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RECURSIVE VALUATION OF DEFAULTABLE SECURITIES AND THE TIMING OF RESOLUTION OF UNCERTAINTY[1]

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We derive the implications of default risk for valuation of securities in an abstract setting in which the fractional default recovery rate and the hazard rate for default may depend on the market value of the instrument itself, or on the market values of other instruments issued by the same entity (which are determined simultaneously). A key technique is the use of backward recursive stochastic integral equations. We characterize the dependence of the market value on the manner of resolution of uncertainty, and in particular give conditions for monotonicity of value with respect to the information filtration.

Introduction. This paper presents a model of defaultable securities in which the fractional recovery upon default and the default hazard rate may depend on the market value of the security itself and possibly on the market values of other securities issued by the firm. If these dependencies are nonlinear, it is shown that, in general, the security value depends on the timing of resolution of uncertainty. Several general pricing relationships are given, extending work by Duffie and Singleton (1994).

In a much different setting Nabar, Stapleton and Subrahmanyam (1988) earlier examined the implications of the timing of resolution of uncertainty for the market value of defaultable claims. Ross (1989) posed as an interesting puzzle the response of a defaultable bond price to an announcement of earlier resolution of uncertainty.

A traditional approach in valuing defaultable debt, going back at least to Black and Scholes (1973), is to take the default time of the firm as literally the first time that the market value of assets is reduced to or below the total market value of liabilities. This approach is taken in much subsequent work, including that of Hull and White (1992, 1995), Longstaff and Schwartz (1993), Nielson, Saá-Requejo and Santa-Clara (1993) and Rendleman (1992).

It some cases, it may be inconvenient to obtain information on the individual components of the firm's balance sheet. In other cases, it may be unrealistic to assume that default is equivalent to a situation in which the market value of assets is reduced to or below that of liabilities. For example, default may arise

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from illiquidity associated with credit restrictions or imperfect information. In any case, public information on the market values of assets and liabilities may be so coarse as to render such a model problematic. In general, default could actually arise well before or after the time at which the market value of assets falls below that of liabilities (if indeed these values could be measured and credibly announced).

Our approach in this paper is not to allow the determination of default to be a strict property of the capital structure of the firm, but rather to allow the time of default to be a stopping time whose arrival intensity is itself a random process that may be given exogenously, or may depend in a specified way on available information such as market rates and prices, including the price of the claim being valued and the prices of other claims issued by the firm, including equity. For example, at a given time, the market is assumed to assess the probability of default over the next instant of time, and may judge this probability to be decreasing in equity value, or equity value relative to accounting measures of liabilities. Likewise, recovery on default may be specified in terms of the relative prices of the issues of the firm, and on other random processes judged in the market place to be relevant.

Similar reduced-form approaches to the valuation of defaultable issues, taking default arrival intensity as a given process, have been undertaken by Artzner and Delbaen (1990, 1992), Duffie and Huang (1994), Duffie and Singleton (1994), Hull and White (1992), Jarrow, Lando and Turnbull (1993), Jarrow and Turnbull (1995), Lando (1993, 1994), Madan and Unal (1993) and Pye (1974).

There are five sections. Section 1 introduces a general model of defaultable claims, the valuation formula of which is derived in Section 2. In Section 3 the model is further specified by letting the default-payoff and hazard-rate processes depend on the price of the defaultable claim itself. Section 4 discusses the resulting dependence of prices on the underlying filtration. Section 5 presents a brief extension in which several defaultable securities issued by the same entity are simultaneously valued.

The Appendix contains proofs.

1. The basic setup. We begin with a standard setup of a filtered probability space $(\Omega, \mathscr{F}, \mathbf{F}, P)$ satisfying the (purely technical) "usual conditions." The underlying time set is the positive real line $[0, \infty)$. The *filtration* $\mathbf{F} = \{\mathscr{F}_t \colon t \geq 0\}$ represents the arrival of information over time. Throughout the paper, equalities involving random variables should be interpreted in the almost-sure sense. (For example, we write $X = Y$ instead of $P[X = Y] = 1$.) For technical background and terminology, the reader may refer to Protter (1990), Jacod and Shiryaev (1987) or Dellacherie and Meyer (1976, 1982).

The probability P is assumed to be an *equivalent martingale measure* in the sense of Harrison and Kreps (1979), under a given *short rate* process r, assumed to be bounded and progressively measurable. As described by Harrison and Kreps (1979), the existence of such a measure is essentially equivalent to the absence of arbitrage. In our setting however, replication arguments, such as those used for the Black–Scholes model, do not easily serve to identify uniquely an equivalent martingale measure. In effect, one cannot necessarily hedge against jumps in value that may occur at default or with the sudden arrival of information on credit quality.

By the definition of an equivalent martingale measure, the price process S of any security with cumulative dividend process D (any adapted process of integrable variation) satisfies

$$(1)\quad S_t = E\left[\int_t^T \exp\left(-\int_t^u r_v\,dv\right)dD_u + \exp\left(-\int_t^T r_v\,dv\right)S_T \;\Big|\; \mathscr{F}_t\right], \qquad t \le T.$$

Equivalently, the discounted gain process

$$\left\{\int_0^t \exp\left(-\int_0^u r_v\,dv\right)dD_u + \exp\left(-\int_0^t r_v\,dv\right)S_t\colon t \ge 0\right\}$$

is a martingale. (Unless otherwise noted, the martingale property will always be with respect to the underlying filtration $\mathbf{F}$.) We will assume throughout that the price of a security at time t is zero if all dividend payments after time t are zero. [This property is not implied by (1).] All prices are taken to be ex-dividend.

We consider a defaultable security that matures at time T, yielding a payoff X at time T, provided there has been no default. The random variable X is assumed to be $\mathscr{F}_T$-measurable and to satisfy $E(|X|^p) < \infty$ for some $p > 1$, fixed throughout the paper. The payoff upon default is described through a predictable process Z, satisfying $E[(\sup_t |Z_t|)^p] < \infty$. (For the formal definition of predictability, see any of the technical references given above.) Default at state ω and time t results in a payoff of $Z(\omega, t)$. Intuitively, predictability means that, were the time of default known, the payoff upon default would be known just prior to default, and would not come as a surprise. We will see below, however, that the time of default may be a surprise.

The following are simple examples of the form of Z that one may wish to consider in practice. Further examples are given later in the paper.

EXAMPLE 1. *Given fractional recovery of "par."* This amounts to a lump-sum settlement on default of $Z(\omega, t) = k$, for some constant k. Brennan and Schwartz (1980), for example, take this approach to the valuation of defaultable convertible bonds. Industry data on default recovery are usually recorded in terms of fractional recovery of par.

EXAMPLE 2. *Given fractional recovery of a default-free version of the same security.* In this case,

$$Z_t = \alpha E\left[\exp\left(-\int_t^T r_s\,ds\right)X \;\Big|\; \mathscr{F}_t\right], \qquad t \in [0, T],$$

for some constant α. For example, a corporate bond may be assumed to recover a fraction of an otherwise identical government (assumed default-free) bond. With a coupon structure, the same idea obviously applies. This is the approach taken by Jarrow, Lando and Turnbull (1993), Jarrow and Turnbull (1995), Lando (1993, 1994) and Madan and Unal (1993). The model implies a lower bound on the price of the claim which is α multiplied by the price of a default-free version of the claim. More generally, the multiplier α can be replaced by a (bounded) predictable stochastic process.

We model the stochastic structure of the default time through an $\mathbf{F}$-stopping time τ valued in $[0, \infty)$. The default time is defined to be the time $\tau \wedge T$, the minimum of τ and T. The event $\{\tau > T\}$ is then the event of no default. Some fairly weak conditions are placed on the stopping time τ. Intuitively, we require that the hazard rate associated with τ, under the equivalent martingale measure, be well defined and bounded; no specific distributional assumptions are made. To make this intuition precise, we introduce the *default indicator function*, $H_t = 1_{\{t \geq \tau\}}$, $t \geq 0$, a stochastic process that is equal to one if default has occurred, and zero otherwise. It is a standard result (due to Doob and Meyer) that H can be decomposed as $H = A + M$, where A is a predictable increasing process, and M is a martingale. We assume that there is a progressively measurable nonnegative bounded process h such that

$$A_t = \int_0^{t\wedge\tau} h_u \, du = \int_0^t h_u 1_{\{u<\tau\}} \, du, \qquad t \geq 0.$$

The process h has the interpretation of a hazard rate under the equivalent martingale measure, since

$$h_t 1_{\{t\leq\tau\}} = \lim_{u\downarrow 0} \frac{E[H_{t+u} - H_t \mid \mathscr{F}_t]}{u} = \lim_{u\downarrow 0} \frac{P[t < \tau \leq t+u \mid \mathscr{F}_t]}{u}.$$

The stopping time τ can also be thought of as the time of the first jump of a point process with intensity h, as described, for example, in Brémaud (1981). If N is a point process with intensity h, then $\{N_t - \int_0^t h_s \, ds\}$ is a martingale. The process H is then defined as $H_t = N_{t\wedge\tau}$, where τ is the time of the first jump of N, and the above decomposition of H follows. The simplest nontrivial example is Poisson arrival, in which h is a deterministic constant and τ is exponentially distributed under the equivalent martingale measure.

For our purposes, it is sufficient to work directly with the equivalent martingale measure P. In other applications, however, one may have to assume that P is not the equivalent martingale measure. The Girsanov–Meyer theorem for semimartingales can then be adapted to show that, subject to conditions, our setup is preserved under an equivalent change of measure, as shown by Artzner and Delbaen (1992). For the Girsanov–Meyer theorem, see, for example, Dellacherie and Meyer (1982), and Brémaud (1981) for the case of point processes.

2. Valuation formulas. With the defaultable-claim model in place, we now provide formulas for its risk-neutral valuation. Our starting point is the pricing formula (1), where D is now assumed to be the cumulative dividend process of the defaultable security. According to the description of the last section, D is given by

$$D_t = \begin{cases} Z_\tau 1_{\{\tau \le t\}}, & \text{for } t < T, \\ Z_\tau 1_{\{\tau \le T\}} + X 1_{\{\tau > T\}}, & \text{for } t \ge T. \end{cases}$$

In terms of the default indicator function, H, this specification can be written as

$$D_t = \int_0^{t \wedge T} Z_u \, dH_u + X 1_{\{\tau > T,\, t \ge T\}}, \qquad t \ge 0. \tag{2}$$

Using the decomposition $H = A + M$ and the assumed integral representation for A, we find that, on the time interval $[0, T)$, the defaultable security's price process, S, is given by

$$\begin{aligned} S_t = E\bigg[& \int_t^T \exp\left(-\int_t^u r_v \, dv \right) Z_u h_u 1_{\{u < \tau\}} \, du \\ & + \exp\left(-\int_t^T r_v \, dv \right) X 1_{\{T < \tau\}} \,\Big|\, \mathscr{F}_t \bigg], \end{aligned} \tag{3}$$

while $S_t = 0$ for $t \ge T$ (since there are no dividends after time T). The formal justification for eliminating the martingale part is provided in Lemma 2 of the Appendix.

Valuation formula (3) has the disadvantage that it explicitly involves the stopping time τ, rather than its hazard rate process. Proposition 1 allows us to replace default, for valuation purposes, with a continuous dividend rate over $[0, T]$ and a payoff X at time T. The proposition makes use of an *RCLL process*, a process with paths that are right continuous and with left limits. Given any RCLL process, Y, we let ΔY be the corresponding jump process, defined by $\Delta Y_t = Y_t - Y_{t-}$, $t \ge 0$, with the convention $Y_{0-} = 0$.

PROPOSITION 1. *Let the RCLL process V be defined by*

$$\begin{aligned} V_t = E\bigg[& \int_t^T Z_u h_u \exp\left(-\int_t^u (r_v + h_v) \, dv \right) du \\ & + X \exp\left(-\int_t^T (r_v + h_v) \, dv \right) \,\Big|\, \mathscr{F}_t \bigg], \qquad t < T, \end{aligned} \tag{4}$$

and $V_t = 0$ for $t \ge T$. Then the claim's price process, S, satisfies

$$S_t = V_t - E\left[\exp\left(-\int_t^\tau r_v \, dv \right) \Delta V_\tau \,\Big|\, \mathscr{F}_t \right] \quad \text{on } \{t < \tau\},\ t \ge 0.$$

If V is predictable, then $S_t = V_t 1_{\{t < \tau\}}$ for all $t \ge 0$.

Recall that equalities of random variables are interpreted in the almost-sure sense. In particular, when we write "$Y_1 = Y_2$ *on* F," for some event F, we mean that $P(F \backslash \{Y_1 = Y_2\}) = 0$.

The expression for V in Proposition 1 can be viewed as the risk-neutral valuation formula for a fictitious security that at each time t pays out dividends at a rate of $Z_t h_t$, under a fictitious short rate process $r + h$.

Clearly, the valuation procedure of Proposition 1 is simpler when V is predictable. For example, this is the case if V is (almost surely) continuous on $(0, T)$. A simple sufficient condition for the continuity of V on $(0, T)$, which is useful in practice, is as follows. Suppose that $\mathbf{F}^c = \{\mathscr{F}_t^c\}$ is a subfiltration of $\mathbf{F}$ with the property that the martingale $\{E[Y \mid \mathscr{F}_t^c]\}$ has a continuous modification for every integrable random variable Y. This notion of a continuous filtration is discussed by Huang (1985), and can be viewed as continuity of the mapping $t \mapsto \mathscr{F}_t^c$ with the space of σ-algebras topologized as in Cotter (1986). A typical example is one in which $\mathbf{F}^c$ is a Brownian filtration. Suppose also that there is a another filtration $\{\mathscr{G}_t\}$ so that $\mathscr{F}_t = \mathscr{F}_t^c \vee \mathscr{G}_t$ for every t. One can then easily check that V, defined by (4), is continuous on $(0, T)$ (up to a modification) if, conditionally on $\mathscr{F}_t^c$, the quadruple (Z, X, h, r) is stochastically independent of $\mathscr{G}_t$, for every $t \in (0, T]$.

3. Recursive valuation. In this section, we further specify the model, by letting the time-t default payoff Z_t or the hazard rate h_t to be (possibly stochastic and time-dependent) functions of the underlying claim price, S_{t-}, just prior to time t. The valuation formulas (3) and (4) then reduce to backward integral recursive equations that give rise to possible price-dependence on the underlying filtration, as discussed in Section 5.

In the remainder of this paper, we assume that the payoff of the claim in case of default is specified by a (measurable) payoff function $p\colon \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$. Upon default at (ω, t), the claim's payoff is $Z(\omega, t) = p(\omega, t, S_{t-}(\omega))$. That is, the payoff upon default possibly depends on the security price just prior to default.

EXAMPLE 3. *Given fractional reduction in value on default.* We can take $p(\omega, t, x) = \phi(\omega, t)x$, where ϕ is a given predictable process. This case is considered by Pye (1974) and Duffie and Singleton (1994). As will be shown in Example 4, the loss in generality is accompanied by a simplification of the valuation model, at least for cases in which the default hazard rate process h is exogenously given. One need not assume that $\phi(\omega, t) \le 1$ for all (ω, t), but this assumption naturally implies no gain in value upon default.

An example involving a nonlinear dependence of p on the security-price argument is discussed in detail by Duffie and Huang (1994), and briefly in Section 4.

The function p is assumed to satisfy the following conditions:

1. $p(\cdot, \cdot, x)$ is adapted to $\mathbf{F}$, for all x.
2. (Uniform Lipschitz condition). There is a constant K such that

$$|p(\omega, t, x) - p(\omega, t, y)| \le K|x - y|, \text{ for all } \omega,\ t,\ x \text{ and } y.$$

3. $p(\omega, t, 0) = 0$ for all ω and t.

Condition (1) states that only available information can be used to determine the default payoff. Condition (2) is purely technical. Finally, condition (3) states that a security that defaults right after it has lost all of its value yields a zero payoff.

In this context, prices will be shown to lie in the space $\mathscr{S}$, consisting of every semimartingale, S, such that $E[(\sup_t |S_t|)^p] < \infty$. (Recall that $p > 1$ is a constant that is fixed throughout the paper.) The following proposition presents an adaptation of valuation formula (3) under the given payoff structure.

PROPOSITION 2. *The claim's price process S is the unique element in $\mathscr{S}$ such that*

$$(5)\qquad S_t = E\left[\int_t^T (p(u, S_u)h_u - r_u S_u)\,du + X1_{\{T<\tau\}} \,\middle|\, \mathscr{F}_t\right], \qquad t < T,$$

and $S_t = 0$ for $t \geq T$.

Uniqueness here is up to indistinguishability. That is, any two solutions S and $\tilde{S}$ of (5) have identical paths with probability 1. Alternatively, the setting of Antonelli (1993), which corresponds to the case $p = 1$, can be used. This case is not covered here.

In certain settings, it is also reasonable to assume that the hazard rate process (and hence the default stopping time) depends on the defaultable security price process. We model such a dependence through a bounded and (product) measurable function $q\colon \Omega \times [0, \infty) \times \mathbb{R} \to \mathbb{R}$, where $q(\cdot, \cdot, x)$ is adapted to $\mathbf{F}$, for every x. We assume that $h(\omega, t) = q(\omega, t, S_{t-}(\omega))$, for all (ω, t). In this case, valuation formula (5) is not helpful, since the stopping time τ is itself a function of S. Instead, we have the following recursive pricing formula, in terms of the function f, defined by $f(\omega, t, x) = (p(\omega, t, x) - x)q(\omega, t, x)$.

In Proposition 3, f uniformly Lipschitz means that condition (2) is satisfied with f in place of p. Condition (2) on p is in fact redundant in this context. Also, condition (3) can be replaced by the weaker $f(\omega, t, 0) = 0$.

PROPOSITION 3. *Suppose that f is uniformly Lipschitz in its price argument. Then there exists a unique V in $\mathscr{S}$ that satisfies*

$$(6)\qquad V_t = E\left[\int_t^T (f(u, V_u) - r_u V_u)\,du + X \,\middle|\, \mathscr{F}_t\right], \qquad t < T,$$

and $V_t = 0$ for $t \geq T$. If V is predictable [*for example, continuous on* $(0, T)$], *or if $\Delta V_\tau = 0$, then the claim's price process is given by $S_t = V_t 1_{\{t<\tau\}}$ for all $t \geq 0$.*

Because of our almost-sure interpretation of equalities and the absolute continuity of A, $\Delta V_\tau = 0$ if V jumps only at a finite number of deterministic times. In some settings of interest, the solution V of (6) is continuous on $(0, T)$. For example, suppose that there are filtrations $\mathbf{F}^c = \{\mathscr{F}_t^c\}$ and $\{\mathscr{G}_t\}$ with $\mathbf{F}^c$ continuous under the Cotter topology, and $\mathscr{F}_t = \mathscr{F}_t^c \vee \mathscr{G}_t$ for all t, exactly as in the discussion following Proposition 1. Then V is continuous on $(0, T)$ if, for every $t \in [0, T]$, the triple (f, r, X) is stochastically independent of $\mathscr{G}_t$, conditionally on $\mathscr{F}_t^c$. [This is an easy consequence of the uniqueness of the solution of (6).]

EXAMPLE 4. Suppose that $f(\omega, t, x) = -\psi(\omega, t)x$, for some adapted bounded process ψ. Let

$$V_t = E\left[\exp\left(-\int_t^T (r_u + \psi_u)\,du\right)X \;\middle|\; \mathscr{F}_t\right], \qquad t < T.$$

Lemma 1 of the Appendix shows that V is the solution of equation (6). For example, in the case in which $q(\omega, t, x) = h(\omega, t)$ and $p(\omega, t, x) = \phi(\omega, t)x$, for all (ω, t, x), we have $\psi = (1-\phi)h$. This example repeats, in a more general setting, the simple valuation model exploited by Duffie and Singleton (1994), allowing standard valuation methods for nondefaultable claims to be applied with default, after substituting the default-adjusted short-rate process $r+\psi$ for the actual short-rate process r. Ramaswamy and Sundaresan (1986) assumed the existence of a default-adjusted short rate process, which in this setting is justified by exogenously given fractional loss rate $1 - \phi$ and hazard rate h. Pye (1974) obtains a similar expression in a discrete-time setting in which r, h and ϕ are deterministic.

4. The effect of a change in the filtration. As stated in the introduction, the dependence of the defaultable claim's default characteristics h and Z on its own price process implies that the pattern of information revelation between times zero and T can affect prices at time zero. In this section we present a general result of price monotonicity with respect to the underlying filtration, and, through a simple example, we show that the monotonicity can be strict.

We compare the prices of a defaultable security in two markets that are distinguished by the amount of available information. The two markets share the same defaultable security, as defined by the primitives p, q and X. While the equivalent martingale measure P and short-rate process r are the same in the two markets, each market has its own information filtration, denoted $\{\mathscr{F}_t^1\}$ and $\{\mathscr{F}_t^2\}$, respectively. Let S^1 and S^2 be the claim's price processes in the respective markets. We will show below that the two price processes are not in general equal. The default-payoff process, Z^i, of the security in market i is defined by $Z_t^i = p(t, S_{t-}^i)$. It follows that Z^1 and Z^2 are not in general the same, unless the payoff p is assumed not to vary with its price argument. Similarly, the hazard-rate process h^i in market i is defined by $h_t^i = q(t, S_{t-}^i)$, and h^1 and h^2 are not in general equal, unless q is assumed not to vary with its price argument. The default times, τ^1 and τ^2, in the two markets are therefore also not necessarily equal.

Suppose for now that $\mathscr{F}_0^1 = \mathscr{F}_0^2$ and $\mathscr{F}_T^1 = \mathscr{F}_T^2$. If p does not depend on its price argument and the default times, τ^i, are equal in the two markets (or q is also independent of its price argument), then it follows from the pricing formula (3) that $S_0^1 = S_0^2$. On the other hand, suppose that the following assumption holds, for example, because either one of Propositions 2 or 3 applies.

ASSUMPTION A. There exist a function $g\colon \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$, satisfying conditions (1) through (3) with g in place of p, and an integrable random variable Y such that, for both $i \in \{1, 2\}$, $S_t^i = V_t^i 1_{\{t<\tau^i\}}$, $t \geq 0$, where V^i is the unique process in $\mathscr{S}$ satisfying

$$V_t^i = E\left[\int_t^T g(u, V_u^i)\, du + Y \,\middle|\, \mathscr{F}_t^i\right], \qquad t < T,$$

and $V_t^i = 0$ for $t \geq T$.

Then the following can be shown: if the default characteristic g is convex in its price argument and $\mathscr{F}_t^1 \subseteq \mathscr{F}_t^2$ for all t, then $S_0^1 \leq S_0^2$. In the case that g is concave in its price argument, the reverse ranking of prices results. In the context of Proposition 2, the convexity or concavity of g in its price argument is equivalent to the corresponding property of p, while in the context of Proposition 3, the convexity or concavity of g in price is the same as that of f.

A refined version of this result is given in the following proposition. Given an event F and time t, we say that $g\colon \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ is *convex* (*concave*) on $F \times [t, T]$, if for every (ω, u) in $F \times [t, T]$, $g(\omega, u, \cdot)$ is convex (concave). The notation $F \cap \mathscr{F}^i$, where F is an event, denotes the class of sets that are obtained as intersections of F and members of $\mathscr{F}^i$.

PROPOSITION 4. *Suppose that Assumption A holds, and that for some time t and event $F \in \mathscr{F}_t^1 \cap \mathscr{F}_t^2$, $F \cap \mathscr{F}_t^1 = F \cap \mathscr{F}_t^2$ and $F \cap \mathscr{F}_u^1 \subseteq F \cap \mathscr{F}_u^2$, for all $u \geq t$. If g is convex (concave) on $F \times [t, T]$, then $S_t^1 \leq S_t^2$ ($S_t^1 \geq S_t^2$) on F.*

The proposition follows from a result on the monotonicity of backward recursive equations with respect to the underlying filtration proved in Skiadas (1996).

Duffie and Huang (1994) consider an extensive example in which concavity of g (in its dependence on V) arises naturally from a difference in the credit quality of the counterparties to a swap or forward contract. Suppose, for example, that there are two counterparties A and B, with respective default characteristics g_A and g_B. As explained by Duffie and Huang (1994), a natural interpretation of the settlement features of the contract implies a valuation model in which Assumption A applies, with

$$g(\omega, t, v) = g_A(\omega, t, v)1_{v\geq 0} + g_B(\omega, t, v)1_{v<0}.$$

Thus, even if g_A and g_B are linear (which is the case that applies if each counterparty has exogenously given hazard rate and fractional loss on default), a difference in the credit quality of the two counterparties implies concavity or convexity of g. Proposition 4 could then be applied to infer that the counterparty of better credit quality would prefer not to advance the public release of information regarding the outcome of audits or credit reviews of the other counterparty. For details, see Duffie and Huang (1994).

We close this section with a simple parametric example of strict price dependence on the timing of resolution of uncertainty.

EXAMPLE 5. In this example the hazard rate and short rate are constants, $h(\omega, t) = h$ and $r(\omega, t) = r \geq 0$, while the payoff function p takes the parametric form $p(\omega, t, x) = ax + bx^l$, for some constants $a \in [0, 1]$, $b \geq 0$ and $l \neq 1$. For simplicity, we assume that X is positive and bounded above by a constant $\bar{X}$ specified below. Provided $b > 0$, p is convex (concave) in x if $l > 1$ ($l < 1$), in which case the claim price is increasing (decreasing) in the underlying information filtration. In the case of $b = 0$, f is linear in x and the claim price does not change with an augmentation of the filtration.

To demonstrate these results in a simple setting, suppose that $\mathbf{F}$ reveals no information up to some time $R < T$, when all information is revealed. That is, for $t < R$, $\mathscr{F}_t$ consists only of events of probability 1 or 0, while for $t \geq R$, $\mathscr{F}_t = \mathscr{F}$. Clearly, the smaller R is, the earlier the resolution of uncertainty and the larger the observed filtration.

The solution of (6) in this setting is straightforward. On the intervals $[0, R)$ and $[R, T)$ the price dynamic reduces to

$$\frac{dV_t}{dt} = \alpha V_t - \beta V_t^l,$$

where $\alpha = (1-a)h + r \geq 0$ and $\beta = bh \geq 0$, while $V_{T-} = X$ and $V_{R-} = E(V_R)$. Since $P\{\tau = R\} = 0$, $\Delta V_\tau = 0$ almost surely, and the conclusion of Proposition 3 holds. For simplicity, we assume $\alpha > 0$; the case of $\alpha = 0$ is similar. The above differential equation can be rearranged as

$$\frac{d \log(|\alpha V_t^{1-l} - \beta|)}{dt} = \alpha(1 - l).$$

If $\beta > 0$, the value $V^* = (\alpha/\beta)^{1/(l-1)}$ is a stationary point, and to guarantee integrability, we assume $X \leq \bar{X} = V^*$. If $\beta = 0$, we set $\bar{X} = \infty$. Integrating, we have

$$V_0^{1-l} = \frac{1}{\alpha}\big(e^{-\alpha(1-l)R}\big(\alpha V_{R-}^{1-l} - \beta\big) + \beta\big).$$

Notice here that $\bar{X} = V^*$ is just a simple bound that works. A sharper bound would be $\bar{X} = V^*(1 - \exp(\alpha(1 - l)T))^{1/(1-l)}$ if $l > 1$, and $\bar{X} = \infty$ if $l \leq 1$.

We can now differentiate both sides with respect to R. Using the fact that $V_{R-} = E(V_R)$ and $dV_R/dR = \alpha V_R - \beta V_R^l$, we obtain

$$\frac{\partial V_0^{1-l}}{\partial R} = b\big(E(V_R)^l - E(V_R^l)\big)\frac{h(1 - l)e^{-\alpha(1-l)R}}{V_{R-}^l}.$$

(To be more precise, we computed here the right derivative, since V is discontinuous at R. By integrating all the way up to T, however, it is easy to show that V_0 is smooth in R.) If $b = 0$, the derivative vanishes, as it should.

For $b \neq 0$, we can apply Jensen's inequality to conclude that the initial price $S_0 = V_0$ is strictly decreasing (increasing) in R if $l > 1$ ($l < 1$).

5. Valuation of interdependent issues. Finally, we consider, for generality, a situation in which the hazard rate for default by the issuer, and the default recovery of a given issue, depend on, among other pieces of information, the prices of other issues of the same firm. These various prices are likewise influenced, and therefore jointly determined. For simplicity, we assume that all interdependent issues default simultaneously.

We now take X, T, p, S and V to be valued in $\mathbb{R}^n$. In particular, for any i, claim i pays X_i at T_i in the event of no default prior to T_i, and otherwise pays $p_i(\omega, t, S_{t-})$ given default at time t, where $p_i\colon \Omega \times [0, T_i] \times \mathbb{R}^n \to \mathbb{R}$. For convenience, we define $p_i(\omega, t, x) = 0$ for $t > T_i$.

The function p is assumed to satisfy the following analogues to the conditions for $n = 1$:

1. $p(\cdot, \cdot, x)$ is adapted to $\mathbf{F}$, for all x.
2. (Uniform Lipschitz condition). There is a constant K such that

$$\|p(\omega, t, x) - p(\omega, t, y)\| \leq K \|x - y\|, \text{ for all } \omega, \ t, \ x \text{ and } y.$$

3. $p_i(\omega, t, x) = 0$ for all ω and t and x with $x_i = 0$.

In this context, prices will be shown to lie in the space $\mathscr{S}^n$, consisting of every semimartingale S valued in $\mathbb{R}^n$, such that $E\left[(\sup_t \|S_t\|)^p\right] < \infty$. The hazard rate process is specified by a (measurable and bounded) function $q\colon \Omega \times [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$, where $q(\cdot, \cdot, x)$ is adapted to $\mathbf{F}$, for every x in $\mathbb{R}^n$. We assume that $h(\omega, t) = q(\omega, t, S_{t-}(\omega))$, for all (ω, t). We have the following adaptation of Proposition 3, in terms of the function $f\colon \Omega \times [0, \max_i T_i] \times \mathbb{R}^n \to \mathbb{R}^n$, defined by $f(\omega, t, x) = (p(\omega, t, x) - x)q(\omega, t, x)$.

PROPOSITION 5. *Suppose that f is uniformly Lipschitz in its price argument. Then there exists a unique V in $\mathscr{S}^n$ that satisfies, for each $i \in \{1, \dots, n\}$,*

$$V_{it} = E\left[\int_t^{T_i} (f_i(u, V_u) - r_u V_{iu})\, du + X_i \,\middle|\, \mathscr{F}_t\right], \qquad t < T_i, \tag{7}$$

with $V_{it} = 0$ for $t > T_i$. If V is predictable [for example, continuous on $(0, T)$], or if $\Delta V_\tau = 0$, then the n claims have a price process valued in $\mathbb{R}^n$ given by $S_t = V_t 1_{\{t<\tau\}}$ for all $t \geq 0$.

APPENDIX

This Appendix contains proofs not contained in the main text. We first present two lemmas that will be of repeated use. We recall once again that all equalities are to be interpreted in the almost-sure sense.

LEMMA 1. *Suppose that B is an adapted RCLL process of integrable variation, and ξ is a progressively measurable bounded process. Then*

$$dY_t = -dB_t - Y_t\xi_t\,dt + dm_t, \qquad t \le T,$$

for some martingale m, if and only if

$$Y_t = E\left[\int_t^T \exp\left(\int_t^u \xi_v\,dv\right)dB_u + \exp\left(\int_t^T \xi_v\,dv\right)Y_T \,\Big|\, \mathcal{F}_t\right], \qquad t \le T.$$

PROOF. Let $R_t = \exp(\int_0^t \xi_v\,dv)$, $t \ge 0$. The above integral equation can be rewritten in terms of R as

$$(8) \qquad Y_tR_t = -\int_0^t R_u\,dB_u + E\left[\int_0^T R_u\,dB_u + R_TY_T \,\Big|\, \mathcal{F}_t\right], \qquad t \le T.$$

Equation (8) can be equivalently written as

$$(9) \qquad d(Y_tR_t) = -R_t\,dB_t + dn_t, \qquad t \le T,$$

for some martingale n. To prove that (8) implies (9), we simply take n to be the (uniformly integrable) martingale corresponding to the second integral in (8). For the converse, we can integrate (9) from t to T, and take conditional expectations with respect to $\mathcal{F}_t$ to recover (8). (The term corresponding to n disappears during this operation, because of the martingale property.) Using now integration by parts, we have

$$d(Y_tR_t) = R_t(dY_t + Y_t\xi_t\,dt), \qquad t \le T.$$

Therefore, (9) is in turn equivalent to $dY_t = -dB_t - Y_t\xi_t\,dt + dm_t$, $t \le T$, where $dm_t = R_t^{-1}\,dn_t$. The lemma follows, after observing that m is a martingale if and only if n is a martingale. □

LEMMA 2. *If Y is predictable and $E[(\sup_t |Y_t|)^p] < \infty$, then $\int Y\,dM$ is a martingale.*

The proof of Lemma 2 is a consequence of Emery's inequality. [See Theorems V.2 and V.3 in Protter (1990).]

PROOF OF PROPOSITION 1. For all $t \ge 0$, we let $\tilde{V}_t = V_t 1_{\{t<T\}} + X1_{\{t \ge T\}}$. Notice that $\tilde{V}$ satisfies the same integral equation as V, but on $[0, T]$ rather than $[0, T)$ only. By Lemma 1 [with $\xi = -(r+h)$ and $dB_t = Z_th_t\,dt$], it follows that

$$d\tilde{V}_t = -(Z_th_t - (r_t + h_t)\tilde{V}_t)\,dt + dm_t, \qquad t \le T,$$

where m is some martingale.

Define now, for any $t \ge 0$, $L_t = 1 - H_t = 1_{\{t<\tau\}}$, and notice that $dL_t = -dH_t = -h_tL_t\,dt - dM_t$. Letting $U_t = \tilde{V}_tL_t$, we have, for all $t \in [0, T]$,

$$\begin{aligned} L_{t-}\,d\tilde{V}_t &= -Z_tL_th_t\,dt + (r_t + h_t)L_t\tilde{V}_t\,dt + L_{t-}\,dm_t \\ &= -Z_t\,dH_t + (r_t + h_t)U_t\,dt + Z_t\,dM_t + L_{t-}\,dm_t. \end{aligned}$$

On the other hand, integration by parts for semimartingales [see, for example, Protter (1990)] implies that

$$\begin{aligned} L_{t-}\, d\tilde{V}_t &= d(\tilde{V}_t L_t) - \tilde{V}_{t-}\, dL_t - \Delta\tilde{V}_t \Delta L_t \\ &= dU_t + U_t h_t\, dt + \Delta\tilde{V}_\tau\, dH_t + \tilde{V}_{t-}\, dM_t. \end{aligned}$$

Combining the two expressions for $L_{t-}\, d\tilde{V}_t$, we obtain

$$dU_t = -(Z_t + \Delta V_\tau)\, dH_t + r_t U_t\, dt + dN_t, \qquad t \le T,$$

where

$$dN_t = (Z_t - V_{t-})\, dM_t + L_{t-}\, dm_t, \qquad t \ge 0.$$

We have used here the fact that $\Delta\tilde{V}_\tau = \Delta V_\tau$, the event $\{\tau = T\}$ being null. Clearly, N is a local martingale.

Assuming for now that N is a true martingale and applying Lemma 1, we obtain

$$(10) \qquad U_t = E\left[\int_t^T \frac{R_t}{R_u}(Z_u + \Delta V_\tau)\, dH_u + \frac{R_t}{R_T} U_T \,\Big|\, \mathscr{F}_t\right], \qquad t \le T,$$

where

$$R_t = \int_0^t r_u\, du, \qquad t \ge 0.$$

Consider first the term involving the jump process of V:

$$\int_t^T \frac{R_t}{R_u} \Delta V_\tau\, dH_u = \frac{R_t}{R_\tau} \Delta V_\tau 1_{\{t<\tau\le T\}} = \frac{R_t}{R_\tau} \Delta V_\tau 1_{\{t<\tau\}},$$

since $\Delta V_t = 0$ for $t > T$. Also, we have $U_T = X 1_{\{\tau > T\}}$, and $V_t = U_t$ on $\{t < \tau\}$. Returning to (10), we can therefore rearrange it to derive the equation:

$$V_t - E\left[\frac{R_t}{R_\tau} \Delta V_\tau \,\Big|\, \mathscr{F}_t\right] = E\left[\int_t^T \frac{R_t}{R_u} Z_u\, dH_u + \frac{R_t}{R_T} X 1_{\{\tau > T\}} \,\Big|\, \mathscr{F}_t\right] \quad \text{on } \{t < \tau\}.$$

Recalling pricing equation (1) and the expression for the claim's dividend process, (2), we see that the right-hand side of the above expression is equal to S_t, and the result follows.

To complete the proof, it remains to show that N is a martingale. The term $\int L_-\, dm$ is clearly a martingale, since L is bounded. That $\int (Z - V_-)\, dM$ is a martingale follows by Lemma 2, with $Y = Z - V_-$, given the assumed integrability condition on Z and the fact that $V \in \mathscr{S}$. [The space $\mathscr{S}$ is defined in Section 2 as the set of every semimartingale, S, such that $E(\sup_t |S_t|^p) < \infty$. The constant $p > 1$ is fixed throughout.] That V is in $\mathscr{S}$ follows from our assumptions on Z, h and r, and Doob's maximal inequality [Protter (1990), Theorem I.20].

In the case that V is predictable, the jump term in the expression for S_t vanishes, since

$$E\left[\frac{\Delta V_\tau}{R_\tau} \;\middle|\; \mathscr{F}_t\right] = E\left[\int_t^T \frac{\Delta V_u}{R_u}\, dH_u \;\middle|\; \mathscr{F}_t\right]$$
$$= E\left[\int_t^T \frac{\Delta V_u}{R_u} h_u L_u \, du \;\middle|\; \mathscr{F}_t\right] = 0.$$

Here we have used the decomposition $H = A + M$ and the fact that almost every path of V, being RCLL, has at most a countable number of jumps. The elimination of the martingale part is again justified by Lemma 2, as in the last paragraph. □

PROOF OF PROPOSITION 2. From equation (3) and the specification of Z, we obtain

$$S_t = E\left[\int_t^T \exp\left(-\int_t^u r_v \, dv\right) p_u(S_{u-}) 1_{\{u<\tau\}} h_u \, du \right.$$
$$\left. + \exp\left(-\int_t^T r_v \, dv\right) X 1_{\{T<\tau\}} \;\middle|\; \mathscr{F}_t\right], \quad t < T.$$

Since almost every path of S has at most countably many discontinuities (a property of RCLL processes), we can substitute S_u for S_{u-} in the above expression. Also, by condition (3) on p, and the fact that prices vanish upon default, we have $p(S_u)1_{\{u<\tau\}} = p(S_u 1_{\{u<\tau\}}) = p(S_u)$. Therefore,

$$S_t = E\left[\int_t^T \exp\left(-\int_t^u r_v \, dv\right) p_u(S_u) h_u \, du \right.$$
$$\left. + \exp\left(-\int_t^T r_v \, dv\right) X 1_{\{T<\tau\}} \;\middle|\; \mathscr{F}_t\right], \qquad t < T.$$

We define $\tilde{S}_t = S_t$ for $t < T$, and $\tilde{S}_T = X 1_{\{T<\tau\}}$, so that $\tilde{S}$ satisfies the last integral equation on the closed interval $[0, T]$. Using Lemma 1 with $\xi = -r$, we obtain

$$d\tilde{S}_t = -(p_t(\tilde{S}_t) h_t - r_t \tilde{S}_t)\, dt + dm_t, \qquad t \le T,$$

for some martingale m. Applying Lemma 1 once again, but with $\xi = 0$ this time, we obtain (5). Existence and uniqueness of a solution to the recursive equation follows from Appendix A of Duffie and Epstein (1992). □

PROOF OF PROPOSITION 3. Existence and uniqueness of a solution to (6) follows from Duffie and Epstein (1992). Under our specification of Z and h, recursion (6) is in fact equivalent to the recursion of Proposition 1. This follows by a now familiar argument. Briefly, let $\tilde{V}_t = V_t$ for $t < T$, and $\tilde{V}_T = X$. Applying Lemma 1 with $\xi = -(r+h)$, we find that the recursion of Proposition 1 is equivalent to the existence of some martingale m such that

$$d\tilde{V}_t = -(Z_t h_t - (r_t + h_t)\tilde{V}_t)\, dt + dm_t, \qquad t \le T.$$

Substituting $p_t(S_t)$ and $q_t(S_t)$ for Z_t and h_t, respectively, and using Lemma 1 once again, but with $\xi = 0$ this time, we obtain (6). [As in the proof of Proposition 2, writing $p_t(S_t)$ instead of $p_t(S_{t-})$ in the integrand is inconsequential, since S has paths that are almost surely continuous. Similarly with q.] Given this equivalence, Proposition 3 becomes a corollary of Proposition 1. □

PROOF OF PROPOSITION 5. The existence of a unique solution V in $\mathscr{S}^n$ to (7) follows from an immediate extension of the contraction argument in Appendix A of Duffie and Epstein (1992) for the case of $n = 1$. As far as the valuation of an individual issue, the existence of a unique solution V to (7) fixes the hazard rate process h through q, and fixes the recovery process Z_i for the ith claim by $Z_{it}(\omega) = p_i(\omega, t, V_{t-}(\omega))$. The proof of Proposition 3 then applies to claim i directly. □

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