Journal of Economic Literature
Vol. XXVII (June 1989), pp. 565–598

Arrow and General Equilibrium Theory*

By DARRELL DUFFIE
Stanford University
and
HUGO SONNENSCHEIN
University of Pennsylvania

We are pleased to acknowledge the helpful comments of John Roberts, Faruk Gul, Salvador Barbera, Dilip Abreu, Vijay Krishna, and two referees. We are also grateful for bibliographic assistance from Matthew Jackson and Remy Chaudhuri and for typing by Jill Fukuhara.

I. *Introduction*

THE GREATEST ACHIEVEMENTS of economic theory concern the determination of value in competitive markets and the extent to which competitive markets lead to an efficient allocation of resources. Kenneth J. Arrow's contribution has been central. The purpose of this essay is to place Arrow's work on general equilibrium theory into perspective and, in particular, to evaluate critically his three major contributions to the theory: his proof, with Gerard Debreu, of the existence of general economic equilibrium; his analysis of the relationship between equilibria and optima; and his extension of equilibrium theory to cover the case of uncertainty. After explaining the contents of Volume 2 of the *Collected Papers* (hence-forth referred to as CPII) and reviewing the genesis of the general equilibrium model, we will take these up in turn. Our discussion will emphasize some of the most contentious questions raised in response to contemporary equilibrium theory: To what extent has the achievement been primarily technical? What have been the major conceptual breakthroughs? What is the importance of the welfare theorems? Do they make a substantive contribution? What is the logical status of a theory of value that admits the possibility of a large number of equilibrium price vectors? Has modern general equilibrium theory helped or hindered the creation of more dynamic theories of price determination? What are the uses of the theory? Has Arrow's emphasis on the mathematical and axiomatic model served us well? What are the unifying themes in his work? Finally, as a tribute to the enduring quality of Arrow's contribution we will discuss where his ideas have taken us and consider some questions on the frontiers that appear especially compelling.

Contents of the Book

In our review of this volume, *General Equilibrium*, the second of six[1] comprising Arrow's

* This is a review of Kenneth J. Arrow. *Collected Papers of Kenneth J. Arrow*. Vol. 2. *General Equilibrium*. Cambridge, MA: Harvard U. Press, Belknap Press, 1983.

[1] These are as follows: Volume 1—*Social Choice and Justice*, Volume 2—*General Equilibrium*, Volume 3—*Individual Choice Under Certainty and Un-*

Collected Papers, we focus on Arrow's contributions in three areas: (*a*) the existence of equilibria, (*b*) the relationship between equilibrium and Pareto optimality, and (*c*) general equilibrium under uncertainty; these contributions occur principally in chapters 4, 2, and 3, respectively. Chapters 1 and 5 complete the segment of Arrow's work from this volume that originally appeared during 1951–54; these are important papers which deal with substitution in the Leontief model of production. The remaining chapters, numbered 6 through 14, were originally published during 1968–81. Chapter 11 is more an excursion into the theory of difference equations than a piece of general equilibrium theory (although it may be argued that the class of difference equations under consideration is of special interest in the study of dynamic economic models). A large portion of Arrow's research on general equilibrium theory between 1954 and 1968 is collected, along with some of the work of his collaborator Leonid Hurwicz, in *Studies in Resource Allocation Processes* (Arrow and Leonid Hurwicz 1977). The best-known work from Arrow and Hurwicz (1977) deals with the stability of economic equilibrium; an example of Arrow's work on stability can also be found in chapter 12 of the volume under review here. While our review does not deal in detail with the work contained in the Arrow-Hurwicz volume, we believe that the research program set forth there is important for understanding Arrow's contribution, so we will give it some attention. The bulk of the material in the remaining chapters (6, 7, 8, 9, 10, and 13) is expository, and illustrates well Arrow's strength in this dimension. (Chapter 10 is joint work with David Starrett.) Notable among these chapters is the published version of Arrow's Nobel lecture of 1972, reprinted as chapter 9. Chapter 14, the last, studies the efficiency of allocations with costly transfers. Beyond Arrow's work in general equilibrium in the *Collected Papers* and in Arrow and Hurwicz (1977), there are the monographs *General Competitive Analysis*, coauthored with Frank Hahn in 1971, and *Essays in the Theory of Risk Bearing* (Arrow 1970).

certainty, Volume 4—*The Economics of Information*, Volume 5—*Production and Capital*, and Volume 6—*Applied Economics*.

Each chapter is actually a lightly edited version of the paper as originally published, with new headnotes "to give the reader some insight into the circumstances that motivated the writing" (CPII, p. vi). While these circumstances are usually interesting, it is sometimes disconcerting not to know (from this volume alone) where the headnotes end and the paper itself begins.

The Development of General Equilibrium Theory

What are the origins of general equilibrium theory? The classical economists (and we have particularly in mind Adam Smith, Ricardo, J. S. Mill, and Marx) had a theory of value that is driven by the cost of production and a zero profit condition. Insofar as they took markets to be related, their work had an aspect of general equilibrium; however, they ignored the influence of demand on value. Arrow and Starrett (1973), with the classicists in mind and Leontief to lean on, provide the following account of how value can be explained without reference to demand.

> There is one primary factor of production. All other goods are produced under conditions of fixed coefficients with one output, the inputs being the primary factor and possibly other produced goods. . . . In symbols let p_i be the price of produced commodity i, v the price of the primary factor, a_{ij} the amount of commodity j used in the production of one unit of commodity i, and b the amount of the primary factor in the production of one unit of commodity i. Then the condition of zero profits is
>
> $$p_i = \sum_j a_{ij}p_j + b_i v,$$
>
> since the right-hand side is the cost of producing one unit of commodity i. As we let i vary over the produced commodities, we have a system of equations in the unknown prices, p_i, v. . . . The classical economists implicitly and Leontief explicitly solved for these prices in terms of v; thus the price of each commodity is a constant multiple of v and the multiple is completely determined by the input-output coefficients a_{ij} and b_i. (p. 228, CPII)

Antoine Cournot (1838) saw clearly the role of demand in the determination of equilibrium in a single market; not until the neoclassicists arrived on the scene, however, does one have a fully integrated multimarket theory of value. The role of Léon Walras in both incorporating demand into the explanation of value and simultaneously taking into account the relationship of markets is central. Joseph Schumpeter (1954) put it most strongly: ". . . the discovery [of economic theory] was not fully made until Walras, whose system of equations, defining (static) equilibria in a system of interdependent quantities, is the Magna Carta of economic theory" (Schumpeter 1954, p. 242).

A refined version of the Walrasian theory survives today as our best expression of the forces that determine relative value. In general, price is not determined by technology alone: A change in tastes will influence the price of a pound of salmon relative to the price of a pound of calf's liver; this will influence the quantity produced of certain wines as well as their price. The price of a bottle of 1934 Yquem is influenced by the weather in that year; it is also influenced by the distribution of wealth. The Walrasian theory has the capacity to explain the influence of taste, technology, and the distribution of wealth and resources on the determination of value. Nothing that came before the Walrasian theory had this capacity. Neither partial equilibrium theory nor theories that depend on technology and resources alone provide as strong an explanation of value. Although, for certain markets, it is possible to explain how price responds to small parameter changes with partial equilibrium reasoning, few economists would contend that this method is adequate when economies are disturbed in a major way. (We have in mind, for example, a major disruption in the capacity to move oil or a large increase in tariffs.) Moreover, if one is to start an investigation into relative values without beginning from given prices, as might be appropriate for the analysis of an economy some decades in the future, then it is quite clear that the Walrasian theory is the most useful conceptual framework available.

Before we turn to the existence of general equilibrium and Arrow's contribution, some further words about the usefulness of general equilibrium theory are in order. The essence of general equilibrium does not preclude aggregation; what is essential is an emphasis on intermarket relations and the requirement that variables are not held fixed in an ad hoc manner. Small general equilibrium models, for example with one set of indifference curves to represent the utility levels in each of two countries, play a significant role in the analysis of trade and taxation issues. Also, the usefulness of general equilibrium theory is not restricted to situations in which we can determine the data of an economy and solve for equilibrium prices. Some of the most important lessons to be learned from the theory are qualitative, such as the conditions under which free markets and exchange lead to an efficient allocation of resources, or which commodities to tax in order to raise revenue most efficiently. Today, the general equilibrium model is not the exclusive province of the high-tech theorist; rather, it is a basic part of the professional economist's tool bag, and one that is increasingly used.

II. *The Existence of General Economic Equilibrium*

A Modern Synthesis of the Walrasian Model and Existence of Equilibria

The existence[2] problem is stated in a simple form as follows. The data of a private ownership economy are tastes, technology, the initial holdings of commodities by consumers, and the ownership of firms. All agents are assumed to take prices as given. The supply function of each firm is assumed to be single-valued (this means that for each vector of input and output prices there is a unique profit-maximizing production plan), and the net aggregate supply of firms is obtained by adding the supplies of individual firms. (By convention, inputs are denoted by negative quantities and outputs by positive quantities.) Each household's income is determined by the value of the household's initial endowment and the value of the profit-maximizing actions of the firms in which the household holds stock; these in turn are both determined by prices. Household aggregate demand, which depends on prices and the distribution of income, is thus seen to depend on prices alone. Finally, household aggregate sup-

[2] Arrow's work on the welfare theorems and equilibrium under uncertainty preceded his work on the existence theorem. We reverse the order of presentation in order to put the more foundational result in its proper place.

ply is the sum of initial endowments. The condition that the difference between aggregate demand and aggregate supply is zero, applied to each commodity, yields the familiar excess-demand system of ℓ equations in the ℓ price variables:

$$z_i(p_1, \ldots, p_\ell) = 0, \quad i = 1,2, \ldots, \ell. \quad (1)$$

The forces of supply and demand, which are defined by the data of the economy, will be in balance at prices $\bar{p}$ if and only if $\bar{p}$ solves (1).[3] In Walrasian theory, value is determined by a solution to (1): *General* equilibrium requires that *all* markets clear. Such an equilibrium price vector $\bar{p}$ depends, through the demand functions of consumers and the supply functions of firms, on the primitive data: tastes, technology, and endowments.

The role of an existence theorem is to insure that, for all economies from a broad class, there will be at least one solution to (1) in nonnegative prices. In the absence of a solution the model does not offer a theory of value.

One sometimes hears that the existence of equilibrium follows from the observation that, because the total quantity of any good sold is necessarily the total quantity purchased, the prices we observe in the actual world are equilibrium prices. General equilibrium theory is an attempt to explain the prices that we observe. The equilibrium prices of our model are taken to *correspond* with the prices that we observe in actual economies, but they are not the same objects as the prices in actual economies. In particular, it is not necessarily true that observed prices are market-clearing. In order to understand the implications of assuming that observed prices are market-clearing, one needs at least a well-articulated theory with sharp criteria for the existence of market-clearing prices. A model may or may not have at least one equilibrium price, and when it does not produce at least one equilibrium price it will not serve to explain prices as devices that clear markets.

[3] Because the decisions of households and firms depend only on relative prices, and because for an arbitrary price vector $(p_1, p_2, \ldots, p_\ell)$ the value of excess demand is zero (this is Walras' law: $\Sigma_i p_i z_i(p_1,p_2, \ldots, p_\ell) = 0$), the system (1) is properly regarded as composed of $\ell - 1$ equations in $\ell - 1$ unknowns.

Existence of Equilibrium Before the Early Fifties

It is usually stated that Walras took the observation that his system contains as many equations as unknowns as a proof of the existence of equilibrium, and this is probably correct.[4] In any case, an adequate proof of the existence of equilibrium requires more than counting equations, and Walras had nothing past this to offer. Indeed, the requisite mathematics for the existence of solutions to nonlinear equations such as (1) were not available to Walras; this came only later with Luitzen Brouwer's (1912) fixed-point theorem.

The formal existence theory begins with the formulation of the Casselian system (Gustav Cassel 1924) and its refinement by Hans Neisser (1932), Heinrich von Stackelberg (1933), Frederick Zeuthen (1933), and Karl Schlesinger (1933–34). This led to the first rigorous proofs of the existence of equilibrium, by Abraham Wald (1933–34, 1934–35, 1936).[5] Wald's published proofs of existence require that aggregate demand be independent of the distribution of income and satisfy the weak axiom of revealed preference in the aggregate.[6] In this Wald is close to the case in which aggregate demand is of the class generated by a single consumer. The requirement that demand has such a special form is extremely restrictive; it has the advantage, however, that it guarantees uniqueness of equilibrium, and Wald appreciated this point. Furthermore, it greatly simplifies the existence proof by allowing one to replace fixed-

[4] Léon Walras understood the possibility of multiple solutions. His example (Walras 1874–77), which is interpreted by Jaffé (in his notes that accompany the English translation, 1954, of Walras) and Schumpeter (1954) to demonstrate that he fully appreciated the possibility of nonexistence, does not make the point in a satisfactory manner (see Takashi Negishi 1987).

[5] Arrow reviews the history of existence theorems in his paper "Economic Equilibrium" for the *International Encyclopedia of the Social Studies* (1968). This is chapter 6 in CPII. See also Weintraub (1983).

[6] The weak axiom of revealed preference requires that there is no pair of distinct commodity bundles x and y such that when x is demanded y can be afforded and when y is demanded x can be afforded. Wald speaks of a result in which demand is generated by utility-maximizing consumers (this does not imply the weak axiom in the aggregate), but the proof was never published.

point theorems by a maximization argument and a separating hyperplane theorem. (The separating hyperplane theorem is stated in Footnote 28.) But, again there is no reason to believe that aggregate demand takes such a special form.

The Modern Existence Theorems

In 1954, Arrow and Gerard Debreu published their proof of the existence of equilibrium for a competitive economy (this is chapter 4 in CPII). Lionel McKenzie (1954) provided a proof at approximately the same time as Arrow and Debreu. David Gale (1955) and Hukukane Nikaido (1956) had their own versions of an existence theorem. We take the point of view that these proofs are primarily technical achievements for which the required mathematical machinery had recently been put in place.[7] We will argue further that the Arrow and Debreu proof involved a substantial amount of technical innovation and that the method of proof has been especially fruitful.

The McKenzie and Arrow-Debreu results were presented at the 1952 Winter Meeting of the Econometric Society. For a model with general demand, these are the first equilibrium existence results communicated to large audiences and out in print. They are rightfully accorded a special place, but one should keep in mind the independence of the Gale and Nikaido treatments and the fact that publication of the Nikaido paper was delayed. Also, the argument used by Nikaido and Gale plays a central role in the classical treatment of existence provided later by Debreu (see Debreu 1959, p. 82). McKenzie, unlike Arrow and Debreu, assumed constant-returns-to-scale in production. This difference vis-à-vis the Arrow-Debreu formulation is superficial, however, because each Arrow-Debreu economy (with possibly regions of strictly decreasing returns to scale) can be embedded in an equivalent constant-returns-to-scale economy (McKenzie 1959). A more substantial difference between the Arrow-Debreu and McKenzie formulations is that Arrow and Debreu start with individual consumers and a distribution of wealth, while McKenzie begins with a continuous aggregate-demand function that is hypothesized to satisfy a boundary condition that cannot be shown, in general, to follow from utility maximization. This is indeed a weakness, or at least an incompleteness, in McKenzie's approach; however, it should not obscure the importance of his contribution.[8]

In their search for conditions on the characteristics of household preferences that are sufficient to ensure that the excess-demand functions in (1) are sufficiently continuous, Arrow and Debreu discovered an important demand discontinuity problem. With monotonic preferences (more of a commodity is better), whenever the price of a commodity is zero, the amount of the commodity demanded will be unbounded (or more precisely, undefined). Furthermore, if the initial endowment is not interior to the consumption set, the quantity demanded can vary continuously with positive prices, but be undefined precisely at a zero price. Even with a one-consumer economy having smooth and strictly convex preferences and no production, equilibrium need not exist if the consumer's initial endowment is on the boundary of the consumption set.

The basic difficulty is illustrated in Figure 1. Indifference curves over nonnegative commodity pairs are labeled I_1, I_2, and I_3; their slopes approach zero as the quantity x_2 of the second good approaches zero. The endowment vector $\bar{x}$ contains a positive amount of only the first commodity. The excess demand for the first commodity is negative when $p_1 \neq 0$ and excess demand approaches zero as p_1 approaches zero. The equilibrium price "wants to be" $p_1 = 0$, in the sense that excess demand approaches 0 as p_1 approaches 0; however, at

[7] See Roy Weintraub (1983), Arrow (1987), and Debreu (1987) for more detailed and personal accounts.

[8] For the record, it must be stated that Arrow and Hahn's reading of McKenzie's contribution, that ". . . if specialized to the case of exchange, it is identical to Wald's" (Arrow and Hahn 1971, p. 51), does McKenzie an injustice. To be specific, McKenzie did not assume, nor do his conditions imply, that aggregate demand satisfies the weak axiom of revealed preference. In a letter to Roy Weintraub (the relevant portion of which is published in Weintraub 1983, pp. 36–37), Arrow later recognizes this distinction. Although McKenzie does limit himself in his existence proof to the case of special (Cobb-Douglas) demand functions, he points out (p. 155) that the techniques are adequate for the general case.

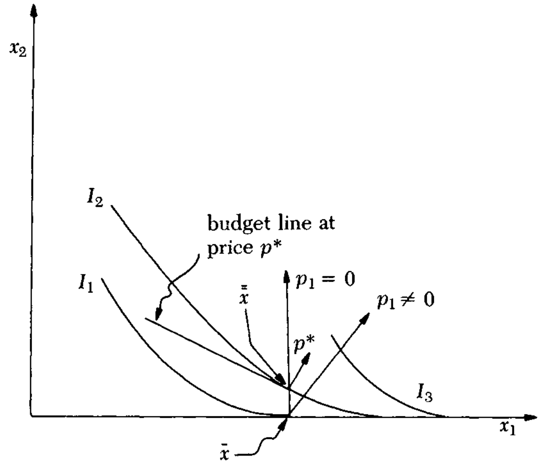


Figure 1. Discontinuity of Demand at a Zero Price

$p_1 = 0$ excess demand is not defined (because more of good 1 is always preferred, and budget feasible). Equilibrium in a one-consumer economy requires that the consumer demands his initial endowment, but in Figure 1 there are no prices at which this is the case. The problem is similar in economies with more than one consumer, and can be resolved by requiring that the initial endowment of each household be interior to the consumption set (see, for example, Negishi 1987, p. 371). (To see why, observe that if the consumer's initial endowment is $\bar{\bar{x}}$, then a low (high) relative price for x_1 leads to a positive (negative) demand for x_1, and that demand varies continuously with price. At the budget line that is drawn, supply equals demand.)

One of the nice contributions of the *Collected Papers* is the accounts that precede each paper. The "demand discontinuity problem" we have just discussed is of historic interest because of its role in the Arrow-Debreu collaboration. Arrow and Debreu began their work on an existence theorem independently. As Arrow writes, "Debreu and I sent our manuscripts to each other and so discovered our common purpose. We also detected the same flaw in each other's work; we had ignored the possibility of discontinuity when prices vary in such a way that some consumers' incomes approach zero. We then collaborated, mostly by correspondence, until we had come to some resolution of this problem" (CPII, p. 59). This resolution was to require, in theorem 1 of their paper, that the initial endowment of each household be interior to its consumption set. (Arrow had faced a difficulty much related to the demand discontinuity problem in his earlier work on the second welfare theorem. More about this appears in Section III.)

The Arrow-Debreu Method

By their own account, the work of Wald had very little direct influence on Arrow and Debreu. It would appear that they each had a feeling for the intrinsic importance of the problem.[9] An appropriate tool was needed, and this was provided by Shizuo Kakutani's (1941) generalization of the classic Brouwer (1912) fixed-point theorem. In Arrow's view, he, Debreu, and McKenzie were simultaneously primed by the work of John Nash (1950) for the use of that theorem. He writes, "It was the paper of John F. Nash, Jr., showing the existence of equilibrium points to games by the use of Kakutani's (1941) fixed-point theorem [equivalent to von Neumann's (1937)], that suggested to several of us the corresponding analysis for the concept of general competitive equilibrium. Gerard Debreu, Lionel McKenzie, and I all followed up this lead independently, each in his own way" (CPII, p. 58).[10] The influ-

[9] There are currents of indirect influence, and of particular interest is one in which Oskar Morgenstern plays a major role. Morgenstern was familiar with the existence question early on. In the mid thirties, Morgenstern put the economist Schlesinger in touch with the mathematician Wald, taking pride in his institute's role in the birth of the first rigorous existence theorem. After he moved to the United States in 1938, Morgenstern worked to keep the importance of the existence question alive. In a review of Hicks' *Value and Capital* Oskar Morgenstern (1941) writes:

> Hicks . . . is systematically incorrect [when he counts equations in order to argue the existence of equilibrium prices] because the determination of a system of equations does not necessarily depend only upon the equality of the number of unknowns with the number of equations. . . . We have as yet such [existence] proofs, only for two systems of equations, those of von Neumann and of Wald.

Finally, although Morgenstern's collaboration with von Neumann did not result from his interest in the Walrasian existence theorem, it led to John Nash's equilibrium result for games, which is the technical starting point for Arrow and Debreu's independent attack on the existence theorem.

[10] Because Arrow and Debreu began their work independently, a substantial amount of reconciliation

ence of Nash on McKenzie is perhaps less clear than what is suggested by Arrow. McKenzie, Gale, and Nikaido approached the existence problem through the device of the aggregate excess-demand function. They made no attempt to accommodate the Nash theorem to their purpose and indeed their arguments may be regarded as having more transparent economic intuition than that provided by Arrow and Debreu. Arrow and Debreu's approach, via an extension of the Nash theorem, is quite subtle, and we will see in a moment that it is especially well suited for extensions of the existence theorem. With this background we now turn to an exposition of the Arrow-Debreu method.

An n-person game is defined by specifying for each agent (*a*) a set of a priori available choices, and (*b*) a real-valued function that specifies the agent's payoff as a function of the n-tuple of choices by all agents. A Nash equilibrium n-tuple of choices has the property that, given the choices of other agents, each agent's payoff is maximized. In economic models, agents may be constrained to pick from sets that depend on the choices of others; for example, a consumer must choose from the set of bundles that he can afford, which indirectly depends on what other agents choose. (For example, what others choose influences prices and prices determine what an agent can afford.) In order to handle cases in which, for each agent there is a set of feasible choices that depend on the choices of others, Debreu formulated the notion of a generalized n-person game. In a generalized n-person game, equilibrium requires that, given the choices of other agents, each agent's payoff is maximized subject to feasibility; furthermore, what is feasible depends on the actions of others. Debreu (1952) proved that generalized games have an equilibrium provided that (1) the feasible sets are convex and vary continuously with the actions of all players, (2) the value of the payoff functions vary continuously with the actions of all players, and (3) for all fixed choices by other players, choices by a player that raise his own payoff form a convex set.[11] At the heart of Debreu's proof is the fixed-point theorem of Kakutani (1941), who was motivated by von Neumann's (1937) work on games.

The Arrow-Debreu proof of the existence of Walrasian equilibrium for an economy proceeds by (1) associating a generalized game with the economy, (2) proving that there exists at least one equilibrium of the generalized game, and (3) demonstrating that in an equilibrium of the generalized game all markets in the economy clear. The continuity and convexity properties required for application of Debreu's result follow from convexity assumptions on production sets and preferences plus (for example) the requirement that endowments be interior to the consumption set.

The above approach to the existence theorem, because it operates at a level prior to the aggregation of supply and demand, is fundamentally different from looking for a solution to the excess-demand function system. From a technical perspective it represents an extremely creative step; in addition, it has great power as a vehicle for extending the existence theorem. A few examples illustrate the latter point. The Walrasian theory of value has been criticized for failing to take into account the fact that an agent may judge quality by price, or be inconsistent in his choices (as when preferences are not transitive), or have preferences that depend on the choices of other agents. The original Arrow-Debreu proof, unlike the proofs that work via the construction of an excess-demand function (such as in McKenzie 1954; Nikaido 1956; Gale 1955; and Debreu 1959), is relatively easily modified to take into account *all* of these ingredients. The generalized game approach allows each agent's payoff to be influenced by the choices of others, and so it applies to economies in which the preferred sets of each consumer vary with the choices of other consumers or producers (as

was essential before they could proceed with their collaboration. This was facilitated by the fact that in their independent starts they had attempted to incorporate Kakutani's theorem by extending Nash's equilibrium theorem for games. Ultimately, as Arrow explains, they ". . . followed more closely Debreu's more elegant formulation, based on the concept of generalized games" (CPII, p. 59). Arrow describes his own formulation on page 59 of the *Collected Papers*. It is now known that the equilibrium existence theorem is equivalent to the Brouwer-Kakutani theorem. This is a consequence of the fact that the class of excess-demand functions is known to have no structure beyond homogeneity and Walras' law.

[11] This is, of course, a loose statement.

would be the case with externalities) or with prices (as when quality is judged by price). Similarly, nontransitivities in preference are admitted by allowing, for each choice by the consumer, a different set of relevant "quasi-indifference curves."[12] As long as these quasi-indifference curves satisfy the standard convexity requirement and vary continuously with one's own choices and the choices of others, the Arrow-Debreu proof (via the Debreu lemma) still goes through. (The result is obtained in a different manner by Andreu Mas-Colell 1974.) Similarly, the Arrow-Debreu approach works well to establish an existence theorem for economies with taxation (obtained in a different manner by Kevin Sontheimer 1971 and John Shoven 1974), or with externalities in production.[13] Although Arrow and Debreu do not mention these extensions, it is difficult to imagine that they did not appreciate the possibility of at least some of them.

Existence Theorems: Their Importance and Their Shortcomings

Here then is the history of value theory that emerges. Before the neoclassicists the conception of the determinants of value was seriously flawed. In particular, multimarket models ignored the influence of demand. Walras made the major advance by providing a model of economies in which value is determined by supply and demand in all markets simultaneously. Over a period of 80 years Walras' ideas were refined, but it was not until Arrow-Debreu, McKenzie, and their contemporaries provided sets of conditions under which general equilibrium must exist that the relevance of the Walrasian theory of value was understood.

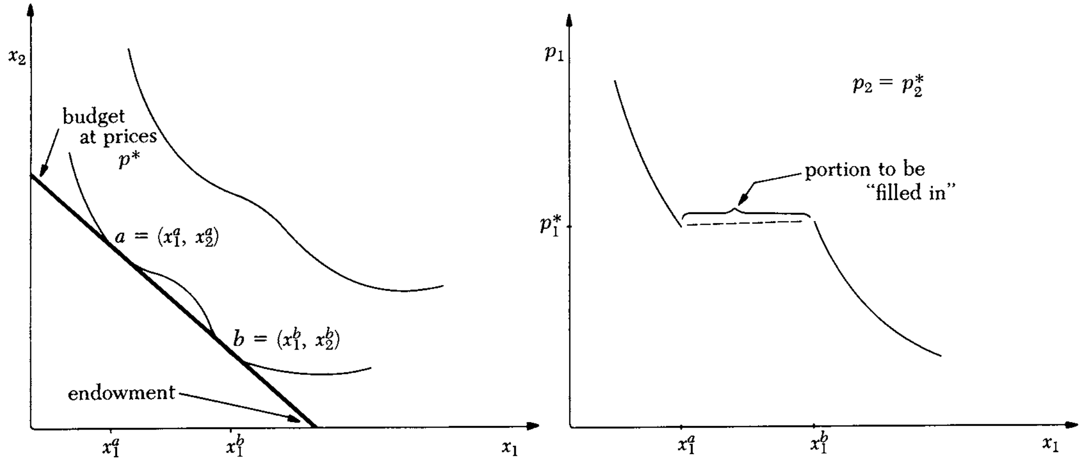
On the positive side, the existence theorems of the 1950s have led to a deep understanding of the conditions under which, for each economy of price-taking agents, there exists at least one vector of market-clearing prices. On the negative side these conditions are seen to be severe, and it can be maintained that they affirm the descriptive irrelevance of the Walrasian model. Furthermore, a case can be made that we have been slow to pay attention to the limitations of the Walrasian theory as a theory of value. For example, the Walrasian theory does not shed light on when price taking applies, or how prices are formed, or which vector of prices will prevail when there is more than one that clears markets. These limitations have been increasingly apparent as the theory has been refined; to our mind their realization should be regarded as one of the major products of modern equilibrium theory. We will close this section on existence by considering some of the major triumphs that have been accomplished since the 1950s as well as some of the major challenges that remain.

The existence theorems, while absolutely fundamental, are primarily technical in nature, and so it is not surprising that many of the major advances since the 1950s have also been technical. Nevertheless, the step forward that we regard as the most significant has required a reformulation of the basic model. To most readers the most unrealistic descriptive assumptions imposed by Arrow and Debreu involve the convexity of preferences and production sets.[14] Both of these assumptions rule out indivisibilities and the latter rules out fixed costs and the resulting U-shaped average cost curves; as a result they would appear to limit the applicability of the Walrasian model greatly. But without them household demand and firm supply can be discontinuous and there might appear to be little possibility for an equilibrium existence theorem. General equilibrium theorists now understand that convexity of preferences and production sets is less critical than was believed. Provided that one is willing to formulate an economy as composed entirely of agents who are infinitesimal relative to the market (we will discuss other reasons for favoring this formulation in a moment), the convexity assumption can be dispensed with. Individual behavior may be discontinuous; in the aggregate, however, there will be continuity. In some form this observation is very old, but was

[12] These are better referred to as "behavior curves" because indifference loses meaning in the absence of transitivity.

[13] For a dissenting opinion on the success of the extension to externalities in production, see McKenzie (1981, pp. 838–39).

[14] The assumptions that handle the "demand discontinuity problem" alluded to before are also not very realistic; however, they have a more technical flavor.



a. Single-Agent Demand b. Aggregate Demand

Figure 2. Continuity of Aggregate Demand with a Continuum of Agents Having Nonconvex Preferences

developed by Michael Farrell (1959) and Jerome Rothenberg (1960), and brought to general equilibrium theory in a manner compatible with the Arrow-Debreu-McKenzie theory by Robert Aumann (1966).[15] Despite Arrow's contribution to this literature (Arrow-Hahn 1971, chapter 7), his statement, "The assumption of convexity cannot be dispensed with in general equilibrium theorems concerning the existence of equilibrium strictly defined" (Arrow 1968, p. 120, CPII) and his emphasis on the role of convexity in his Nobel lecture (Arrow 1973; CPII, chapter 9) have suggested a reluctance to let go of convexity. In fact, the following strong statement is entirely in order: With a continuum of infinitesimal agents, convexity of the aggregate production set and of preferences is simply not required, nor is it implied.[16]

[15] Of course, global increasing returns to scale (or more modestly, situations in which efficient scale is reached at a level of output which is noninfinitesimal relative to the total size of the market) remains a problem. Also, we do not deny the descriptive reality of the latter situation.

[16] In the absence of external economies, convexity of the aggregate production set is implied by the assumption that firms are infinitesimal; however, the fact that convexity of the production set is not essential is underscored by the fact that with external economies the aggregate production set will, in general, not be convex (John Chipman 1970) and nevertheless Walrasian equilibrium is ensured to exist.

Figures 2*a* and 2*b* help us to explain how the existence of equilibrium is demonstrated in the absence of the assumption of convex preferences. We will see how the continuity of demand obtains in the aggregate even when it fails for individual agents. In Figure 2*a* we have described the preferences and endowment of a single agent. At prices p^* demand is not single-valued; as a result, the excess-demand correspondence (depicted in 2*b*) jumps at this price. This discontinuity disappears when there is a continuum of infinitesimal agents. To understand why, consider an economy in which there is a continuum of infinitesimal agents with the initial endowment and preferences given in 2*a*. If at p^*, 3/4 of the agents are assigned the bundle *a* and 1/4 of the agents are assigned the bundle *b*, then all agents will be maximizing utility and mean demand will be 3/4 of the way from *b* to *a*. By considering proportions other than 3 to 1, all the points between *a* and *b* are filled in and the result is an excess-demand function (correspondence) with sufficient continuity to play its part in the existence of equilibrium. With the realization that convexity need not be assumed when the Walras model is formulated with a continuum of infinitesimal agents, the conditions needed for existence appear to be quite modest indeed. Moreover, it is for the case in which agents are small (infinitesimal) relative to the market that we find the most convincing theoretical support for the Walrasian hypothesis of price-taking behavior.[17] (The strong statement would be that the Walrasian model makes sense only with a continuum of infinitesimal agents, and in this case convexity is not necessary.)

There is little question in our mind that the early existence theorems were the start of a chain of reasoning that has led to a deeper appreciation of the role of size (as opposed to decreasing marginal rates of substitution and the absence of some regions of increasing returns to scale) for determining when it is that the Walrasian theory of value applies. When there are a large number of consumers in each market and when the size of firms is endogenously determined to be small, then price taking is more plausible and the need for convexity, as an assumption, is diminished.[18] In short, for competitive analysis the conditions under which size is small (or the assumption of small size) should be regarded as a more basic hypothesis than convexity.

Another direction in which there has been substantial technical advance deserves special mention. This concerns the extension of Walrasian analysis to economies with an unbounded number of a priori available commodities. These arise naturally when there are differentiated products or when commodities are dated (see, for example, Truman Bewley 1972 and Mas-Colell 1974). A technique pioneered by Negishi (1960) and developed by Arrow and Hahn (1971, chapter 5) is playing a major role in the analysis. Prominent examples are Mas-Colell (1986) as well as Mas-Colell and Scott Richard (1987). It is also here that one finds some of the most intriguing conceptual problems (such as those faced by Marcus Berliant 1984).

What are the principal challenges that lie ahead for the existence theory? Arrow points out (CPII, p. 107) that the term *equilibrium*, as it has been used in economics, has two aspects. These are determinateness and the balance of forces. Determinateness in turn has two aspects. These are existence and uniqueness. A severe limitation of the Arrow-Debreu-McKenzie theory concerns the possibility of multiple equilibria. We have stated that the theory is adequate to determine prices as a function of primitives: tastes, technology, and the distribution of wealth. Because an economy may have many Walrasian equilibria, this statement is at the very least misleading. Even in a highly stylized two-commodity exchange economy consisting of two individuals with homogeneous utility functions, many different equilibrium prices are possible. In fact, the equilibrium price set may be an essentially arbitrary subset of the set of relative prices.[19] The assumptions on primitives necessary to ensure uniqueness of prices are indeed very strong,[20] and the best general results enable us to say only that the case of a finite number of equilibria is generic.[21]

[17] For a review of the literature on game theoretic foundations for price-taking behavior, see Mas-Colell (1980). There are many results that support the Walrasian model when agents are small relative to the market: Of particular historical and substantive importance are the Francois Edgeworth (1881), Debreu and Herbert Scarf (1963), and Aumann (1964) studies relating the core and the Walrasian equilibria. There are also models that support the Walrasian analysis when there are few agents (see, for example, Pradeep Dubey 1982 and Leo Simon 1984), but the conclusion that the Walrasian model is appropriately applied in these cases is far from clear. Price taking is perhaps the defining assumption of the Walrasian theory, and there is no question that it has served us well; however, we are at risk when we apply the Walrasian theory to situations in which price taking is not strongly supported. Extensions of the Walrasian model have been proposed to take us beyond the case of price-taking behavior (see especially Negishi 1961 and Arrow and Hahn 1971, chapter 6); however, the theory of value has had no more than limited success in this direction.

[18] In several contexts Arrow speaks of the problems caused for the hypothesis of price taking by the thinness of markets. We will have more to say about this in our discussion of the welfare theorems. (See especially Footnote 31.)

[19] This is related to the fact that the class of excess-demand functions generated by economies has no structure beyond Walras' law and homogeneity.

[20] Arrow and Hahn (1971, chapter 9) take up the question of uniqueness. (One might add to their discussion the substantial literature that deals with the case in which an economy acts as a single agent.) They offer no reason to believe that an arbitrary economy will satisfy the sufficient conditions for uniqueness that they discuss.

[21] Debreu (1970) has shown that (under conditions) except perhaps for a closed subset of measure zero in the space of economies, every economy has a finite set of equilibria. Even this result appears to break down for economies with an infinite number of commodities (Mas-Colell 1975) or with incomplete markets (Yves Balasko and David Cass 1986; John Geanakoplos and Mas-Colell 1985).

It follows from the preceding observation that the Walrasian theory and the existence theorems do not tell us how to relate tastes, technology, and the distribution of wealth to a single set of relative values. Rather, they tell us that there is at least one vector (and possibly many more) of relative values compatible with the data of the model. In the absence of uniqueness, the comparative statics of how prices and allocations will change with a change in the parameter values is not a well-defined exercise. The finiteness result alluded to above may be of some help here, but what is really needed is a completion of the Walrasian theory that describes the particular choices that are made from the equilibrium set. Such a completion will almost surely require a theory that deals explicitly with the adjustment to equilibrium. If forces are not in balance, what changes will take place in order to bring them into balance?[22]

This brings to mind Arrow's work with Hurwicz (Arrow and Hurwicz 1958) and with Block and Hurwicz (Arrow, Henry Block, and Hurwicz 1959) on tâtonnement adjustment processes. In the tâtonnement theory the specification of an initial condition is necessary in order to determine the path that an economy follows to a particular equilibrium. One starts with an initial price vector, and the relative price of a commodity is raised or lowered depending on whether there is excess demand or supply for the good. In fact, the process may not converge (David Gale 1963 and Herbert Scarf 1960) and from what we now know about the structure of excess-demand functions, there is little that one can say about the paths generated by tâtonnement dynamics. (In particular, arbitrary orbits can be created.) Moreover, the tâtonnement does not comfortably correspond to the manner in which we see most prices adjusted in the actual world, and it is most likely for this reason that few would argue today that it is a useful way to select from Walrasian equilibria. (This fact should not diminish the historical importance of the tâtonnement literature as a first try at a rigorous theory that is explicit about the adjustment to equilibrium.)

The completion of equilibrium theory that is necessary in order to generate a determinate theory of value almost surely requires that we come to grips with the manner in which individual buyers and sellers meet, learn, and propose terms of trade. While models are being advanced in which some of this goes on—for example, the search models of Peter Diamond (1981) and its descendants, and the bargaining foundation for Walrasian theory provided by Douglas Gale (1986)—a good understanding of how an economy selects among Walrasian equilibria may require a breakthrough similar in magnitude to what was provided by Walras. Perhaps the fix-price models (these explain rationing at disequilibrium prices) and the bargaining models (these allow agents to pick prices) will provide ingredients of a theory in which the choice of one from the set of Walrasian prices is explained, but they have a long way to go. An equilibrium existence theory should aim toward determinateness. It is also severely limited unless it is embedded in a dynamic theory. The challenge of finding the appropriate ingredients for such a theory of value remains quite open.[23]

Related to the multiplicity of equilibrium is the empirical looseness of the Walrasian theory. However weak the empirical consequences of consumer-demand theory, they are a powerhouse when compared to their general equilibrium counterparts, and all of this has been brought out most clearly as a consequence of the Arrow-Debreu-McKenzie theory. As we have said, in the absence of uniqueness of equilibrium, comparative statics is not well defined; furthermore, the same lack of restrictive structure for the class of market-generated excess-demand functions that makes the multiplicity of equilibrium easy to obtain suggests that even

[22] The following analogy seems appropriate. The theory of gravitation tells us that a ball propelled into an empty room will come to rest only on the floor (and if the floor is not level it will—almost always—come to rest at one of the finite number of local minima on the floor). In order to determine which of the local minima the ball will reach, gravitation must be supplemented by theories that explain the speed and direction that a ball takes on the rebound. The complete theory will have as variables the initial directed velocity of the ball, the composition of the room and ball, and the shape of the room.

[23] Game theory suffers from a similar indeterminateness. One might expect that the notion of markets being in balance is more likely to lead to determinacy than is the notion of Nash equilibrium for games, but this is difficult to make precise.

when equilibrium is unique the comparative statics is quite arbitrary.[24]

III. *The Basic Theorems of Classical Welfare Economics*

The basic theorems of classical welfare economics concern the equivalence between Pareto optimal allocations and Walrasian equilibrium allocations. (The former are defined by the condition that it is not possible to raise any agent's utility without lowering the utility of at least one other agent and the latter by the condition that they are supported by market-clearing prices.) The genesis of the notion of Pareto optimality can be found in the work of Edgeworth (1881) and Pareto (1909). The classical treatment of the equivalence of Pareto optima and Walrasian allocations, via the calculus, is found in the work of Abba Lerner (1934) and Oscar Lange (1942).[25]

Under mild conditions Pareto optimal allocations are identified as solutions to the program: Maximize individual 1's utility subject to resource and technological constraints and to the condition that each other agent's utility is fixed at a prescribed level. With differentiability this yields the condition that for each pair of commodities the marginal rates of substitution be set equal across individuals and that they coincide with the marginal rate of product transformation. Walrasian equilibrium requires utility maximization for households and profit maximization in the production sector relative to a common vector of commodity prices. With differentiability, the former requirement means that for each pair of commodities the marginal rate of substitution must be set equal to the commodity price ratio and the latter means that this price ratio must be equal to the marginal rate of transformation. These arguments demonstrate that (with differentiability) Pareto optimality and Walrasian equilibrium are characterized by the same set of marginal conditions and form the basis for the "marginal-this-equals-marginal-that" proof of the basic welfare theorems.

The equivalence between optima and equilibria is separated into two parts. The first welfare theorem concerns the Pareto optimality of Walrasian equilibrium allocations. It offers the modern expression of Adam Smith's declaration that individuals acting in their own interest will promote the social good. At its basis are the Walrasian model and the Edgeworth-Pareto notion of optimality. The second welfare theorem concerns the possibility of obtaining the efficiency benefits of perfect competition while at the same time maintaining influence over the distribution of income. The theorem provides conditions under which, to each Pareto optimum, there is an assignment of units of account (income) and prices so that the given optimum is a Walrasian equilibrium. The theorem is meant to suggest a separation between efficiency concerns (these are to be taken care of by a decentralized process involving flexible prices, free entry, and so on) and equity concerns (which are to be taken care of by lump sum transfers of income).

The modern version of the welfare theorems is due to Arrow (1951a, CPII, chapter 2) and Debreu (1951). From a technical point of view their contribution was influenced by the work of John von Neumann and Oskar Morgenstern (1947) on game theory, Harold Kuhn and Albert Tucker's (1951) generalization of the classical Lagrange theorem, and Tjalling Koopmans' (1951) activity analysis of production. Arrow paid special attention to the fact that the treatment of the basic theorems finessed corner solutions by assuming interior optima. In his celebrated paper he writes:

> It turns out that, broadly speaking, the optimal properties of the competitive price system remain even when social optima are achieved at corner maxima. In a sense, the role of prices in allocation is more fundamental than the equality of marginal rates of substitution or transformation, to which it is usually subordinated. From a mathematical point of view, the

[24] Here we are taking a hard line, and one with which the workers in computational general equilibrium theory might be expected to disagree (see, for example, John Whalley 1976). If one specifies particular functional forms, then equilibrium may be unique and comparative statics quite definite. But the strength of the conclusions will come from the strength of the forms that are imposed, not from the Walrasian theory per se.

[25] Samuelson's account (Paul Samuelson 1947, pp. 203–19) of the history of the welfare theorems is most worthwhile. Of particular interest is his listing of the variety of ways in which it has been argued that perfect competition represents an optimal situation.

trick is the replacement of methods of differential calculus by the use of elementary theorems in the theory of convex bodies. . . . (p. 18)

It is interesting to note that, in their emphasis on convex analysis, neither Arrow nor Debreu made any remarks whatsoever concerning the fundamental difference in the conditions necessary to establish the basic theorems of welfare economics. Both listed a set of maintained assumptions that include convexity, and proceeded to prove the two theorems under these assumptions. Today proper fuss is made of the fact that the conditions under which optima are equilibria are distinguished from the conditions under which equilibria are optima by the dependence of the former on convexity. This distinction is sometimes considered to be one of the major contributions by Arrow and Debreu to the basic theorems, but in their pioneering contributions it lies somewhat below the surface.

Before turning to the statement and proof of the welfare theorems a further point should be made. Both Arrow and Debreu approached the welfare theorems in a highly axiomatic manner. This is particularly noticeable in Arrow's more developed treatment. Alternative sets of conditions are considered, the proofs are rigorous, and the mathematics is separated from the interpretation. One significant aspect of Arrow's contribution is the style of presentation. Also, the particular axiomatization put forth and developed further in the work with Debreu (1954) has had a very major role in the development of the canonical model of perfect competition.

The argument of the first welfare theorem is essentially algebraic, while the second welfare theorem has a more geometric proof. For simplicity, we recall the arguments for the case of a single firm and under the condition that households prefer more of each commodity to less. The proof that we present of the first theorem is essentially the same as that given by Arrow and Debreu independently. Surely, it is one of the simplest arguments in all of economic analysis, although the result that is established—the Pareto optimality of Walrasian equilibrium—is perhaps the central theorem of price theory. One must also keep in mind that the argument is miles apart from the marginal-this-equals-marginal-that demonstration, and moreover, is more general.

Proof of the First Welfare Theorem

We first sketch out the essential idea of the proof, starting with a Walrasian allocation relative to prices p. If an alternative allocation makes one household better off and keeps the remaining households as well off, then (in the absence of externalities) it must assign each household a bundle of market value no less than what it receives in the Walrasian allocation (and one household must be assigned a bundle of higher valuation).[26] In order to obtain such an alternative allocation, the value (at prices p) of what is produced must be raised. But in a Walrasian allocation the value of what is produced is maximized (relative to prices p) and so the alternative allocation is not feasible, proving efficiency of the Walrasian allocation.

Now we give a more formal proof, taking only the pure exchange case for simplicity. The economy is defined by the preferences of each agent over nonnegative bundles and the initial endowment ω^i of each agent ($i = 1,2, \ldots, n$). An *allocation* is a nonnegative bundle for each agent, $(x^1,x^2, \ldots, x^n)$; the allocation is *feasible* if and only if $x^1 + \cdots + x^n \leq \omega^1 + \cdots + \omega^n$. Suppose that $(x^1,x^2, \ldots, x^n)$ is a Walrasian allocation relative to the nonnegative price vector p. We will argue that there is no feasible allocation $(\bar{x}^1,\bar{x}^2, \ldots, \bar{x}^n)$ for the economy that keeps each household as well off and makes some agents better off than in the equilibrium.

Suppose (in order to show a contradiction) that $(\bar{x}^1,\bar{x}^2, \ldots, \bar{x}^n)$ is such an allocation and assume without loss of generality that it is the first agent that is made better off. By the definition of utility maximization (and in the absence of externalities), agent 1 must not be able to afford $\bar{x}^1$ at prices p; that is, $p \cdot \bar{x}^1 > p \cdot \omega^1$. In fact, because each agent prefers more of each commodity to less, in order to maintain its utility each must spend at least its income; that is $p \cdot \bar{x}^i \geq p \cdot \bar{\omega}^i$ for all i. Summing these inequalities yields $p \cdot (x^1 + \cdots + x^n) > p \cdot (\omega^1 + \cdots + \omega^n)$, but this contradicts the feasibility of $(\bar{x}^1,\bar{x}^2, \ldots, \bar{x}^n)$. Thus the first welfare theorem

[26] With externalities each consumer's utility may depend on the comsumption of others, so it is possible for all households to be assigned a bundle of lower valuation and yet have every household's utility increase.

is proved, and the proof is easily extended to the case of production.

The preceding argument is strikingly simple. It uses only the definitions of equilibrium and optima and a bit of addition. In addition to its simplicity, the treatment by Arrow and Debreu of the first welfare theorem completely changes one's conception of what is important for determining when equilibria are optima. We now realize that equilibria are Pareto optima because an allocation that Pareto improves on a Walrasian allocation must have a higher value than the Walrasian allocation. The fact that, in an equilibrium, marginal rates of substitution are lined up (marginal valuations are equated) is neither necessary nor (without convexity and more) sufficient for optimality. A fine way to appreciate why the contribution of Arrow and Debreu is much more than what can comfortably be called "an extension of a known result" is via an excursion into the world of Samuelson's pure consumption loan model (1958). Here we find an example of an economy with an infinite number of both agents and commodities, and in which the unique Walrasian equilibrium is not Pareto optimal. One's appreciation of the Arrow-Debreu treatment is enhanced by the fact that it leads one immediately to the condition that determines when Walrasian equilibria are optimal in the pure consumption loan model.[27] Because it is based on a deeper understanding of the relation between optima and equilibria than implicit in first-order conditions for optimality, the method introduced by Arrow and Debreu produces substantive results.

We follow David Cass and Menahem Yaari (1966) and consider a specification in which an agent is born in each period $i = 1, 2, \ldots$ and lives for two periods. The agent born in period i is called agent i and his preferences are represented by $U^i(x^i_i, x^i_{i+1}) = x^i_i x^i_{i+1}$, where x^i_j is the amount of commodity consumed in period j by agent i. Each agent i ($i = 1,2, \ldots$) has an initial endowment of 3/4 in period i and 1/4 in period $i + 1$. There is no production. It is easy to see that equilibrium requires that agent 1 demand his initial endowment because nobody else holds or cares for period 1 consumption. Furthermore, if all agents with index less than j must demand their initial endowment in an equilibrium, then agent j must also demand his or her initial endowment. At the initial endowment the marginal utility for consumption by agent i in period i is $MU_i = 1/4$; likewise, $MU_{i+1} = 1/3$. The marginal rate of substitution of each agent is

$$\frac{MU_i}{MU_{i+1}} = \frac{1/4}{3/4} = \frac{1}{3},$$

and so in an equilibrium the price ratios must be $p_1/p_2 = 1/3$, $p_2/p_3 = 1/3$, and so on. In other words, relative prices must be $1,3,9, \ldots$, and it is easily verified that these prices support autarky as an equilibrium. The equality between marginal rates of substitution and price ratios suggests that this equilibrium allocation is an optimum. But this is not the case, because if agent $i + 1$ gives 1/4 of a unit of commodity $i + 1$ to agent i ($i = 1,2, \ldots$), then the first agent's utility rises from 3/16 [= (3/4)(1/4)] to 3/8 [= (3/4)(1/2)], while utility rises in remaining generations from 3/16 to 1/4 [= (1/2)(1/2)].

The Arrow-Debreu proofs give the appropriate warning. In order for an allocation $\bar{x}$ to Pareto dominate (in the sense that all agents are made better off) the Walrasian allocation x strongly (relative to the prices p) it must be the case that the value of each agent's part of $\bar{x}$ is greater than the value of that agent's part of x, which is in turn equal to the value of the agent's endowment. With an infinite number of commodities and agents there is no contradiction here because both $p \cdot \Sigma_i x^i$ and $p \cdot \Sigma_i \bar{x}^i$ may be infinite even though $p \cdot x^i < p \cdot \bar{x}^i < \infty$ for each i. Then, it is possible that $\Sigma_i x^i = \Sigma_i \bar{x}^i$ (each side of the equality is an infinite sum) and this is precisely what happens in the specified example. When the value of the equilibrium allocation is finite, as is the case in the above economy when the endowments are altered to be 1/4 when young and 3/4 when old (implying a marginal rate of substitution of 3, and hence prices $p_1 = 1$, $p_2 = 1/3$, $p_3 = 1/9$, $\ldots$), Arrow and Debreu's reasoning applies and the initial allocation is an equilibrium and also optimal.

The Second Welfare Theorem

The second welfare theorem states that, under appropriate conditions, there corresponds

[27] It is not unreasonable to regard these conditions as more technical than fundamental; however, models in which equilibria are not optima invite further analysis and potentially suggest a role for government intervention.

to each Pareto optimal $(x^1,x^2, \ldots, x^n,y)$ a price vector p so that: (*a*) any bundle that raises the *i*th household utility must cost more than $p \cdot x^i$ for $i = 1,2, \ldots, n$, and (*b*) the firm can make no profit higher than $p \cdot y$. (For simplicity we continue with the case of a single firm.) In other words, under appropriate conditions each Pareto optimal allocation is Walrasian. Arrow's proof of the theorem is geometric, and although the elements of his analysis survive in current treatments, the outline of his presentation is sufficiently different from what one sees today that it is interesting to recall.

First, Arrow considers the case of a single household (see Figure 3). The transformation set T embodies the resource constraint ω and the production possibilities of the firm sector; the indifference curves represent the preferences of the household. The allocation (x,y) is optimal because no point in the set B of points above the indifference curve through x can be produced from the resources available and the production possibilities. Under the assumption that T and B are convex there exists a line separating T and B and this defines the price system that is required by the theorem.[28]

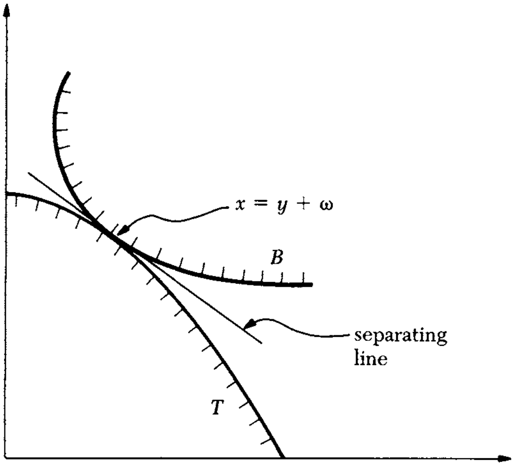


Figure 3. The Second Welfare Theorem for a Single Agent

Next, Arrow takes up the case of several households. In order to exposit his analysis we consider a standard diagram with a transformation set T (again, this embodies the resource constraint and the production set of the firm) and two individuals, one with dashed and one with solid indifference curves (see Figure 4). The allocation (x^1,x^2,y) is optimal. For $k = 1$, or $k = 2$, Arrow defines T_k to be ". . . the set of all possible bundles which individual k can secure for himself if he is given complete charge of the distribution of goods subject only to the conditions that the distribution be compatible with the production possibilities and at the same time not bring any other individual to a position in which the latter is worse off than he would be under the given optimal distribution" (CPII, p. 20). By the definition of optimality, x^k is maximal for the k^{th} agent in T_k (for $k = 1,2$). Arrow observes that, from the case (just considered) of single household optima, ". . . it could be deduced that there is a set of prices for each individual such that utility maximization under a budget constraint would lead him to choose the given optimal point." "However," Arrow continues, "a stronger statement can be made; the same set of prices will do for all individuals." The algebra of the argument in fact shows that if the price system p is defined by the fact that it separates T_1 from the region above the (solid) indifference curve through x^1 (this region is denoted by B_1), then these prices also separate T_2 from the region above the dashed indifference curve through x^2 (this region is denoted by B_2).[29]

Two key assumptions of the theorem concern the convexity of the transformation set and the convexity of the regions above indifference curves. (These are automatic if we posit a continuum of infinitesimal agents.) Arrow's original treatment remains essentially definitive. Although it is somewhat special in its reliance

[28] If there are ℓ commodities, then T and B will be in ℓ-dimensional space. If two convex sets in ℓ-dimensional space do not intersect, then the separating hyperplane theorem states that there exists an $\ell - 1$ dimensional hyperplane separating these sets. Such a hyperplane defines the required ℓ relative prices. That is, there exist prices $p = (p_1, \ldots, p_\ell)$ such that $p \cdot x \geq p \cdot x'$ for all x in T and x' in B.

[29] Some economic intuition for the argument is obtained by realizing that both the pair (T_1, B_1) and the pair (T_2, B_2) contain all directions in which some agent's utility can be raised and to which production can be moved. A more precise understanding comes from the fact that the price system p^i defined by separating T_i from B_i must separate $T_i - B_i$ from the origin $(i = 1,2)$. (The set $T_i - B_i$ contains precisely those points of the form $t_i - b_i$, with t_i in T_i and b_i in B_i.) However, $T_1 - B_1$ and $T_2 - B_2$ are identical; thus $p^1 = p^2$.

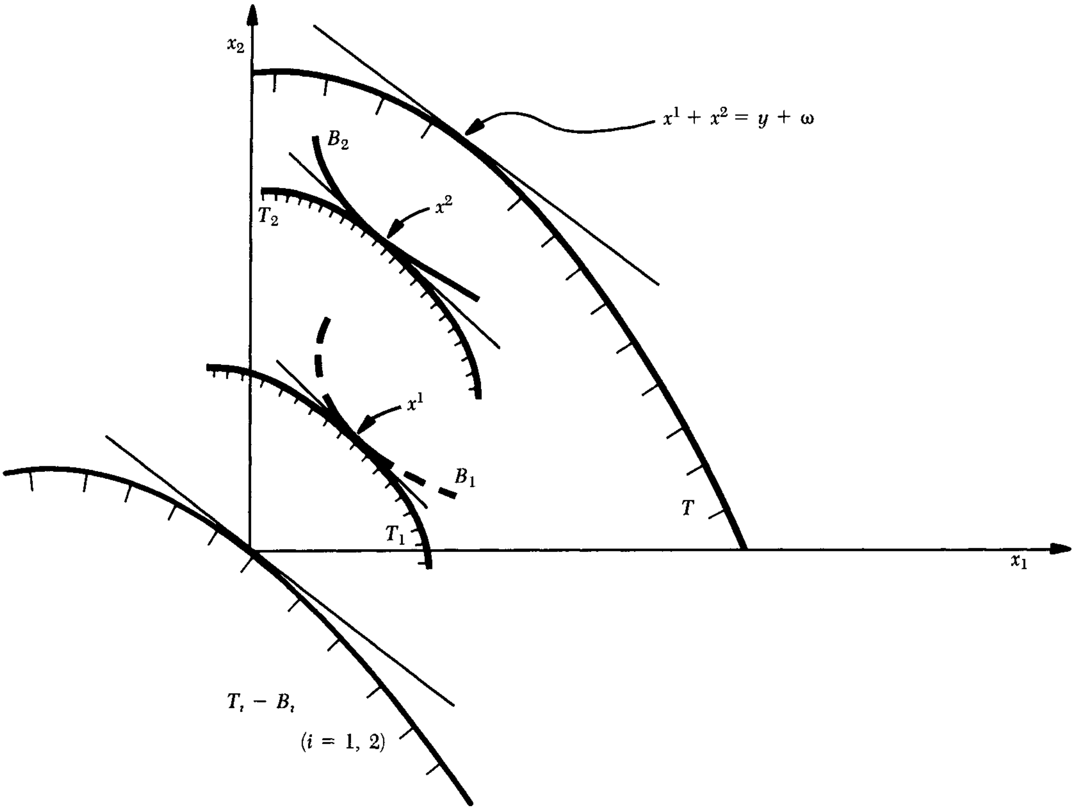


Figure 4. The Second Welfare Theorem with Two Agents

on strictly convex preferences (this rules out indifference curves with flat spots), Arrow anticipated with remarkable accuracy the variety of subtle details to deal with in cases of nonmonotonic preferences, lack of free disposal, and optima that are on the boundary of the consumption set. Of special note is the so-called exceptional case of a Pareto optimum on the boundary of the consumption set that is not a Walrasian equilibrium. (In the one-agent economy of Figure 1, the endowment is an optimal allocation, but there is no price at which it is a Walrasian allocation. To see this, observe that at positive prices for the first commodity there is positive excess demand for the second commodity, and at a zero price for the first commodity there is "infinite" excess demand for that commodity; in other words, there are no prices that make the optimum an equilibrium. Arrow's classic example of an optimum that is not an equilibrium [CPII, p. 39] is for a two-person, two-commodity Edgeworth Box economy.) Debreu's treatment of the second welfare theorem appears at first simpler than Arrow's, and the basic ingredients in the two approaches are undeniably the same. Arrow's treatment, however, because it explicitly comes to grips with the problem of boundary optima, bliss points, and so on, is the more developed product.

There is no doubt that the emphasis on the second welfare theorem in Arrow's "Extension of the Basic Theorems . . ." (CPII, chapter 2) is strong. For the question that he had in mind in that paper, ". . . the Separating Hyperplane Theorem supplied the answer" (CPII, p. 14). More than twenty years later, in his Nobel lecture, Arrow emphasized the second theorem and explained in some detail the separation theorem for convex sets. Regarding the first theorem he writes only: "A by-product of the investigation was the proof of the converse theorem: A competitive equilibrium is always Pareto-efficient, and this theorem is true without any convexity assumption" (CPII, p. 216). As we have noted, the observation that convexity is unnecessary is not made in "An Extension . . ." So strong is the emphasis on conditions under which the conclusion of the second theorem holds that convexity is among the maintained hypotheses of the paper. Certainly, the second welfare theorem is the more mathematically interesting of the two classical propositions, but is it the more important? Despite his emphasis on the second theorem, Arrow is perhaps torn. He writes, "From the point of view of policy, the most important consequence . . . [is] that the use of the price system will lead to a socially optimal allocation of resources" (CPII, p. 18). This is, of course, the first theorem. It is tempting to conjecture that the proof of the first theorem that Arrow and Debreu devised was so trivial that it was a little while before they realized what they had: no convexity, no separating hyperplane theorem, but the kernel of Adam Smith's declaration concerning the invisible hand in a few short lines.[30]

The Welfare Theorems: Interpretation and New Directions

Arrow's work on the welfare theorems should be understood in the context of what has been called the "New Welfare Economics." As Arrow writes, "The hope . . . is that the problems of social welfare can be divided into two parts: a preliminary social value judgment as to the distribution of welfare followed by a detailed division of commodities taking interpersonal comparisons made by the first step as given" (CPII, pp. 40, 41). He regarded his earlier work on social choice, the celebrated general possibility theorem (Arrow 1951b), to be relevant to the first part of the problem; in "An Extension . . ." he was dealing with the second part. For the new welfare economics, ideas such as decentralization (agents' actions depend only on their own characteristics) and informational efficiency are central. From this perspective the first welfare theorem is perhaps best read in the following manner: If Pareto optimality is our only concern (in particular, with no concern for the distribution of income), then for economies without externalities or public goods the competitive mechanism is a satisfactory *decentralized* method for allocating resources. But caution is in order. As was the case with the existence theorems, the first welfare theorem does *not* tell us when perfect competiton will arise, or more specifically when the freedom to bargain, exchange, enter a market, and so on will create incentives for agents to act as if prices are beyond their control.[31] The following version of the first welfare theorem, in its attention to incentives, provides an especially useful expression of Adam Smith's declaration regarding the invisible hand. It adds to the usual analysis the Walrasian existence theorem, the notion of infinitesimal agents, and noncooperative foundations for price taking (see Footnote 17). This is a modern synthesis, which should be understood to go well beyond the early work of Arrow and Debreu.

PROPOSITION. *For externality-free economies in which all agents are infinitesimal, there is theoretical support for the assertion that individual agents, each free to bargain, exchange, and make the deals that are best for themselves, will be forced to act as if they are price takers. Furthermore, in the presence of mild technical assumptions, such economies have at least one Walrasian (price-taking) equilibrium and each of these is Pareto optimal. Finally, in this equi-*

[30] The importance for competitive analysis of convexity theory and the separation theorem for convex sets is strengthened by the work of Debreu and Scarf (1963) and Aumann (1964) on the relation between core and Walrasian allocations. One should note that the proof of the second welfare theorem sits inside of the proof of these theorems. To be precise, the separating hyperplane theorem and the arguments by Arrow and Debreu that follow its application in the second welfare theorem supply half of the answer to the relation between core and Walrasian equilibrium. This was of course an unpredictable consequence of the early efforts of Arrow and Debreu, but it is a significant aspect of their contribution.

[31] An example of a context in which Arrow expresses sensitivity to this point concerns the interpretation of externalities as ordinary commodities. He writes, "Externalities can be regarded as ordinary commodities, and all of the formal theory of competitive equilibrium is valid, including its optimality" (CPII, p. 146). But he notes that with this interpretation, "Markets for externalities usually involve small numbers of buyers and sellers. . . . Even if competitive equilibrium could be defined, there would be no force driving the system to it: we are in the realm of imperfect competiton."

librium the actions of agents will be decentralized in the sense that they will depend only on their own characteristics and the common prices.[32]

As soon as we go beyond Pareto optimality by adding distributional concerns, that is, as soon as we bite the bullet of interpersonal comparisons of utility, then perfect competition is no longer a satisfactory method for allocating resources. In a private ownership economy with perfect competition, "thems that have, get." Arrow recalls lectures about the relative efficiency of methods used to alter the distribution of income. He speaks of " . . . informal efficiency arguments [which] hinged on the idea that under rent control people were buying the wrong kind of housing, say, excessively large apartments" (CPII, p. 213). The second welfare theorem is a response to these efficiency arguments and suggests the possibility of a separation between efficiency concerns and equity concerns. The separation is achieved by transferring units of account to satisfy equity concerns and then allowing the decentralized functioning of competitive markets to take care of efficiency.

To what extent can competitive markets be used to achieve efficient allocations that are also desirable with respect to income distribution? Arrow has much to say about the mathematics of the second welfare theorem, but almost nothing to say concerning his belief in the use of competitive markets to reach the dual objectives of efficiency and favorable income distribution. Let us start by giving a best modern statement of the second welfare theorem.

PROPOSITION. *For externality free economies in which all agents are infinitesimal, there is theoretical support for the assertion that individual agents, each free to bargain, exchange, and make the deals that are best for themselves, will be forced to act as if they are price takers. Furthermore, in the presence of mild technical assumptions (not including convexity because agents are infinitesimal) every Pareto optimum can be achieved as a Walrasian (price-taking) equilibrium after a suitable redistribution of income. Finally, in equilibrium the actions of agents are decentralized in that they will depend only on the agents' own characteristics.*

As a policy tool for achieving efficient allocations that are also desirable with respect to income distribution, the second welfare theorem faces a (well-known) difficulty. Even when there is a continuum of infinitesimal agents, the process of redistributing units of account creates incentive problems (income transfers distort the labor-leisure tradeoff, wealth taxes distort the incentives to accumulate wealth). All optima can be achieved in a decentralized manner by income transfers plus the Walrasian mechanism (perfect competition); however, the implementation of such a regime requires each agent to ignore the nonnegligible effect of his labor supply on the amount of taxes that he must pay. When one properly takes into account the efficiency costs of transfers, one's preferences among price controls, commodity taxes, and income tax schemes become less clear. Even the decentralization of the above process may be called into question in the sense that the set of available optima cannot be known in advance without centralized knowledge and computation.

What is left of the second welfare theorem is the idea that, if we ignore the incentive effects of transfers, then the search among optima for distributionally desirable allocations can be carried out by transfers followed by flexible prices.[33] But there is no reason to believe that incentive effects associated with processes of income redistribution are small and this limits the relevance of the result for policy. The response has been a line of work that is very much in the spirit of the new welfare economics and yet goes beyond the second welfare theorem by taking incentive effects and decentralization into account. For a world in which the distribution of abilities is known (workers pro-

[32] Keep in mind the negative reading of the theorem: This argument regarding the benefits of the invisible hand requires the absence of both externalities and significant increasing returns to scale. The actual world has a fair amount of both.

[33] Another reading of the theorem is that objections concerning the results of competition (again, externalities and monopolistic elements aside) can be seen as objections to the distribution of income. This reading is more passive in that it does not suggest that transfers will or should be implemented.

duce different amounts of output per unit of labor supplied), but in which agents can be taxed only according to their incomes (and not their abilities), James Mirrlees (1971) formulated and solved the problem of determining the optimal tax schedule relative to given social preferences over the distribution of income. While one can hardly claim that the seeds of Mirrlees' work can be found in the results of Arrow and Debreu, the clarity and rigor of the Arrow-Debreu formulation and the precision of their results invited further research on the new welfare economics. Also, one might mention that the idea that workers privately know their abilities and self-select their work levels bears an important relation to Arrow's (1963) discussion of markets with adverse selection. Mirrlees (1971) took the further step of maximizing social preferences subject to an incentive constraint, that is, subject to allocations that are individually optimal given the tax schedule.

A more direct descendant of Arrow's work on social choice and the welfare theorems, and in particular of Arrow's work with Leonid Hurwicz, is an extraordinary literature that deals with the possibility of achieving social optima subject to the constraint that agents act in a decentralized manner and in their own interest. Here, rather than being a given of the problem, the institutions of exchange become an object of choice.[34] With respect to incentive aspects, ideas from game theory have played a major role. We have learned that even with public goods there exist decentralized incentive-respecting mechanisms that produce partial equilibrium optima (as shown, for example, by Theodore Groves 1973 and Edward Clarke 1971).[35] Groves and John Ledyard (1977) were the first to extend the analysis to general equilibrium; however, this comes with some loss in decentralization. With respect to the decentralization of information and informational efficiency, Arrow and Hurwicz' paper "Decentralization and Computation in Resource Allocation" (1960) is a most influential early contribution, and the theory of teams (Jacob Marschak and Roy Radner 1972) adds a statistical perspective to our understanding of the possibilities of decentralization. The foundational approach to the problem of decentralization and informational efficiency starts with the definition of these terms. The pioneering work is due to Hurwicz (1960); some especially interesting further results are due to Kenneth Mount and Stanley Reiter (1974).

Without any question Arrow's pioneering efforts have had an important role in the development of a framework for analyzing institutional arrangements that range from auctions (Myerson 1985), to voting schemes (Allan Gibbard 1973 and Mark Satterthwaite 1973, 1975), to hierarchical structures for a firm (Geanakoplos and Paul Milgrom 1985), to arrangements between a principal and his agent (Stephen Ross 1973). Arrow's work on the classical welfare theorems is best appreciated as a part of a research agenda that includes the general possibility theorem and his interest in decentralization and incentives.

IV. *Uncertainty in General Equilibrium*

In 1952 Arrow presented the first general equilibrium theory of the allocation of uncertain consumption.[36] As opposed to the significant

[34] A recent exposition of the state of this literature is provided by Leonid Hurwicz (1985). Hurwicz own sustained contribution deserves special attention. Work by Arrow and Hurwicz individually, and several joint papers on the subject, are reprinted in *Studies in Resource Allocation Processes* (1977) and are not contained in the *Collected Papers*.

[35] Two of the most significant contributions are due to Myerson and Maskin, both students of Arrow. Their approaches are summarized in Eric Maskin (1985) and Roger Myerson (1985).

[36] Maurice Allais (1953) simultaneously presented a model of equilibrium under uncertainty represented by normally distributed random variables. In this and other regards, Allais' results did not establish a general paradigm in the way Arrow's did. The historical notes in this volume of Arrow's works show that Arrow was personally well poised to incorporate uncertainty into general equilibrium theory, given his background as a student of statistics under Hotelling (his supervisor) and Abraham Wald. Influenced especially by Wald's (1950) foundational developments in statistics and by Leonard Savage's (1954) new treatment of decisions under uncertainty, Arrow adopted the emerging viewpoint of uncertainty in decision making: One can treat uncertainty in terms of a set of *states* of the world, say $(1,2, \ldots, S)$, one of which is to occur. In general, a decision maker acts before learning which of these states will occur so as to maximize the utility of outcomes that depend

technical contributions of Arrow's work on the problems of existence and efficiency of equilibrium allocations, the achievements of his paper *The Role of Securities in the Optimal Allocation of Risk Bearing*[37] are entirely conceptual and interpretational. None of the key results call for a new proof.[38] The ideas in the paper may nevertheless represent Arrow's most influential contribution to equilibrium theory. Despite the title of the paper, its stated goal is broader: to extend "the theory of the optimal allocation of resources under conditions of certainty . . . to conditions of subjective uncertainty" (CPII, p. 48). Arrow accomplishes his goal using two fundamentally different market allocation schemes. First, he introduces the notion of *contingent commodities*, which allows the treatment of uncertain consumption within the usual general equilibrium theory of allocation. As an alternative allocation mechanism, Arrow then introduces the combined use of financial security markets and spot commodity markets, which is distinct from the usual general equilibrium model in that trade takes place at two distinct points in time. Securities are traded before uncertainty about the state of the world is resolved; after the true state is revealed, agents collect their security dividends and trade on spot commodity markets. A central observation of the paper is that with perfect foresight of spot market prices the two allocation schemes, contingent commodity markets on the one hand and "dynamic" security and spot market trading on the other, lead to the same allocations. In particular, the use of security markets allows a sparse market structure: Rather than complete markets for all commodities in all future states of the world, it is enough (for the purpose of obtaining efficient allocations) to have spot market trading and enough security markets to generate any income stream. Indeed, this is an incipient version of the modern theory of financial asset pricing, which shows that one can avoid setting up markets for a large variety of contingent claims by allowing repeated trading of a sparse set of securities.

After explaining in some detail Arrow's treatment of contingent-commodity market equilibrium and security markets equilibrium, we will present an important criticism of the latter theory by discussing the perfect foresight assumption on which it depends. Finally, we will explain how Arrow's insights lead directly to the literature on the spanning role of financial securities and asset pricing theory. This literature represents one of the most significant and compelling uses of a recent advance in economic theory.

Contingent Commodities

Arrow's idea of *contingent commodities* is simple. In general equilibrium theory, one speaks of a bundle $(x_1, x_2, \ldots, x_\ell)$ of ℓ goods. Classical examples of a good include commodities such as corn or wheat. With these two goods alone, $\ell = 2$. Suppose, however, that there are also two possible states of the world, labeled *rain* and *shine*, in which consumption of both corn and wheat may occur. Only one

on the states of the world. For a given model, any variable (such as a price, quantity, parameter, or action) is treated as a *random variable*, meaning a function assigning a particular outcome to each state of the world. Arrow brought this perspective on individual decision making to bear on the interaction of agents in the general equilibrium model. Quoting from Arrow's notes (CPII, p. 47), "I was led by the Wald-Savage viewpoint to consider an elementary decision as one that took a unit value for one state of nature and zero elsewhere; thus all general decisions could be regarded as bundles of elementary decisions."

[37] This paper appears with a new introduction as chapter 3 of the *Collected Papers*, Volume 2. It appeared originally in French in 1953 in the proceedings of the Colloque sur les Fondments et Applications de la Theorie du Risque en Econometrie of the Centre Nationale de la Recherche Scientifique (Paris). These proceedings were published under the title *Économetrie*, Number 42. As Arrow writes in his *Collected Papers*, the original 1953 paper was translated into French with Arrow's help from his English manuscript by the Institut des Sciences Economiques Appliquées. Rather than translating back into English, the *Review of Economic Studies* published Arrow's original English manuscript in 1964. In 1971, a different version appeared in Arrow's *Essays in the Theory of Risk Bearing*. Thus four different published versions now exist!

[38] The only argument calling for a formal proof is one showing that, if an agent's preferences are convex and represented by a von Neumann–Morgenstern utility function u, then u is quasi-concave. This result is provided in order to interpret Arrow's assumption that preferences are of this von Neumann–Morgenstern variety. In a recent paper, Arrow (1987) discusses the role of the von Neumann–Morgenstern assumption in this setting, and how it has been relaxed by others such as Allais (1953) and Debreu (1951).

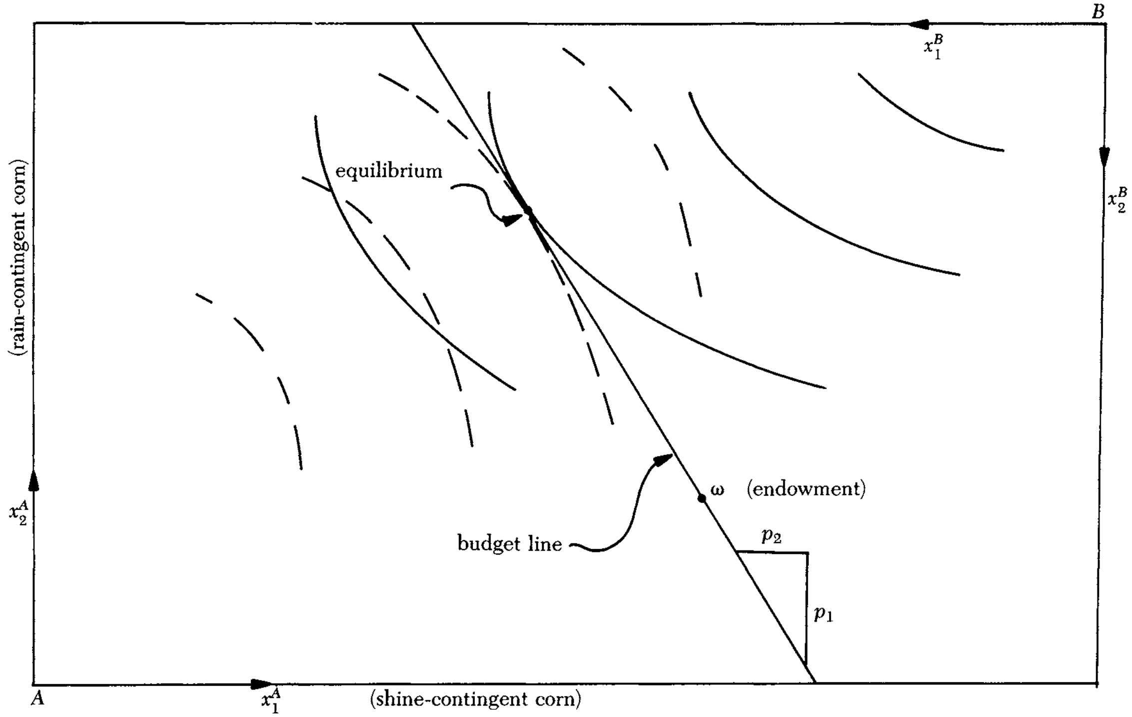


Figure 5. Edgeworth's Box for State-Contingent Commodities

of these states will actually occur, but there is uncertainty as to which one. All together, in Arrow's model, there are then $\ell = 4$ goods: corn contingent on rain, corn contingent on shine, wheat contingent on rain, and wheat contingent on shine. These four goods are examples of *contingent commodities*. At prices (p_1,p_2,p_3,p_4) for these four goods (in the same order), one could purchase a unit of corn contingent on rain for p_1. In return, one receives one unit, say a bushel, of corn if and only if it rains. Any particular state-contingent consumption plan can be purchased by combining the four individual contingent commodities. At the announced prices, for example, one could purchase a bushel of corn with certainty at a cost of $p_1 + p_2$.

The idea of contingent-commodity markets is most easily illustrated in the case of a single commodity, say corn, and two states of the world, say rain and shine. The Edgeworth box drawn in Figure 5 shows two agents' preferences for two goods: corn contingent on rain and corn contingent on shine. The height of the box indicates the total amount of corn available if it should rain; the width represents the amount of corn available in the other contingency, shine. Each point in the box represents a particular allocation of these total endowments to the two agents, measuring the allocation of agent A from the bottom left corner, and the allocation of agent B from the top right corner. The point of original endowments is indicated by ω. Each line through ω indicates the set of budget-feasible allocations for each agent corresponding to a particular pair of prices for the two contingent commodities. The slope of such a budget line is the ratio of the prices of the two goods, as shown. The solid curved lines indicate the preferences of agent A; any allocation to the right and above a solid curved line is preferred by A to any point below and to the left. Likewise, the dashed curved lines indicate the preferences of agent B, increasing to the bottom and left. Given a particular budget line, agent A will choose the allocation on that budget line lying on the solid preference curve closest to the top-right corner

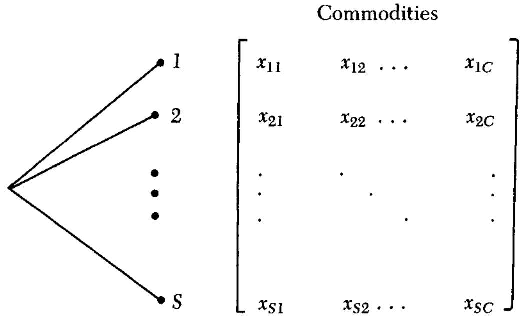


Figure 6. State-Contingent Commodities

of the box. This would imply tangency of the budget line to the solid preference curve at the point of A's optimal allocation. The budget line drawn in Figure 5 corresponds to an equilibrium.

The equilibrium shown is an "efficient allocation of risk bearing," using Arrow's words. If the two agents do not trade the contingent commodities, but rather allow the true state to be revealed without trading, they both miss the opportunity to insure themselves against the risk of a small endowment of corn. (Once the true state is revealed, there is obviously no incentive to trade because both agents then hold the same commodity.) Assuming the two agents have differentiable concave utility functions, we know that the allocation of risk (in this case) is Pareto efficient if and only if the ratio of marginal utility for rain-contingent corn to marginal utility for shine-contingent corn is the same for the two agents. That is to say, an allocation point is efficient if and only if the indifference curves of the two agents have the same slope at the allocation point. This condition is true because both agents find it optimal to equate their ratios of marginal utilities to the ratio of the corresponding prices, the slope of the budget line.

In general, with C commodities and S states of the world, there are SC goods, called *state-contingent commodities,* available for trade. A bundle $x = (x_{sc})$ of state-contingent commodities is illustrated in Figure 6. The unit price for delivery of commodity c contingent on state s is denoted p_{sc}, and the list of all state-contingent commodity prices is denoted $p = (p_{sc})$. Likewise, for a particular allocation of the available state-contingent commodities to the various agents, the number of units of commodity c allocated to agent i contingent on state s is denoted x^i_{sc}, and the whole bundle of state-contingent commodities for agent i is denoted x^i. The market value of x^i at the given prices p is $\Sigma_{s=1}^{S} \Sigma_{c=1}^{C} p_{sc}x^i_{sc}$, which is denoted $p \cdot x^i$. The bundle x^i is optimal for agent i at the given prices if, for any strictly preferred bundle $\hat{x}^i$, we have $p \cdot \hat{x}^i > p \cdot x^i$. A contingent-commodity market equilibrium[39] consists of a feasible allocation $(x^1, \ldots, x^n)$ of contingent commodities as well as contingent-commodity prices p, such that, for each i, the bundle x^i is optimal for agent i.

Efficiency and Equilibrium Under Uncertainty

When Arrow speaks of the "allocation of risk bearing," he refers to the allocation of contingent commodities, in the usual sense of the allocation of available goods. An "optimal allocation of risk bearing" thus means a Pareto efficient allocation of contingent commodities. In describing that part of his paper dealing with contingent commodities, Arrow writes, ". . . it is briefly argued that, if there exist markets for claims on all commodities, the competitive system will lead to an optimal allocation under certain hypotheses" (CPII, p. 48). This statement is properly argued by invoking the first welfare theorem, but this is not in fact what Arrow shows. Rather, Arrow uses the second welfare theorem to show that an efficient allocation of contingent commodities can be viewed as an equilibrium allocation, at some prices for state–contingent commodity prices p. That is, if the agents' preferences are strictly monotone and convex[40] and if $(x^1, \ldots, x^n)$ is an efficient allocation, then there exist contingent commodity prices $p = (p_{sc})$ such that, if any agent i strictly prefers the state–contingent commodity bundle $\hat{x}^i$ to the allocated bundle x^i, then $p \cdot \hat{x}^i > p \cdot x^i$. Given the wonderful conceptual leap made by Arrow in defining state–contingent commodities, there is really nothing to demonstrate. This is precisely the second wel-

[39] We are speaking here of a compensated equilibrium; that is, $(x^1, \ldots, x^n, p)$ is a compensated equilibrium provided it is an equilibrium relative to the endowments $(x^1, \ldots, x^n)$.

[40] This is certainly the case if preferences are represented by von Neumann–Morgenstern utility functions of the form $E[u(x)]$, where u is strictly monotone and quasi-concave.

TABLE 1
A SECURITY-SPOT MARKET EXAMPLE

	Dividends		(Corn, Wheat)		
	d_{1s}	d_{2s}	$\bar{p}_s$	x_s	$\bar{p}_s \cdot x_s$
s = *rain*	3	2	(1,1)	(4,6)	10
s = *shine*	3	6	(2,1)	(1,4)	6

fare theorem[41] itself, and not an application or extension of it.

One need not stop at state–contingent extensions of the notion of an economic good; Debreu (1953; 1959, chapter 7) allowed the definition of a good to include the state, time, and location of its use, in addition to its physical characteristics. As Debreu notes, this yields a model "formally identical with the theory of certainty." Here we have one of the premier triumphs of the separation of the mathematics from the interpretation that we mentioned in Section III. Arrow obtained an important new result by reinterpreting the available mathematics.

The Introduction of Securities to the General Equilibrium Model

We can illustrate Arrow's introduction of security markets into the general equilibrium model by starting with a simple example. Consider the problem faced by an agent in a model with two states of the world, *rain* and *shine*, and two commodities, *corn* and *wheat*. There are four contingent commodities, *but they are not for sale.* Rather there are two securities for sale; security number one promises a dividend of 3 units of account (say dollars) with certainty; security number 2 promises a dividend d_2 of \$2 continent on rain and \$6 contingent on shine, as indicated in Table 1. The agent may purchase any portfolio of these two securities before the weather is revealed. Once the weather is known, the dividends received from the portfolio can be spent on spot commodity markets. The spot commodity prices are also shown in Table 1; the price of wheat is always \$1 per bushel, while the price of corn is \$1 per bushel if rain and \$2 if shine.

Suppose the agent's plan is to eat 4 bushels of corn and 6 of wheat if it rains, and if it shines, 1 bushel of corn and 4 bushels of wheat, as shown in Table 1 under the consumption plan x. At the spot commodity prices shown in the table under $\bar{p}$, this consumption plan requires \$10 in rain and \$6 in shine. The portfolio $y = (y_1, y_2)$ of securities required to finance this consumption plan, y_1 shares of d_1 and y_2 shares of d_2, therefore must solve the equations:

$$\begin{aligned} y_1 3 + y_2 2 &= 10 \\ y_1 3 + y_2 6 &= 6. \end{aligned} \tag{2}$$

The solution is $y_1 = 4$ and $y_2 = -1$. That is, the agent finances the consumption plan x by purchasing 4 shares of security 1 and short selling[42] 1 share of security 2. Given *shine*, for example, the agent collects \$6 in dividends, just that required to pay for the *shine*–contingent consumption plan. Clearly any consumption plan x could be financed by a combination of security trading and spot trading. In this way, contingent-commodity markets can generally be replaced in this manner by security and spot markets, provided there is a complete set of security markets.[43]

Security-Spot Market Equilibrium

Now let us return to the general case of S states of the world with C commodities con-

[41] Arrow curiously neglected to cite his own treatment of the second welfare theorem (Arrow 1951a, CPII, chapter 2), published only the year before, which is necessary for the result in this generality, and merely referred to the work of Lange (1942) and Samuelson (1947).

[42] To short sell a security is to receive (rather than pay) its price in return for the obligation to pay (rather than receive) its dividends.

[43] By a complete set of security markets, we mean at least as many linearly independent security dividend vectors as states of the world. If there are N securities defined by dividend vectors $d_1, d_2, \ldots, d_N$, we can let D denote the S by N matrix whose (s,n)-element is d_{ns}. Then equation (2) is in general replaced by a system of S equations in n unknowns. (The portfolio of shares $y = (y_1, \ldots, y_N)$ of securities is chosen to solve an equation of the form $Dy = w$, where $w = (w_1, \ldots, w_S)$ is the vector of required contingent dividend income.) The system has a solution if D is a matrix of rank S, or equivalently, if there are S linearly independent security dividend vectors.

TABLE 2
ARROW SECURITY DIVIDENDS

State	d_1	d_2	...	d_S
1	1	0	...	0
2	0	1	...	0
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
S	0	0	...	1

sumed in each state. Arrow assumed the existence of markets for S securities; as illustrated in Table 2, the dividend payoff d_s of the sth security is taken for simplicity to be 1 if state occurs, and 0 otherwise.

It is known by all agents in advance of any trading, and before the true state is revealed, that if state s occurs, then the spot price of commodity c is some fixed number $\bar{p}_{sc}$. This is the rational expectations assumption of perfect foresight; we will later give a critical discussion of this strong assumption. Before the true state is revealed, the securities are traded at positive prices $q = (q_1, \ldots, q_S)$. A budget-feasible portfolio is a bundle $y = (y_1, \ldots, y_S)$ of the S securities whose total market value $q \cdot y = q_1y_1 + \cdots + q_Sy_S$ is no greater than zero.

Agent i is defined by an endowment $\omega = (\omega^i_{sc})$ of state-contingent commodities and preferences over state-contingent commodities: $x^i \succ_i \bar{x}^i$ means that agent i strictly prefers the state-contingent commodity bundle $x^i = (x^i_{sc})$ to the bundle $\hat{x}^i$. Given the security and spot prices, q and $\bar{p}$, a budget-feasible plan for agent i is a budget-feasible portfolio y^i and a state-contingent commodity bundle $x^i = (x^i_{sc})$ such that, for any state s,

$$\bar{p}_s \cdot x^i_s \leq \bar{p}_s \cdot \omega^i_s + y^i_s, \qquad (3)$$

where $\bar{p}_s = (\bar{p}_{s1}, \ldots, \bar{p}_{sC})$ is the vector of spot commodity prices in state s, and where the vectors x^i_s and ω^i_s are analogously defined. The left-hand side of (3) is the spot market value of the planned consumption purchases in state s; the right-hand side is the sum of the spot market value of the endowment in state s plus the dividends accruing to the security portfolio y in state s. A budget-feasible plan (x^i,y^i) for agent i is an optimal plan for agent i provided there is no budget-feasible plan $(\hat{x}^i,\hat{y}^i)$ such that $\hat{x}^i \succ_i x^i$.

The collection $[(x^1,y^1), \ldots, (x^n,y^n),\bar{p},q]$ is a security-spot market equilibrium if, for each i, the plan (x^i,y^i) is optimal for agent i given the prices $\bar{p}$ and q, and if security markets clear $(y^1 + \cdots + y^n = 0)$ and spot markets clear $(x^1 + \cdots + x^n = 0)$.

The notion of security-spot market equilibrium suggests that it will be optimal for agent i, once the true state is revealed, actually to carry through with the budget-feasible plan to consume the bundle $x^i_s = (x^i_{s1}, \ldots, x^i_{sC})$. The von Neumann–Morgenstern expected utility assumption allows one to make sense out of such a utility comparison in state s between this bundle x^i_s and any other commodity bundle, and moreover has the property that the originally chosen bundle x^i_s is indeed still optimal given the occurrence of state s.

The Second Welfare Theorem for Security Markets

How do security markets support an efficient endowment allocation $(\omega^1, \ldots, \omega^n)$ of state-contingent commodities? Arrow had already shown that this allocation can be supported by setting up SC different contingent-commodity markets. As we saw earlier, by invoking the second welfare theorem, Arrow showed that there are contingent-commodity prices p (that is, a unit price p_{sc} for delivery of commodity c contingent on state s) such that

$$\hat{x}^i \succ_i \omega^i \quad \text{implies} \quad p \cdot \hat{x}^i > p \cdot \omega^i. \qquad (4)$$

To complete the demonstration it is sufficient to observe that any contingent-commodity equilibrium allocation is also a security-spot market equilibrium allocation. In fact, the contingent-commodity and security-spot market allocations are identical, and this fact does not depend on the conditions needed for the second welfare theorem. Arrow's equations are adequate to establish this equivalence. Based on knowledge of these contingent-commodity prices, he proceeded to construct a security-spot market equilibrium also supporting the same allocation as follows:

The spot price for commodity c in state s is set to be $\bar{p}_{sc} = p_{sc}/q_s$, where q_s is any strictly positive price[44] for the sth security. The endowment allocation ω^i is clearly budget-feasible (at any prices) for agent i with the security portfolio $\bar{y}^i = 0$, because it requires no trade to achieve ω^i.

We claim that $[(\omega^1,\bar{y}^1), \ldots, (\omega^n,\bar{y}^n),\bar{p},q]$ is in fact a security-spot market equilibrium. Because this is obviously a market-clearing allocation of securities and contingent commodities, in order to prove our claim one must show only that $(\omega^i,\bar{y}^i)$ is an optimal plan for agent i at the prices $(\bar{p},q)$. Suppose agent i strictly prefers to consume the bundle $\hat{x}^i = (\hat{x}^i_{sc})$ of contingent commodities. If $\hat{y}^i = (\hat{y}^i_1, \ldots, \hat{y}^i_S)$ is a security portfolio satisfying the corresponding budget-feasibility equations

$$\bar{p}_s \cdot \hat{x}^i_s \leq \bar{p}_s \cdot \omega^i_s + \hat{y}^i_s$$

for all s, then

$$\hat{y}^i_s \geq \sum_c \bar{p}_{sc} (\hat{x}^i_{sc} - \omega^i_{sc})$$

for all s. The cost of the shares of the sth security are then

$$q_s\hat{y}_s \geq q_s \sum_c \bar{p}_{sc}(\hat{x}^i_{sc} - \omega^i_{sc}) = \sum_c p_{sc}(\hat{x}^i_{sc} - \omega^i_{sc}).$$

The total cost of the required portfolio $\hat{y}^i = (\hat{y}^i_1, \ldots, \hat{y}^i_S)$ of securities is therefore

$$q \cdot \hat{y}^i = \sum_s q_s y_s \geq \sum_s \sum_c p_{sc}(\hat{x}^i_{sc} = x^i_{sc})$$
$$= p \cdot \hat{x}^i - p \cdot x^i.$$

But this total portfolio cost must be strictly positive, using relation (4). The portfolio $\hat{y}^i$ required to finance $\hat{x}_i$ is therefore not budget-feasible. Thus $[(\omega^1,\bar{y}^1), \ldots, (\omega^n,\bar{y}^n),\bar{p},q]$ is indeed a security-spot market equilibrium supporting the efficient allocation $(x^1, \ldots, x^n)$. Thus, any contingent-commodity equilibrium allocation is also a security-spot market allocation, and the second welfare theorem for security markets is proved.

[44] Arrow implicitly assumed the existence of an additional security, inside money, that is, a security paying a dividend of one in every state, selling initially for \$1 (bearing no interest). In this case, the absence of arbitrage requires that $q_1 + \cdots + q_S = 1$, as remarked by Arrow (CPII, p. 52). For example, one could take q_s to be a commonly held subjective probability that state s occurs. There is no need, however, to bring money or probabilities into the story.

Arrow saw substantial significance in the ability of security markets to support an efficient allocation. In his words, "Socially, the significance of the theorem is that it permits economizing on markets; only $S + C$ markets are needed to achieve the optimal allocation, instead of the SC markets implied in Theorem 1" [which pertains to contingent-commodity markets] (CPII, p. 52).

Once again, one can easily be confused by Arrow's application here of the second welfare theorem juxtaposed with his words suggesting the first welfare theorem: "Under certain hypotheses, the allocation of risk-bearing by competitive security markets is in fact optimal" (CPII, p. 48). The first welfare theorem is not actually applied, although it seems to carry more significance than the second in this setting. For example, beginning as we do above with an efficient allocation, any set of security markets, complete or incomplete, is consistent with the existence of an equilibrium in security and spot markets supporting the given allocation. This can be seen by reviewing the arguments made in the above (complete markets) case, and noting that they apply a fortiori if some (or all) of the security markets are missing. (Of course, the result is stronger when agents are allowed access to complete markets.) In order to show that any security-spot market equilibrium is Pareto efficient (the first welfare theorem), however, complete markets are required, as shown by Oliver Hart (1975).

Rational Expectations and Perfect Foresight

Along with the introduction of security markets, Arrow makes the first explicit use of the so-called perfect foresight assumptions regarding equilibrium price expectations.[45] In Arrow's model each agent prepares a complete catalog of future states, with identical (and correct!) beliefs regarding the spot prices that will prevail in each. Although computationally unreasona-

[45] John Hicks' (1939) model of the effect of price expectations on current prices and market equilibrium in *Value and Capital* was "temporary" in nature (and "dynamic" in the same sense as Arrow's); future prices had an influence on current prices, but the nature of this dependence was given outside of the model. Arrow sought to bring price expectations into a general interdependent equilibrium system.

ble, one may argue that this can always be done in principle by appropriately defining the states.[46] The essential theoretical difficulty of the perfect foresight rational expectations assumption is not the "complete contingent foresight" assumption, but rather the associated coordination problem: How is it that each agent happens to arrive at the *same* catalog of spot commodity prices, state by state? What would each agent have to know to compute these prices? (For a "worst case," consider a situation in which there are multiple-spot market equilibria.) Coordination would require, for example, a super Walrasian auctioneer who knows tastes and technology and computes and then announces all future spot prices, state by state, *before* the true state is revealed and the commodities themselves are put up for sale. From a descriptive point of view, this is silly, and Arrow's security-spot market equilibrium deserves to be criticized for its cavalier treatment of how agents arrive at their expectations. But one should be careful not to overdo this criticism. To the extent that the alternative invites ad hoc specification regarding beliefs of future prices, the general equilibrium model is not closed. Not surprisingly, current modeling of general equilibrium under uncertainty still adheres mainly to Arrow's perfect foresight assumption, as with the standard multiperiod models of security market equilibria of "plans, prices, and price expectations," beginning with Roy Radner (1967, 1972).

The idea of an equilibrium in which individual expectations concerning future prices are mutually consistent and also consistent with equilibrium is useful, and one can logically isolate the problem of how these expectations are formed. To summarize, beyond the introduction of contingent commodities and uncertainty, which is the start of all analysis of value and distribution under uncertainty, Arrow is also responsible for introducing the idea of fulfilled expectations equilibrium to close the notion of equilibrium in situations in which agents must form expectations regarding future prices.[47]

Further Development: Multiperiod Extensions of Spanning

Arrow's notion of the spanning role of financial securities set the stage for dramatic multiperiod extensions and important applications in finance. The *span* of a given set of securities is the set of cash flows that can be generated by trading the securities. The more frequently a given set of securities is traded, the greater is its span, and the effect can be dramatic, leading all the way to modern financial asset pricing theory in continuous-time models.

Suppose we reconsider the simple numerical example drawn from Table 1, embedded in the second period of the event tree illustrated in Figure 7. All together, there are now four states of the world and eight contingent commodities. One can therefore obtain the effect of complete markets with eight contingent-commodity markets, or with four security markets and two spot markets, following the reasoning applied ear-

[46] Suppose, for example, that the spot commodity prices p_s for state s are not in fact known with perfect foresight, but rather, the agent in question believes there may be two different prices, p_s' and p_s'', that could occur in state s. Then state s could be replaced with two states, s' and s'', with respective spot prices $p_{s'}$ and $p_{s''}$. One could follow this path of reasoning until all uncertainty concerning a particular state is resolved. In summary, if the set of states is truly, as supposed, a complete description of uncertainty for a given agent, then that agent can have no uncertainty regarding what happens in a specific state. This may lead to an infinite number of states, but extensions of the Walrasian model (such as those of Bewley 1972 and Mas-Colell 1986) have been built to cope with the infinite-dimensional problem. It had been uncommon, at least until the advent of "sun spot" models (Cass and Karl Shell 1983), to model states of the world determined by "endogenous" entities such as prices, as well as the more conventional "exogenous" technology and preference "shocks," such as weather. The consideration of states defined by endogenous variables such as prices is now a commonly used and convenient device.

[47] A further problem for Arrow's analysis of uncertainty concerns differences in information available to different agents. This is a case in which criticism of the theory has been as important as the theory itself, and nobody would doubt that Arrow has been his own most influential critic. Asymmetric information leads to the nonexistence of the markets that are required for an efficient allocation of risk bearing. In Arrow's mind this leads to market failure and a role for government action (Arrow 1963). Also, one should keep in mind the essential thinness of markets caused by typical information asymmetries. An extreme view would hold that informational asymmetries are so prevalent and significant as to destroy the descriptive relevance of the Walrasian model.

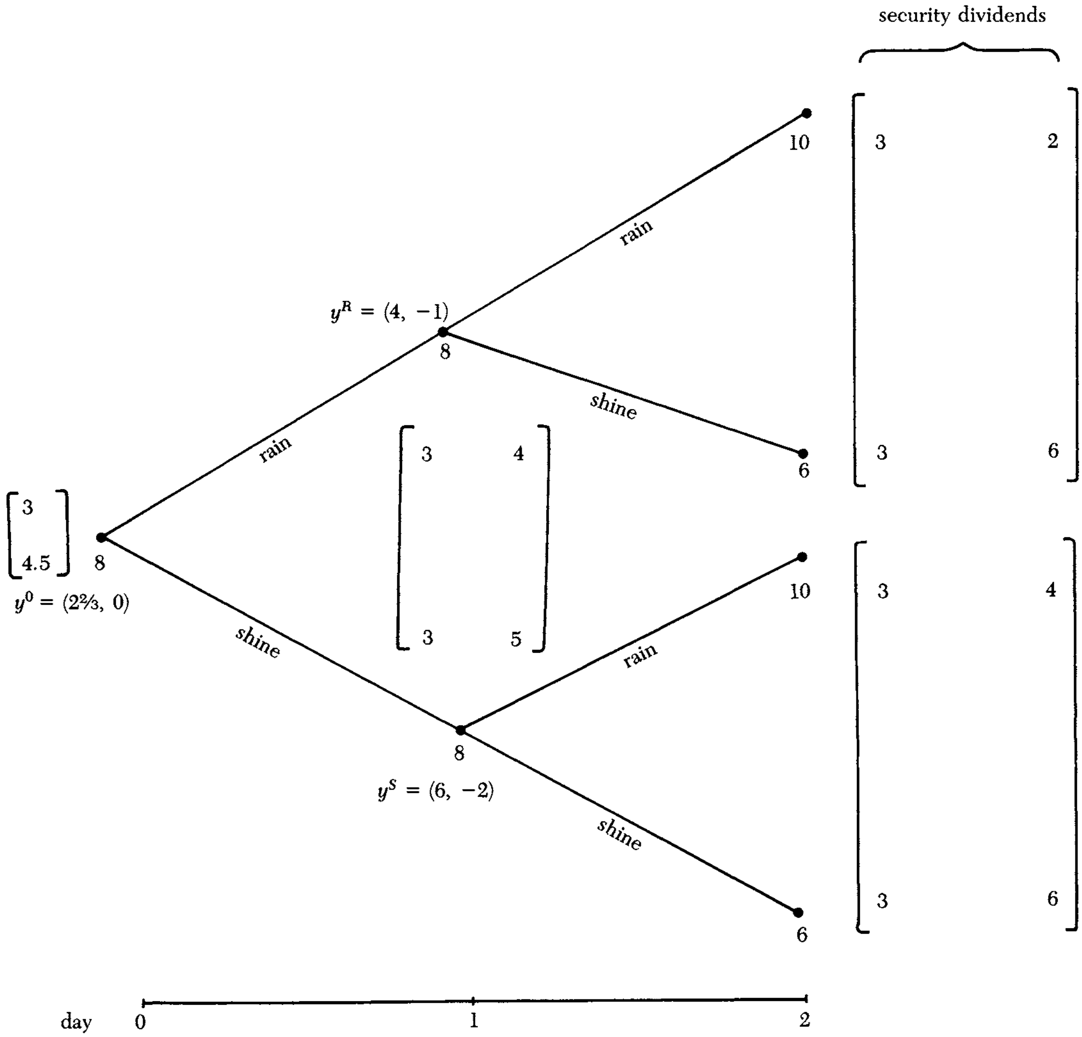


Figure 7. A Dynamic Trading Strategy with Intermediate Revelation of Information

lier. As the number of time periods grows, it might seem that the number of markets required for an efficient allocation of contingent commodities[48] would grow exponentially. But the fact that some information is revealed at each time allows one to obtain the effect of complete markets with only two security markets (plus spot commodity markets), which operate (in the two-period setting) as follows:

1. securities are traded at time zero before any weather is known,
2. the first day's weather is revealed as *rain* or *shine*,
3. securities are traded again,
4. another day of weather is revealed, and
5. agents collect the security dividends and trade on spot commodity markets.

For simplicity, we assume that spot commodity prices and the agent's state-contingent–commodity plan depend only on the second day's weather, so that the terminal dividends re-

[48] One should think in terms of the first, not second, welfare theorem, in light of our remarks concerning Hart (1975).

quired to finance the weather-contingent consumption plan x shown in Table 1 are just as calculated in Table 1: \$10 if rain and \$6 if shine. These required dividends are shown again on the terminal nodes of the event tree in Figure 7. To their right are shown the dividends paid by the two securities: Given rain on the first day, the dividend of the risky security on the second day is \$2 given rain, \$6 given shine (as in Table 1); while given shine on the first day, the dividend of the risky security on the second day is \$4 given rain, \$6 given shine. The riskless security pays \$3 with certainty, as shown.

We recall from the earlier one-period example that the portfolio $y^R = (4, -1)$ of securities (4 of the riskless security; -1 of the risky security) must be purchased on day one, given rain, in order to finance the given consumption plan. Assuming all states are equally likely and, for simplicity, that the price of any security is its conditional expected dividend, the security prices on the first day given rain are $q^R = (3,4)$. Thus $q^R \cdot y^R =$ \$8 are required on day one given rain to finance the consumption. Given shine on the first day, the required portfolio y^S turns out to be $(6, -2)$, while the security prices (the expected dividends) are $q^S = (3,5)$, for a portfolio cost of $q^S \cdot y^S =$ \$8. Thus, whether rain or shine, \$8 is needed on day one to finance the consumption plan; this can be achieved by investing \$8 in the riskless security on day zero.

This dynamic programming style of analysis allows us to construct a dynamic security trading strategy that finances any possible consumption plan. In this case ($S = 4$ states), two security markets and two spot markets are thus an effective substitute for $SC = 8$ contingent-commodity markets. Adding yet another round of trade allows the same two securities to generate, in the same dynamic sense, the set of contingent claims to income in $S = 2^3 = 8$ states of the economy. With n rounds of trade, 2 securities can generate a 2^n-dimensional space of contingent claims, and so on. With many rounds of trade, then, the economy of market structure that Arrow suggested becomes dramatic indeed.

Modern Financial Asset Pricing and the Black-Scholes Formula

As remarked by Mark Rubinstein (1987), "the genesis of the modern approach to the valuation of derivative assets can be traced to a paper ["The Role of Securities . . ."] by Kenneth Arrow" (p. 80). That is, going beyond the use of security markets for implementation of consumption allocations, one can apply Arrow's ideas so as to obtain easy-to-use formulas for the prices of certain securities relative to the prices of others.

In the previous example, we showed that one can construct a security trading strategy whose net effect is to require an investment of \$8 at day zero and to yield a payoff on day two of \$10 contingent on rain and \$6 contingent on shine. Suppose some financial exchange offers for trade a new security with a day-two dividend of \$10 contingent on rain and \$6 contingent on shine. If this new security trades on day zero at some price other than \$8, there must be an arbitrage. To see this, suppose for example that the new security sells for \$8.50. An arbitrageur could sell one share of this new security, immediately invest \$8 of the sale proceeds into the riskless security, and trade this portfolio on day one in exchange for the portfolio $y^R = (4, -1)$ if rain, or $y^S = (6, -2)$ if shine. Having short-sold one share of the new security, the arbitrageur is required to pay (to the purchasing agent) that security's dividends on day two: \$10 if rain, \$6 is shine. But this is precisely the dividend stream accruing on that day to the portfolio (y^R or y^S) of the other two securities formed on day one. The net effect of this arbitrage strategy is an initial profit of \$0.50 (and no further cash flow). The arbitrage could be scaled up for large profits (neglecting transactions costs). Similarly, if the initial price of the new security is less than \$8, the opposite stragety is an arbitrage. Thus, the unique arbitrage-free price of the new security is \$8. As long as any agent is nonsatiated, the arbitrage-free price of any security must be its market value.

There is nothing special about the particular contingent-dividend stream \$10 if rain at, \$6 if shine. Any dividend stream can be generated by an appropriate trading strategy at some initial cost, which is the unique arbitrage-free price of that dividend steam. For a popular example, consider an option on the risky (\$2 if rain, \$6 if shine) security, with an exercise price of \$4; the option entitles its holder the right (but not the obligation) on day two to pay \$4 in exchange for the risky security's dividends. Clearly, the option is exercised only if the dividend of the risky security is \$6, so the net dividends accruing to the holder of the option are \$2 if shine (zero if rain). The reader can check that the unique arbitrage-free price of the option on day zero is \$1. This can be verified by once again constructing a portfolio trading strategy that pays \$2 if shine on day two, and zero otherwise; the required initial portfolio sells for \$1, so the option must sell initially for \$1 in order to preclude arbitrage. The easier way to find the arbitrage price of the option is first to calculate the probabilities of the four states (0.25, 0.25, 0.25, 0.25). The unique arbitrage-free price of any security is its expected payoff; in this case, the option price is $0.25 \times \$2 + 0.25 \times \$2 = \$1$.

While it may seem to the uninitiated that the example must be quite special if the arbitrage-free price of any security is merely its expected payoff, this is in fact always the case, under an appropriate choice of numeraire, by changing one's probability assessments. This is like "relativity": Expectations are relative to the individual agent, but any agent's expectations can be distorted so that expected security payoffs are equal to security prices. This was shown in great generality by J. Michael Harrison and David Kreps (1979), following the lead of others.[49] Although abstract, this is now a standard approach used by investment banks to price financial assets.

Without a doubt, the most famous example of an arbitrage-free pricing is the Black-Scholes option pricing formula. Although Fisher Black and Myron Scholes (1973) developed their formula in a continuous-time setting, John Cox, Stephen Ross, and Mark Rubinstein (1979) provided a proof of the formula by applying the same discrete-time reasoning used in our numerical example,[50] and by taking limits as the length of a time interval approaches zero and the number of time intervals approaches infinity. In a continuous-time model, one can actually generate an infinite-dimensional space of contingent dividend streams by trading as few as two assets, albeit continually. The Black-Scholes option pricing formula has caused a significant change in the actual behavior of options markets. It has even become common for option traders to carry calculators programmed to compute quickly the Black-Scholes option pricing formula. The continuous-time spanning properties of securities applied in the Black-Scholes setting, combined with and building on Arrow's model of the role of securities in supporting efficient allocations under uncertainty, has recently been the basis for a continuous-time general equilibrium theory yielding new theoretical insights and new asset pricing theories, as seen for example in the results of Breeden (1979), Cox, Ingersoll, and Ross (1985), and Chi-Fu Huang (1987). It is not easy to point to a case in which a recent development in basic economic theory has been employed in as influential and practical a manner.

V. *Conclusion*

General equilibrium theory offers the best available answer to the fundamental questions of economics: What determines relative value? Under what conditions do competitive markets lead to an efficient allocation of resources? To be sure the theory is quite imperfect, but as a conceptual tool it serves us extremely well. The contemporary axiomatic formulation of general equilibrium theory was substantially put into place in the early 1950s, and the work of Kenneth Arrow (with Debreu) is at the center of this achievement. The cumulative effect of this work is a more complete and well-specified statement of the model of Walrasian equilibrium theory than any that had come before. This has led to a better understanding of which conditions are essential for the theory and which are not, and has opened the door to significant extensions.

The existence theorems per se are primarily

[49] Cox, Ross, and Rubinstein (1979) were the first to show that the Black-Scholes option pricing formula can be obtained by calculating the expected payoff of the option under a change of probability assessments. In fact, the idea of pricing a security by calculating its expected payoff, using as probabilities the prices of "Arrow securities" (taking the probability of state s to be the price q^s of a security paying \$1 in state s, suitably normalizing so that $\Sigma_s q^s = 1$) can even be found in the appendix Arrow added to the version of his "Role of Securities" paper in *Essays in the Theory of Risk Bearing* (1970).

[50] This reasoning was developed in a suitable form in an early edition of William Sharpe (1985).

a technical achievement; however, one should keep in mind the fundamental position that they occupy in the theory. We have taken issue with Arrow's continued emphasis on convexity as the essential assumption; we prefer to emphasize the condition that agents are small relative to the market. (Regions of increasing returns are a problem for the theory when they extend to output levels that are significant in terms of total demand. Similarly, asymmetric information and externalities are a problem for the theory when they cause markets to be too thin.)

Arrow's approach to the welfare theorems and the independent treatment by Debreu are radically different in method than what came before. Again, the work is important for its role in providing axioms of competitive theory. Although the work has its technical aspect, there is also a large substantive contribution: As a result of the work of Arrow and Debreu the separateness of the two welfare theorems was brought into sharp focus and the conditions necessary for their conclusions understood. We have tried to view Arrow's extension of the classical theorems in the light of both the "New Welfare Economics" and Arrow's continuing interest in issues of decentralization and incentives. We have taken a strong position with respect to the significance of the second welfare theorem. When used to defend the superiority of flexible prices *cum* transfers to commodity taxation, the theorem is weak on incentive grounds. There are more satisfactory readings of the result, however, and from a technical perspective the efficiency price characterization of optima serves us well. From the start of Arrow's work in general equilibrium theory and welfare economics, we see the commitment to a bold research program: the axiomatization of methods for passing from individual preferences to social goals and the study of mechanisms for achieving those social goals in a decentralized manner. Arrow's work on social choice is the cornerstone of the first part of the research program, while his contribution to the second part, in particular with Hurwicz, has played a major role. Arrow's attention to incentive issues comes later; here too, however, his influence has been great.[51]

The extension by Arrow of general equilibrium theory to include the case of uncertainty represents one of the great moments in economic theory. The substance of the achievement includes a theory of the allocation of risk and the pricing of risky assets. Also at the core is an idea for reconciling equilibrium with expectations via the notion of fulfilled expectations equilibrium. Beyond this there are the beginnings of a theory in which financial security markets and rational expectations of future spot prices together display the need for complete markets for contingent commodities. This is a nascent version of the theory of dynamic trading strategies and the modern theory and practice of financial asset pricing. Not only has Arrow's theory of general equilibrium under uncertainty (and his criticisms of the theory) opened whole new areas of investigation, it has also been of enormous practical significance.

The work on general equilibrium theory is by any standard remarkable. Arrow wrote one paper on each subject: The Welfare Theorems, Existence of Equilibrium, and General Equilibrium with Uncertainty. The methods, the technique, the substance, and the interpretation have all been extremely influential. And the achievement neither begins nor ends with this work: Before it comes "The General Possibility Theorem" and after it "Uncertainty and the Welfare Economics of Medical Care" (Arrow 1963). Keep in mind that this review covers but one of six volumes!

[51] In addition to the paper mentioned in Footnote 35, Arrow's articulation of moral hazard and adverse selection (Arrow 1963), and also the technical link between the Gibbard-Satterthwaite theorem (Gibbard 1973; Satterthwaite 1975) and his general possibility theorem should be kept in mind.

References

ALLAIS, MAURICE. "Géneralization des theories de l'équilibre économique general et du rendement social au cas de risque," *Économetrie. Colloques internationaux centre national de la recherche scientifique*, 1952, Paris, 1953, *40*, pp. 81–109.

ARROW, KENNETH. "An Extension of the Basic Theorems of Classical Welfare Economics," in *Proceedings of the second Berkeley symposium on mathematical statistics and probability.* Ed: JERZY NEYMAN. Berkeley: U. of California Press, 1951a, pp. 507–32.

———. *Social choice and individual values.* New Haven: Yale U. Press, 1951b.

———. "Le rôle des valeurs boursières pour la repartition la meillure des risques," *Econometrie.*

Colloques internationaux centre national de la recherche scientifique, Paris, 1952, *40,* pp. 41–47; discussion, pp. 47–48, C.N.R.S., Paris, 1953. English translation: 1953, *Rev. Econ. Stud.,* Apr. 1964, *31,* pp. 91–96.

———. "Toward a Theory of Price Adjustment," in *The allocation of economic resources.* Ed.: MOSES ABRAMOVITZ. Stanford, CA: Stanford U. Press, 1959, pp. 41–51.

———. "Uncertainty and the Welfare Economics of Medical Care," *Amer. Econ. Rev.,* Dec. 1963, *53,* pp. 941–73.

———. "Economic Equilibrium," in *International encyclopedia of the social sciences,* Ed.: D. SILLS. NY: Crowell Collier & Macmillan, 1968, pp. 376–89.

———. *Essays in the theory of risk bearing.* Chicago: Markham; London: North-Holland, 1970.

———. "General Economic Equilibrium: Purpose, Analytic Techniques, Collective Choice," in *Les prix nobel en 1972.* Ed.: NOBEL FOUNDATION. Stockholm: Nobel Foundation, 1973, pp. 206–31.

———. "Risk Allocation and Information: Some Recent Theoretical Developments," *Geneva Papers on Risk and Insurance,* 1978, *8,* pp. 5–19.

———. "Behavior Uncertainty and Its Implications for Policy." Technical Report 399, IMSSS, Stanford U., 1983a.

———. *Collected papers of Kenneth J. Arrow.* Vol. 2: *General equilibrium.* Cambridge: Belknap Press of Harvard U. Press, 1983b.

———. *Rational choice.* Chicago: U. of Chicago Press, 1987.

ARROW, KENNETH; BLOCK, HENRY AND HURWICZ, LEONID. "On the Stability of the Competitive Equilibrium II," *Econometrica,* Jan. 1959, *27,* pp. 82–109.

ARROW, KENNETH AND DEBREU, GERARD. "Existence of an Equilibrium for a Competitive Economy," *Econometrica,* July 1954, *22,* pp. 265–90.

ARROW, KENNETH AND HAHN, FRANK. *General competitive analysis.* San Francisco: Holden-Day, 1971.

ARROW, KENNETH AND HURWICZ, LEONID. "On the Stability of the Competitive Equilibrium I," *Econometrica,* Oct. 1958, *26,* pp. 522–52.

———. "Decentralization and Computation in Resource Allocation," in *Essays in economics and econometrics.* Ed.: RALPH W. PFOUTS. Chapel Hill: U. of North Carolina Press, pp. 34–104.

———. *Studies in resource allocation processes.* Cambridge and NY: Cambridge U. Press, 1977.

ARROW, KENNETH AND STARRETT, DAVID. "Cost-Theoretical and Demand-Theoretical Approaches to the Theory of Price Determination," in *Carl Menger and the Austrian school of economics.* Eds.: JOHN HICKS AND WILHELM WEBER. Oxford: Clarendon Press, 1973, pp. 129–48.

AUMANN, ROBERT. "Markets with a Continuum of Traders," *Econometrica,* Apr. 1964, *32,* pp. 39–50.

———. "Existence of Competitive Equilibria in Markets with a Continuum of Traders," *Econometrica,* Jan. 1966, *34,* pp. 1–17.

BALASKO, YVES AND CASS, DAVID. "The Structure of Financial Equilibrium: I. Exogenous Yields and Unrestricted Participation." Working Paper 85–23R, Center for Analytic Research in Economics and the Social Sciences, U. of Pennsylvania, 1986.

BERLIANT, MARCUS. "A Characterization of the Demand for Land," *J. Econ. Theory,* Aug. 1984, *33,* pp. 289–300.

BEWLEY, TRUMAN. "Existence of Equilibria in Economies with Infinitely Many Commodities," *J. Econ. Theory,* July 1972, *4*(3), pp. 514–40.

BLACK, FISHER AND SCHOLES, MYRON. "The Pricing of Options and Corporate Liabilities," *J. Polit. Econ.,* May–June 1973, *81*(3), pp. 637–54.

BREEDEN, DOUGLAS. "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities," *J. Finan. Econ.,* Sept. 1979, *7*(3), pp. 265–96.

BROUWER, LUITZEN EGBERTUS. "Invarianz des *n*-dimensionalen Gebiets," *Mathematische Annalen,* 1912, *71,* pp. 305–13.

CASS, DAVID AND SHELL, KARL. "Do Sunspots Matter?" *J. Polit. Econ.,* Apr. 1983, *91*(2), pp. 193–227.

CASS, DAVID AND YAARI, MENAHEM. "A Re-Examination of the Pure Consumption Loans Model," *J. Polit. Econ.,* Aug. 1966, *74,* pp. 353–67.

CASSEL, GUSTAV. *The theory of social economy.* NY: Harcourt, Brace, Co., 1924.

CHAMBERLIN, EDWARD. *The theory of monopolistic competition,* 7th ed. Cambridge: Harvard U. Press, 1956.

CHIPMAN, JOHN. "External Economies of Scale and Competitive Equilibrium," *Quart. J. Econ.,* May 1970, *84*(3), pp. 347–85.

CLARKE, EDWARD. "Multipart Pricing of Public Goods," *Public Choice,* 1971, *11,* pp. 17–33.

COURNOT, ANTOINE AUGUSTIN. *Recherches sur les principes mathématiques de la théorie des richesses.* Paris: Librarie des Sciences Politiques et Sociales, M. Rivière & cie. 1838. Trans. by NATHANIEL T. BACON, *Researches into the mathematical principles of the theory of wealth.* NY: Macmillan, 1897.

COX, JOHN: INGERSOLL, JONATHAN E., JR. AND ROSS, STEPHEN A. "A Theory of the Term Structure of Interest Rates," *Econometrica,* Mar. 1985, *53*(2), pp. 385–408.

COX, JOHN; ROSS, STEPHEN AND RUBINSTEIN, MARK. "Option Pricing: A Simplified Approach," *J. Finan. Econ.,* Sept. 1979, *7*(3), pp. 229–63.

DEBREU, GERARD. "The Coefficient of Resource Utilization," *Econometrica,* July 1951, *19,* pp. 273–92.

———. "A Social Equilibrium Existence Theorem," *Proceedings of the National Academy of Sciences,* 1952, *38,* pp. 886–93.

———. "Une Economie de l'Incertain." Unpub., Electricité de France, 1953.

———. *Theory of value.* Cowles Foundation Monograph 17, New Haven, CT: Yale U. Press, 1959.

______. "Economies with a Finite Set of Equilibria," *Econometrica*, May 1970, *38*, pp. 387–92.

______. *Mathematical economics.* Cambridge: Cambridge U. Press, 1983.

______. "Existence of General Equilibrium," in *The new Palgrave: A dictionary of economics.* Eds.: JOHN EATWELL, MURRAY MILGATE AND PETER NEWMAN. London: Macmillan, 1987, pp. 216–19.

DEBREU, GERARD AND SCARF, HERBERT. "A Limit Theorem on the Core of an Economy," *Int. Econ. Rev.*, Sept. 1963, *4*, pp. 235–46.

DIAMOND, PETER. "Mobility Costs, Frictional Unemployment, and Efficiency," *J. Polit. Econ.*, Aug. 1981, *89*(4), pp. 798–812.

______. "Aggregate Demand Management in Search Equilibrium," *J. Polit. Econ.*, Oct. 1982, *90*(5), pp. 881–94.

DUBEY, PRADEEP. "Price-Quantity Strategic Market Games," *Econometrica*, Jan. 1982, *50*(1), pp. 111–26.

EDGEWORTH, FRANCOIS YSIDRO. *Mathematical psychics.* London: C. Kegan Paul, 1881.

FARRELL, MICHAEL JAMES. "The Convexity Assumption in the Theory of Competitive Markets," *J. Polit. Econ.*, Aug. 1959, *67*, pp. 377–91.

GALE, DAVID. "The Law of Supply and Demand," *Mathematica Scandinavia*, 1955, *4*, pp. 143–46.

______. "General Equilibrium for Linear Models." Unpub., Brown U., 1957.

______. "A Note on Global Instability of Competitive Equilibrium," *Naval Research Logistics Quarterly*, 1963, *10*, pp. 81–87.

GALE, DOUGLAS. "Bargaining and Competition," *Econometrica*, July 1986, *54*(4), pp. 785–818.

GEANAKOPLOS, JOHN AND MAS-COLELL, ANDREU. "Real Indeterminacy with Financial Assets." Unpub., Cowles Foundation, Yale U., 1985.

GEANAKOPLOS, JOHN AND MILGROM, PAUL. "A Theory of Hierarchies Based on Limited Managerial Attention." Working Paper, Yale U., 1985.

GIBBARD, ALLAN. "Manipulation of Voting Schemes: A General Result," *Econometrica*, July 1973, *41*(4), pp. 587–601.

GREEN, JERRY. "The Nonexistence of Informational Equilibria," *Rev. Econ. Stud.*, Oct. 1977, *44*(3), pp. 451–63.

GROVES, THEODORE. "Incentives in Teams," *Econometrica*, July 1973, *41*, pp. 617–31.

GROVES, THEODORE AND LEDYARD, JOHN. "Optimal Allocation of Public Goods: A Solution to the 'Free-Rider' Problem," *Econometrica*, May 1977, *45*(4), pp. 783–809.

HARRISON, J. MICHAEL AND KREPS, DAVID. "Martingales and Arbitrage in Multiperiod Securities Markets," *J. Econ. Theory*, June 1979, *20*(3), pp. 381–408.

HART, OLIVER. "On The Optimality of Equilibrium when the Market Structure Is Incomplete," *J. Econ. Theory*, Dec. 1975, *11*(3), pp. 418–43.

HELLWIG, MARTIN. "On the Aggregation of Information in Competitive Markets," *J. Econ. Theory*, June 1980, *22*(3), pp. 477–98.

HICKS, JOHN. *Value and capital.* Oxford: Clarendon Press, 1939.

HILDENBRAND, WERNER. *Core and equilibria of a large economy.* Princeton, NJ: Princeton U. Press, 1974.

HUANG, CHI-FU. "An Intertemporal General Equilibrium Asset Pricing Model: The Case of Diffusion Information," *Econometrica*, Jan. 1987, *55*(1), pp. 117–42.

HURWICZ, LEONID. "Optimality and Informational Efficiency in Resource Allocation Processes," in *Mathematical methods in the social sciences.* Ed.: KENNETH ARROW, SAMUEL KARLIN, AND PATRICK SUPPES. Stanford: Stanford U. Press, 1960, pp. 27–46.

______. "Social Goals And Social Organization: A Perspective," in *Social goals and social organization: Essays in memory of Elisha Pazner.* Eds.: LEONID HURWICZ, DAVID SCHMEIDLER, AND HUGO SONNENSCHEIN. Cambridge: Cambridge U. Press, 1985, pp. 1–16.

KAKUTANI, SHIZUO. "A Generalization of Brouwer's Fixed-Point Theorem," *Duke Math. J.*, 1941, *8*, pp. 451–59.

KOOPMANS, TJALLING. *Activity analysis of allocation and production.* NY: Wiley, 1951.

KREPS, DAVID. "A Note on 'Fulfilled Expectations' Equilibria," *J. Econ. Theory*, Feb. 1977, *14*(1), pp. 32–43.

KUHN, HAROLD AND TUCKER, ALBERT. "Nonlinear Programming," in *Proceedings of the second Berkeley symposium on mathematical statistics and probability.* Ed.: JERZY NEYMAN. Berkeley: U. of California Press, 1951, pp. 481–92.

LANGE, OSCAR. "The Foundations of Welfare Economics," *Econometrica*, July–Oct. 1942, *10*, pp. 215–28.

LERNER, ABBA. "The Concept of Monopoly and the Measurement of Monopoly Power," *Rev. Econ. Stud.*, Jan. 1934, *1*, pp. 157–75.

MARSCHAK, JACOB AND RADNER, ROY. *Economic theory of teams.* New Haven: Yale U. Press, 1972.

MAS-COLELL, ANDREU. "An Equilibrium Existence Theorem without Complete or Transitive Preferences," *J. Math. Econ.*, 1974, *1*, pp. 237–46.

______. "A Model of Equilibrium with Differentiated Commodities," *J. Math. Econ.*, 1975, 2, pp. 263–96.

______. "Noncooperative Approaches to the Theory of Perfect Competition: Presentation," *J. Econ. Theory*, Apr. 1980, *22*(2), pp. 121–35.

______. "The Price Equilibrium Existence Problem in Topological Vector Latices," *Econometrica*, Sept. 1986, *54*(5), pp. 1039–53.

MAS-COLELL, ANDREU AND RICHARD, SCOTT. "A New Approach to the Existence of Equilibria in Vector Lattices." Unpub., Dept. of Economics, Harvard U., 1987.

MASKIN, ERIC. "The Theory of Implementation in Nash Equilibrium," in *Social goals and social organization.* Eds.: LEONID HURWICZ, DAVID SCHMEIDLER, AND HUGO SONNENSCHEIN. Cam-

bridge: Cambridge U. Press, 1985, pp. 173–204.
McKenzie, Lionel. "On Equilibrium in Graham's Model of World Trade and Other Competitive Systems," *Econometrica*, Apr. 1954, *22*, pp. 147–61.
______. "Competitive Equilibrium with Dependent Consumer Preferences," in *The second symposium on linear programming*. Ed.: National Bureau of Standards and Dept. of the Air Force. Washington, DC, 1955.
______. "On the Existence of General Equilibrium for a Competitive Market," *Econometrica*, Jan. 1959, *27*, pp. 54–71.
______. "The Classical Theorem on Existence of Competitive Equilibrium," *Econometrica*, June 1981, *49*(4), pp. 819–41.
Merton, Robert. "Theory of Rational Option Pricing," *Bell J. Econ. Manage. Sci.*, Spring 1973, *4*(1), pp. 141–83.
Mirrlees, James. "An Exploration in the Theory of Optimum Income Taxation," *Rev. Econ. Stud.*, June 1971, *38*(114), pp. 175–208
Moore, John. "The Existence of 'Compensated Equilibrium' and the Structure of the Pareto Efficiency Frontier," *Int. Econ. Rev.*, 1975, *16*, pp. 267–300.
Morgenstern, Oskar. "Professor Hicks on Value and Capital," *J. Polit. Econ.*, June 1941, *49*, pp. 361–93.
Mount, Kenneth and Reiter, Stanley. "The Informational Size of Message Spaces," *J. Econ. Theory*, June 1974, *18*(2), pp. 161–92.
Myerson, Roger. "Bayesian Equilibrium and Incentive Compatibility: An Introduction," in *Social goals and social organization*. Ed.: Leonid Hurwicz, David Schmeidler, and Hugo Sonnenschein. Cambridge: Cambridge U. Press. 1985, pp. 229–59.
Nash, John. "Equilibrium Points in *N*-Person Games," *Proceedings of the National Academy of Sciences*, 1950, *36*, pp. 48–49.
Negishi, Takashi. "Welfare Economics and Existence of an Equilibrium for a Competitive Economy," *Metroeconomica*, Aug.–Dec. 1960, *12*, pp. 92–97.
______. "Monopolistic Competition and General Equilibrium," *Rev. Econ. Stud.*, June 1961, *28*, pp. 196–201.
______. "On the Non-existence of Equilibrium: From Thornton to Arrow," in *Arrow and the ascent of modern economic theory*. Ed.: George Feiwel. NY: NYU Press, 1987, pp. 361–74.
Neisser, Hans. "Lohnhöhe and Beschäftigungsgrad im Marktgleichgewicht," *Weltwirtsch. Arch.*, 1932, *36*, pp. 413–55.
von Neumann, John. "Über ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes," *Ergebnisse eines mathematischen Kolloquiums*, 1937, *8*, pp. 73–78. English Translation in *Rev. Econ. Stud.* 1945, *13*, pp. 1–9.
von Neumann, John and Morgenstern, Oskar. *Theory of games and economic behavior*. 2nd ed. Princeton, NJ: Princeton U. Press, 1947.
Nikaido, Hukukane. "On the Classical Multilateral Exchange Problem," *Metroeconomica*, Aug. 1956, *8*, pp. 135–45.
Pareto, Vilfredo. *Manuel d'economie politique*. Paris: Girard & Briere, 1909.
Radner, Roy. "Equilibre des Marchés a Terme et au Comptant en Cas d'Incertitude," *Cahiers d'Économétrie*, 1967, *4*, pp. 35–52.
______. "Existence of Equilibrium of Plans, Prices, and Price Expectations in a Sequence of Markets," *Econometrica*, Mar. 1972, *40*(2), pp. 289–303.
Roberts, John. "An Equilibrium Model with Involuntary Employment at Flexible, Competitive Prices and Wages," *Amer. Econ. Rev.*, Dec. 1987, *77*(5), pp. 856–74.
Ross, Stephen. "The Economic Theory of Agency: The Principal's Problem," *Amer. Econ. Rev.*, May 1973, *63*(2), pp. 134–39.
Rothenberg, Jerome. "Non-convexity, Aggregation, and Pareto Optimality," *J. Polit. Econ.*, Oct. 1960, *68*, pp. 435–68.
Rubinstein, Ariel and Wolinsky, Asher. "Decentralized Trading, Strategic Behavior and the Walrasian Outcome." Technical Report 497, IMSSS, Stanford U., 1986.
Rubinstein, Mark. "Derivative Asset Analysis," *J. Econ. Perspectives*, Fall 1987, *1*(2), pp. 73–93.
Samuelson, Paul. *Foundations of economic analysis*. Cambridge: Harvard U. Press, 1947.
______. "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," *J. Polit. Econ.*, Dec. 1958, *66*, pp. 467–82.
Satterthwaite, Mark. "The Existence of a Strategy Proof Voting Procedure; a Topic in Social Choice Theory." PhD thesis, U. of Wisconsin, Madison, 1973.
______. "Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions," *J. Econ. Theory*, Apr. 1975, *10*(2), pp. 187–217.
Savage, Leonard. *Foundations of statistics*. NY: Dover, 1954.
Scarf, Herbert. "Some Examples of Global Instability of the Competitive Equilibrium," *Int. Econ. Rev.*, Sept. 1960, *1*, pp. 157–72.
Schlesinger, Karl. "Über die Produktionsgleichungen der ökonomischen Wertlehre," *Ergebnisse eines mathematischen Kolloquiems*, 1933–34, *6*, pp. 10–11.
Schumpeter, Joseph. *History of economic analysis*. NY: Oxford U. Press, 1954.
Sharpe, William. *Investments*. 3rd ed. Englewood Cliffs, NJ: Prentice-Hall, 1985.
Shoven, John. "A Proof of the Existence of a General Equilibrium with *ad Valorem* Taxes," *J. Econ. Theory*, May 1974, *8*(1), pp. 1–25.
Simon, Leo. "Bertrand, the Cournot Paradigm and the Theory of Perfect Competition," *Rev. Econ. Stud.*, Apr. 1984, *51*(2), pp. 209–30.
Sontheimer, Kevin. "The Existence of International Trade Equilibria with Trade Tax-Subsidy Distor-

tions," *Econometrica,* Nov. 1971, *39*(6), pp. 1015–35.

VON STACKELBERG, HEINRICH. "Zwei Kritische Bemerkurgen zur Preistheorie Gustav Cassels," *Z. Nationalökon.*, 1933, *4*, pp. 456–72.

STARR, ROSS. "Quasi-Equilibria in Markets with Nonconvex Preferences," *Econometrica,* Jan. 1969, *37*(1), pp. 25–38.

VARIAN, HAL. "The Arbitrage Principle in Financial Economics," *J. Econ. Perspectives,* Fall 1987, *1*(2), pp. 55–72.

WALD, ABRAHAM. "Über die eindeutige positive Lösbarkeit der neuen Produktionsgleichungen," *Ergebnisse eines Mathematischen Kolloquiums,* 1933–34, *7*, pp. 1–6.

______. "Über die Produktionsgleichungen der ökonomische Wertlehre," *Ergebnisse eines Mathematischen Kolloquiums,* 1934–35, *7*, pp. 1–6.

______. "Über eine Gleichungssysteme der Mathematischen Ökonomie," *Z. Nationalökon.*, 1936, *7*, pp. 637–70. English Translation: in *Econometrica,* Oct. 1951, *19*, pp. 368–403.

______. *Statistical decision functions.* NY: Wiley, 1950.

WALRAS, LÉON. *Eléments d'économie politique pure.* 4th ed. Lausanne: L. CORBAZ, 1874–77. English translation of the definitive edition by W. JAFFÉ, *Elements of pure economics.* London: Allen & Unwin, 1954.

WEINTRAUB, E. ROY. "On the Existence of a Competitive Equilibrium: 1930–1954," *J. Econ. Lit.*, Mar. 1983, *21*(1), pp. 1–39.

WHALLEY, JOHN. "Some General Equilibrium Analysis Applied to Fiscal Harmonization in the European Community," *Eur. Econ. Rev.*, Oct. 1976, *8*(3), pp. 291–312.

ZEUTHEN, FREDERICK LUDWIG BANG. "Das Prinzip der Knappheit, technische Kombination, und Ökonomische Qualität," *Z. Nationalökon.*, 1933, *4*, pp. 1–24.