t^\tau \mathcal{D}G ds + \int_t^\tau G_x \eta dB_s, \tag{7}$$

where

$$\mathcal{D}f \equiv f_t + f_x \nu + \frac{1}{2} \text{tr}[\eta^\top f_{xx} \eta].$$

Subtracting (7) from (5), using (6), for  $t \leq \tau \leq T$ ,

$$\int_t^\tau [\mathcal{D}G - a_s \mathcal{D}\mathcal{S} - a_s \delta_s - b_s r_s + d_s] ds + \int_t^\tau [G_x \eta - a_s \mathcal{S}_x \eta] dB_s = 0.$$

This is satisfied if and only if each integrand is identically zero (almost everywhere). Thus,

$$G_x \eta = a_s \mathcal{S}_x \eta, \tag{8}$$

$$\mathcal{D}G - a_s \mathcal{D}\mathcal{S} - a_s \delta_s - b_s r_s + d_s = 0. \tag{9}$$

Eqs. (5) and (8) are satisfied by

$$a_t = G_x \mathcal{S}_x^{-1},$$

$$b_t = G - G_x \mathcal{S}_x^{-1} \mathcal{S}.$$

Now, substituting for  $a$  and  $b$  in (9),

$$\begin{aligned} & -RG + G_x \mathcal{S}_x^{-1} (R\mathcal{S} - \Delta - \mathcal{S}_t - \frac{1}{2} \text{tr}[\eta^\top \mathcal{S}_{xx} \eta]) + G_t \\ & + \frac{1}{2} \text{tr}[\eta^\top G_{xx} \eta] + D = 0, \end{aligned} \tag{10}$$

with the terminal boundary condition  $G(x, T) = H(x)$ . By the Feynman–Kac formula [see, for example, Duffie (1988)], the unique solution to (10) is given by (2). Moreover, the solution is indeed  $C^{2,1}$ , as supposed. Thus all of the steps above are justified. We have shown that the dividend stream  $\{d_s\}$  and terminal payoff  $h_T$  can be generated by the trading strategy  $(a, b)$  with an investment of  $G(X_t, t)$  at time  $t$ . This proves Lemma 1 (given the obvious arbitrage arguments).

The following related lemma will be used to price resettled contingent claims.

*Lemma 2. Suppose there is no arbitrage. The market value,  $j_t$ , at time  $t$  of a security that has a terminal (time  $T$ ) payoff of  $[\mathcal{P}(X_T, T) + \int_t^T D(X_s, s) ds] \exp(\int_t^T r_s ds)$  is*

$$j_t = J(X_t, t) = E_t \left[ \mathcal{P}(\hat{X}_T, T) + \int_t^T D(\hat{X}_s, s) ds \right], \tag{11}$$

where  $\hat{X}_t$  is defined by eqs. (3)–(4).

The proof is similar to that of Lemma 1. Specifically, assume that  $(a, b)$  is a self-financing trading strategy in the risky assets and riskless borrowing that replicates the asset. For  $t \leq \tau \leq T$ , suppose the price of the asset at time  $\tau$  is

$$j_\tau = \beta_\tau \left( J(X_\tau, \tau) + \int_t^\tau d_s ds \right) = a_\tau S_\tau + b_\tau \quad (12)$$

$$= J(X_t, t) + \int_t^\tau a_s dS_s + \int_t^\tau (a_s \delta_s + b_s r_s) ds, \quad (13)$$

where  $\beta_\tau \equiv \exp(\int_t^\tau r_s ds)$ . By Ito's Lemma,

$$dj_\tau = \left[ \beta_\tau \mathcal{D}J + Jr_\tau \beta_\tau + r_\tau \beta_\tau \int_t^\tau d_s ds + \beta_\tau d\tau \right] d\tau + \beta_\tau J_x \eta dB_\tau. \quad (14)$$

From (13),

$$dj_\tau = [a_\tau (\mathcal{D}\mathcal{S} + \delta) + b_\tau r_\tau] d\tau + a_\tau \mathcal{S}_x \eta dB_\tau. \quad (15)$$

Equating the  $dB_\tau$  terms in (15) and (14),  $a_\tau = \beta_\tau J_x \mathcal{S}_x^{-1}$ . Substituting for  $a_\tau$  in (12),

$$b_\tau = \beta_\tau J + \beta_\tau \int_t^\tau d_s ds - \beta_\tau J_x \mathcal{S}_x^{-1} \mathcal{S}. \quad (16)$$

Dividing by  $\beta_\tau$  and equating the  $d\tau$  terms in (14) and (15),

$$\mathcal{D}J + Jr + D = J_x \mathcal{S}_x^{-1} (\mathcal{D}\mathcal{S} + \Delta) + Jr - rJ_x \mathcal{S}_x^{-1} \mathcal{S}.$$

Expanding  $\mathcal{D}J$  and using the definition of  $\mu$ ,

$$J_x \mu + J_t + \frac{1}{2} \text{tr}(\eta^\top J_{xx} \eta) + D = 0, \quad (17)$$

which has the assumed solution (10) given the boundary condition  $J(x, T) = \mathcal{P}(x, T)$ . Hence  $j_\tau$  is indeed the price of the asset at time  $\tau$ , proving Lemma 2.

### Continuously resettled claims

In a discrete-time setting, a claim that pays dividends  $d_t$  in period  $t$  and is marked to changes in the 'price process'  $\{Q_t\}$  is actually a contingent claim whose market value is always zero, and which pays  $d_t + Q_t - Q_{t-1}$  at period  $t$ . These pays (or collects) may also be left to accumulate with interest in a

margin account until the position in this claim is offset, or until expiration of the claim, say at period  $T$ . The terminal resettlement price  $Q_T$  is set (either by contract design or arbitrage considerations) to some terminal value, say  $P_T$ , which may be the price of a related asset.

In continuous time, given the dividend rate process,  $d_t$ , instantaneous interest rate process,  $r_t$ , and the continuous resettlement price process,  $Q_t$ , a continuously resettled position process,  $\theta_t$ , generates a margin account whose value  $W_t$  is determined by an arbitrary initial investment  $W_0$  and

$$dW_t = (r_t W_t + \theta_t d_t) dt + \theta_t dQ_t. \tag{18}$$

Since the terminal value  $W_T$  belongs to the position holder, the initial investment  $W_0$  is the arbitrage-free price at time 0 of a claim to  $W_T$  at time  $T$ . Similarly, at any time  $t$ ,  $W_t$  is the arbitrage-free price of a claim to  $W_T$  at time  $T$ .

*Theorem 1. Suppose there is no arbitrage. The resettlement price,  $Q_t$ , of a continuously resettled claim that pays dividends at the rate  $d_t$  and is marked at the terminal date  $T$  to  $P_T$ , is also equal to the current market value of an asset that has the terminal (time  $T$ ) payoff*

$$\left[ \int_t^T d_s ds + P_T \right] \exp \left[ \int_t^T r_s ds \right]. \tag{19}$$

To see this [a generalization of Proposition 2 from Cox, Ingersoll, and Ross (1981)], let the margin account balance at time  $t$  be  $W_t = Q_t$ . For the position process

$$\theta_\tau = \exp \left[ \int_t^\tau r_s ds \right], \tag{20}$$

we have

$$\begin{aligned} dW_\tau &= W_\tau r_\tau d\tau + \theta_\tau dQ_\tau + \theta_\tau d_\tau d\tau \\ &= W_\tau r_\tau d\tau + \exp \left[ \int_t^\tau r_s ds \right] (dQ_\tau + d_\tau d\tau). \end{aligned} \tag{21}$$

Consider the process  $Y_\tau$  defined by

$$Y_\tau = W_\tau - \left[ Q_\tau + \int_t^\tau d_s ds \right] \exp \left[ \int_t^\tau r_s ds \right], \quad t \leq \tau \leq T.$$

Using Ito's Lemma and simplifying, we obtain  $dY_\tau = Y_\tau r_\tau d\tau$ , with initial condition  $Y_t = 0$ . By inspection,  $Y_\tau = 0$  for all  $\tau$ . Thus, for  $t \leq \tau \leq T$ ,

$$W_\tau = \left[ Q_\tau + \int_t^\tau d_s ds \right] \exp \left[ \int_t^\tau r_s ds \right].$$

Since  $P_T = Q_T$ ,  $Q_t$  is indeed the value at time  $t$  of an asset that has the terminal payout given by (19). This proves the theorem. (The smoothness conditions in section 2 play no role in the proof.)

*Corollary 1. Under the conditions described, the continuous resettlement price process is given by*

$$Q_t = \mathcal{Q}(X_t, t) = E_t \left[ \mathcal{P}(\hat{X}_T, T) + \int_t^T D(\hat{X}_s, s) \right], \quad (22)$$

where  $\hat{X}$  is defined by (3)–(4).

This follows immediately from Lemma 2.

The futures price at time  $t$  for delivery of  $P_T$  at time  $T$  is the resettlement price  $Q_t$  for the corresponding claim, with zero dividends ( $d = 0$ ). The corresponding forward price, on the other hand, is that 'price'  $L_t$  for which a claim to  $P_T - L_t$  at time  $T$  has a market value of zero at time  $t$ .

*Corollary 2. Under the conditions described, the futures and forward prices ( $F_t$  and  $L_t$ , respectively) for delivery of  $P_T$  at time  $T$  are given at any time  $t$  by*

$$F_t = \mathcal{F}(X_t, t) = E_t \left[ \mathcal{P}(\hat{X}_T, T) \right], \quad (23)$$

$$L_t = \frac{E_t \left( \mathcal{P}(\hat{X}_T, T) \exp \left[ - \int_t^T R(\hat{X}_\tau, \tau) d\tau \right] \right)}{E_t \left( \exp \left[ - \int_t^T R(\hat{X}_\tau, \tau) d\tau \right] \right)}. \quad (24)$$

Eq. (23) follows immediately from Corollary 1, while (24) follows from Lemma 1 when we impose the well-known no-arbitrage condition that the forward price is the price at time  $t$  of an asset that pays out the value of the underlying asset at time  $T$ , divided by the price  $Z_{t,T}$  of a bond paying one unit of account with certainty at time  $T$ . [See Proposition 1 in Cox, Ingersoll, and Ross (1981).]

*Corollary 3. Under the conditions described, the resettlement price of a (continuously resettled) futures option, with exercise price  $K$  and expiration date*



$\tau < T$ , is given at time  $t$ ,  $t \leq \tau$ , by

$$O_t = \mathcal{O}(X_t, t) = E_t\left(\left[\mathcal{F}(\hat{X}_\tau, \tau) - K\right]^+\right). \tag{25}$$

Although the regularity conditions in section 2 call for smooth functions, the lack of smoothness of  $x \mapsto (\mathcal{F}(x, \tau) - K)^+$  at  $\mathcal{F}_{(x, \tau)} = K$  has no effect on the result [see, for example, Duffie (1988, sect. 22)].

#### 4. Discussion and examples

##### 4.1. Futures vs forward prices

From our expressions for  $F_t$  and  $L_t$ ,

$$F_t \leq L_t \Leftrightarrow \text{cov}_t\left(\mathcal{I}^i(\hat{X}_T, T), \exp\left[-\int_t^T R(\hat{X}_\tau, \tau) d\tau\right]\right) \geq 0. \tag{26}$$

The forward and futures prices are equal, in particular, when interest rates are constant. The following result is reminiscent of a similar discrete-time finding by Cox, Ingersoll, and Ross (1981).

*Corollary 4.* Suppose that the risky assets pay no dividends, and that  $\mathcal{P}(\hat{X}_T, T)$  is uncorrelated with  $\exp[-\int_t^T R(\hat{X}_\tau, \tau) d\tau]$ . Then

$$F_t = L_t = P_t / Z_{t, T}, \tag{27}$$

where  $Z_{t, T} = E_t(\exp[-\int_t^T R(\hat{X}_\tau, \tau) d\tau])$  is the price at time  $t$  of a discount bond maturing at time  $T$ .

This follows from Lemma 1.

##### 4.2. Asset prices as state variables

In the analysis above, for generality, the vector of asset prices is assumed to be a (smooth) function of the underlying state variables. If this function is invertible, we can without loss of generality take the state variables to be the asset prices themselves. In this situation, our expressions simplify substantially. Eq. (1) reduces to

$$\mu(x, t) = R(x, t)x - \Delta(x, t), \tag{1'}$$

and the partial differential eq. (10), satisfied for the price  $G$  of a contingent

claim, is equivalent to

$$-rG + G_x(Rx - \Delta) + G_t + \frac{1}{2} \text{tr}[\eta^\top G_{xx}\eta] + D = 0. \quad (10')$$

Using this form, we can analytically calculate futures and forward prices in special cases. Suppose, for example, that there is only one risky asset, with price  $S_t$ , and the short-term riskless rate  $r_t = r$ , a constant. Then, for dividend  $D = 0$ , eq. (10) simplifies to

$$-rG + (rx - \Delta) \frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + \frac{1}{2} \eta^2 \frac{\partial^2 G}{\partial x^2} = 0. \quad (28)$$

For a futures contract delivering the underlying risky asset, the relevant boundary condition is that the futures price at maturity must equal the spot price, that is,  $G(x, T) = x$ ,  $x \in \mathbb{R}$ . We can solve (28) when the dividend payout rate  $\Delta$  takes on certain forms.

#### 4.3. Examples

(a)  $\delta_\tau = -c(\tau)$  (deterministic cost of storage), for some function  $c$  on  $[0, T]$ .

It can be verified by direct calculation that the solution to (28) is the usual cost-of-carry formula for the futures price

$$F_t = G(S_t, t) = \int_t^T e^{r(T-s)} c(s) ds + e^{r(T-t)} S_t. \quad (29)$$

(b)  $\delta_\tau = \alpha S_\tau$ , for some  $\alpha$  (constant dividend yield).

The futures price is given by

$$F_t = e^{(r-\alpha)(T-t)} S_t. \quad (30)$$

(c)  $\delta_\tau = \bar{\alpha} S_t$ , with  $\tau \geq t$  (constant dividend rate).

The futures price is given by

$$F_t = S_t \left[ e^{r(T-t)} \left( 1 - \frac{\bar{\alpha}}{r} \right) + \frac{\bar{\alpha}}{r} \right]. \quad (31)$$

In comparing (b) and (c), note that

$$e^{r(T-t)} \left( 1 - \frac{\alpha}{r} \right) + \frac{\alpha}{r} \begin{cases} > e^{(r-\alpha)(T-t)} & \text{if } 0 < \alpha < r, \\ < e^{(r-\alpha)(T-t)} & \text{if } \alpha > r, \\ = e^{(r-\alpha)(T-t)} & \text{if } \alpha = 0 \text{ or } r. \end{cases}$$

**4.4. Continuously resettled futures options**

In addition to our simplifying assumptions above, suppose that  $\eta(x, t) = \sigma x$  for some constant  $\sigma$ . We could solve (28) directly, but there is a simpler method. Suppose there is a constant dividend yield,  $\alpha$ , as in example (b). Then, using Corollary 3 and substituting for  $F$  from (30), eq. (25) yields the continuously resettled futures option price

$$\begin{aligned}
 O_t &= E_t \left[ \left( e^{(r-\alpha)(T-\tau)} \hat{S}_\tau - K \right)^+ \right] \\
 &= e^{r(T-t) - \alpha(T-\tau)} E_t \left[ e^{-r(\tau-t)} \left( \hat{S}_\tau - e^{-(r-\alpha)(T-\tau)} K \right)^+ \right]. \tag{32}
 \end{aligned}$$

From Lemma 1, the value of a European call option on the underlying asset, with exercise price  $e^{-(r-\alpha)(T-\tau)} K$  and expiration date  $\tau$ , is

$$C_t = E_t \left[ e^{-r(\tau-t)} \left( \hat{S}_\tau - e^{-(r-\alpha)(T-\tau)} K \right)^+ \right], \tag{33}$$

which is calculated by a simple adjustment to the Black-Scholes (1973) formula. Comparing (32) and (33),

$$\begin{aligned}
 O_t &= e^{r(T-t) - \alpha(T-\tau)} C_t \\
 &= e^{r(T-t) - \alpha(T-\tau)} S_t \Phi(d_1) - K \Phi(d_2) \\
 &= e^{\alpha(\tau-t)} F_t \Phi(d_1) - K \Phi(d_2), \tag{34}
 \end{aligned}$$

where  $\Phi(\cdot)$  is the standard normal cumulative distribution function,

$$\begin{aligned}
 d_1 &= \frac{\log(S_t/K) + r(T-t) - \alpha(T-\tau) + \frac{1}{2}\sigma^2(\tau-t)}{\sigma\sqrt{\tau-t}} \\
 &= \frac{\log(F_t/K) + (\alpha + \frac{1}{2}\sigma^2)(\tau-t)}{\sigma\sqrt{\tau-t}},
 \end{aligned}$$

$$d_2 = d_1 - \sigma\sqrt{\tau-t}.$$

In contrast, under the same assumptions, a conventional (European) option on the futures contract without resettlement has (by Lemma 1) the market value

$$E_t \left[ e^{-r(\tau-t)} \left( \mathcal{F}(\hat{X}_\tau, \tau) - K \right)^+ \right] = e^{-r(\tau-t)} O_t, \tag{35}$$

to be compared with (25). Black (1976) derives (35) in the special case  $T = \tau$ ,  $\alpha = 0$ . Ramaswamy and Sundaresan (1985) derive an equivalent result.

*4.5. Alternative derivation of Corollary 1*

We can derive Corollary 1 in a more direct manner, using arguments similar to those used in our proof of Lemma 1.

To do this, assume that  $\mathcal{Q}: \mathbb{R}^k \times \mathbb{R} \rightarrow \mathbb{R}$  is  $C^{2,1}$  and that there exists some self-financing trading strategy  $(a, b)$  in the risky asset and riskless borrowing that replicates the asset. Then, by the definition of a self-financing strategy and since the asset's value is always zero,

$$\begin{aligned}
 0 &= a_\tau S_\tau + b_\tau \beta_\tau, \quad t \leq \tau \leq T, \\
 &= \int_t^\tau a_s dS_s + \int_t^\tau (a_s \delta_s + b_s r_s) ds - \int_t^\tau dQ_s - \int_t^\tau d_s ds. \tag{36}
 \end{aligned}$$

By Ito's Lemma,

$$\mathcal{S}(X_\tau, \tau) = \mathcal{S}(X_t, t) + \int_t^\tau \mathcal{D}\mathcal{S} ds + \int_t^\tau \mathcal{S}_x \eta dB_s, \tag{37}$$

$$\mathcal{Q}(X_\tau, \tau) = \mathcal{Q}(X_t, t) + \int_t^\tau \mathcal{D}\mathcal{Q} ds + \int_t^\tau \mathcal{Q}_x \eta dB_s. \tag{38}$$

Subtracting (38) from (36), using (37),

$$\int_t^\tau (\mathcal{D}\mathcal{Q} - a_s \mathcal{D}\mathcal{S} - a_s \delta_s - b_s r_s + d_s) ds + \int_t^\tau (\mathcal{Q}_x \eta - a_s \mathcal{S}_x \eta) dB_s = 0,$$

$t \leq \tau \leq T.$

This is satisfied only if each integrand is identically zero. Thus,

$$\begin{aligned}
 \mathcal{Q}_x \eta &= a_s \mathcal{S}_x \eta, \\
 \mathcal{D}\mathcal{Q} - a_s \mathcal{D}\mathcal{S} - a_s \delta_s - b_s r_s + d_s &= 0.
 \end{aligned}$$

Solving for  $a_s$  and  $b_s$  as in the proofs of Lemmas 1 and 2, we see that  $\mathcal{Q}(x, t)$  must satisfy the partial differential equation

$$\begin{aligned}
 \mathcal{Q}_x \mathcal{S}_x^{-1} \left( R\mathcal{S} - \Delta - \mathcal{S}_t - \frac{1}{2} \text{tr}[\eta^\top \mathcal{S}_{xx} \eta] \right) + \mathcal{Q}_t \\
 + \frac{1}{2} \text{tr}[\eta^\top \mathcal{Q}_{xx} \eta] + D = 0,
 \end{aligned}$$

with the terminal boundary condition  $Q(x, T) = \mathcal{P}(x, T)$ . By the Feynman–Kac formula, the solution is given by (22). Moreover, the solution is indeed  $C^{2,1}$ , as assumed.

While this method produces the right answer, it is incomplete. We have used arbitrage arguments to get our asset prices, saying that if we have a trading strategy that produces exactly the same cash inflows and outflows as an existing asset, the price of the asset must equal the initial cost of the trading strategy, or else we can make an arbitrage profit. This does not work here. If  $Q$  does not follow the process we derived, (36) does not tell us how to make an arbitrage profit, since the term in  $Q$  appears on the right-hand side of the equation.

In other words, this second approach shows only that the suggested price  $Q_t$  is consistent with the absence of arbitrage. It does not show that any other price implies an arbitrage opportunity.

#### 4.6. Generality

Although most of the results in this paper can be extended to a general stochastic setting, we have chosen to present them in a simpler Markovian setting, this yielding somewhat more concrete formulas for trading strategies and prices. The reader is invited to normalize the asset price vector  $S_t$  by  $\exp(\int_0^t r_s ds)$  and apply Girsanov's Theorem for the general ('path-dependent') case, as suggested by Harrison and Kreps (1979).

Within the Markovian setting, the strong smoothness conditions on the primitive functions described in section 2 can easily be relaxed, as indicated for instance in Duffie (1988), so long as the Feynman–Kac formula applies.

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