Dynamic Directed Random Matching*

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Abstract

We develop a general and unified model in which a continuum of agents conduct directed random searches for counterparties. Our results provide the first probabilistic foundation for static and dynamic models of directed search (including the matching-function approach) that are common in search-based models of financial markets, monetary theory, and labor economics. The agents' types are shown to be independent discrete-time Markov processes that incorporate the effects of random mutation, random matching with match-induced type changes, and with the potential for enduring partnerships that may have randomly timed break-ups. The multi-period cross-sectional distribution of types is shown to be deterministic and is calculated using the exact law of large numbers.

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1 Introduction

The economics literature is replete with models that assume independent random matching among a continuum of agents.[1] The agents in these models are frequently motivated to conduct "directed search," that is, to focus their searches toward those types of counterparties that offer greater gains from interaction, or toward those types that are less costly to find. For example, Rogerson, Shimer, and Wright (2005) describe cases in which "search is directed – i.e., workers do not encounter firms completely at random but try to locate those posting attractive terms of trade." Our central marginal contribution is to provide a mathematical foundation for the existence and properties of directed search models.

Independent directed random matching, which includes the popular "matching-function" approach, is the key to achieving tractability in many search-based models of financial markets, monetary theory, and labor economics.

Previous work on mathematical foundations for random matching considers only search that is "undirected," in the sense that, conditional on a match by a given agent at a given time, the probability that the match is with a particular "target" type of agents is merely the fraction of agents of the target type. Directed search can arise, for example, when one side of a market posts terms of trade that are especially attractive to specific types of agents.

Despite heavy reliance in the economics literature on models of independent directed search,[2] until now there has actually been no demonstration of the existence of such search models, nor of the assumed aggregate behavior of these models that is supposedly based on the law of large numbers. This paper demonstrates the existence and properties of general models of static and dynamic independent directed search, thus placing a complete mathematical foundation under the directed-search models assumed in the prior literature. Our results include new features and properties that may be useful in future research.

Earlier foundational work on random matching in a dynamic setting, which we review in Section 5, also presumes that partnerships break up immediately after matching. Here, we allow for the potential of enduring partnerships, which may have randomly timed break-ups. In order to meet the objectives of this paper, a completely new methodology is required, for

[1]Hellwig (1976) is the first, to our knowledge, to have relied on the effect of the exact law of large numbers for random pairwise matching in a market. Other examples include Binmore and Samuelson (1999), Currarini, Jackson and Pin (2009), Duffie, Gârleanu, and Pedersen (2005), Green and Zhou (2002), Kiyotaki and Wright (1989), Lagos and Rocheteau (2009), Vayanos and Weill (2008), and Weill (2007).

[2]Among the many applications of directed search in the economics literature, in addition to those cited elsewhere in this paper, are the models of Acemoglu and Shimer (1999), Albrecht, Gautier, and Vroman (2006), Burdett, Shi, and Wright (2001), Camera and Selcuk (2009), Eeckhout and Kircher (2010), Faig and Jerez (2005), Guerrieri, Shimer, and Wright (2010), Kiyotaki and Lagos (1993), Li, Rocheteau and Weill (2012) McAfee (1993), Menzio (2007), Moen (1997), Peters (1991), Shi (2002), and Watanabe (2010).

both static and dynamic settings.[3]

We first consider a static setting in which search is "directed," in the sense that the probability q_{kl} that an agent of type k is matched to an agent of type l can vary with the respective types k and l, from some type space S. We first show, in Theorem 1, the existence of directed random matching in which counterparty types are independent across agents. It follows from the exact law of large numbers that the proportion of type-k agents matched with type-l agents is almost surely $p_k q_{kl}$, where p_k is the proportion of type-k agents in the population. By allowing the matching probabilities $\{q_{kl}\}_{k,l \in S}$ to depend on the underlying cross-sectional type distribution p, we also encompass the "matching-function" approach that has frequently been applied in the labor literature, as surveyed by Petrongolo and Pissarides (2001) and Rogerson, Shimer, and Wright (2005), as well as over-the-counter models of trade in financial markets, as in Maurin (2015).

In typical dynamic settings for random matching, once two agents are matched, their types change according to some deterministic or random rule. For example, when an unemployed worker meets a firm with a vacant job, the worker's type changes to "employed." When a prospective buyer and seller meet, their status as asset owners can change, and they can learn information from each other. Random mutation of agent types is also a common model feature, allowing for shocks to preferences, productivity, or endowments.[4]

In practice, and in an extensive part of the literature, once a pair of agents is matched, they may stay matched for some time. Typical examples include the relationships between employer and employee, or between two agents that take time to bargain over their terms of trade.[5] In this paper, we develop the first mathematical model for independent random matching that allows for potentially enduring partnerships.

Our general model of independent dynamic directed random matching incorporates the effects of random mutation, random matching with match-induced type changes, and enduring partnerships. The agents' types are shown to be independent discrete-time Markov chains. By the exact law of large numbers in the dynamic setting, the multi-period cross-sectional distribution of agents' types is deterministic, and has a period-to-period update mapping that coincides with the transition function of the law of the Markov chain for individual agent types. For the special time-homogeneous case, we obtain a stationary joint cross-sectional distribution of agent types, incorporating both unmatched agent types and pairs of currently matched

[3] See the discussions in the first two paragraphs of Subsection E.1 on the proof of the static results, and the second paragraph of Subsection E.2 on the proof of the dynamic results, respectively.

[4] See, for example, Duffie, Gârleanu, and Pedersen (2005) and Lester, Postlewaite and Wright (2012).

[5] See, for example, Acemoglu and Wolitzky (2011), Andolfatto (1996), Diamond (1982), Mortensen and Pissarides (1994), Tsoy (2014), and the references in the surveys of Petrongolo and Pissarides (2001), Rogerson, Shimer, and Wright (2005) and Wright et al. (2017).

types. This stationary cross-sectional distribution coincides with the stationary probability distribution of the individual agent type processes. Many previously studied search-based models of money, over-the-counter financial markets, and labor markets have relied on these and other properties, which we demonstrate here for the first time.

We illustrate the applications of our model of directed random matching with four examples taken, respectively, from Duffie, Malamud and Manso (2014) in financial economics; Kiyotaki and Wright (1989) and Matsuyama, Kiyotaki and Matsui (1993) in monetary economics; and Andolfatto (1996) in labor economics. These examples show how our model can be used to provide rigorous foundations for typical random-matching models used in these respective literatures.

The remainder of the paper is organized as follows. Section 2 is a brief guide to our main results in an easily accessible form. In Section 3, we describe a static model of independent directed random matching, including an existence result as well as an application to a typical over-the-counter financial market model. In order to capture the effect of enduring partnerships, we must separately treat legacy and newly matched pairs of agents. In particular, we keep track of agents and their matched partners at each step (mutation, matching, and type changing), in every time period. Because the treatment of enduring partnerships is considerably more involved, its exposition is postponed to Appendix A. In Section 4, we treat the relatively simpler case of a dynamical system with random mutation, directed random matching, and match-induced type changing, but without enduring partnerships. This section includes results covering the existence and exact law of large numbers for a dynamical system with Markov conditional independence. Appendix B contains the remaining illustrative examples of applications of our main results, to models of monetary and labor economics.

The proofs of the results on the exact law of large numbers and stationarity for a general dynamic directed random matching are given in Appendix C.[6] The proofs for the existence results for static and dynamic directed random matching make extensive use of tools from nonstandard analysis, of which a brief introduction is provided in Appendix D.[7] Those proofs are located in Appendix E. Section 5 offers a discussion of the prior foundational mathematical research on random matching models, and some concluding remarks.

2 Guide to the Main Results

We first offer a brief guide to the main results at an informal level, unburdened by many technical details that we postpone to later sections.

[6]Nonstandard analysis is not needed in the proofs of those results.

[7]The reader can also be referred to the first three chapters of Loeb and Wolff (2015) for basic nonstandard analysis.

We emphasize throughout the key effects of the exact law of large numbers. This law is largely responsible for the popularity of random-matching models, because of the tractability associated with deterministic, and explicitly computable, quantities of matches between given types of agents. In multi-period settings, key additional tractability is obtained via the deterministic and explicitly computable evolution of the cross-sectional distribution of agent types. For example, consider the stochastic dynamic programming problem faced by a given agent in an economy with interacting agents, whose respective types change randomly over time through various shocks, including those induced by matching. Without the effect of the exact law of large numbers, the state variable for a given agent's problem would need to include not only that agent's current type, but also the randomly evolving cross-sectional distribution of types of all other agents. In many settings, the high dimensionality of the resulting state variable would rule out any reasonable progress toward a tractable solution. However, with independent random matching and an application of the exact law of large numbers, a given agent can safely assume that the cross-sectional distribution of types of the other agents evolves over time deterministically (almost surely). This leaves a fixed-point problem, of finding agent-level policy rules that are consistent in equilibrium with optimality by each agent. In this paper, however, we take agent-level policy rules as given. We also provide supporting assumptions for stationarity, under which the cross-sectional distribution of types is actually constant and deterministic, further simplifying the analysis.

In the context of random-matching models, the independence of matching outcomes is generally viewed as a behavioral assumption. That is, when agents conduct searches without explicit coordination, independence has been viewed as a natural assumption.

2.1 The exact law of large numbers

We fix a probability space $(\Omega, \mathcal{F}, P)$. An element of Ω is a state of the world. A measurable subset B of Ω (that is, an element of $\mathcal{F}$) is an event, whose probability is $P(B)$. The agent space is an atomless probability space $(I, \mathcal{I}, \lambda)$. An element of I represents an agent. The mass of some measurable subset A of agents is $\lambda(A)$. Because the total mass of agents is 1, we can also treat $\lambda(A)$ as the fraction of the agents that are in A. In fact, we can take I to be the unit interval $[0, 1]$ and λ to be an extension of the Lebesgue measure.[8]

In order to obtain the exact law of large numbers (ELLN) for a collection $\{f_i : i \in I\}$ of agent-level random variables, we model such a collection as a function $f : I \times \Omega \to \mathbb{R}$ that is measurable with respect to a sufficiently rich set of measurable subsets of $I \times \Omega$, denoted

[8] For measure-theoretic reasons, however, we need the set $\mathcal{I}$ of measurable sets of agents to be richer than the usual Lebesgue measurable sets. We also follow the convention that all probability spaces are countably additive and complete, unless otherwise noted.

$\mathcal{I} \boxtimes \mathcal{F}$, that extends the usual product σ-algebra $\mathcal{I} \otimes \mathcal{F}$. The required properties of $\mathcal{I} \boxtimes \mathcal{F}$ are given in the next section. The usual product σ-algebra $\mathcal{I} \otimes \mathcal{F}$ is not satisfactory for this purpose. We will also use a weaker version of the notion of independence of the agent-level random variables. An $\mathcal{I} \boxtimes \mathcal{F}$-measurable function f from $I \times \Omega$ to $\mathbb{R}$ is said to be essentially pairwise independent if for λ-almost all $i \in I$, the agent-level random variables f_i and f_j are independent for λ-almost all $j \in I$. As explicitly shown,[9] this condition is weaker than the usual conditions of mutual independence (any finite collection of random variables are independent) and pairwise independence (any pair of random variables are independent). The weaker condition allows one to state a more general version of the exact law of large numbers for independent random matching.[10] Unless otherwise noted, by "independence," we mean "essential pairwise independence," throughout this paper.

From the exact law of large numbers of Sun (2006, Corollary 2.10) (or see Lemma 1 below), if f is $\mathcal{I} \boxtimes \mathcal{F}$-measurable, integrable (in that $\int_I E(|f_i|)\, d\lambda(i)$ is finite), and essentially pairwise independent, then

$$\int_I f_i \, d\lambda(i) = \int_I E(f_i)\, d\lambda(i) \quad \text{almost surely.} \tag{1}$$

For example, if the agent-level random variables $\{f_i : i \in I\}$ are not only pairwise independent, but also have the same probability distribution with a finite expectation, then (1) implies that the cross-sectional average outcome $\int_I f_i \, d\lambda(i)$ is almost surely equal to the expected outcome for any agent, $E(f_i)$.

2.2 Static directed random matching

Each agent has some type in $S = \{1, 2, \ldots, K\}$. These types are assigned by some measurable function, $\alpha : I \to S$. The initial fraction of type-k agents is thus $p_k = \lambda(\{i : \alpha(i) = k\})$. The cross-sectional type distribution $p = (p_k)$ is thus an element of the space Δ of probability distributions on S.

A random matching is a function $\pi : I \times \Omega \to I$ that assigns a unique randomly chosen agent $\pi(i)$ to agent i. In the event that $\pi(i) = i$, agent i is not matched. Otherwise, $\pi(i)$ is the agent to whom i is matched. We will consider matchings with the property that any agent of type k is matched to an agent of type l with some given "directed-matching" probability q_{kl}, for any $(k, l) \in S^2$. Of course, these parameters (q_{kl}) must satisfy $\sum_{l \in S} q_{kl} \leq 1$. That is, for any agent i of type k, we have $q_{kl} = P(g_i = l)$, where $g(i) = \alpha(\pi(i))$ denotes the type of the agent to whom i is matched. In the event that i is not matched, we denote $g(i) = J$.

[9] See Footnote 13.

[10] On the other hand, Hammond and Sun (2006) shows that the essential versions of pairwise and mutual independence are equivalent even in the conditional setting.

A special case is uniform random matching, which means that $q_{kl} = p_l$. For reasons given in the Introduction, a rich body of prior research requires more generality than uniform matching.

For the specified matching probabilities (q_{kl}) to be feasible, we must have

$$p_k \, q_{kl} = p_l \, q_{lk}, \tag{2}$$

because the left and right hand sides are both equal to the expected total quantity of matches of agents of type k with agents of type l.

A random matching π is said to be independent if the associated types $\{g_i : i \in I\}$ are essentially pairwise independent. In this case, it follows immediately from the exact law of large numbers that the quantity $\lambda(\{i : \alpha(i) = k, g(i) = l\})$ of agents of type k that are matched to agents of type l is almost surely equal to the expected quantity $p_k q_{kl}$. One of our main results in Section 4 states that for any given initial distribution $p = (p_k)$ of types and any feasible matching probabilities (q_{kl}), there exists an initial type function α, a random matching π, and an associated $\mathcal{I} \boxtimes \mathcal{F}$-measurable process g for partners' types satisfying these key properties. We will show additional useful properties of such a directed random matching model.

2.3 Matching functions

Proposition 1 and Theorem 1 of Section 4 also provide a rigorous probabilistic foundation for the "matching-function" approach that is widely used in the literature of search-based labor markets. Matching functions allow the probabilities of matching to be directed and to depend on an endogenously determined cross-sectional distribution of types.

In models of search-based labor markets, it is typical to suppose that firms and workers are characterized by their types. A commonly used modeling device in this setting is a matching function $m_{kl} : [0,1] \times [0,1] \to [0,1]$ that specifies the quantity of type-k agents that are matched with type-l agents, given any proportions of type-k agents and type-l agents. (See Petrongolo and Pissarides (2001) for a survey of the matching-function approach.) Clearly one must require that for any k and l in S and any p in Δ,

$$m_{kl}(p_k, p_l) = m_{lk}(p_l, p_k), \quad \sum_{r \in S} m_{kr}(p_k, p_r) \le p_k. \tag{3}$$

Let $q_{kl} = m_{kl}(p_k, p_l)/p_k$ for $p_k \neq 0$, and let $q_{kl} = 0$ for $p_k = 0$. Then the requirements for a matching probability function are satisfied by (q_{kl}). By our results in Section 4, there exists an independent directed random matching π with parameters (p, q). Moreover, for any types k and l, the mass $\lambda(\{i : \alpha_i = k, g_i = l\})$ of agents of type k that are matched to agents of type l is almost surely

$$p_k q_{kl} = m_{kl}(p_k, p_l),$$

as specified by the given matching function. This means that any matching function satisfying Equation (3) can be realized through independent directed random matching, almost surely. For the special case of only two types of agents (say, types 1 and 2), any nonnegative matching function $m(p_1, p_2)$ with $m(p_1, p_2) \leq \min(p_1, p_2)$ can be realized through independent directed random matching. For this, one can simply take $q_{12} = m(p_1, p_2)/p_1$ and $q_{21} = m(p_1, p_2)/p_2$. More general cases are considered in Footnote 31.

A common parametric specification is the Cobb-Douglas matching function, for which

$$m_{UV}(p_U, p_V) = A\, p_U^\alpha\, p_V^\beta,$$

for parameters α and β in $(0,1)$, and a non-negative scaling parameter A. We emphasize that for some parameters α, β, and A, the inequality $A\, p_U^\alpha\, p_V^\beta \leq \min(p_U, p_U)$ may fail for some $(p_U, p_V) \in \Delta$. In that case, one can let $m(p_U, p_V) = \min(A\, p_U^\alpha p_V^\beta, p_U, p_V)$.

2.4 Markovian mutation and match-induced type changes

We now extend to a multi-period setting with time periods $0, 1, \ldots$ Typically, models used in the literature allow for the following additional probabilistic specifications:

- Before random matching occurs in each period, a random mutation causes an agent of type k to become an agent of type l with a given probability b_{kl}.
- At any matching between agents of types k and l, the agent that was of type k becomes an agent of type r with probability $\nu_{kl}(r)$. Likewise, the agent that was of type l becomes an agent of type r with probability $\nu_{lk}(r)$.

The complete list of model parameters is thus (p^0, q, b, ν), where the initial type distribution p^0 and the matching probabilities $q = (q_{kl})$ are as described above for the static model. In the more general model of Section 4, we allow the parameters (q, b, ν) to vary with the time period.

In each period, the mutations, random matchings, and match-induced type changes are assumed to be conditionally independent across agents, in the essential-pairwise sense. The initial types $\{\alpha_i^0 : i \in I\}$ are assumed to be essentially pairwise independent, which includes the special case of deterministic initial types.

In period n, after any mutation and match-induced types changes that have occurred in that period, let α_i^n denote the type of agent i and let $p_k^n = \lambda(\{i : \alpha_i^n = k\})$ denote the fraction of agents of type k. Let $\check{p}^n$ be the expected type distribution $\mathbb{E}(p^n)$. The initial conditions α^0 and $\check{p}^0 \in \Delta$ are given. The objective is to calculate the probability distributions and other properties of the agent-level type process $\alpha_i = \{\alpha_i^0, \alpha_i^1, \ldots\}$, as well the cross-sectional type distribution process $p = \{p^0, p^1, \ldots\}$.

In Section 4, we show that the cross-sectional distribution p^n of agent types in period n is almost-surely deterministic. We also demonstrate that, almost surely:

1. $p^n = \Gamma(p^{n-1})$, where $\Gamma : \Delta \to \Delta$ is explicitly computed.[11]

2. For λ-almost every agent i, the agent's type process $\alpha_i = \{\alpha_i^1, \alpha_i^2, \ldots\}$ is a Markov chain with the probability transition function Γ. That is, letting $w_i^n \in \Delta$ denote the probability distribution of α_i^n, we have $w_i^{n+1} = \Gamma(w_i^n)$.

3. The agent-level type processes $\{\alpha_i : i \in I\}$ are essentially pairwise independent.

4. From the exact law of large numbers and the above three results, it follows that the cross-sectional type distribution p^n is the same as its expectation $\ddot{p}^n$ with probability one. In addition, if almost every agent i has the same initial type probability distribution $\ddot{p}^0$, then $w_i^n = \ddot{p}^n$ for almost every agent i. That is, we can always arrange for the probability distribution of each agent's type to coincide with the cross-sectional distribution of types. (In Section 4, we state this equivalence at the level of distributions on sample paths in S^∞.)

5. There exists a stationary distribution p^*, defined by $p^* = \Gamma(p^*)$. Thus, if p^* is the initial probability distribution of α_i^0 for almost every agent i, then for almost every agent i, in every time period n, the type α_i^n of agent i has a probability distribution $w_i^n = p^*$ equal to the cross-sectional type distribution $p^n = p^*$.

Theorem 2 provides additional characterization of the close relationship in this Markovian setting between agent-level type probability distributions and cross-sectional type distributions. We later generalize to allow for enduring matchings, by which a pair of agents, once matched, may stay paired for some duration whose probability distribution can depend on their respective types, in a sense that we make precise.

3 Static Directed Random Matching

This section begins the statement of our results at a more complete level. We start with some mathematical preliminaries. Then a static model of directed random matching is formally given in Subsection 3.2, where we present the exact law of large numbers, the existence of independent directed random matching, and an illustrative application to a model of over-the-counter financial markets.

[11] Letting $\eta_r(p) = 1 - \sum_{l \in S} p_l q_{lr}$ and $\bar{p}_k(p) = \sum_{l \in S} p_l b_{lk}$, we have

$$\Gamma_r(p) = p_r \eta_r(\bar{p}(p)) + \sum_{k,l \in S} \bar{p}_k(p) q_{kl}(\bar{p}(p)) \nu_{kl}(r).$$

3.1 Mathematical preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space. The agent space is an atomless probability space $(I, \mathcal{I}, \lambda)$.

While a continuum of independent random variables, one for each of a large population such as I, can be formalized as a mapping on $I \times \Omega$, such a function can never be measurable with respect to the completion of the usual product σ-algebra $\mathcal{I} \otimes \mathcal{F}$, except in the trivial case in which almost all of the random variables are constants.[12] As in Sun (2006), we shall therefore work with an extension of the usual product probability space that retains the crucial Fubini property.

Definition 1 *A probability space $(I \times \Omega, \mathcal{W}, Q)$ extending the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ is said to be a Fubini extension of this product space if, for any real-valued Q-integrable function f on $(I \times \Omega, \mathcal{W})$,*

(1) For λ-almost all $i \in I$, $f_i = f(i, \cdot)$ is integrable on $(\Omega, \mathcal{F}, P)$.

(2) For P-almost all $\omega \in \Omega$, $f_\omega = f(\cdot, \omega)$ is integrable on $(I, \mathcal{I}, \lambda)$.

(3) $\int_{I \times \Omega} f \, dQ = \int_I (\int_\Omega f_i \, dP) \, d\lambda = \int_\Omega (\int_I f_\omega \, d\lambda) \, dP$.

To reflect the fact that the probability space $(I \times \Omega, \mathcal{W}, Q)$ has $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ as its marginal spaces, as required by the Fubini property, it will be denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

The Fubini extension could include a sufficiently rich collection of measurable sets to allow applications of the exact law of large numbers that we shall need. An $\mathcal{I} \boxtimes \mathcal{F}$-measurable function f will be called a "process," each f_i will be called a random variable of this process, and each f_ω will be called a sample function of the process. As shown in Section 2 of Sun (2006), a sufficient condition for proving the exact law of large numbers is the condition of essential pairwise independence. A formal definition is as follows.[13]

[12] See, for example, Proposition 2.1 in Sun (2006).

[13] Here we state the definition of essential pairwise independence using a complete separable metric space X for the sake of generality; in particular, a finite space or an Euclidean space is a complete separable metric space. Fix any $i \in I$. If the singleton set $\{i\}$ is measurable in $\mathcal{I}$, then it is clear that $\{i\}$ has measure zero (since $(I, \mathcal{I}, \lambda)$ is atomless). Note that a singleton set is not necessarily measurable in a general measurable space. However, such measurability follows from the atomless property and the convention that a probability space is complete. In particular, one can take, for each $n \geq 1$, a $\mathcal{I}$-measurable partition $\{A_k^n\}_{k=1}^{2^n}$ of I with $\lambda(A_k^n) = 1/2^n$ such that $A_k^n = A_{2k-1}^{n+1} \bigcup A_{2k}^{n+1}$ for $1 \leq k \leq 2^n$. For any $n \geq 1$, there is a unique k_n such that $i \in A_{k_n}^n$, which implies that $i \in \bigcap_{n=1}^\infty A_{k_n}^n$. Since $\bigcap_{n=1}^\infty A_{k_n}^n$ has measure zero and $(I, \mathcal{I}, \lambda)$ is complete, the singleton set $\{i\}$ is in $\mathcal{I}$ with measure zero. If the pairwise independence condition holds for f, namely, for any $i \neq j \in I$, f_i and f_j are independent, then for each $i \in I$, f_i and f_j are independent except for j in the λ-null set $\{i\}$, which means that f satisfies the condition of essential pairwise independence. The usual condition of mutual independence (any finite collection of random variables are independent) is even stronger than pairwise independence.

Definition 2 *(Essential pairwise independence)* *An $\mathcal{I} \boxtimes \mathcal{F}$-measurable process f from $I \times \Omega$ to a complete separable metric space X is said to be essentially pairwise independent if for λ-almost all $i \in I$, the random variables f_i and f_j are independent for λ-almost all $j \in I$.*

3.2 Static directed random matching

We follow the notation in Subsection 3.1. Let $S = \{1, 2, \ldots, K\}$ be a finite space of agent types and $\alpha : I \to S$ be an $\mathcal{I}$-measurable type function, mapping individual agents to their types. For any k in S, we let $p_k = \lambda(\{i : \alpha(i) = k\})$ denote the fraction of agents of type k. We can view $p = (p_k)_{k \in S}$ as an element of the space Δ of probability measures on S. Because $(I, \mathcal{I}, \lambda)$ has no atoms, for any type distribution $p \in \Delta$, one can find an $\mathcal{I}$-measurable type function with distribution p.

A function $q : S \times S \to \mathbb{R}_+$ is a matching probability function for the type distribution p if, for any k and l in S,

$$p_k \, q_{kl} = p_l \, q_{lk}, \quad \sum_{r \in S} q_{kr} \leq 1. \tag{4}$$

The matching probability q_{kl} specifies the probability that an agent of type k is matched to an agent of type l. Thus $\eta_k = 1 - \sum_{l \in S} q_{kl}$ is the associated no-matching probability for an agent of type k.

Definition 3 *Let α, p, and q be given as above, and J a special type representing no-matching.*

(i) A full matching ϕ is a one-to-one mapping from I onto I such that, for each $i \in I$, $\phi(i) \neq i$ and $\phi(\phi(i)) = i$.

(ii) A (partial) matching ψ is a mapping from I to I such that for some subset B of I, the restriction $\psi|_B$ of ψ to B is a full matching on B, and the restriction $\psi|_{I \setminus B}$ of ψ to $I \setminus B$ is the identity mapping. This means that agent i in B is matched to another agent $\psi(i)$ in B, whereas any agent i not in B is unmatched, in that $\psi(i) = i$.

(iii) A random matching π is a mapping from $I \times \Omega$ to I such that

(a) π_ω is a matching for each $\omega \in \Omega$.

(b) $g(i, \omega) = \begin{cases} \alpha(\pi(i, \omega)) & \text{if } \pi(i, \omega) \neq i \\ J & \text{if } \pi(i, \omega) = i \end{cases}$
is measurable from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to $S \cup \{J\}$.

(iv) A random matching π from $I \times \Omega$ to I is directed, with parameters (p, q) satisfying condition (4), if for λ-almost every agent i of type k, $P(g_i = J) = \eta_k$ and $P(g_i = l) = q_{kl}$.

(v) *A random matching π is said to be independent if the associated type process g is essentially pairwise independent.*

For an agent $i \in I$ who is matched, the random variable $g_i = g(i, \cdot)$ is the type of her matched partner. Part *(iv)* of the definition thus means that for λ-almost every agent i of type k, her probability of being matched with a type-l agent is q_{kl}, while her no-matching probability is η_k.

The following result is a direct application of the exact law of large numbers. In particular, letting $I_k = \{i \in I : \alpha(i) = k\}$, the result follows from Theorem 2.8 of Sun (2006) (see Lemma 1 below) by working with the process $g^{I_k} = g|_{I_k \times \Omega}$ on the rescaled agent space I_k, where $g|_{I_k \times \Omega}$ is the restriction of g to $I_k \times \Omega$.

Proposition 1 *Let π be an independent directed random matching with parameters (p, q). Then, for P-almost every $\omega \in \Omega$, we have*

(i) *For $k \in S$, $\lambda(\{i \in I : \alpha(i) = k, g_\omega(i) = J\}) = p_k \eta_k$.*

(ii) *For any $(k, l) \in S \times S$, $\lambda(\{i : \alpha(i) = k, g_\omega(i) = l\}) = p_k q_{kl}$.*

Let κ be the probability measure on $S \times (S \cup \{J\})$ defined by letting $\kappa(k, l) = p_k q_{kl}$ for any $(k, l) \in S \times S$ and $\kappa(k, J) = p_k \eta_k$ for $k \in S$. Proposition 1 says that the cross-sectional joint type distribution of (α, g_ω) is κ with probability one.

Now we state our main existence result for the static setting.

Theorem 1 *For any type distribution p on S and any matching probability function q for p, there exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ on which is defined a type function α and an independent directed random matching*[14] *π with parameters (p, q), which is measure preserving in the sense that for each $\omega \in \Omega$, $\lambda(\pi_\omega^{-1}(A)) = \lambda(A)$ for any $A \in \mathcal{I}$.*

The proof of Theorem 1 will be given in Subsection E.1 for the case of a Loeb measure space of agents via the method of nonstandard analysis.[15] Since the unit interval and the class of Lebesgue measurable sets with the Lebesgue measure provide the archetype for models of economies with a continuum of agents, the next proposition (proved in Subsection E.3) shows that one can take an extension of the classical Lebesgue unit interval as the agent space for the construction of an independent directed random matching.

[14] As shown in the last paragraph in the proof of Lemma 7, one can take a subset $\tilde{I}$ of I such that $\lambda(I \setminus \tilde{I}) = 0$, and the random types $\{g_i\}_{i \in \tilde{I}}$ as constructed there satisfy a stronger independence condition in the sense that any finitely many random variables from that collection are mutually independent.

[15] A simple treatment of nonstandard analysis is given in Appendix D. We note that the proof of Theorem 1 is substantially different from the corresponding existence result for the case of "undirected" search in Duffie and Sun (2007); see the first paragraph of Subsection E.1.

Proposition 2 *For any type distribution p on S and any matching probability function q for p, there exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ such that:*

1. *The agent space $(I, \mathcal{I}, \lambda)$ is an extension of the Lebesgue unit interval $(L, \mathcal{L}, \chi)$.*

2. *There is defined on the Fubini extension a type function α and an independent directed random matching*[16] *π with parameters (p, q).*

The following example provides an illustrative application of Theorem 1 and Proposition 1 to a model of over-the-counter financial markets.

Example 1 In Duffie, Malamud and Manso (2014), the economy is populated by a continuum of risk-neutral agents. There are M different types of agents that differ according to the quality of their initial information, their preferences for the asset to be traded, and the likelihoods with which they meet each of other types of agents for trade. The proportion of type-l agents is m_l, where $l = 1, \ldots, M$. Any agent of type l is randomly matched with some other agent with probability $\lambda_l \in [0, 1)$. This counterparty is of type-r with probability κ_{lr}. In the present context, we can take the matching probability $q_{lr} = \lambda_l \kappa_{lr}$ for any l and r in S. Theorem 1 guarantees the existence of independent directed random matching with the given parameters m_l, q_{lr}. Proposition 1 implies that the total quantity of matches of agents of a given type l with the agents of a given type r is almost surely $m_l \lambda_l \kappa_{lr} = m_r \lambda_r \kappa_{rl}$. (See page 7 of Duffie, Malamud and Manso (2014).)

4 Dynamic Directed Random Matching

In this section we show how to construct a dynamical system that incorporates the effects of random mutation, directed random matching, and match-induced type changes with time-dependent parameters. We first define such a dynamical system in Subsection 4.1. The key condition of Markov conditional independence is formulated in Subsection 4.2. Based on that condition, we state in Subsection 4.3 an exact law of large numbers for such a dynamical system. The section ends with the existence of Markov conditionally independent dynamic directed random matching.

4.1 Definition of dynamic directed random matching

As in Section 3, we fix an atomless probability space $(I, \mathcal{I}, \lambda)$ representing the space of agents, a sample probability space $(\Omega, \mathcal{F}, P)$, and a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Let

[16] In addition, there exists a sub-σ-algebra $\mathcal{I}'$ of $\mathcal{I}$ and a Fubini extension $(I \times \Omega, \mathcal{I}' \boxtimes \mathcal{F}, \lambda \boxtimes P)$ such that $\mathcal{I}' \boxtimes \mathcal{F} \subseteq \mathcal{I} \boxtimes \mathcal{F}$ and π is an independent directed random matching with parameters (p, q) on $(I \times \Omega, \mathcal{I}' \boxtimes \mathcal{F}, \lambda \boxtimes P)$, which is measure preserving in the sense that for each $\omega \in \Omega$, $\lambda(\pi_\omega^{-1}(A)) = \lambda(A)$ for any $A \in \mathcal{I}'$. See the penultimate paragraph of the proof of Proposition 5 in Subsection E.3.

$S = \{1, 2, \ldots, K\}$ be a finite set of types and let J be a special type representing no-matching. We shall define a discrete-time dynamical system $\mathbb{D}_0$ with the property that at each integer time period $n \geq 1$, agents first experience a random mutation and then random matching with directed probabilities. Finally, any pair of matched agents are randomly assigned new types whose probabilities may depend on the pair of prior types of the two agents.

At period $n \geq 1$, each agent of type $k \in S$ first experiences a random mutation, becoming an agent of type l with a given probability b^n_{kl}, with $\sum_{r \in S} b^n_{kr} = 1$. At the second step, every agent conducts a directed search for counterparties. In particular, for each $(k, l) \in S \times S$, the directed matching probability is determined by a function q^n_{kl} on the space of type distributions Δ, with the property that, for all k and l in S, the function that maps the type distribution $p \in \Delta$ to $p_k q^n_{kl}(p)$ is continuous and satisfies, for all $p \in \Delta$,

$$p_k \, q^n_{kl}(p) = p_l \, q^n_{lk}(p) \quad \text{and} \quad \sum_{r \in S} q^n_{kr}(p) \leq 1. \tag{5}$$

The intention is that, if the population type distribution in the current period is p, then an agent of type k is matched to some agent whose type is l with probability $q^n_{kl}(p)$. Thus, $\eta^n_k(p) = 1 - \sum_{l \in S} q^n_{kl}(p)$ is the associated probability of no match. When an agent of type k is matched at time n to an agent of type l, the agent of type k becomes an agent of type r with probability $\nu^n_{kl}(r)$, where $\sum_{r \in S} \nu^n_{kl}(r) = 1$. The primitive model parameters are (b, q, ν).

Let α^0 be the initial S-valued type process on the Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. For each time period $n \geq 1$, the agents' types after the random mutation step are given by a process h^n from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to S. Then, a random matching is described by a function π^n from $I \times \Omega$ to I. The end-of-period types are given by a process α^n from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to S. Thus the post-mutation type function h^n satisfies

$$P(h^n_i = l \,|\, \alpha^{n-1}_i = k) = b^n_{kl}. \tag{6}$$

For the directed random matching step, let g^n be an $\mathcal{I} \boxtimes \mathcal{F}$-measurable function defined by $g^n(i, \omega) = h^n(\pi^n(i, \omega), \omega)$, with the property that for any type $k \in S$, for λ-almost every i and P-almost every $\omega \in \Omega$,

$$P(g^n_i = l \,|\, h^n_i = k, \check{p}^n) = q^n_{kl}(\check{p}^n(\omega)), \tag{7}$$

where $\check{p}^n(\omega) = \lambda(h^n_\omega)^{-1}$ is the post-mutation type distribution realized in state ω. The end-of-period agent type function α^n satisfies, for λ-almost every agent i,

$$P(\alpha^n_i = r \,|\, h^n_i = k, g^n_i = J) = \delta_k(r) \text{ and } P(\alpha^n_i = r \,|\, h^n_i = k, g^n_i = l) = \nu^n_{kl}(r), \tag{8}$$

where $\delta_k(r)$ is 1 if $k = r$ and is zero otherwise. Thus, we have inductively defined the properties required of a dynamical system $\mathbb{D}_0$ that incorporates the specified effects of random mutation, directed random matching, and match-induced type changes with given parameters (b, q, ν).

4.2 Markov conditional independence (MCI)

We now add independence conditions on the dynamical system $\mathbb{D}_0$, along the lines of those in Duffie and Sun (2007, 2012). The idea is that each of the just-described steps (mutation, random matching, match-induced type changes) are conditionally independent across almost all agents.

We say that the dynamical system $\mathbb{D}_0$ is Markov conditionally independent (MCI) if, for λ-almost every i and λ-almost every j, for every period $n \geq 1$, and for all types k and l in S, the following four properties apply:

- Initial independence: α_i^0 and α_j^0 are independent.
- Markov and conditionally independent mutation:

$$P(h_i^n = k, h_j^n = l \mid \alpha_i^0, \ldots, \alpha_i^{n-1}; \alpha_j^0, \ldots, \alpha_j^{n-1}) = P(h_i^n = k \mid \alpha_i^{n-1}) P(h_j^n = l \mid \alpha_j^{n-1}).$$

- Markov and conditionally independent random matching:

$$P(g_i^n = k, g_j^n = l \mid \alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n; \alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n) = P(g_i^n = k \mid h_i^n) P(g_j^n = l \mid h_j^n).$$

- Markov and conditionally independent matched-agent type changes:

$$\begin{aligned} & P(\alpha_i^n = k, \alpha_j^n = l \mid \alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n, g_i^n; \alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n, g_j^n) \\ = \;\; & P(\alpha_i^n = k \mid h_i^n, g_i^n) P(\alpha_j^n = l \mid h_j^n, g_j^n). \end{aligned}$$

4.3 The exact law of large numbers for MCI dynamical systems

We define a sequence Γ^n of mappings from Δ to Δ such that, for each $p \in \Delta$,

$$\Gamma_r^n(p_1, \ldots, p_K) = \bar{p}_r^n(p) \eta_r^n(\bar{p}^n(p)) + \sum_{k,l \in S} \bar{p}_k^n(p) q_{kl}^n(\bar{p}^n(p)) \nu_{kl}^n(r),$$

where $\bar{p}_k^n(p) = \sum_{l \in S} p_l b_{lk}^n$ for $k \in S$.

The following theorem presents an exact law of large numbers for the agent type processes at the end of each period, and gives a recursive calculation for the cross-sectional joint agent type distribution p^n at the end of period n.

Theorem 2 *A Markov conditionally independent dynamical system $\mathbb{D}_0$ with parameters (b, q, ν), for random mutation, directed random matching and match-induced type changes, satisfies the following properties.*

(1) *For each time* $n \geq 1$*, let* $p^n(\omega) = \lambda(\alpha^n_\omega)^{-1}$ *be the realized cross-sectional type distribution at the end of the period* n*. The expectation* $\mathbb{E}(p^n)$ *is given by*

$$\mathbb{E}(p^n_r) = \Gamma^n_r(\mathbb{E}(p^{n-1})) = \bar{p}^n_r \eta^n_r(\bar{p}^n) + \sum_{k,l \in S} \bar{p}^n_k q^n_{kl}(\bar{p}^n)\nu^n_{kl}(r),$$

where $\bar{p}^n_k = \sum_{l \in S} \mathbb{E}(p^{n-1}_l) b^n_{lk}$.

(2) *For* λ*-almost every agent* i*, the type process* $\{\alpha^n_i\}_{n=0}^\infty$ *of agent* i *is a Markov chain with transition matrix* z^n *at time* $n-1$ *defined by*

$$z^n_{kl} = \eta^n_l(\bar{p}^n) b^n_{kl} + \sum_{r,j \in S} b^n_{kr} q^n_{rj}(\bar{p}^n)\nu^n_{rj}(l).$$

(3) *For* λ*-almost every* i *and* λ*-almost every* j*, the Markov chains* $\{\alpha^n_i\}_{n=0}^\infty$ *and* $\{\alpha^n_j\}_{n=0}^\infty$ *are independent.*[17]

(4) *For* P*-almost every state* ω*, the cross-sectional type process* $\{\alpha^n_\omega\}_{n=0}^\infty$ *is a Markov chain with transition matrix* z^n *at time* $n-1$*.*

(5) *For* P*-almost every state* ω*, at each time period* $n \geq 1$*,* $p^n(\omega) = \lambda(\alpha^n_\omega)^{-1}$*, and the realized cross-sectional type distribution after random mutation* $\lambda(h^n_\omega)^{-1}$ *is* $\bar{p}^n$*.*

(6) *If there is some fixed* $\ddot{p}^0 \in \Delta$ *that is the probability distribution of the initial type* α^0_i *of agent* i *for* λ*-almost every* i*, then the probability distribution* $\zeta = \ddot{p}^0 \otimes (\otimes_{n=1}^\infty z^n)$ *on* S^∞ *is equal to the sample-path distribution of the Markov chain* $\alpha_i = \{\alpha^n_i\}_{n=0}^\infty$ *for* λ*-almost every agent* i*. For* P*-almost every state* $\omega \in \Omega$*,* ζ *is also the cross-sectional joint distribution* $\lambda\alpha_\omega^{-1}$ *of the sample paths of agents' realized type process.*

(7) *Suppose that the parameters* (b, q, ν) *are time independent. Then there exists a type distribution* $p^* \in \Delta$ *such that* p^* *is a stationary distribution for any Markov conditionally independent dynamical system* $\mathbb{D}_0$ *with parameters* (b, q, ν)*, in the sense that for every period* $n \geq 0$*, the realized cross-sectional type distribution* p^n *at time* n *is* p^* P*-almost surely. all of the relevant Markov chains are time homogeneous with a constant transition matrix* z^1 *having* p^* *as a fixed point. In addition, if the initial type process* α^0 *is i.i.d. across agents, then, for* λ*-almost every agent* i*,* $P(\alpha^n_i)^{-1} = p^*$ *for any period* $n \geq 0$*.*

4.4 Existence of MCI dynamic directed random matching

The following theorem provides for the existence of a Markov conditionally independent (MCI) dynamical system with random mutation, random matching, and match-induced type changes.

[17] Two Markov chains with a state space S are said to be independent if they are independent as random variables taking values in S^∞.

Theorem 3 *For any primitive model parameters (b, q, ν) and for any type distribution $\ddot{p}^0 \in \Delta$, there exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ on which is defined a dynamical system $\mathbb{D}_0$ with random mutation, random matching, match-induced type changes, that is Markov conditionally independent with these parameters (b, q, ν), and with the initial cross-sectional type distribution p^0 that is $\ddot{p}^0$ with probability one. In addition, for any $n \geq 1$, π^n is measure preserving in the sense that for each $\omega \in \Omega$, $\lambda((\pi^n_\omega)^{-1}(A)) = \lambda(A)$ for any $A \in \mathcal{I}$. These properties can be achieved with an initial type process α^0 that is deterministic, or i.i.d. across agents.*[18]

With the next proposition, we show that the agent space $(I, \mathcal{I}, \lambda)$ can be an extension of the classical Lebesgue unit interval $(L, \mathcal{L}, \chi)$. That is, we can take $I = L = [0, 1]$ with a σ-algebra $\mathcal{I}$ that contains the Lebesgue σ-algebra $\mathcal{L}$, and so that the restriction of λ to $\mathcal{L}$ is the Lebesgue measure χ.

Proposition 3 *Fixing any model parameters (b, q, ν) and any initial cross-sectional type distribution $\ddot{p}^0 \in \Delta$, there exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ such that:*

(1) The agent space $(I, \mathcal{I}, \lambda)$ is an extension of the Lebesgue unit interval $(L, \mathcal{L}, \chi)$.

(2) There is defined on the Fubini extension a dynamical system $\mathbb{D}_0$ that is Markov conditionally independent with the parameters (b, q, ν), where the initial cross-sectional type distribution p^0 is $\ddot{p}^0$ with probability one.

(3) These properties can be achieved with an initial type process α^0 that is deterministic, or i.i.d. across agents.[19]

5 Discussion

We finish with a discussion of the prior literature on the mathematics of random matching, and then offer some concluding comments about our main new results and some immediately available extensions of our results to more general type spaces, changing population sizes, or background "macro-economic" processes that cause random changes in the evolution of the cross-sectional type distributions.

As mentioned in Section 2, when agents conduct searches without explicit coordination, it is reasonable to impose the assumption of independence of their searches. However, Footnote

[18]This means that the process α^0 is essentially pairwise independent, and α^0_i has distribution $\ddot{p}^0$ for λ-almost all $i \in I$.

[19]In addition, there exists a sub-σ-algebra $\mathcal{I}'$ of $\mathcal{I}$ and a Fubini extension $(I \times \Omega, \mathcal{I}' \boxtimes \mathcal{F}, \lambda \boxtimes P)$ such that $\mathcal{I}' \boxtimes \mathcal{F} \subseteq \mathcal{I} \boxtimes \mathcal{F}$, the dynamical system $\mathbb{D}_0$ on $(I \times \Omega, \mathcal{I}' \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is Markov conditionally independent, and for any $n \geq 1$, π^n is measure preserving in the sense that for each $\omega \in \Omega$, $\lambda((\pi^n_\omega)^{-1}(A)) = \lambda(A)$ for any $A \in \mathcal{I}'$. See the penultimate paragraph of the proof of Proposition 5 in Subsection E.3.

4 of McLennan and Sonnenschein (1991) showed the non-existence of a type-free static random full matching that satisfies a number of desired conditions, when the agent space is taken to be the unit interval with the Borel σ-algebra and Lebesgue measure. That problem was resolved through the construction of an independent type-free static random full matching with a suitable agent space, as in Duffie and Sun (2007).[20] Xiang Sun (2016) extended the results on independent static random partial matching in Duffie and Sun (2007) from finite type spaces to general type spaces. Duffie and Sun (2007) and Duffie and Sun (2012) go beyond the static case to present the existence of, and the exact law of large numbers for, discrete-time independent random matching.

When the independence assumption for random matching is not required, one can construct many non-independent random full matchings with some desired matching properties, even for finitely many agents.[21]

Based on the classical asymptotic law of large numbers, Boylan (1992) constructed a random full matching for a countable population. Gilboa and Matsui (1992) presented a particular matching model of two countable populations with a countable number of encounters in the time interval $[0, 1)$, where both the agent space $\mathbb{N}$ and the sample space are endowed with purely finitely additive measures. A non-independent random full matching was constructed in Alós-Ferrer (1999) for a given type function on the population space $[0, 1]$ by rearranging the intervals in $[0, 1]$ through measure-preserving mappings. Rather than relying on particular examples of a non-independent random matching that have certain desired matching properties, Duffie and Sun (2012) instead proved the exact law of large numbers for general independent random matchings.

An independent random matching automatically involves a process with a continuum of independent random variables. The classical Kolmogrov consistency theorem (see, for example, Bogachev (2007, p. 95)) implies the existence of a continuum of independent real-valued random variables. Following Doob (1937), some measure extension leads to the fact that almost all sample functions differ from an arbitrarily given function h (whether measurable or not) at only countably many points.[22] This implies that one can claim that the sample functions (and thus the sample means and distributions) are essentially arbitrary.[23] Though it is not

[20]See the main theorem of Podczeck and Puzzello (2012) for another construction. It is clear that an independent type-free static random full matching also provides a model of independent static random full matching for general types; see Duffie and Sun (2012) and Podczeck and Puzzello (2012).

[21]See Subsection 3.2 of Molzon and Puzzello (2010). However, the requirement that all the agents are matched randomly then leads to correlation among the finitely many agents. It therefore does not make sense to consider independent random matching in the finite-agent setting. For a detailed discussion of the literature on "non-independent" random matching, see Section 6 of Duffie and Sun (2012).

[22]See Subsection 6.1 in Sun (2006) for the detailed discussion.

[23]That is, the sample means and distributions of an *i.i.d.* process can be essentially equal to the theoretical means and distribution, respectively, or any other arbitrarily given means and distribution. See Judd (1985)

possible to require an *i.i.d.* process to have essentially constant sample means and distributions at the coalitional level for an agent space based on the Lebesgue unit interval,[24] one can simply use the transfer principle[25] in nonstandard analysis to restate the classical law of large numbers for a sequence of *i.i.d.* random variables in a nonstandard model, and thus claim the existence of a process with the required properties.[26] The essential difficulty associated with a continuum of independent random variables is that independence and joint measurability with respect to the usual measure-theoretic product are never compatible with each other except for the trivial case in which almost all of the random variables are constants.[27] In Sun (2006), various versions of the exact law of large numbers, and their converses in both static and dynamic settings, are proved by direct application of simple measure-theoretic methods in the framework of a Fubini extension. Such a framework is adopted in this paper (1) to construct static and dynamic models of independent directed random matching that incorporate the effects of random mutation, random matching with match-induced type changes, and with the potential for enduring partnerships that may have randomly timed break-ups, and (2) to study various properties of a general independent directed random matching via the exact law of large numbers.[28]

All of the papers on random matching mentioned above address the case of "undirected" search, in the sense that the matching probabilities are proportional to the population sizes of the matched agents with given types. The main purpose of this paper is to provide a suitable search-based model of markets in which agents can direct their searches, causing relatively higher per-capita matching probabilities with specific types of counterparties. Although models with directed search are common in the literatures covering money, labor markets, and

and Sun (2006, p. 53).

[24]See Feldman and Gilles (1985).

[25]See Proposition 7 in Subsection D.2.

[26]See, for example, Hersh and Greenwood (1975), Keisler (1977), Hurd and Loeb (1985), Anderson (1991), and Footnote 46 below for the transfer of a sequence of *i.i.d.* random variables and relevant results. Green (1994) also provided an alternative construction for the existence of such a process with the required property. Note that the agent space as considered in those papers are not the Lebesgue type space while the relevant sample probability space is countably additive. On the other hand, one can still claim the existence of an *i.i.d.* process to have any kind of sample means/distributions at the coalitional level based on the usual Lebesgue agent space and some purely *finitely additive* sample probability space; see Berti, Gori and Rigo (2012). Given the various possibilities in claiming the stability of sample functions in specific examples, what one needs is to find a suitable measure-theoretic condition to guarantee aggregate sample stability under independence.

[27]See Proposition 2.1 in Sun (2006).

[28]Dynamic independent random matching, as considered in this paper, needs a specific construction based on the framework of a Fubini extension, which allows us to apply the static and dynamic versions of exact law of large numbers shown in such a general framework in Sun (2006). However, this construction does not fit any earlier examples on the existence of an *i.i.d.* process having essentially constant sample means/distributions. In particular, the condition of $*$-independence (see Hurd and Loeb (1985) and Footnote 46 below) holds for the transfer of an *i.i.d.* sequence of random variables to the nonstandard model. As noted in Footnote 19 of Duffie and Sun (2007), $*$-independence, which is much stronger than the usual independence condition, is not satisfied even in the simple setting of static random full matching.

over-the-counter financial markets, prior work has simply assumed that the exact law of large numbers would lead to a deterministic cross-sectional distribution of agent types, and that this distribution would obey certain properties. We provide a model that justifies this assumed behavior, down to the basic level of random contacts between specific individual agents. We provide the resulting transition distribution for the Markov processes for individual agents' types, and for the aggregate cross-sectional distribution of types in the population, and show the close relationship between these two objects.

By incorporating directed search, we are also able to provide the first rigorous probabilistic foundation for the notion of a "matching function" that is heavily used in the search literature of labor economics.

A secondary objective of our paper is to allow for random matching with enduring partnerships. The durations of these partnerships can be random or deterministic, and can be type dependent. Earlier work providing mathematical foundations for random matching presumes that partnerships break up immediately after matching. Enduring partnerships are crucial for search-based labor-market models, such as those cited in Footnote 5, in which there are episodes of employment resulting from a match between a worker and a firm, eventually followed by a randomly timed separation.[29] In some of these models, separation is *i.i.d.* across periods of employment. This is the case, for example, in Cho and Matsui (2013), Merz (1999), Pissarides (1985), Shi and Wen (1999), Shimer (2005), and Yashiv (2000), among many other papers. In other cases, the separation probability depends on the vintage of the match, and can depend on the quality of the match between the worker and the firm. Since the separation probabilities in our general model depend on the types of the matched agents, our results can cover such cases by introducing new types.

We have verified that our results can be extended under mild revisions of the proofs to settings in which agents have countably many types, and can enter and exit (for example, through "birth" and "death"), allowing for a total population size that is changing over time without a fixed bound, as in Yashiv (2000).[30] It is also straightforward to allow for a background Markov process that governs the parameters determining probabilities for mutation, matching, and type change (as well as enduring match break-ups). In this case, the background Markov state causes aggregate uncertainty, but conditional on the path of the background state, the cross-sectional distribution of population types evolves deterministically, almost surely.

[29]Further such situations arise in the models of Cho and Matsui (2013), Flinn (2006), Haan, Ramey and Watson (2000), Hall (2005), Hosios (1990), Merz (1995), Merz (1999), Mortensen (1982), Pissarides (1985), Shimer (2005), Shi and Wen (1999), and Yashiv (2000).

[30]Our results cover cases in which there is a fixed bound for the total population size in all time periods. In such cases, one can simply introduce a new type to represent the inactive agents and re-scale the total population size.

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Appendices

The following appendices allow matches that result in enduring partnerships, offer some illustrative examples of applications of our main results to models from the literature on monetary and labor economics, provide a brief introduction to nonstandard analysis, and present proofs of the results.

A Dynamic Directed Random Matching with Enduring Partnerships

This appendix extends the model of dynamic directed random matching found in Section 4 so as to allow for enduring partnerships and for correlated type changes of matched agents. Unlike the more basic model of Section 4, in order to capture the effect of enduring partnerships we now must consider separate treatments of existing matched pairs of agents and newly formed matched pairs of agents.

We first define such a dynamical system in Subsection A.1. The key condition of Markov conditional independence is formulated in Subsection A.2. Based on that condition, Subsection A.3 presents an exact law of large numbers for such a dynamical system. Subsection A.4 provides results covering the existence of Markov conditionally independent dynamical system with directed random matching and with partnerships that have randomly timed breakups. We will show that all of our results can be obtained for an agent space that is a Loeb measure space as constructed in nonstandard analysis, or is an extension of the classical Lebesgue unit interval. This section has self-contained notation. In particular, some of the notation used in this section may have a meaning that differs from its usage in Section 4.

The exact law of large numbers and stationarity for independent dynamic directed random matching, as stated in Theorem 2 in Section 4, is a special case of Theorem 4 and Proposition 4 in Subsection A.3. The existence of independent dynamic directed random matching, as stated in Theorem 5 and Proposition 5 in Subsection A.4, extend respectively Theorem 3 and Proposition 3 in Section 4. Hence, the proofs of Theorems 2, 3, and Proposition 3 are omitted. The proofs of Theorem 4 and Proposition 4 do not use nonstandard analysis, and are given in Appendix C. The proofs of Theorem 5 and Proposition 5 need nonstandard analysis, and are presented in Subsections E.2 and E.3 respectively, after a brief introduction to nonstandard analysis in Appendix D.

A.1 Definition of dynamic directed random matching with enduring partnerships

As in Sections 3 and 4, we fix an atomless probability space $(I, \mathcal{I}, \lambda)$ representing the space of agents, a sample probability space $(\Omega, \mathcal{F}, P)$, and a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Let $S = \{1, 2, \ldots, K\}$ be a finite set of types and let J be a special type representing no-matching.

The "extended type" space is $\hat{S} = S \times (S \cup \{J\})$. An agent with an extended type of the form (k, l) has underlying type $k \in S$ and is currently matched to another agent of type $l \in S$. If the agent's extended type is instead of the form (k, J), then the type-k agent is "unmatched." The space $\hat{\Delta}$ of extended type distributions is the set of probability distributions $\hat{p}$ on $\hat{S}$ satisfying $\hat{p}(k, l) = \hat{p}(l, k)$ for all k and l in S.

Each time period is divided into three steps: mutation, random matching, match-induced type changing with break-up. We now introduce the primitive parameters governing each of these steps.

At the first (mutation) step of time period $n \geq 1$, each agent of type $k \in S$ experiences a random mutation, becoming an agent of type l with a given probability b^n_{kl}, a parameter of the model. By definition, for each type k we must have $\sum_{l \in S} b^n_{kl} = 1$.

At the second step, any currently unmatched agent conducts a directed search for counterparties. For each $(k, l) \in S \times S$, let q^n_{kl} be a function on $\hat{\Delta}$ into $\mathbb{R}_+$ with the property that for all k and l in S, the function $\hat{p}_{kJ} q^n_{kl}(\hat{p})$ is continuous in $\hat{p} \in \hat{\Delta}$ and satisfies, for any $\hat{p}$ in $\hat{\Delta}$,

$$\hat{p}_{kJ}\, q^n_{kl}(\hat{p}) = \hat{p}_{lJ}\, q^n_{lk}(\hat{p}) \quad \text{and} \quad \sum_{r \in S} q^n_{kr}(\hat{p}) \leq 1. \tag{9}$$

Whenever the underlying extended type distribution is $\hat{p}$, the probability[31] that an unmatched agent of type k is matched to an unmatched agent of type l is $q^n_{kl}(\hat{p})$. Thus, $\eta^n_k(\hat{p}) = 1 - \sum_{l \in S} q^n_{kl}(\hat{p})$ is the no-matching probability for an unmatched agent of type k.

At the third step, each currently matched pair of agents of respective types k and l (including those who have just been paired at the matching step) breaks up with probability θ^n_{kl}, where

$$\theta^n_{kl} = \theta^n_{lk}. \tag{10}$$

If a matched pair of agents of respective types k and l stays in their partnership, they become a pair of agents of types r and s, respectively, with a specified probability $\sigma^n_{kl}(r, s)$, where

$$\sum_{r,s \in S} \sigma^n_{kl}(r, s) = 1 \text{ and } \sigma^n_{kl}(r, s) = \sigma^n_{lk}(s, r) \tag{11}$$

[31] Let φ be any continuous function from $\hat{\Delta}$ to itself. Assume that: (1) for any $k, l \in S$, $\varphi_{kl}(\hat{p}) \geq \hat{p}_{kl}$; (2) for any $\hat{p} \in \hat{\Delta}$, $\varphi(\hat{p})$ and $\hat{p}$ have the same marginal measure on S, that is, for any $k \in S$, $\sum_{r \in S \cup \{J\}} \varphi_{kr}(\hat{p}) = \sum_{r \in S \cup \{J\}} \hat{p}_{kr}$. If $\hat{p}$ in $\hat{\Delta}$ is the underlying extended type distribution for the agents, then $\varphi(\hat{p})$ represents the extended type distribution after matching. For any $k, l \in S$, let $q_{kl}(\hat{p}) = (\varphi_{kl}(\hat{p}) - \hat{p}_{kl}) / \hat{p}_{kJ}$ if $\hat{p}_{kJ} > 0$ and $q_{kl}(\hat{p}) = 0$ if $\hat{p}_{kJ} = 0$. Then, the function q satisfies the continuity condition as well as Equation (9), as required for a matching probability function. In fact, any matching probability function can be obtained in this way. For the special case that all of the matched agents break up at the end of each period, we need only consider continuous functions from Δ to $\hat{\Delta}$. Let ϕ be any such continuous function with the property that for any $p \in \Delta$, $\varphi(p)$ has the marginal measure p on S. That is, for any $k \in S$, $\sum_{l \in S \cup \{J\}} \varphi_{kl}(p) = p_k$. For any $k, l \in S$, let $q_{kl}(p) = \phi_{kl}(p)/p_k$ if $p_k > 0$ and $q_{kl}(p) = 0$ if $p_k = 0$. Then, the function q satisfies the continuity condition as well as Equation (5), as required for a matching probability function. Again, any matching probability function for this particular setting can be obtained in this way.

for any $k, l, r, s \in S$. The second identity is merely a labeling symmetry condition. If a matched pair of agents of respective types k and l breaks up, the agent of type k becomes an agent of type r with probability $\varsigma_{kl}^n(r)$, where

$$\sum_{r \in S} \varsigma_{kl}^n(r) = 1. \tag{12}$$

We now give an inductive definition of the properties defining a dynamical system $\mathbb{D}$ for the behavior of a continuum population of agents experiencing, at each time period: random mutations, matchings, and match-induced type changes with break-up. We later state conditions under which such a system exists. The state of the dynamical system $\mathbb{D}$ at the end of each integer period $n \geq 0$ is defined by a pair $\Pi^n = (\alpha^n, \pi^n)$ consisting of:

- An agent type function $\alpha^n : I \times \Omega \to S$ that is $\mathcal{I} \boxtimes \mathcal{F}$-measurable. The corresponding end-of-period type of agent i is $\alpha^n(i, \omega) \in S$.
- A random matching $\pi^n : I \times \Omega \to I$, describing the end-of-period agent $\pi^n(i)$ to whom agent i is currently matched, if agent i is currently matched. If agent i is not matched, then $\pi^n(i) = i$. The associated partner-type function $g^n : I \times \Omega \to S \cup \{J\}$ provides the type
$$g^n(i) = \begin{cases} \alpha^n(\pi^n(i)) & \text{if } \pi^n(i) \neq i \\ J & \text{if } \pi^n(i) = i \end{cases}$$
of the agent to whom agent i is matched, if agent i is matched, and otherwise specifies $g^n(i) = J$. As a matter of definition, we require that g^n is $\mathcal{I} \boxtimes \mathcal{F}$-measurable.

We take the initial condition $\Pi^0 = (\alpha^0, \pi^0)$ of $\mathbb{D}$ as given. The initial condition may, if desired, be deterministic (constant across Ω). The joint cross-sectional extended type distribution $\hat{p}^n$ at the end of period n is $\lambda(\beta^n)^{-1}$, where $\beta^n = (\alpha^n, g^n)$ is the extended type process. That is, when $\omega \in \Omega$ is a sample realization, $\hat{p}^n_\omega(k, l)$ is the fraction of the population at the end of period n that has type k and is matched to an agent of type l. Likewise, $\hat{p}^n_\omega(k, J)$ is the fraction of the population that is of type k and is not matched.

For the purpose of the inductive definition of the dynamical system $\mathbb{D}$, we suppose that $\Pi^{n-1} = (\alpha^{n-1}, \pi^{n-1})$ has been defined for some $n \geq 1$, and define $\Pi^n = (\alpha^n, \pi^n)$ as follows.

Mutation. The post-mutation type function $\bar{\alpha}^n$ is $\mathcal{I} \boxtimes \mathcal{F}$-measurable, and satisfies, for any k_1, k_2, l_1, and l_2 in S, for any $r \in S \cup \{J\}$, and for λ-almost-every agent i,

$$P(\bar{\alpha}_i^n = k_2, \bar{g}_i^n = l_2 \,|\, \alpha_i^{n-1} = k_1, g_i^{n-1} = l_1) = b^n_{k_1 k_2} b^n_{l_1 l_2} \tag{13}$$

$$P(\bar{\alpha}_i^n = k_2, \bar{g}_i^n = r \,|\, \alpha_i^{n-1} = k_1, g_i^{n-1} = J) = b_{k_1 k_2}^n \delta_J(r), \tag{14}$$

where $\delta_J(r)$ is one if $r = J$, and zero otherwise. Equation (13) means that a paired agent and her partner mutate independently. The post-mutation partner-type function $\bar{g}^n$ is defined by $\bar{g}^n(i,\omega) = \bar{\alpha}^n(\pi^{n-1}(i,\omega),\omega)$, for any $\omega \in \Omega$. We assume that $\bar{g}^n$ is $\mathcal{I} \boxtimes \mathcal{F}$-measurable. The post-mutation extended-type function is $\bar{\beta}^n = (\bar{\alpha}^n, \bar{g}^n)$. The post-mutation extended type distribution that is realized in state $\omega \in \Omega$ is $\check{p}^n(\omega) = \lambda \left(\bar{\beta}_\omega^n\right)^{-1}$.

Matching. Let $\bar{\pi}^n : I \times \Omega \to I$ be a random matching with the following properties.

(i) For each state $\omega \in \Omega$, let $A^\omega = \{i : \pi^{n-1}(i,\omega) \neq i\}$ be the set of agents who are matched. We have

$$\bar{\pi}_\omega^n(i) = \pi_\omega^{n-1}(i) \text{ for } i \in A^\omega, \tag{15}$$

meaning that those agents who were already matched at the end of period $n-1$ remain matched (to the same partner) at this step, which implies that the post-matching partner-type function $\bar{\bar{g}}^n$, defined by

$$\bar{\bar{g}}^n(i,\omega) = \begin{cases} \bar{\alpha}^n(\bar{\pi}^n(i,\omega),\omega) & \text{if } \bar{\pi}^n(i,\omega) \neq i \\ J & \text{if } \bar{\pi}^n(i,\omega) = i, \end{cases}$$

satisfies

$$P(\bar{\bar{g}}_i^n = r \,|\, \bar{\alpha}_i^n = k, \bar{g}_i^n = l) = \delta_l(r), \tag{16}$$

for any k and l in S and any $r \in S \cup \{J\}$, where $\delta_c(d)$ is zero if $c \neq d$ and is one if $c = d$.

(ii) $\bar{\bar{g}}^n$ is $\mathcal{I} \boxtimes \mathcal{F}$-measurable.

(iii) Given the post-mutation extended type distribution $\check{p}^n$, an unmatched agent of type k is matched to a unmatched agent of type l with conditional probability $q_{kl}^n(\check{p}^n)$, in that, for λ-almost every agent i and P-almost every ω,

$$P(\bar{\bar{g}}_i^n = l \,|\, \bar{\alpha}_i^n = k, \bar{g}_i^n = J, \check{p}^n) = q_{kl}^n(\check{p}^n(\omega)), \tag{17}$$

which also implies that

$$P(\bar{\bar{g}}_i^n = J \,|\, \bar{\alpha}_i^n = k, \bar{g}_i^n = J, \check{p}^n) = \eta_k^n(\check{p}^n(\omega)). \tag{18}$$

The extended type of agent i after the random matching step is $\bar{\bar{\beta}}_i^n = (\bar{\alpha}_i^n, \bar{\bar{g}}_i^n)$.

Type changes of matched agents with break-up. This step determines an end-of-period random matching π^n, an $\mathcal{I} \boxtimes \mathcal{F}$-measurable agent type function α^n, and an $\mathcal{I} \boxtimes \mathcal{F}$-measurable partner-type function g^n so that we have $g^n(i,\omega) = \alpha^n(\pi^n(i,\omega),\omega)$ for all $(i,\omega) \in I \times \Omega$, and so that, for λ-almost every agent i and for any $k_1, k_2, l_1, l_2 \in S$ and $r \in S \cup \{J\}$,

$$\pi^n(i) = \begin{cases} \bar{\pi}^n(i), & \text{if } \pi^n(i) \neq i \\ i, & \text{if } \pi^n(i) = i \end{cases} \tag{19}$$

$$P(\alpha_i^n = l_1, g_i^n = r \mid \bar{\alpha}_i^n = k_1, \bar{\bar{g}}_i^n = J) = \delta_{k_1}(l_1)\, \delta_J(r) \tag{20}$$

$$P(\alpha_i^n = l_1, g_i^n = l_2 \mid \bar{\alpha}_i^n = k_1, \bar{\bar{g}}_i^n = k_2) = (1 - \theta_{k_1 k_2}^n)\sigma_{k_1 k_2}^n(l_1, l_2) \tag{21}$$

$$P(\alpha_i^n = l_1, g_i^n = J \mid \bar{\alpha}_i^n = k_1, \bar{\bar{g}}_i^n = k_2) = \theta_{k_1 k_2}^n \varsigma_{k_1 k_2}^n(l_1). \tag{22}$$

Equation (19) says that agent i cannot change her partner if she remains matched. Equation (20) mean that unmatched agents stay unmatched without changing types. Equations (21) and (22) specify the type changing probabilities for a pair of matched agents who stay together or break up. The extended-type function at the end of the period is $\beta^n = (\alpha^n, g^n)$.

Thus, we have inductively defined the properties of a dynamical system $\mathbb{D} = (\Pi^n)_{n=1}^{\infty}$ incorporating the effects of random mutation, directed random matching, and match-induced type changes with break-up, consistent with given parameters $(b, q, \theta, \sigma, \varsigma)$. The initial condition Π^0 of $\mathbb{D}$ is unrestricted. We next turn to the key Markovian independence properties for such a system, and then to the exact law of large numbers and existence of a dynamical system with these properties.

A.2 Markov conditional independence

We now add independence conditions on the dynamical system $\mathbb{D} = (\Pi^n)_{n=0}^{\infty}$, along the lines of those in Duffie and Sun (2007, 2012), and Section 4. The idea is that each of the just-described steps (mutation, random matching, and match-induced type changes with break-up) are conditionally independent across almost all agents. In the following definition, we will refer to objects, such as the intermediate-step extended type functions $\bar{\beta}^n$ and $\bar{\bar{\beta}}^n$, that were constructed in the previous subsection.

We say that the dynamical system $\mathbb{D}$ is Markov conditionally independent (MCI) if, for λ-almost every i and λ-almost every j, for every period $n \geq 1$, and for all $k_1, k_2 \in S$, and $l_1, l_2 \in S \cup \{J\}$, the following five properties apply:

- Initial independence: β_i^0 and β_j^0 are independent.

- Markov and independent mutation:

$$P\left(\bar{\beta}_i^n = (k_1, l_1), \bar{\beta}_j^n = (k_2, l_2) \;\middle|\; (\beta_i^t)_{t=0}^{n-1}, (\beta_j^t)_{t=0}^{n-1}\right)$$
$$= P\left(\bar{\beta}_i^n = (k_1, l_1) \;\middle|\; \beta_i^{n-1}\right) P\left(\bar{\beta}_j^n = (k_2, l_2) \;\middle|\; \beta_j^{n-1}\right). \tag{23}$$

- Markov and independent random matching:

$$P\left(\bar{\bar{\beta}}_i^n = (k_1, l_1), \bar{\bar{\beta}}_j^n = (k_2, l_2) \;\middle|\; \bar{\beta}_i^n, \bar{\beta}_j^n, (\beta_i^t)_{t=0}^{n-1}, (\beta_j^t)_{t=0}^{n-1}\right)$$
$$= P\left(\bar{\bar{\beta}}_i^n = (k_1, l_1) \;\middle|\; \bar{\beta}_i^n\right) P\left(\bar{\bar{\beta}}_j^n = (k_2, l_2) \;\middle|\; \bar{\beta}_j^n\right). \tag{24}$$

- Markov and independent matched-agent type changes with break-up:

$$P\left(\beta_i^n = (k_1, l_1), \beta_j^n = (k_2, l_2) \;\middle|\; \bar{\bar{\beta}}_i^n, \bar{\bar{\beta}}_j^n, (\beta_i^t)_{t=0}^{n-1}, (\beta_j^t)_{t=0}^{n-1}\right)$$
$$= P\left(\beta_i^n = (k_1, l_1) \,|\, \bar{\bar{\beta}}_i^n\right) P\left(\beta_j^n = (k_2, l_2) \,|\, \bar{\bar{\beta}}_j^n\right). \tag{25}$$

A.3 The exact law of large numbers for MCI dynamical systems with enduring partnerships

For each period $n \geq 1$, we define a mapping Γ^n from $\hat{\Delta}$ to $\hat{\Delta}$ by

$$\begin{aligned}
\Gamma_{kl}^n(\hat{p}) &= \sum_{(k_1,l_1)\in S^2} \tilde{p}_{k_1 l_1}^n (1-\theta_{k_1 l_1}^n)\sigma_{k_1 l_1}^n(k,l) + \sum_{(k_1,l_1)\in S^2} \tilde{p}_{k_1 J}\, q_{k_1 l_1}^n(\tilde{p}^n)(1-\theta_{k_1 l_1}^n)\sigma_{k_1 l_1}^n(k,l) \\
\Gamma_{kJ}^n(\hat{p}) &= \tilde{p}_{kJ}\eta_k^n(\tilde{p}) + \sum_{(k_1,l_1)\in S^2} \tilde{p}_{k_1 l_1}^n \theta_{k_1 l_1}^n \varsigma_{k_1 l_1}^n(k) + \sum_{(k_1,l_1)\in S^2} \tilde{p}_{k_1 J}^n\, q_{k_1 l_1}^n(\tilde{p}^n)\theta_{k_1 l_1}^n \varsigma_{k_1 l_1}^n(k),
\end{aligned}$$

where $\tilde{p}_{kl}^n = \sum_{(k_1,l_1)\in S^2} \hat{p}_{k_1 l_1} b_{k_1 k}^n b_{l_1 l}^n$ and $\tilde{p}_{kJ}^n = \sum_{l\in S} \hat{p}_{lJ} b_{lk}^n$.

The following theorem, which extends Theorem 4.3, presents an exact law of large numbers for the joint agent-partner type processes at the end of each period. The result also provides a recursive calculation of the cross-sectional joint agent-partner type distribution $\hat{p}^n$ at the end of period n.

Theorem 4 *Let $\mathbb{D}$ be a dynamical system with random mutation, random matching, and match-induced type changes with break-up whose parameters are $(b, q, \theta, \sigma, \varsigma)$. If $\mathbb{D}$ is Markov conditionally independent, then:*

(1) For each time period $n \geq 1$, the expected cross-sectional extended type distributions $\tilde{p}^n = \mathbb{E}(\check{p}^n)$ after the mutation step and $\mathbb{E}(\hat{p}^n)$ at the end of the period are given by, respectively, $\mathbb{E}(\check{p}_{kl}^n) = \sum_{k_1,l_1\in S} \mathbb{E}(\hat{p}_{k_1 l_1}^{n-1}) b_{k_1 k}^n b_{l_1 l}^n$ and $\mathbb{E}(\check{p}_{kJ}^n) = \sum_{l\in S} \mathbb{E}(\hat{p}_{lJ}^{n-1}) b_{lk}^n$, and by $\mathbb{E}(\hat{p}^n) = \Gamma^n(\mathbb{E}(\hat{p}^{n-1}))$.

(2) For λ-almost every agent i, the extended-type process $\{\beta_i^n\}_{n=0}^{\infty}$ is a Markov chain in $\hat{S}$ whose transition matrix z^n at time $n-1$ is given by

$$
\begin{aligned}
z^n_{(kJ)(k'J)} &= b^n_{kk'}\eta^n_{k'}(\tilde{p}^n) + \sum_{k_1,l_1,\in S} b^n_{kk_1} q^n_{k_1l_1}(\tilde{p}^n)\theta^n_{k_1l_1}\varsigma^n_{k_1l_1}(k'), \\
z^n_{(kl)(k'J)} &= \sum_{k_1,l_1\in S} b^n_{kk_1} b^n_{ll_1}\theta^n_{k_1l_1}\varsigma^n_{k_1l_1}(k'), \\
z^n_{(kJ)(k'l')} &= \sum_{k_1,l_1\in S} b^n_{kk_1} q^n_{k_1l_1}(\tilde{p}^n)(1-\theta^n_{k_1l_1})\sigma^n_{k_1l_1}(k',l'), \\
z^n_{(kl)(k'l')} &= \sum_{k_1,l_1\in S} b^n_{kk_1} b^n_{ll_1}(1-\theta^n_{k_1l_1})\sigma^n_{k_1l_1}(k',l'). \qquad (26)
\end{aligned}
$$

(3) For λ-almost every i and λ-almost every j, the Markov chains $\{\beta_i^n\}_{n=0}^{\infty}$ and $\{\beta_j^n\}_{n=0}^{\infty}$ are independent.

(4) For P-almost every $\omega \in \Omega$, the cross-sectional extended-type process $\{\beta_\omega^n\}_{n=0}^{\infty}$ is a Markov chain[32] *with transition matrix z^n at time $n-1$.*

(5) For P-almost all $\omega \in \Omega$, at each time period $n \geq 1$, the realized cross-sectional extended type distribution after random mutation $\lambda(\bar{\beta}_\omega^n)^{-1}$ is equal to its expectation $\tilde{p}^n$, and the realized cross-sectional extended type distribution at the end of period n, $\hat{p}^n(\omega) = \lambda(\beta_\omega^n)^{-1}$, is equal to its expectation $\mathbb{E}(\hat{p}^n)$.

(6) If there is some fixed $\ddot{p}^0 \in \hat{\Delta}$ that is the probability distribution of the initial extended type β_i^0 of agent i for λ-almost every i, then for λ-almost every i the Markov chain $\beta_i = \{\beta_i^n\}_{n=0}^{\infty}$ has the sample-path probability distribution $\xi = \ddot{p}^0 \otimes (\otimes_{n=1}^{\infty} z^n)$ on the space $\hat{S}^{\infty}$. Moreover, in this case, $\xi = \lambda(\beta_\omega)^{-1}$ for P-almost every ω. That is, for any measurable rectangle $A = \prod_{n=0}^{\infty} A_n \subseteq \hat{S}^{\infty}$ of sample paths, the probability $\xi(A)$ is equal, for P-almost every $\omega \in \Omega$, to the fraction $\lambda(\{i : \beta_\omega(i) \in A\})$ of agents whose extended type process has a sample path in A in sample realization ω.

For the time-independent case, in which the parameters $(b, q, \theta, \sigma, \varsigma)$ do not depend on the time period $n \geq 1$, the following proposition shows the existence of a stationary extended type distribution.

Proposition 4 *Suppose that the parameters $(b, q, \theta, \sigma, \varsigma)$ are time independent. Then there exists an extended-type distribution $\hat{p}^* \in \hat{\Delta}$ that is a stationary distribution for any MCI dynamical system $\mathbb{D}$ with parameters $(b, q, \theta, \sigma, \varsigma)$, in the sense that:*

[32] For a given sample realization $\omega \in \Omega$, $\{\beta_\omega^n\}_{n=0}^{\infty}$ is defined on the agent space $(I, \mathcal{I}, \lambda)$, which is a probability space itself. Thus, $\{\beta_\omega^n\}_{n=0}^{\infty}$ can be viewed as a discrete-time process.

(1) For every $n \geq 0$, the realized cross-sectional extended-type distribution $\hat{p}^n$ at time n is $\hat{p}^*$ P-almost surely;

(2) All of the relevant Markov chains in Theorem 4 are time homogeneous with a constant transition matrix z^1 having $\hat{p}^*$ as a fixed point;

(3) If the initial extended type process β^0 is i.i.d. across agents, then, for λ-almost every i, the extended type distribution of agent i at any period $n \geq 0$ is $P\left(\beta_i^n\right)^{-1} = \hat{p}^*$.

A.4 Existence of MCI dynamic directed random matching with enduring partnerships

The following theorem provides for the existence of a Markov conditionally independent (MCI) dynamical system with random mutation, random matching, and match-induced type changes with break-up. Theorem 3 is a special case.

Theorem 5 *For any primitive model parameters $(b, q, \theta, \sigma, \varsigma)$ and for any extended type distribution $\ddot{p}^0 \in \hat{\Delta}$, there exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ on which is defined a dynamical system $\mathbb{D} = (\Pi^n)_{n=0}^{\infty}$ with random mutation, random matching, and match-induced type changes with break-up, that is Markov conditionally independent with these parameters $(b, q, \theta, \sigma, \varsigma)$, and with the initial cross-sectional extended type distribution $\hat{p}^0$ being $\ddot{p}^0$ with probability one. In addition, for any $n \geq 1$, π^n and $\bar{\pi}^n$ are measure preserving in the sense that for each $\omega \in \Omega$, $\lambda((\pi_\omega^n)^{-1}(A)) = \lambda((\bar{\pi}_\omega^n)^{-1}(A)) = \lambda(A)$ for any $A \in \mathcal{I}$. These properties can be achieved with an initial condition Π^0 that is deterministic, or alternatively with an initial extended type process β^0 that is i.i.d. across agents.*[33]

In the next proposition, we show that the agent space $(I, \mathcal{I}, \lambda)$ in Theorem 5 can be an extension of the classical Lebesgue unit interval $(L, \mathcal{L}, \chi)$. That is, we can take $I = L = [0, 1]$ with a σ-algebra $\mathcal{I}$ that contains the Lebesgue σ-algebra $\mathcal{L}$, and so that the restriction of λ to $\mathcal{L}$ is the Lebesgue measure χ.

Proposition 5 *Fixing any model parameters $(b, q, \theta, \sigma, \varsigma)$ and any initial cross-sectional extended type distribution $\ddot{p}^0 \in \hat{\Delta}$, there exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ such that:*

(1) The agent space $(I, \mathcal{I}, \lambda)$ is an extension of the Lebesgue unit interval $(L, \mathcal{L}, \chi)$.

[33] This means that the process β^0 is essentially pairwise independent, and that β_i^0 has distribution $\ddot{p}^0$ for λ-almost every agent i.

(2) There is defined on the Fubini extension a dynamical system $\mathbb{D} = (\Pi^n)_{n=0}^{\infty}$ that is Markov conditionally independent with the parameters $(b, q, \theta, \sigma, \varsigma)$, where the initial cross-sectional extended type distribution $\hat{p}^0$ is $\ddot{p}^0$ with probability one.

(3) These properties can be achieved with an initial condition Π^0 that is deterministic, or alternatively with an initial extended type process β^0 that is i.i.d. across agents.[34]

B Illustrative applications in monetary and labor economics

This appendix provides three example applications, which are designed to illustrate how our results provide a mathematical foundation for the dynamic matching models used in monetary economics and labor economics. The first example is from Kiyotaki and Wright (1989) and Kehoe, Kiyotaki and Wright (1993). The second example is from Matsuyama, Kiyotaki and Matsui (1993). The last example treats the labor-market matching model of Andolfatto Andolfatto (1996), a setting that calls for enduring matches of the sort considered in Appendix A.

B.1 Kiyotaki-Wright: Model A

As in Model A of Kiyotaki and Wright (1989), three indivisible goods are labeled 1, 2, and 3. There is a continuum of agents of unit total mass. A given type of agent consumes good k and can store one unit of good l, for some $l \neq k$. This type is denoted $\langle k, l \rangle$. The economy is thus populated by agents of 6 distinct types $\langle 1,2 \rangle, \langle 1,3 \rangle, \langle 2,1 \rangle, \langle 2,3 \rangle, \langle 3,1 \rangle$, and $\langle 3,2 \rangle$, which form our type space[35] S. In order to avoid confusion over differences in terminology[36] with Kiyotaki and Wright (1989), we say that an agent who consumes good k has "trait" k. There are equal proportions of agents with the three respective traits. In each period n, every agent is randomly matched with some other agent. When matched, two agents decide whether or not to trade. If there is no trade between the matched pair, they keep their goods. If there is a trade, and if the agent who consumes good k gets good k from the other, then that agent immediately consumes good k and produces one unit of good $k+1$ (modulo 3), so that his type becomes $\langle k, k+1 \rangle$ (modulo 3, as needed). If there is a trade and an agent with trait k gets good l for $l \neq k$, then his type becomes $\langle k, l \rangle$. Kiyotaki and Wright (1989) and Kehoe, Kiyotaki and

[34] The statement in Footnote 19 is still valid in this more general case. For the convenience of the reader, we repeat it here. There exists a sub-σ-algebra $\mathcal{I}'$ of $\mathcal{I}$ and a Fubini extension $(I \times \Omega, \mathcal{I}' \boxtimes \mathcal{F}, \lambda \boxtimes P)$ such that $\mathcal{I}' \boxtimes \mathcal{F} \subseteq \mathcal{I} \boxtimes \mathcal{F}$, the dynamical system $\mathbb{D}$ on $(I \times \Omega, \mathcal{I}' \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is Markov conditionally independent, and for any $n \geq 1$, π^n and $\bar{\pi}^n$ are measure preserving in the sense that for each $\omega \in \Omega$, $\lambda((\pi^n_\omega)^{-1}(A)) = \lambda((\bar{\pi}^n_\omega)^{-1}(A)) = \lambda(A)$ for any $A \in \mathcal{I}'$. See the penultimate paragraph of the proof of this proposition in Subsection E.3.

[35] It is clear that our results, which are stated earlier in terms of $S = \{1, 2, \ldots, K\}$, hold for any finite type space with appropriate notational change.

[36] In Kiyotaki and Wright (1989), the meaning of "type" is different from that in our present paper.

Wright (1993) consider this matching model with both stationary and non-stationary trading strategies.

We can use our model of dynamic directed random matching in Section 4 to give a mathematical foundation for the matching models in Kiyotaki and Wright (1989) and Kehoe, Kiyotaki and Wright (1993) by choosing suitable parameters (b, q, ν) governing random mutation, random matching and match-induced type changing. At period n, let $b^n_{\langle k_1, l_1 \rangle \langle k_2, l_2 \rangle} = \delta_{k_1}(k_2)\delta_{l_1}(l_2)$ be the mutation probabilities, and let $q^n_{\langle k_1, l_1 \rangle \langle k_2, l_2 \rangle}(p) = p_{\langle k_2, l_2 \rangle}$ be the matching probabilities for $p \in \Delta$. We will need to specify the match-induced type changing probabilities in both cases.

First, a stationary (pure) trading strategy in Kiyotaki and Wright (1989, p. 931) is described by some $\tau : \{1, 2, 3\} \times \{1, 2, 3\} \to \{0, 1\}$ that implies a trade, $\tau_k(l, r) = 1$, if a trait-k agent actually wants to trade good l for good r, and results in no trade, $\tau_k(l, r) = 0$, otherwise. Thus τ determines the match-induced type changes. Because the consumption traits of agents do not change, the type of a matched agent cannot change to a type with a different trait. Thus, for the type-changing probability ν^n of an agent with trait k_1, the probability for the target types is concentrated on only two types, $\langle k_1, k_1 + 1 \rangle$ and $\langle k_1, k_1 + 2 \rangle$. This means that it suffices to define the type-changing probability for only the target type $\langle k_1, k_1 + 1 \rangle$. Suppose that an agent i of type $\langle k_1, k_1 + 1 \rangle$ is matched with an agent j of type $\langle k_2, l_2 \rangle$. For $l_2 = k_1 + 1$, there is no need to trade. When $l_2 = k_1$ and there is a trade, agent i will consume good k_1, produces a unit of good $k_1 + 1$, and keeps the same type $\langle k_1, k_1 + 1 \rangle$. (This applies trivially for the no-trade case.) When $l_2 = k_1 + 2$, the probability $\nu_{\langle k_1, k_1+1 \rangle \langle k_2, l_2 \rangle}(\{\langle k_1, k_1 + 1 \rangle\})$ that agent i has a type change is the probability of no trade between agents i and j. The probability of having a trade between agents i and j is $\tau_{k_1}(k_1 + 1, l_2)\tau_{k_2}(l_2, k_1 + 1)$. We therefore have

$$\nu^n_{\langle k_1, k_1+1 \rangle \langle k_2, l_2 \rangle}(\{\langle k_1, k_1 + 1 \rangle\}) = \begin{cases} 1 & \text{if } l_2 \neq k_1 + 2 \\ 1 - \tau_{k_1}(k_1 + 1, l_2)\tau_{k_2}(l_2, k_1 + 1) & \text{if } l_2 = k_1 + 2. \end{cases}$$

By similar arguments,

$$\nu^n_{\langle k_1, k_1+2 \rangle \langle k_2, l_2 \rangle}(\{\langle k_1, k_1 + 1 \rangle\}) = \begin{cases} 0 & \text{if } l_2 = k_1 + 2 \\ \tau_{k_1}(k_1 + 2, l_2)\tau_{k_2}(l_2, k_1 + 2) & \text{if } l_2 \neq k_1 + 2. \end{cases}$$

Next, we consider the case of non-stationary trading strategies, as in Sections 3 and 6 of Kehoe, Kiyotaki and Wright (1993). Suppose that $(s_1(n), s_2(n), s_3(n))$ is a time-dependent mixed strategy at period n, where $s_k(n)$ is the probability that an agent with trait k trades good $k+1$ for $k+2$. Based on $(s_1(n), s_2(n), s_3(n))$, one can compute the probability $P^n_{\langle k_1, k_2 \rangle}(k_3)$ that an agent of type $\langle k_1, k_2 \rangle$ trades for good k_3.

We must define the match-induced type changing probabilities corresponding to the given time-dependent mixed strategy $(s_1(n), s_2(n), s_3(n))$. Suppose that an agent of type $\langle k_1, k_1 + 1 \rangle$ is matched with an agent of type $\langle k_2, l_2 \rangle$. For cases with $l_2 = k_1$ or $l_2 = k_1 + 1$, the type changing probability $\nu^n_{\langle k_1,k_1+1\rangle\langle k_2,l_2\rangle}(\{\langle k_1,k_1+1\rangle\}) = 1$. When $l_2 = k_1+2$, the probability that the match leads to a trade is $P^n_{\langle k_1,k_1+1\rangle}(l_2)P^n_{\langle k_2,l_2\rangle}(k_1+1)$. Thus

$$\nu^n_{\langle k_1,k_1+1\rangle\langle k_2,l_2\rangle}(\{\langle k_1,k_1+1\rangle\}) = \begin{cases} 1 & \text{if } l_2 \neq k_1+2 \\ 1 - P^n_{\langle k_1,k_1+1\rangle}(l_2)P^n_{\langle k_2,l_2\rangle}(k_1+1) & \text{if } l_2 = k_1+2. \end{cases}$$

Similarly,

$$\nu^n_{\langle k_1,k_1+2\rangle\langle k_2,l_2\rangle}(\{\langle k_1,k_1+1\rangle\}) = \begin{cases} 0 & \text{if } l_2 = k_1+2 \\ P^n_{\langle k_1,k_1+2\rangle}(l_2)P^n_{\langle k_2,l_2\rangle}(k_1+2) & \text{if } l_2 \neq k_1+2. \end{cases}$$

B.2 Matsuyama, Kiyotaki and Matsui

Our next example is from Matsuyama, Kiyotaki and Matsui (1993). Here, agents are divided into two groups. Agents are more likely to be matched to a counterparty of their own group than to a counterparty of a different group.

The economy is populated by a continuum of infinitely-lived agents of unit total mass. Agents are from two regions, Home and Foreign. Let $r \in (0,1)$ be the size of the Home population. There are $K \geq 3$ kinds of indivisible commodities. Within each region, there are equal proportions of agents with the K respective traits. An agent with trait k derives utility only from consumption of commodity k. After he consumes commodity k, he is able to produce one and only one unit of commodity $k+1$ (mod K) costlessly, and can also store up to one unit of his production good costlessly. He can neither produce nor store other types of goods.

In addition to the commodities described above, there are two distinguishable fiat monies without intrinsic worth, which we call the Home currency and the Foreign currency. Each currency is indivisible and can be stored costlessly in amounts of up to one unit by any agent, provided that the agent does not carry his production good or the other currency. This implies that, at any date, the inventory of each agent consists of one unit of the Home currency, one unit of the Foreign currency, or one unit of his production good, but does not include more than one of these objects in total at any one time.

For some $\beta \in (0,1)$, in each period n, a Home agent is matched to a Home agent with probability r, and is matched to a Foreign agent with probability $\beta(1-r)$. The probability with which he is not matched is thus $(1-\beta)(1-r)$. Similarly, a Foreign agent is matched to a Home agent with probability βr, is matched to a Foreign agent with probability $(1-r)$, and is unmatched with probability $(1-\beta)r$.

The type space S is the set of ordered tuples of the form (a,b,c), where $a \in \{H,F\}$, $b \in \{1,\ldots,K\}$, and $c \in \{g,h,f\}$. Here, H represents Home, F represents Foreign, g represents good, h represents Home currency, and f represents Foreign currency.

An agent chooses a trading strategy to maximize his expected discounted utility, taking as given the strategies of other agents and the distribution of inventories. Matsuyama, Kiyotaki and Matsui (1993) focused on pure strategies that depend only on an agent's nationality and the objects that he and his counterparty have as inventories. Thus, the Home agent's (pure) trading strategy can be described simply as

$$\tau_{ab}^H = \begin{cases} 1 & \text{if he agrees to trade object } a \text{ for object } b \\ 0 & \text{otherwise,} \end{cases}$$

where a and b are in $\{g, h, f\}$. The Foreign agent's trading strategy can similarly be described as

$$\tau_{ab}^F = \begin{cases} 1 & \text{if he agrees to trade object } a \text{ for object } b \\ 0 & \text{otherwise.} \end{cases}$$

For example, $\tau_{gf}^H = 0$ means that a Home agent does not agree to trade his production good for the Foreign currency, while $\tau_{hg}^F = 1$ means that a Foreign agent agrees to trade the Home currency for his consumption good.

We can apply our model of dynamic directed random matching with immediate break-up in Section 4 to give a mathematical foundation for the matching model in Matsuyama, Kiyotaki and Matsui (1993) by choosing suitable time-independent parameters (b, q, ν) governing random mutation, random matching, and match-induced type changing. To this end, we take mutation probabilities

$$b_{(a_1,b_1,c_1)(a_2,b_2,c_2)} = \delta_{a_1}(a_2)\delta_{b_1}(b_2)\delta_{c_1}(c_2).$$

For a given cross-sectional agent type distribution $p \in \Delta$, the directed matching probabilities are

$$q_{(a_1,b_1,c_1)(a_2,b_2,c_2)}(p) = \begin{cases} p_{(a_2,b_2,c_2)} & \text{if } a_1 = a_2 \\ \beta \cdot p_{(a_2,b_2,c_2)} & \text{if } a_1 \neq a_2. \end{cases}$$

Because the nationalities and consumption traits of agents do not change, a matched agent cannot change to a type with a different nationality or trait. Thus, for the type changing probability ν of an agent with nationality a_1 and trait b_1, search is directed to the three counterparty types (a_1, b_1, g), (a_1, b_1, f) and (a_1, b_1, h).

Suppose that agent i is of type (a_1, b_1, g) and is matched with agent j, who has type (a_2, b_2, c_2). The probability that agent i changes type to (a_1, b_1, h) is $\nu_{(a_1,b_1,g)(a_2,b_2,c_2)}(\{(a_1, b_1, h)\})$. The good carried by an agent of type (a_1, b_1, g) must be $b_1 + 1$. For $b_2 \neq b_1 + 1 \pmod K$, the good that agent i carries is not the consumption good of agent j, which means that there is no trade, so the probability $\nu_{(a_1,b_1,g)(a_2,b_2,c_2)}(\{(a_1, b_1, h)\})$ is 0. When $c_2 \neq h$, agent i cannot get the Home currency from j, so $\nu_{(a_1,b_1,g)(a_2,b_2,c_2)}(\{(a_1, b_1, h)\})$ is also 0. When $b_2 = b_1 + 1$ and $c_2 = h$, $\nu_{(a_1,b_1,g)(a_2,b_2,c_2)}(\{(a_1, b_1, h)\})$ is the probability that agent i trades with an agent with the type of agent j, which is $\tau^{a_1}_{gh} \cdot \tau^{a_2}_{hg}$. We therefore have

$$\nu_{(a_1,b_1,g)(a_2,b_2,c_2)}\left(\{(a_1,b_1,h)\}\right) = \begin{cases} \tau^{a_1}_{gh} \cdot \tau^{a_2}_{hg} & \text{if } b_2 \equiv b_1 + 1 \pmod{K} \text{ and } c_2 = h \\ 0 & \text{otherwise.} \end{cases}$$

The following type-change probabilities can be obtained by similar arguments:

$$\nu_{(a_1,b_1,g)(a_2,b_2,c_2)}\left(\{(a_1,b_1,f)\}\right) = \begin{cases} \tau^{a_1}_{gf} \cdot \tau^{a_2}_{fg} & \text{if } b_2 \equiv b_1 + 1 \pmod{K} \text{ and } c_2 = f, \\ 0 & \text{otherwise} \end{cases}$$

$$\nu_{(a_1,b_1,g)(a_2,b_2,c_2)}\left(\{(a_1,b_1,g)\}\right) = 1 - \nu_{(a_1,b_1,g)(a_2,b_2,c_2)}\left(\{(a_1,b_1,h)\}\right) - \nu_{(a_1,b_1,g)(a_2,b_2,c_2)}\left(\{(a_1,b_1,f)\}\right)$$

$$\nu_{(a_1,b_1,h)(a_2,b_2,c_2)}\left(\{(a_1,b_1,g)\}\right) = \begin{cases} \tau^{a_1}_{hg} \cdot \tau^{a_2}_{gh} & \text{if } b_2 \equiv b_1 - 1 \pmod{K} \text{ and } c_2 = g \\ 0 & \text{otherwise} \end{cases}$$

$$\nu_{(a_1,b_1,h)(a_2,b_2,c_2)}\left(\{(a_1,b_1,f)\}\right) = \begin{cases} \tau^{a_1}_{hf} \cdot \tau^{a_2}_{fh} & c_2 = f \\ 0 & \text{otherwise} \end{cases}$$

$$\nu_{(a_1,b_1,h)(a_2,b_2,c_2)}\left(\{(a_1,b_1,h)\}\right) = 1 - \nu_{(a_1,b_1,h)(a_2,b_2,c_2)}\left(\{(a_1,b_1,g)\}\right) - \nu_{(a_1,b_1,h)(a_2,b_2,c_2)}\left(\{(a_1,b_1,f)\}\right)$$

$$\nu_{(a_1,b_1,f)(a_2,b_2,c_2)}\left(\{(a_1,b_1,g)\}\right) = \begin{cases} \tau^{a_1}_{fg} \cdot \tau^{a_2}_{gf} & \text{if } b_2 \equiv b_1 - 1 \pmod{K} \text{ and } c_2 = g \\ 0 & \text{otherwise} \end{cases}$$

$$\nu_{(a_1,b_1,f)(a_2,b_2,c_2)}\left(\{(a_1,b_1,h)\}\right) = \begin{cases} \tau^{a_1}_{fh} \cdot \tau^{a_2}_{hf} & \text{if } c_2 = h \\ 0 & \text{otherwise} \end{cases}$$

$$\nu_{(a_1,b_1,f)(a_2,b_2,c_2)}\left(\{(a_1,b_1,f)\}\right) = 1 - \nu_{(a_1,b_1,f)(a_2,b_2,c_2)}\left(\{(a_1,b_1,h)\}\right) - \nu_{(a_1,b_1,f)(a_2,b_2,c_2)}\left(\{(a_1,b_1,g)\}\right).$$

B.3 Matching in labor markets with multi-period employment episodes

This example is taken from Andolfatto (1996), whose Section 1 considers a discrete-time labor-market-search model. The agents are workers and firms. Each firm has a single job position. Section 2 of Andolfatto (1996) works with stationary distributions. We can use the model of dynamic directed random matching with enduring partnership developed in Appendix A to capture the search process leading to Equation (1) of Andolfatto (1996) in the stationary setting.

The agent type space is $S = \{E, U, A, V, D\}$. Here, E and U represent, respectively, employed workers and unemployed workers while A, V and D represent active, vacant and dormant jobs respectively. Dormant job positions are neither matched with a worker nor immediately open. The proportion of agents that are workers is $w > 0$.

At the beginning of each period, each vacant firm may mutate to a dormant job and each dormant job may mutate to a vacant job. Let $\check{p}_{UJ}$ and $\check{p}_{VJ}$ be the respective proportions of unemployed workers and vacant firms after the mutation step. In the stationary setting, the quantity $M(\check{p}_{VJ},\, e\cdot\check{p}_{UJ})$ of new job matches in a given period is governed by a continuous aggregate matching function $M:[0,1]\times\mathbb{R}_+\to[0,1]$ that incorporates[37] the search effort e applied by each worker seeking employment with $M(\check{p}_{VJ},\, e\cdot\check{p}_{UJ})\le\min\{\check{p}_{VJ},\check{p}_{UJ}\}$. Job-worker pairs that have existed for at least one period are assumed to break up with probability $\bar{\theta}$ in each period. Newly formed pairs cannot break up in the current period. While a job-worker pair maintain their partnership, their current types (A,E) do not change. On the other hand, if they break up, the job becomes vacant and the worker becomes unemployed.

Equation (1) in Andolfatto (1996) in the stationary setting is

$$E^*=(1-\bar{\theta})E^*+M(V^*,e\cdot(w-E^*)),\tag{27}$$

where E^* and V^* are the respective fractions of employed workers and vacant jobs in the particular case.[38]

Viewed in terms of our model in Appendix A, the corresponding time-independent parameters are given as follows. Vacant firms could mutate to dormant, and vice versa. Workers and active firms do not mutate. For any k and l in S, let

$$b_{kl}=\begin{cases}\frac{1-w-E^*-V^*}{1-w-E^*} & \text{if } k=V \text{ or } D \text{ and } l=D\\ \frac{V^*}{1-w-E^*} & \text{if } k=V \text{ or } D \text{ and } l=V\\ \delta_k(l) & \text{otherwise.}\end{cases}$$

Matching occurs only between unemployed workers and vacant jobs. The matching probabilities are defined as follows. For any k and l in S, define

$$q_{kl}(\check{p})=\begin{cases}\frac{M(\check{p}_{VJ},e\cdot\check{p}_{UJ})}{\check{p}_{UJ}} & \text{if } (k,l)=(U,V) \text{ and } \check{p}_{UJ}>0\\ \frac{M(\check{p}_{VJ},e\cdot\check{p}_{UJ})}{\check{p}_{VJ}} & \text{if } (k,l)=(V,U) \text{ and } \check{p}_{UJ}>0\\ 0 & \text{otherwise.}\end{cases}$$

Next, we consider the step of type changing with break-up. For any $k,l,r,s\in S$, we have

$$\theta_{kl}=\begin{cases}\bar{\theta} & \text{if } (k,l)=(E,A) \text{ or } (A,E)\\ 0 & \text{otherwise}\end{cases}\tag{28}$$

$$\sigma_{kl}(r,s)=\begin{cases}\delta_E(r)\delta_A(s) & \text{if } k=U \text{ and } l=V\\ \delta_A(r)\delta_E(s) & \text{if } k=V \text{ and } l=U\\ \delta_k(r)\delta_l(s) & \text{otherwise}\end{cases}\tag{29}$$

[37] The mass of workers is assumed to be one in Andolfatto (1996). Since the matching function in Andolfatto (1996) is assumed to have constant returns to scale, one can re-scale the total worker-firm population to be one, with a proportion w of agents being workers.

[38] See (P6$'$) on page 120 of Andolfatto (1996) for the steady state equation with the Cobb-Douglas matching function.

$$\varsigma_{kl}(r) = \begin{cases} \delta_U(r) & \text{if } k = E \text{ and } l = A \\ \delta_V(r) & \text{if } k = A \text{ and } l = E \\ \delta_k(r) & \text{otherwise.} \end{cases} \tag{30}$$

Equation (28) means that an employed worker has probability $\bar{\theta}$ of losing her job. When two agents are newly matched in the current period, the worker-firm types change from (U, V) to (E, A). For those paired agents who were matched in a previous period, their types do not change while they stay together. Finally, the worker-firm pair of types from (E, A) to (U, V) when they break up. Equations (29) and (30) express these ideas.

Taking the equilibrium search effort e as given, Theorem 4 and Proposition 4 imply that any stationary type distribution satisfies

$$\hat{p}^*_{EA} = \Gamma(\hat{p}^*)_{EA}. \tag{31}$$

We take a stationary type distribution $\hat{p}^*$ corresponding to the given fractions of employed workers and vacant jobs E^* and V^* as in Equation (27), which means that $\hat{p}^*_{EA} = E^*$ and $\hat{p}^*_{VJ} = V^*$. By the formulas above the statement of Theorem 4, we obtain that

$$\Gamma_{EA}(\hat{p}^*) = \tilde{p}_{EA}(1 - \bar{\theta}) + \tilde{p}_{UJ} q_{UV}(\tilde{p})$$

$$\tilde{p}_{UJ} = \hat{p}^*_{UJ} = w - \hat{p}^*_{EA} = w - E^*$$

$$\tilde{p}_{VJ} = \hat{p}^*_{VJ} b_{VV} + \hat{p}^*_{DJ} b_{DV} = \hat{p}^*_{VJ} b_{VV} + (1 - w - \hat{p}^*_{EA} - \hat{p}^*_{VJ}) b_{DV} = V^*.$$

Substituting the above terms into Equation (31), we derive

$$E^* = \hat{p}^*_{EA} = (1 - \bar{\theta})E^* + M\left(V^*, \, e \cdot (w - E^*)\right).$$

Thus the stationary distribution of employed workers and vacant jobs considered in Andolfatto (1996) can be derived from our model of dynamic directed random matching with enduring partnership with appropriate parameters.

C Proofs of Theorem 4 and Proposition 4

Before proving Theorem 4, we need a few lemmas.

First, we state the following general version of the exact law of large numbers in Sun (2006) as a lemma here for the convenience of the reader.[39]

[39] Part (2) of the lemma is part of Theorem 2.8 in Sun (2006). That theorem actually shows that the statement in Part (2) here is equivalent to the condition of essential pairwise independence. While Parts (1), (3) and (4) of the lemma are special cases of Part (2), they are stated respectively in Corollary 2.9, Theorem 2.12 and Corollary 2.10 of Sun (2006).

Lemma 1 *Let f be a measurable process from a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to a complete separable metric space X.*

1. *For P-almost all $\omega \in \Omega$, the sample distribution λf_ω^{-1} of the sample function f_ω is the same as the distribution $(\lambda \boxtimes P) f^{-1}$ of the process.*[40]

2. *For any $A \in \mathcal{I}$ with $\lambda(A) > 0$, let f^A be the restriction of f to $A \times \Omega$, λ^A and $\lambda^A \boxtimes P$ the probability measures rescaled from the restrictions λ and $\lambda \boxtimes P$ to $\{D \in \mathcal{I} : D \subseteq A\}$ and $\{C \in \mathcal{I} \boxtimes \mathcal{F} : C \subseteq A \times \Omega\}$ respectively. Then for P-almost all $\omega \in \Omega$, the sample distribution $\lambda^A (f^A)_\omega^{-1}$ of the sample function $(f^A)_\omega$ is the same as the distribution of $(\lambda^A \boxtimes P)(f^A)^{-1}$ of the process f^A.*

3. *If there is a distribution μ on X such that for λ-almost all $i \in I$, the random variable f_i has distribution μ, then the sample function f_ω (or $(f^A)_\omega$) also has distribution μ for P-almost all $\omega \in \Omega$.*

4. *If X is the real line $\mathbb{R}$ and f is integrable on $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, then for P-almost all $\omega \in \Omega$, $\int_I f_\omega d\lambda = \int_{I \times \Omega} f d\lambda \boxtimes P$.*

By viewing a discrete-time stochastic process taking values in X as a random variable taking values in X^∞, Lemma 1 implies the following exact law of large numbers for a continuum of discrete-time stochastic processes, which is formally stated in Theorem 2.16 in Sun (2006).

Corollary 1 *Let f be a mapping from $I \times \Omega \times \mathbb{N}$ to a complete separable metric space X such that for each $n \geq 0$, $f^n = f(\cdot, \cdot, n)$ is an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process. Then, for λ-almost all $i \in I$, $\{f_i^n\}_{n=0}^\infty$ is a discrete-time stochastic process. Assume that the stochastic processes $\{f_i^n\}_{n=0}^\infty$, $i \in I$ are essentially pairwise independent, i.e., for λ-almost all $i \in I$, λ-almost all $j \in I$, the random vectors $(f_i^0, \ldots, f_i^n)$ and $(f_j^0, \ldots, f_j^n)$ are independent for all $n \geq 0$. Then, for P-almost all $\omega \in \Omega$, the empirical process $f_\omega = \{f_\omega^n\}_{n=0}^\infty$ has the same finite-dimensional distributions as that of $f = \{f^n\}_{n=0}^\infty$, i.e. $(f_\omega^0, \ldots, f_\omega^n)$ and $(f^0, \ldots, f^n)$ have the same distribution for any $n \geq 0$.*

To prove that the agents' extended type processes are essentially pairwise independent in Lemma 3 below, we need the following elementary lemma, which is Lemma 5 in Duffie and Sun (2012).

[40] Here, $(\lambda \boxtimes P) f^{-1}$ is the distribution ν on X such that $\nu(B) = (\lambda \boxtimes P)(f^{-1}(B))$ for any Borel set B in X; λf_ω^{-1} is defined similarly.

Lemma 2 *Let ϕ_m be a random variable from $(\Omega, \mathcal{F}, P)$ to a finite space A_m, for $m = 1, 2, 3, 4$. If the random variables ϕ_3 and ϕ_4 are independent, and if, for all $a_1 \in A_1$ and $a_2 \in A_2$,*

$$P(\phi_1 = a_1, \phi_2 = a_2 \mid \phi_3, \phi_4) = P(\phi_1 = a_1 \mid \phi_3)P(\phi_2 = a_2 \mid \phi_4), \tag{32}$$

then the two pairs of random variables (ϕ_1, ϕ_3) and (ϕ_2, ϕ_4) are independent.

The following lemma is useful for applying the exact law of large numbers for discrete time processes in Theorem 2.16 of Sun (2006) (see Corollary 1) to our setting.

Lemma 3 *Assume that the dynamical system $\mathbb{D}$ is Markov conditionally independent. Then, the discrete time processes $\{\beta_i^n\}_{n=0}^{\infty}, i \in I$, are essentially pairwise independent. In addition, for each fixed $n \geq 1$, the random variables $\bar{\beta}_i^n, i \in I$ ($\bar{\bar{\beta}}_i^n, i \in I$) are also essentially pairwise independent.*

Proof. Let E be the set of all $(i, j) \in I \times I$ such that Equations (23), (24) and (25) hold for all $n \geq 1$. Then, by grouping countably many null sets together, we obtain that for λ-almost all $i \in I$, λ-almost all $j \in I$, $(i, j) \in E$, i.e., for λ-almost all $i \in I$, $\lambda(E_i) = \lambda(\{j \in I : (i, j) \in E\}) = 1$.

We can use induction to prove that for any fixed $(i, j) \in E$, if $(\beta_i^0, \ldots, \beta_i^n)$ and $(\beta_j^0, \ldots, \beta_j^n)$ are independent for $n \geq 0$, then so are the pairs $\bar{\beta}_i^n$ and $\bar{\beta}_j^n$, $\bar{\bar{\beta}}_i^n$ and $\bar{\bar{\beta}}_j^n$ for $n \geq 1$. The case of $n = 0$ is simply the assumption of initial independence in Subsection A.2. Suppose that it is true for the case $n-1$. That is, if $(\beta_i^0, \ldots, \beta_i^{n-1})$ and $(\beta_j^0, \ldots, \beta_j^{n-1})$ are independent, then so are the pairs $\bar{\beta}_i^{n-1}$ and $\bar{\beta}_j^{n-1}$, $\bar{\bar{\beta}}_i^{n-1}$ and $\bar{\bar{\beta}}_j^{n-1}$. Then, the Markov conditional independence condition and Lemma 2 imply that $(\beta_i^0, \ldots, \beta_i^{n-1}, \bar{\beta}_i^n)$ and $(\beta_j^0, \ldots, \beta_j^{n-1}, \bar{\beta}_j^n)$ are independent, so are the pairs $(\beta_i^0, \ldots, \beta_i^{n-1}, \bar{\beta}_i^n, \bar{\bar{\beta}}_i^n)$ and $(\beta_j^0, \ldots, \beta_j^{n-1}, \bar{\beta}_j^n, \bar{\bar{\beta}}_j^n)$, and $(\beta_i^0, \ldots, \beta_i^{n-1}, \bar{\beta}_i^n, \bar{\bar{\beta}}_i^n, \beta_i^n)$ and $(\beta_j^0, \ldots, \beta_j^{n-1}, \bar{\beta}_j^n, \bar{\bar{\beta}}_j^n, \beta_j^n)$. Hence, the random vectors $(\beta_i^0, \ldots, \beta_i^n)$ and $(\beta_j^0, \ldots, \beta_j^n)$ are independent for all $n \geq 0$, which means that $\{\beta_i^n\}_{n=0}^{\infty}$ and $\{\beta_j^n\}_{n=0}^{\infty}$ are independent. It is also clear that if, for each $n \geq 1$, the random variables $\bar{\beta}_i^n$ and $\bar{\beta}_j^n$ are independent, then so are $\bar{\bar{\beta}}_i^n$ and $\bar{\bar{\beta}}_j^n$. The desired result follows. ∎

The following lemma shows how to compute the expected cross-sectional extended type distributions $\mathbb{E}(\hat{p}^n)$ and $\mathbb{E}(\check{p}^n)$.

Lemma 4 *The following hold:*

1. *For each $n \geq 1$, $\mathbb{E}(\hat{p}^n) = \Gamma^n(\mathbb{E}(\hat{p}^{n-1}))$.*

2. *For each $n \geq 1$, the expected cross-sectional extended type distribution $\tilde{p}^n = \mathbb{E}(\check{p}^n)$ immediately after random mutation at time n, satisfies $\mathbb{E}(\check{p}_{kl}^n) = \sum_{k_1, l_1 \in S} \mathbb{E}(\hat{p}_{k_1 l_1}^{n-1}) b_{k_1 k}^n b_{l_1 l}^n$ and $\mathbb{E}(\check{p}_{kJ}^n) = \sum_{l \in S} \mathbb{E}(\hat{p}_{lJ}^{n-1}) b_{lk}^n$.*

Proof. Fix any $k, l \in S$. Equations (13) and (14) imply respectively that for any $k_1, l_1 \in S$,

$$P\left(\bar{\beta}_i^n = (k, J) \,|\, \beta_i^{n-1} = (k_1, l_1)\right) = 0, \text{ and } P\left(\bar{\beta}_i^n = (k, l) \,|\, \beta_i^{n-1} = (k_1, J)\right) = 0. \tag{33}$$

The Fubini property will be used extensively in the computations below. We shall illustrate its usage in Equation (34). It then follows from the Fubini property and Equations (13) and (33) that

$$\begin{aligned}
\tilde{p}_{kl}^n &= \int_\Omega \lambda\left(i \in I : \bar{\beta}_\omega^n(i) = (k, l)\right) dP(\omega) = \int_I P\left(\bar{\beta}_i^n = (k, l)\right) d\lambda(i) \\
&= \int_I \sum_{k_1, l_1 \in S} P\left(\bar{\beta}_i^n = (k, l), \beta_i^{n-1} = (k_1, l_1)\right) d\lambda(i) \\
&= \int_I \sum_{k_1, l_1 \in S} P\left(\bar{\beta}_i^n = (k, l) \,|\, \beta_i^{n-1} = (k_1, l_1)\right) P\left(\beta_i^{n-1} = (k_1, l_1)\right) d\lambda(i) \\
&= \sum_{k_1, l_1 \in S} \mathbb{E}(\hat{p}_{k_1 l_1}^{n-1}) b_{k_1 k}^n b_{l_1 l}^n.
\end{aligned} \tag{34}$$

By Equations (14) and (33), we obtain that

$$\begin{aligned}
\tilde{p}_{kJ}^n &= \int_I P\left(\bar{\beta}_i^n = (k, J)\right) d\lambda(i) = \int_I \sum_{k_1 \in S} P\left(\bar{\beta}_i^n = (k, J), \beta_i^{n-1} = (k_1, J)\right) d\lambda(i) \\
&= \int_I \sum_{k_1 \in S} P\left(\bar{\beta}_i^n = (k, J) \,|\, \beta_i^{n-1} = (k_1, J)\right) P\left(\beta_i^{n-1} = (k_1, J)\right) d\lambda(i) \\
&= \sum_{k_1 \in S} \int_I b_{k_1 k}^n P\left(\beta_i^{n-1} = (k_1, J)\right) d\lambda(i) \\
&= \sum_{k_1 \in S} \mathbb{E}(\hat{p}_{k_1 J}^{n-1}) b_{k_1 k}^n.
\end{aligned} \tag{35}$$

By Lemma 3, $\bar{\beta}^n$ is essentially pairwise independent process. The exact law of large numbers in Lemma 1 implies that $\check{p}^n(\omega) = \mathbb{E}(\check{p}^n) = \tilde{p}^n$ for P-almost all $\omega \in \Omega$. Combining with Equations (17) and (18), we can obtain that for any $l \in S$,

$$P(\bar{\bar{g}}_i^n = l \,|\, \bar{\alpha}_i^n = k, \bar{g}_i^n = J) = q_{kl}^n\left(\tilde{p}^n\right), \text{ and } P(\bar{\bar{g}}_i^n = J \,|\, \bar{\alpha}_i^n = k, \bar{g}_i^n = J) = \eta_k^n\left(\tilde{p}^n\right). \tag{36}$$

It follows from Equations (20) and (21) that

$$\begin{aligned}
\mathbb{E}(\hat{p}_{kl}^n) &= \int_I P\left(\beta_i^n = (k, l)\right) d\lambda(i) = \int_I \sum_{k_1, l_1 \in S} P\left(\beta_i^n = (k, l), \bar{\bar{\beta}}_i^n = (k_1, l_1)\right) d\lambda(i) \\
&= \int_I \sum_{k_1, l_1 \in S} P\left(\beta_i^n = (k, l) \,|\, \bar{\bar{\beta}}_i^n = (k_1, l_1)\right) P\left(\bar{\bar{\beta}}_i^n = (k_1, l_1)\right) d\lambda(i) \\
&= \int_I \sum_{k_1, l_1 \in S} (1 - \theta_{k_1 l_1}^n) \sigma_{k_1 l_1}^n(k, l) P\left(\bar{\bar{\beta}}_i^n = (k_1, l_1)\right) d\lambda(i) \\
&= \sum_{k_1, l_1 \in S} (1 - \theta_{k_1 l_1}^n) \sigma_{k_1 l_1}^n(k, l) \int_I P\left(\bar{\bar{\beta}}_i^n = (k_1, l_1)\right) d\lambda(i).
\end{aligned} \tag{37}$$

Equations (16) and (36) imply that

$$\begin{aligned}
&\int_I P\left(\bar{\bar{\beta}}_i^n=(k,l)\right)d\lambda(i)=\int_I\sum_{k_1,l_1\in S}P\left(\bar{\bar{\beta}}_i^n=(k,l)\,|\,\bar{\beta}_i^n=(k_1,l_1)\right)P\left(\bar{\beta}_i^n=(k_1,l_1)\right)d\lambda(i)\\
&\qquad+\int_I\sum_{k_1\in S}P\left(\bar{\bar{\beta}}_i^n=(k,l)\,|\,\bar{\beta}_i^n=(k_1,J)\right)P\left(\bar{\beta}_i^n=(k_1,J)\right)d\lambda(i)\\
&=\int_I P\left(\bar{\bar{\beta}}_i^n=(k,l)\,|\,\bar{\beta}_i^n=(k,l)\right)P\left(\bar{\beta}_i^n=(k,l)\right)d\lambda(i)\\
&\qquad+\int_I P\left(\bar{\bar{\beta}}_i^n=(k,l)\,|\,\bar{\beta}_i^n=(k,J)\right)P\left(\bar{\beta}_i^n=(k,J)\right)d\lambda(i)\\
&=\tilde{p}_{kl}^n+q_{kl}^n\left(\tilde{p}^n\right)\tilde{p}_{kJ}^n.
\end{aligned}\tag{38}$$

By substituting Equation (38) into Equation (37), we can express $\mathbb{E}(\hat{p}_{kl}^n)$ in terms of $\mathbb{E}(\check{p}^n)$ as

$$\mathbb{E}(\hat{p}_{kl}^n)=\sum_{k_1,l_1\in S}\tilde{p}_{k_1l_1}^n(1-\theta_{k_1l_1}^n)\sigma_{k_1l_1}^n(k,l)+\sum_{k_1,l_1\in S}\tilde{p}_{k_1J}^n q_{k_1l_1}^n\left(\tilde{p}^n\right)(1-\theta_{k_1l_1}^n)\sigma_{k_1l_1}^n(k,l).\tag{39}$$

Similarly, Equations (20) and (22) imply the second and third identities while Equations (36) and (38) imply the last identity in the following equation:

$$\begin{aligned}
&\mathbb{E}(\hat{p}_{kJ}^n)=\int_I P\left(\beta_i^n=(k,J)\right)d\lambda(i)\\
&=\int_I P\left(\beta_i^n=(k,J),\bar{\bar{\beta}}_i^n=(k,J)\right)d\lambda(i)+\int_I\sum_{k_1,l_1\in S}P\left(\beta_i^n=(k,J),\bar{\bar{\beta}}_i^n=(k_1,l_1)\right)d\lambda(i)\\
&=\int_I P\left(\bar{\bar{\beta}}_i^n=(k,J)\right)d\lambda(i)+\int_I\sum_{k_1,l_1\in S}\theta_{k_1l_1}^n\varsigma_{k_1l_1}^n(k)P\left(\bar{\bar{\beta}}_i^n=(k_1,l_1)\right)d\lambda(i)\\
&=\int_I P\left(\bar{\bar{\beta}}_i^n=(k,J)\,|\,\bar{\beta}_i^n=(k,J)\right)P\left(\bar{\beta}_i^n=(k,J)\right)d\lambda(i)\\
&\quad+\sum_{k_1,l_1\in S}\theta_{k_1l_1}^n\varsigma_{k_1l_1}^n(k)\int_I P\left(\bar{\bar{\beta}}_i^n=(k_1,l_1)\right)d\lambda(i)\\
&=\tilde{p}_{kJ}^n\eta_k^n(\tilde{p}^n)+\sum_{k_1,l_1\in S}\tilde{p}_{k_1l_1}^n\theta_{k_1l_1}^n\varsigma_{k_1l_1}^n(k)+\sum_{k_1,l_1\in S}\tilde{p}_{k_1J}^n q_{k_1l_1}^n(\tilde{p}^n)\theta_{k_1l_1}^n\varsigma_{k_1l_1}^n(k)
\end{aligned}\tag{40}$$

By combining Equations (34), (35),(39) and (40), we obtain that $\mathbb{E}(\hat{p}^n)=\Gamma^n(\mathbb{E}(\hat{p}^{n-1}))$. ■

The following lemma shows the Markov property of the agents' extended type processes.

Lemma 5 *Suppose the dynamical system $\mathbb{D}$ is Markov conditional independent. Then, for λ-almost all $i\in I$, the extended type process for agent i, $\{\beta_i^n\}_{n=0}^{\infty}$, is a Markov chain with transition matrix z^n at time $n-1$.*

Proof. Fix $n\geq 1$; by summing over all the $(k_2,l_2)\in\hat{S}$ in Equation (23), we obtain that for λ-almost all $i\in I$,

$$P\left(\bar{\beta}_i^n=(k_1,l_1)\mid(\beta_i^t)_{t=0}^{n-1}\right)=P\left(\bar{\beta}_i^n=(k_1,l_1)\mid\beta_i^{n-1}\right).\tag{41}$$

By grouping countably many null sets together, we know that for λ-almost all $i \in I$, Equation (41) holds for all $n \geq 1$.

Similarly, Equations (24) and (25) imply that for λ-almost all $i \in I$,

$$P\left(\bar{\bar{\beta}}_i^n = (k_1, l_1) \,|\, \bar{\beta}_i^n, (\beta_i^t)_{t=0}^{n-1}\right) = P\left(\bar{\bar{\beta}}_i^n = (k_1, l_1) \,|\, \bar{\beta}_i^n\right)$$
$$P\left(\beta_i^n = (k_1, l_1) \,|\, \bar{\bar{\beta}}_i^n, (\beta_i^t)_{t=0}^{n-1}\right) = P\left(\beta_i^n = (k_1, l_1) \,|\, \bar{\bar{\beta}}_i^n\right)$$

hold for all $n \geq 1$. A simple computation shows that for λ-almost all $i \in I$,

$$P(\beta_i^n = (k_1, l_1) \mid \beta_i^0, \ldots, \beta_i^{n-1}) = P(\beta_i^n = (k_1, l_1) \mid \beta_i^{n-1})$$

for all $a_1 \in S$, $r_1 \in S \cup \{J\}$ and $n \geq 1$. Hence, for λ-almost all $i \in I$, agent i's extended type process $\{\beta_i^n\}_{n=0}^{\infty}$ is a Markov chain.

By combining Equations (34), (35) and (39), we can obtain that

$$\begin{aligned}
\mathbb{E}(\hat{p}_{kl}^n) \quad = \quad & \sum_{k_1, l_1, k' \in S} b_{k'k_1}^n q_{k_1 l_1}^n(\tilde{p}^n)(1 - \theta_{k_1 l_1}^n)\sigma_{k_1 l_1}^n(k, l)\mathbb{E}(\hat{p}_{k'J}^{n-1}) \\
& + \sum_{k_1, l_1, k', l' \in S} b_{k'k_1}^n b_{l'l_1}^n (1 - \theta_{k_1 l_1}^n)\sigma_{k_1 l_1}^n(k, l)\mathbb{E}(\hat{p}_{k'l'}^{n-1}).
\end{aligned}$$

Since the transition probabilities $z_{(k'l')(kl)}^n$ and $z_{(k'J)(kl)}^n$ from time $n-1$ to time n are the respective coefficients of $\mathbb{E}(\hat{p}_{k'l'}^{n-1})$ and $\mathbb{E}(\hat{p}_{k'J}^{n-1})$ for any $k, l, k', l' \in S$, we can obtain that

$$\begin{aligned}
z_{(k'l')(kl)}^n \quad &= \quad \sum_{k_1, l_1 \in S} b_{k'k_1}^n b_{l'l_1}^n (1 - \theta_{k_1 l_1}^n)\sigma_{k_1 l_1}^n(k, l) \\
z_{(k'J)(kl)}^n \quad &= \quad \sum_{k_1, l_1 \in S} b_{k'k_1}^n q_{k_1 l_1}^n(\tilde{p}^n)(1 - \theta_{k_1 l_1}^n)\sigma_{k_1 l_1}^n(k, l),
\end{aligned}$$

which follow the corresponding formulas in Equation (26). Similarly, by combining Equations (34), (35) and (40), we can obtain that

$$\begin{aligned}
\mathbb{E}(\hat{p}_{kJ}^n) \quad = \quad & \sum_{k' \in S} b_{k'k}^n \eta_k^n(\tilde{p}^n)\mathbb{E}(\hat{p}_{k'J}^{n-1}) + \sum_{k_1, l_1, k', l' \in S} b_{k'k_1}^n b_{l'l_1}^n \theta_{k_1 l_1}^n \varsigma_{k_1 l_1}^n(k)\mathbb{E}(\hat{p}_{k'l'}^{n-1}) \\
& + \sum_{k_1, l_1, k' \in S} b_{k'k_1}^n q_{k_1 l_1}^n(\tilde{p}^n)\theta_{k_1 l_1}^n \varsigma_{k_1 l_1}^n(k)\mathbb{E}(\hat{p}_{k'J}^{n-1}).
\end{aligned}$$

Since the transition probabilities $z_{(k'l')(kJ)}^n$ and $z_{(k'J)(kJ)}^n$ from time $n-1$ to time n are the respective coefficients of $\mathbb{E}(\hat{p}_{k'l'}^{n-1})$ and $\mathbb{E}(\hat{p}_{k'J}^{n-1})$ for any $k, k', l' \in S$, we can obtain that

$$\begin{aligned}
z_{(k'l')(kJ)}^n \quad &= \quad \sum_{k_1, l_1 \in S} b_{k'k_1}^n b_{l'l_1}^n \theta_{k_1 l_1}^n \varsigma_{k_1 l_1}^n(k) \\
z_{(k'J)(kJ)}^n \quad &= \quad b_{k'k}^n \eta_k^n(\tilde{p}^n) + \sum_{k_1, l_1 \in S} b_{k'k_1}^n q_{k_1 l_1}^n(\tilde{p}^n)\theta_{k_1 l_1}^n \varsigma_{k_1 l_1}^n(k),
\end{aligned}$$

which follow the corresponding formulas in Equation (26). ■

Now, for each $n \geq 1$, we view each β^n as a random variable on $I \times \Omega$. Thus $\{\beta^n\}_{n=0}^{\infty}$ is a discrete-time stochastic process.

Lemma 6 *Assume that the dynamical system $\mathbb{D}$ is Markov conditionally independent. Then, $\{\beta^n\}_{n=0}^{\infty}$ is also a Markov chain with transition matrix z^n at time $n-1$.*

Proof. We can compute the transition matrix of $\{\beta^n\}_{n=0}^{\infty}$ at time $n-1$ by using Lemma 5 and the Fubini property. Fix any $k_1, k_2 \in S$ and any $l_1, l_2 \in S \cup \{J\}$. We have

$$\begin{aligned}
&(\lambda \boxtimes P)(\beta^n = (k_2, l_2), \beta^{n-1} = (k_1, l_1)) \\
&\qquad = \int_I P(\beta_i^n = (k_2, l_2) \mid \beta^{n-1} = (k_1, l_1)) P(\beta_i^{n-1} = (k_1, l_1))\, d\lambda(i) \\
&\qquad = \int_I z^n_{(k_1 l_1)(k_2 l_2)} P(\beta^{n-1} = (k_1, l_1))\, d\lambda(i) \\
&\qquad = z^n_{(k_1 l_1)(k_2 l_2)} \cdot (\lambda \boxtimes P)(\beta^{n-1} = (k_1, l_1)),
\end{aligned} \tag{42}$$

which implies that $(\lambda \boxtimes P)(\beta^n = (k_2, l_2) \mid \beta^{n-1} = (k_1, l_1)) = z^n_{(k_1 l_1)(k_2 l_2)}$.

Next, for any $n \geq 1$, and for any $(a^0, \ldots, a^{n-2}) \in (S \times (S \cup \{J\}))^{n-1}$, we have

$$\begin{aligned}
&(\lambda \boxtimes P)\left((\beta^0, \ldots, \beta^{n-2}) = (a^0, \ldots, a^{n-2}), \beta^{n-1} = (k_1, l_1), \beta^n = (k_2, l_2)\right) \\
&\quad = \int_I P\left((\beta_i^0, \ldots, \beta_i^{n-2}) = (a^0, \ldots, a^{n-2}), \beta_i^{n-1} = (k_1, l_1), \beta_i^n = (k_2, l_2)\right) d\lambda(i) \\
&\quad = \int_I P\left(\beta_i^n = (k_2, l_2) \mid \beta_i^{n-1} = (k_1, l_1)\right) P\left((\beta_i^0, \ldots, \beta_i^{n-2}) = (a^0, \ldots, a^{n-2}), \beta_i^{n-1} = (k_1, l_1)\right) d\lambda(i) \\
&\quad = z^n_{(k_1 l_1)(k_2 l_2)} \cdot (\lambda \boxtimes P)((\beta^0, \ldots, \beta^{n-2}) = (a^0, \ldots, a^{n-2}), \beta^{n-1} = (k_1, l_1)),
\end{aligned} \tag{43}$$

which implies that

$$(\lambda \boxtimes P)(\beta^n = (k_2, l_2) \mid (\beta^0, \ldots, \beta^{n-2}) = (a^0, \ldots, a^{n-2}), \beta^{n-1} = (k_1, l_1)) = z^n_{(k_1 l_1)(k_2 l_2)}.$$

Hence the discrete-time process $\{\beta^n\}_{n=0}^{\infty}$ is indeed a Markov chain with transition matrix z^n at time $n-1$. ■

Proof of Theorem 4: Properties (1), (2), and (3) of the theorem are shown in Lemmas 4, 5, and 3 respectively.

By the exact law of large numbers for discrete time processes in Corollary 1, we know that for P-almost all $\omega \in \Omega$, $(\beta_\omega^0, \ldots, \beta_\omega^n)$ and $(\beta^0, \ldots, \beta^n)$ (viewed as random vectors) have the same distribution for all $n \geq 1$. Since, as noted in Lemma 6, $\{\beta^n\}_{n=0}^{\infty}$ is a Markov chain with transition matrix z^n at time $n-1$, so is $\{\beta_\omega^n\}_{n=0}^{\infty}$ for P-almost all $\omega \in \Omega$. Thus property (4) is shown.

Since the processes $\bar{\beta}^n$ and β^n are essentially pairwise independent as shown in Lemma 3, the exact law of large numbers in Lemma 1 implies that at time period n, for P-almost all $\omega \in \Omega$, the realized cross-sectional extended type distribution after the random mutation, $\breve{p}^n(\omega) = \lambda(\bar{\beta}^n_\omega)^{-1}$ is the expected cross-sectional extended type distribution $\mathbb{E}(\breve{p}^n)$, and the realized cross-sectional extended type distribution at the end of period n, $\hat{p}^n(\omega) = \lambda(\beta^n_\omega)^{-1}$ is the expected cross-sectional extended type distribution $\mathbb{E}(\hat{p}^n)$. Thus, property (5) is shown.

Assume that there exists $\ddot{p}^0 \in \hat{\Delta}$ such that $P\left(\beta^0_i\right)^{-1} = \ddot{p}^0$ holds for λ-almost every $i \in I$. The exact law of large numbers in Lemma 1 implies that $\ddot{p}^0 = \mathbb{E}(\hat{p}^0)$. For λ-almost all $i \in I$, since the transition matrix of $\{\beta^n_i\}_{n=1}^\infty$ is $\{z^n\}_{n=1}^\infty$, the Markov chains $\{\beta^n_i\}_{n=0}^\infty$ induce the same distribution on $\hat{S}^\infty$ as ξ. For P-almost all $\omega \in \Omega$, the Markov chains $\{\beta^n_\omega\}_{n=0}^\infty$ induce the same distribution on $\hat{S}^\infty$ as ξ. Thus, property (6) is shown. ∎

Proof of Proposition 4: Given that the parameters $(b, q, \theta, \sigma, \varsigma)$ are time independent, the mapping Γ^n from $\hat{\Delta}$ to $\hat{\Delta}$ in Subsection A.3 is time independent, and will simply be denoted by Γ. By the continuity assumption in the sentence above Equation (9), $\hat{p}_{kJ} q^n_{kl}(\hat{p})$ is continuous in $\hat{p} \in \hat{\Delta}$ for any $k, l \in S$. For any $k_1, l_1 \in S$, since $\tilde{p}_{k_1 J} = \sum_{r \in S} \hat{p}_{rJ} b^n_{rk_1}$ is continuous in $\hat{p} \in \hat{\Delta}$, we can also obtain that $\tilde{p}_{k_1 J}\, q^n_{k_1 l_1}(\tilde{p})$ is continuous in $\hat{p} \in \hat{\Delta}$. Therefore, Γ is a continuous function from $\hat{\Delta}$ to itself. By Brower's Fixed Point Theorem, Γ has a fixed point $\hat{p}^*$. In this case, $\mathbb{E}(\hat{p}^n) = \hat{p}^*$, $z^n = z^1$ for all $n \geq 1$. Hence the Markov chains $\{\beta^n_i\}_{n=0}^\infty$ for λ-almost all $i \in I$, $\{\beta^n\}_{n=0}^\infty$, $\{\beta^n_\omega\}_{n=0}^\infty$ for P-almost all $\omega \in \Omega$ are time-homogeneous.

If the initial extended type process β^0 is *i.i.d.*, then the extended type distribution of agent i at time $n = 0$ is $P\left(\beta^0_i\right)^{-1} = \hat{p}^*$ for λ-almost every $i \in I$. By (6) of Theorem 4, for any $n \geq 1$, β^n_i induce the same distribution on $\hat{S}$ for λ-almost all $i \in I$. Therefore, for any $n \geq 1$, $P\left(\beta^n_i\right)^{-1} = \hat{p}^*$ for λ-almost all $i \in I$.

D A Brief Introduction to Nonstandard Analysis

In order to summarize background knowledge for the proofs of Theorems 1 and 5, this section presents basic nonstandard analysis by adopting some material from Loeb and Wolff (2015) and several other related results. First, a simple construction of the nonstandard number system that extends the usual ordered field of real numbers is given in Subsection D.1. A more general framework of nonstandard analysis is then presented in Subsection D.2. The key constructions of Loeb measure spaces and Loeb transition probabilities are introduced in Subsections D.3 and D.4 respectively. The crucial relevant result is the so-called Fubini property for Loeb product and transition probabilities. In the final part of this section, we discuss some motivation for using hyperfinite agent spaces in settings like those for random matching.[41]

D.1 Non-standard number system

First, we extend the ordered field of real numbers $\mathbb{R}$ to an ordered field ${}^*\mathbb{R}$ that contains infinitesimals. To this end, we introduce the concept of a free ultrafilter.

Definition 4 *A free ultrafilter on the set* $\mathbb{N}$ *of positive integers is a collection* $\mathcal{U} \subseteq \mathcal{P}(\mathbb{N}) = \{A : A \subseteq \mathbb{N}\}$ *such that*

1. $\emptyset \notin \mathcal{U}$.

2. $A \in \mathcal{U}$ *and* $B \in \mathcal{U} \Longrightarrow A \cap B \in \mathcal{U}$.

3. $A \subseteq \mathbb{N}$ *and* $A \notin \mathcal{U} \Longrightarrow \mathbb{N} \backslash A \in \mathcal{U}$.

4. A *is a finite subset of* $\mathbb{N} \Longrightarrow \mathbb{N} \backslash A \in \mathcal{U}$.

Fix a free ultrafilter $\mathcal{U}$. One can define a set function ι on the power set $\mathcal{P}(\mathbb{N})$ of $\mathbb{N}$ such that $\iota(A) = 1$ if $A \in \mathcal{U}$, and $\iota(A) = 0$ if $A \notin \mathcal{U}$. It is easy check that ι is a finitely additive probability measure on $\mathcal{P}(\mathbb{N})$. If a property holds on some set $A \in \mathcal{U}$, then the property holds with ι-probability one; we can simply say that the property holds almost everywhere, expressed for brevity as "a.e.".

Two sequences $\langle r_i \rangle$ and $\langle s_i \rangle$ of real numbers are said to be equivalent if $r_i = s_i$ a.e., which means $\{i \in \mathbb{N} : r_i = s_i\} \in \mathcal{U}$. We write $[\langle r_i \rangle]$ for the equivalence class containing the sequence $\langle r_i \rangle$, and we use ${}^*\mathbb{R}$ to denote the collection of such equivalence classes. The set ${}^*\mathbb{R}$ is called the set of nonstandard real numbers, or the "hyperreal" numbers. Such a construction using an ultrafilter is called an ultrapower construction.[42] We note that the set $\mathbb{R}$ of real numbers is embedded in the set of nonstandard real numbers ${}^*\mathbb{R}$ via the map $c \to [\langle c \rangle]$, where $\langle c \rangle$ is the constant sequence with term $c \in \mathbb{R}$. We write ${}^*c$ for $[\langle c \rangle]$, but later drop the star for convenience. In contrast to hyperreal numbers in ${}^*\mathbb{R}$, the numbers in $\mathbb{R}$ are also called standard real numbers.

The summation and multiplication operations $+$, $\cdot$ and the absolute value function $|\cdot|$ together with the "less than" order relation $<$ for ${}^*\mathbb{R}$ are defined as follows.

Definition 5 *Given real sequences* $\langle r_i \rangle$ *and* $\langle s_i \rangle$, *we set*

[41] For a comprehensive treatment of nonstandard analysis, see Chapters 1, 2, 3 and 6 in Loeb and Wolff (2015). For the details on Loeb transition probabilities and the associated Fubini property, see Section 5 of Duffie and Sun (2007).

[42] Though the set ${}^*\mathbb{R}$ of nonstandard real numbers depends on the underlying ultrafilter, the particular choice of such an ultrafilter is not an issue. When we consider applications of nonstandard analysis, what we use are some general properties of nonstandard models, such as the Transfer Principle in Proposition 7 and the Countable Saturation Principle in Proposition 8 below.

1. $[\langle r_i \rangle] + [\langle s_i \rangle] = [\langle r_i + s_i \rangle]$.

2. $[\langle r_i \rangle] \cdot [\langle s_i \rangle] = [\langle r_i \cdot s_i \rangle]$.

3. $|[\langle r_i \rangle]| = [\langle |r_i| \rangle]$.

4. $[\langle r_i \rangle] < [\langle s_i \rangle]$ *if* $r_i < s_i$ *a.e.*

It is easy to check that the operations $+$ and $\cdot$, as well as $|\cdot|$ and the ordering $<$, are independent of the choices of the representing sequences. The structure $({}^*\mathbb{R}, +, \cdot, <)$ forms an ordered field that extends the ordered field $(\mathbb{R}, +, \cdot, <)$.

For any $r \in {}^*\mathbb{R}$, r is **infinite** (or **unlimited**) if $|r| > n$ for every standard positive integer $n \in \mathbb{N}$; r is **finite** (or **limited**) if $|r| < n$ for some $n \in \mathbb{N}$; and r is **infinitesimal** if $|r| < \frac{1}{n}$ for every $n \in \mathbb{N}$. Recall that for $r = [\langle r_i \rangle] \in {}^*\mathbb{R}$ and $c \in \mathbb{R}$, $|r| < c$ ($|r| > c$) means that $|r_i| < c$ ($|r_i| > c$) holds a.e.

For $x, y \in {}^*\mathbb{R}$, we say that x and y are infinitesimally close or infinitely close if $x - y$ is infinitesimal and in that case we write $x \simeq y$. The equivalence class for $\simeq$ containing x is called the monad of x, written as $monad(x)$. That is, $monad(x) = \{y \in {}^*\mathbb{R} : y \simeq x\}$.

If $\rho \in {}^*\mathbb{R}$ is finite, then the unique real number c with $\rho \simeq c$ is called the standard part of ρ. We write $c = \mathrm{st}(\rho)$ or $c = {}^\circ\rho$.

Let ${}^*\mathbb{N} = \{[\langle r_i \rangle] : r_i \in \mathbb{N} \text{ a.e.}\} \subseteq {}^*\mathbb{R}$ be the set of hyperfinite integers, and ${}^*\mathbb{N}_\infty$ the set of unlimited hyperfinite integers. We have the following proposition.

Proposition 6 *For any unlimited hyperfinite integer* $r = [\langle r_i \rangle] \in {}^*\mathbb{N}_\infty$*, the set* $\{r' \in {}^*\mathbb{N} : r' \leq r\}$*, which is also denoted by* $\{1, 2, \ldots, r\}$*, has the cardinality of the continuum.*

Proof: Let $\mathbb{R}^\infty$ be the set of sequences of standard real numbers. Since $\mathbb{R}^\infty$ has the cardinality of the continuum, the cardinality of ${}^*\mathbb{N}$ is therefore at most the cardinality of the continuum, which implies that the cardinality of $A = \{1, 2, \ldots, r\}$ is also at most the cardinality of the continuum.

Let $B = \{\frac{1}{r}, \frac{2}{r}, \ldots, \frac{r}{r}\}$. It is clear that A and B have the same cardinality. For any standard real number $c \in [0, 1]$, let $r' = \lfloor c \cdot r \rfloor = [\langle h_i \rangle] \in {}^*\mathbb{R}$, where h_i is the integer part of the standard real number $c \cdot r_i$. From now on, we may drop the multiplication symbol $\cdot$ when there is no confusion. It is clear that $r' \in {}^*\mathbb{N}$ and $cr - 1 < r' \leq cr$, which implies that $\frac{r'}{r} \in B$ and $c - \frac{1}{r} < \frac{r'}{r} \leq c$. Note that $\frac{1}{r}$ is infinitesimal, and $\mathrm{st}(\frac{r'}{r}) = c$. Therefore, $\mathrm{st}(\cdot)$ is a surjection from B to $[0, 1]$, which implies that the cardinality of B is at least the cardinality of the continuum. Hence, the cardinality of A is also at least the cardinality of the continuum. Combining with the conclusion of the above paragraph, we know that A has the cardinality of the continuum.
■

D.2 General framework of nonstandard analysis

To develop the general framework of nonstandard analysis, we need to work with the concept of superstructure. Fix a set X containing $\mathbb{R}$. Let $V_0(X) = X$, and for each positive integer $n \in \mathbb{N}$, let $V_n(X) = V_{n-1}(X) \cup \mathcal{P}(V_{n-1}(X))$, where $\mathcal{P}(V_{n-1}(X))$ is the power set of $V_{n-1}(X)$. The superstructure over X is the set $V(X) = \cup_{n=0}^{\infty} V_n(X)$. Entities in X are said to be of rank 0, and for $n \geq 1$, entities in $V_n(X) \backslash V_{n-1}(X)$ are said to be of rank n.

For $a, b \in V_n(X)$, one can define an ordered pair (a, b) as the set $\{\{a\}, \{a, b\}\}$, which is an element in $V_{n+2}(X)$. With the definition of ordered pairs, one can then define the Cartesian product of two sets in $V(X)$, as well as relations and functions in $V(X)$. For $k \geq 3$, one can define ordered k-tuples $(a_1, a_2, \ldots, a_k)$ as $\{(1, a_1), (2, a_2), \ldots, (k, a_k)\}$. The k-tuple versions of Cartesian products, relations and functions in $V(X)$ can be similarly defined. In fact, the superstructure can be used to cover basically all of the relevant mathematical structures that are useful for applications.

We now describe the construction of formal statements, or "formulas," in a formal language $\mathcal{L}_X$ about the superstructure $V(X)$. Given X, the language $\mathcal{L}_X$ for the superstructure $V(X)$ over X has the following symbols:

1. Connectives: $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$.

2. Quantifiers: $\forall$, $\exists$.

3. Parentheses: $[\ ]$, $(\ )$, $\langle\ \rangle$.

4. Constant Symbols: At least one name for each entity in $V(X)$.

5. Variable Symbols: A fixed collection of symbols representing variables.

6. Equality Symbol: Denotes equality for elements of X, and set equality otherwise.

7. Set membership: $\in$.

The above symbols serve as the "alphabet" of the language $\mathcal{L}_X$. A fixed set of variable symbols together with other symbols in $\mathcal{L}_X$ will lead to a well-defined collection of formal syntactical statements.

Definition 6 *A formula of $\mathcal{L}_X$ is built up inductively with the following rules:*

(a) If $x_1, \cdots, x_n$, x, and y are either constants or variables, then the following are called atomic formulas: $x \in y$, $x = y$; $(x_1, \cdots, x_n) \in y$; $(x_1, \cdots, x_n) = y$; $((x_1, \cdots, x_n), x) \in y$; $((x_1, \cdots, x_n), x) = y$.

(b) If Φ and Θ are formulas, so are $(\neg\Phi)$, $(\Phi \wedge \Theta)$, $(\Phi \to \Theta)$, $(\Phi \vee \Theta)$, and $(\Phi \leftrightarrow \Theta)$.

(c) If x is a variable symbol and y is either a variable symbol or a constant symbol and Φ is a formula, then $(\forall x \ \in \ y)\Phi$ and $(\exists x \in y)\Phi$ are formulas.

The logical connectives $\neg$, $\vee$, $\wedge$, $\to$, $\leftrightarrow$ have the usual meanings in terms of the satisfiability of formulas connected by them. For example, $(\neg\Phi)$ means that Φ is not satisfied while $\vee$, $\wedge$ mean "or", "and" respectively. For the formulas $(\forall x \ \in \ y)\Phi$ and $(\exists x \in y)\Phi$, the scope of the quantifies $\forall, \exists$ is Φ. One can define the scope of a quantifier within a formula inductively.

Definition 7 *A variable x is free in a formula Φ if it is not within the scope of any quantifier for x. A closed formula in $\mathcal{L}_X$ is a formula without free variables.*

Fix a free ultrafilter $\mathcal{U}$. Given $\langle a_i \rangle$ and $\langle b_i \rangle$, both in the space X^∞ of sequences in X, are said to be equivalent if $a_i = b_i$ a.e. For any $c \in X$, let ${}^*c = [\langle c, c, \ldots \rangle]$ be the equivalence class of sequences in X^∞ that contains the constant sequence $\langle c, c, \ldots \rangle$. For any sequence $\{A_i\}_{i=1}^\infty$ of sets in $V_n(X) \backslash X$ for some $n \geq 1$, define the set $[\langle A_i \rangle] = \{[\langle x_i \rangle] : x_i \in A_i \text{ a.e.}\}$. For $A \in V(X)$, let ${}^*A = [\langle A, A, A, \ldots \rangle]$. In particular, ${}^*X$ is the set of equivalent classes of sequences in X^∞.

Definition 8 *If Φ is a formula in $\mathcal{L}_X$, the $*$-transform of Φ, denoted ${}^*\Phi$, is the formula in $\mathcal{L}_{{}^*X}$ that is obtained by replacing each constant c in Φ with ${}^*c$.*

The following result is a basic tool in nonstandard analysis.[43]

Proposition 7 (Transfer Principle) *If Φ is a closed formula in $\mathcal{L}_X$ that is true for $V(X)$, then ${}^*\Phi$ is true for $V({}^*X)$.*

All entities in $V(X)$ and entities in $V({}^*X)$ of the form ${}^*b$, for some $b \in V(X)$, are called **standard**. An entity a in $V({}^*X)$ is called **internal** if for some set $b \in V(X)$, $a \in {}^*b$. All other entities in $V({}^*X)$ are called **external**. For any internal set A in $V({}^*X)$, one can always find a sequence $\{A_i\}_{i=1}^\infty$ of sets in $V_n(X) \backslash X$ for some $n \geq 1$ such that A is the set of equivalence classes $\{[\langle a_i \rangle] : a_i \in A_i \text{ a.e.}\}$. If any kind of internal operations are applied to internal sets, one still obtains internal sets; see, for example, Theorem 2.8.4 in Loeb and Wolff (2015). In particular, if A and B are internal, then so are $A \cup B$, $A \cap B$, $A \backslash B$, and $A \times B$. An internal function is a function whose graph is internal.

For $B \in V(X) \backslash X$, let $\mathcal{P}_F(B)$ denote the finite subsets of B. An element $A \in {}^*\mathcal{P}_F(B)$ will be called a **hyperfinite** set. In particular, A is the set of equivalence classes $\{[\langle b_i \rangle] : b_i \in$

[43] For a detailed proof, readers are referred to Sections 2.2-2.5 of Loeb and Wolff (2015).

B_i a.e.} for some sequence $\langle B_i \rangle$ of finite subsets of B. The internal cardinality of A is simply the hyperinteger $[\langle |B_i| \rangle]$, where $|B_i|$ is the cardinality of the finite set B_i.

The following is an important uniformity principle that transforms a local property expressed by finite intersections to a global property described by the intersection of all the sets in the sequence. The proof is taken from page 199 of Khan and Sun (1997).

Proposition 8 (Countable Saturation Principle) *For a sequence of nonempty internal sets,* $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \ldots$, *we have* $\cap_{n\in\mathbb{N}}^{\infty} A_n \neq \emptyset$.

Proof. For any $n \in \mathbb{N}$, since A_n is internal, there exists $B_n \in V(X) \setminus X$ such that $A_n \in {}^*B_n = [\langle B_n, B_n, \ldots \rangle]$. Then there exists a sequence $\{A_{ni}\}$ of sets such that $A_{ni} \in B_n$ for any $i \in \mathbb{N}$ and $A_n = [\langle A_{n1}, A_{n2}, \ldots \rangle]$. Let $I_n = \{i \geq n : A_{1i} \supseteq \cdots \supseteq A_{ni} \neq \emptyset\}$. Then for all $n \in \mathbb{N}$, $I_n \in \mathcal{U}$, $I_n \supseteq I_{n+1}$, and $\cap_{n\in\mathbb{N}} I_n = \emptyset$. This implies that for any $i \in I_1$, $n(i) = \max\{n \in \mathbb{N} : i \in I_n\}$ is well defined. For $i \in I_1$, since $i \in I_{n(i)}$, we know that $A_{n(i)i} \neq \emptyset$. Pick b_i from $A_{n(i)i}$, and note that $i \in I_n$ implies that $n(i) \geq n$, and hence $b_i \in A_{n(i)i} \subseteq A_{ni}$. Thus $\{i \in I_1 : b_i \in A_{ni}\} \supseteq I_n \in \mathcal{U}$. By defining b_i to be some point in A_1 if i is not in I_1, we obtain that $[\langle b_i \rangle] \in A_n$. Since n is arbitrary, the proof is finished.

D.3 Construction of hyperfinite Loeb spaces

Fix any unlimited hyperfinite integer M. Let $\Lambda = \{1, 2, \ldots, M\}$, and $\mathcal{C}$ be the internal power set of all the internal subsets of Λ. Let $w : \Lambda \to {}^*\mathbb{R}_+$ be an internal function such that $\sum_{i\in\Lambda} w(i) = 1$.

Define an internal finitely-additive measure from $\mathcal{C}$ to ${}^*[0, 1]$ such that $\mu(A) = \sum_{i\in A} w(i)$ for any $A \in \mathcal{C}$. Then $(\Lambda, \mathcal{C}, \mu)$ is called a hyperfinite internal probability space. If $w(i) \equiv \frac{1}{M}$ for all $i \in \Lambda$, then $(\Lambda, \mathcal{C}, \mu)$ is called a hyperfinite counting probability space.

We let $\mathrm{st}(\mu)$ be the function from $\mathcal{C}$ into $\mathbb{R}_+$ defined by $\mathrm{st}(\mu)(A) = \mathrm{st}(\mu(A))$ for any $A \in \mathcal{C}$. It is clear that $\mathrm{st}(\mu)$ is a finitely-additive measure on the algebra $\mathcal{C}$. The important point is that $\mathrm{st}(\mu)$ is a countably-additive measure on the algebra $\mathcal{C}$. To see this, consider a sequence $A_1 \supseteq A_2 \supseteq \ldots$ of internal sets in $\mathcal{C}$ such that $\cap_{n\in\mathbb{N}} A_n = \emptyset$. The countable saturation principle implies the existence of $m \in \mathbb{N}$ such that $A_m = \emptyset$. It is thus clear that $\lim_{n\to\infty} \mathrm{st}(\mu)(A_n) = 0$. By the well-known Caratheodory's extension theorem (see, for example, Loeb (2016, p. 181)), $\mathrm{st}(\mu)$ can be extended to a measure μ_L on the σ-algebra $\sigma(\mathcal{C})$ that is generated by $\mathcal{C}$. By including all μ_L-null subsets, we obtain a standard complete probability space $(\Lambda, L_\mu(\mathcal{C}), \mu_L)$, which is called a Loeb measure space.

D.4 Transition probabilities

Let $(I, \mathcal{I}_0, \lambda_0)$ be a hyperfinite internal probability space for which $\mathcal{I}_0$ is the internal power set on some hyperfinite set I. Let Ω be a hyperfinite internal set with $\mathcal{F}_0$ its internal power set. Let P_0 be an internal function from I to the space of hyperfinite internal probability measures on $(\Omega, \mathcal{F}_0)$, which is called an internal transition probability. For $i \in I$, denote the hyperfinite internal probability measure $P_0(i)$ on $(\Omega, \mathcal{F}_0)$ by P_{0i}.

It is clear that the Cartesian product $I \times \Omega$ is a hyperfinite set. Let $\mathcal{I}_0 \otimes \mathcal{F}_0$ be the internal power set on $I \times \Omega$. Define a hyperfinite internal probability measure τ_0 on $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0)$ by letting $\tau_0(\{(i,\omega)\}) = \lambda_0(\{i\}) P_{0i}(\{\omega\})$ for $(i,\omega) \in I \times \Omega$. The measure τ_0 will be called the product transition probability of the measure λ_0 and the transition probability P_0. Let $(I, \mathcal{I}, \lambda)$, $(\Omega, \mathcal{F}_i, P_i)$, and $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \tau)$ be the Loeb spaces corresponding respectively to $(I, \mathcal{I}_0, \lambda_0)$, $(\Omega, \mathcal{F}_0, P_{0i})$, and $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \tau_0)$. The collection $\{P_i : i \in I\}$ of Loeb measures will be called a Loeb transition probability, and denoted by P. The measure τ will be called the Loeb product transition probability of the measure λ and the Loeb transition probability P. We shall also denote τ_0 by $\lambda_0 \otimes P_0$ and τ by $\lambda \boxtimes P$.

The following result presents a generalized Fubini theorem for a Loeb transition probability, which is proved in Section 5 of Duffie and Sun (2007).[44]

Proposition 9 *Let f be a real-valued integrable function on $(I \times \Omega, \sigma(\mathcal{I}_0 \otimes \mathcal{F}_0), \tau)$. Then, (1) $f_i = f(i, \cdot)$ is $\sigma(\mathcal{F}_0)$-measurable for each $i \in I$ and integrable on $(\Omega, \sigma(\mathcal{F}_0), P_i)$ for λ-almost all $i \in I$; (2) $\int_\Omega f_i(\omega)\, dP_i(\omega)$ is integrable on $(I, \sigma(\mathcal{I}_0), \lambda)$; (3) $\int_I \int_\Omega f_i(\omega)\, dP_i(\omega)\, d\lambda(i) = \int_{I\times\Omega} f(i,\omega)\, d\tau(i,\omega)$.*

If P_{0i} does not depend on i, then $\tau = \lambda \boxtimes P$ is called the Loeb product measure. The corresponding measure space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is called the Loeb product space. In this case, the symmetric position of the two probability spaces respectively on I and Ω implies that the properties as stated in Proposition 9 also hold when the iterated integral is taken in different order. It is clear that $\mathcal{I} \boxtimes \mathcal{F}$ contains the usual product σ-algebra $\sigma(\mathcal{I}_0) \otimes \sigma(\mathcal{F}_0)$. Thus, the Loeb product space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a Fubini extension.[45] Such a result in the special case was shown by Keisler (1977); see also Loeb and Wolff (2015, p. 214).

[44] For simplicity, we only state the result in terms of the σ-algebra $\sigma(\mathcal{I}_0 \otimes \mathcal{F}_0)$. One can also re-state the result to the case when the underlying measure space is the completion of $(I \times \Omega, \sigma(\mathcal{I}_0 \otimes \mathcal{F}_0), \tau)$.

[45] Assume that both λ and P are atomless. Proposition 8.4.5 in Loeb and Wolff (2015, Chapter 8) (by the third author of this paper) indicates that $\mathcal{I} \boxtimes \mathcal{F}$ is a strict extension of $\sigma(\mathcal{I}_0) \otimes \sigma(\mathcal{F}_0)$. As noted in Proposition 8.4.1 of the same chapter, one can also construct $\mathcal{I} \boxtimes \mathcal{F}$-measurable processes which are essentially pairwise independent with any given variety of distributions.

D.5 Why hyperfinite agent spaces work

As noted in Subsection D.2, a hyperfinite set can be viewed as an equivalence class of a sequence of finite sets. The transfer principle also indicates that hyperfinite sets preserve properties of finite sets. Thus, an idealized model based on a hyperfinite agent space captures the asymptotic nature of the large finite phenomenon being modeled.[46] Khan and Sun (1997) call this property "asymptotic implementability."

A typical hyperfinite set is of the form $\{1, 2, \ldots, r\}$ for some unlimited hyperinteger r. It is equivalent to work with $T = \{\frac{1}{r}, \frac{2}{r}, \ldots, \frac{r}{r}\}$. Since the standard parts of elements in T are the real numbers in $[0, 1]$, a limit model based on T may be reduced to a model based on $[0, 1]$, provided that the relevant mappings on T have the essential continuity property in the sense that for almost all $t \in T$, those points infinitely close to t will have infinitely close values in the relevant target spaces.[47] In the case of independent random matching, as considered in this paper, the random types across the agent space are always discontinuous because of the cross-agent independence assumption. Hence, it is not possible to study independent random matching via the classical Lebesgue unit interval.

E Proofs of the Existence Results

The main existence results in this paper are Theorems 1 and 5, which are proved in Subsections E.1 and E.2 respectively. Subsection E.3 presents the proofs of Propositions 2 and 5.

E.1 Proof of Theorem 1

In the proof of Theorem 2.6 in Duffie and Sun (2007) for the existence of independent static random partial matching, after one chooses the unmatched agents randomly, the measure on the space of all the matchings on the set of matched agents is the hyperfinite counting probability, which treats the matched agents symmetrically regardless of their types. In this paper, since the matching probabilities depend on the types of both the agents and their partners, such a symmetric treatment of agents is not possible. The idea underlying the proof of Theorem 1

[46]Consider a simple example that transfers the classical law of large numbers to the hyperfinite setting. A hyperfinite sequence $\{X_i\}_{i=1}^n$ ($n \in \mathbb{N}_\infty$) of internal random variables from an internal probability space $(\Omega, \mathcal{F}_0, P_0)$ to ${}^*[-c, c]$ with a positive standard real number c is said to be $*$-independent if $P_0(X_1 \le a_1, \ldots, X_n \le a_n) = \prod_{i=1}^n P_0(X_i \le a_i)$ holds for any internal sequence $\{a_i\}_{i=1}^n$ of hyperreal numbers. Let $(I, \mathcal{I}_0, \lambda_0)$ be the hyperfinite counting probability space on $\{1, \ldots, n\}$. Suppose that X_i, $1 \le i \le n$ are $*$-independent with a common mean m and variance σ^2. The Chebyshev Inequality says that $P_0\left(|\frac{X_1+\cdots+X_n}{n} - m| \ge \frac{1}{n^{1/3}}\right) \le \frac{\sigma^2}{n^{1/3}}$. Note that $\frac{1}{n^{1/3}}$ and $\frac{\sigma^2}{n^{1/3}}$ are infinitesimals. Thus, for P-almost all $\omega \in \Omega$, $\int_I X_\omega(i) d\lambda_0 = \frac{X_1(\omega)+\cdots+X_n(\omega)}{n} \simeq m$, that is, $\int_I {}^\circ X_\omega(i) d\lambda = {}^\circ m$, where λ and P are the corresponding Loeb measures. A similar equality holds for any set in $\mathcal{I}_0$ in place of I.

[47]See Loeb and Wolff (2015, p. 190) on the characterization of Lebesgue measurability on a hyperfinite Loeb counting probability space.

is as follows. The set of type-k agents to be matched with type-l agents (denoted by A_{kl}) is chosen randomly according to the matching probabilities. One needs to make sure that (1) for $k < l$, A_{kl} and A_{lk} must have exactly the same internal cardinality so that the agents between them can be matched, and (2) the internal cardinality of A_{kk} must be an even hyperinteger so that agents in A_{kk} can be matched to each other. The key point is to guarantee that the construction leads an independent directed random matching with given parameters.

In dynamic directed random matching with enduring partnership as considered in this paper, only the unmatched agents will conduct directed random searches for counterparties while those existing paired agents will not participate in the search process. Thus, in order to handle the matching step for the inductive definition of dynamic directed random matching, Lemma 7 below allows the existence of both pre-matched agents and unmatched agents so that pre-matched agents remain matched to the same partners and unmatched agents may search for counterparties. In contrast, Theorem 2.6 in Duffie and Sun (2007) considers only the case in which all the agents are unmatched.

Let $I = \{1, \ldots, \hat{M}\}$ be a hyperfinite set with $\hat{M}$ an unlimited hyperfinite integer in ${}^*\mathbb{N}_\infty$, $\mathcal{I}_0$ the internal power set on I, and λ_0 the hyperfinite counting probability measure on $\mathcal{I}_0$ with $\lambda_0(A) = |A|/|I|$ for any $A \in \mathcal{I}_0$, where $|A|$ is the internal cardinality of $|A|$. The corresponding Loeb counting probability space $(I, \mathcal{I}, \lambda)$ is our space of agents. Following Definition 3, an internal partial matching ψ from I to I is an internal mapping from I to I such that $\psi(\psi(i)) = i$ for any $i \in I$. When $\psi(i) \neq i$ ($\psi(i) = i$), agent i is matched with agent $\psi(i)$ (agent i is not matched). When $\psi(i) \neq i$ for each $i \in I$, ψ is said to be an internal full matching on I. For a given hyperfinite internal probability space $(\Omega, \mathcal{F}_0, P_0)$, an internal random (partial) matching π is an internal mapping from $I \times \Omega$ to I such that π_ω is an internal partial matching for each $\omega \in \Omega$.

The following lemma will be used to prove both Theorems 1 and 5.

Lemma 7 *As above, let $(I, \mathcal{I}_0, \lambda_0)$ be the hyperfinite counting probability space with its Loeb space $(I, \mathcal{I}, \lambda)$. Then, there exists a hyperfinite internal set Ω with its internal power set $\mathcal{F}_0$ such that for any initial internal type function α^0 from I to S and initial internal partial matching π^0 from I to I with*

$$g^0(i) = \begin{cases} \alpha^0(\pi^0(i)) & \text{if } \pi^0(i) \neq i \\ J & \text{if } \pi^0(i) = i, \end{cases}$$

and internal extended type distribution $\hat{\rho} = \lambda_0 \left(\alpha^0, g^0\right)^{-1}$, and for any internal matching probability function q from $S \times S$ to ${}^*\mathbb{R}_+$ with $\sum_{r \in S} q_{kr} \leq 1$ and $\hat{\rho}_{kJ} q_{kl} \simeq \hat{\rho}_{lJ} q_{lk}$ (i.e., $\hat{\rho}_{kJ} q_{kl} - \hat{\rho}_{lJ} q_{lk}$ is an infinitesimal) for any $k, l \in S$, there exists an internal random matching π from $I \times \Omega$ to I and an internal probability measure P_0 on $(\Omega, \mathcal{F}_0)$ with the following properties.

(i) Let $H = \{i : \pi^0(i) \neq i\}$. Then $P_0\left(\{\omega \in \Omega : \pi_\omega(i) = \pi^0(i) \text{ for any } i \in H\}\right) = 1$.

(ii) Let g be the internal mapping from $I \times \Omega$ to $S \cup \{J\}$, defined by

$$g(i,\omega) = \begin{cases} \alpha^0(\pi(i,\omega)) & \text{if } \pi(i,\omega) \neq i \\ J & \text{if } \pi(i,\omega) = i, \end{cases}$$

for any $(i,\omega) \in I \times \Omega$. Then, for any $k, l \in S$, $P_0(g_i = l) \simeq q_{kl}$ for λ-almost every agent $i \in I$ satisfying $\alpha^0(i) = k$ and $\pi^0(i) = i$.

(iii) Denote the corresponding Loeb probability spaces of the internal probability spaces $(\Omega, \mathcal{F}_0, P_0)$ and $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes P_0)$ respectively by $(\Omega, \mathcal{F}, P)$ and $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. The mapping g is an essentially pairwise independent process from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to $S \cup \{J\}$.

To reflect their dependence on (α^0, π^0, q), π and P_0 are also denoted $\pi_{(\alpha^0,\pi^0,q)}$ and $P_{(\alpha^0,\pi^0,q)}$.

Proof. The proof consists of four steps. The first step is to allow the initially unmatched type-k agents to randomly choose the types of their partners according to the matching probabilities. However, for a sample realization, the set of type-k agents to be matched with type-l agents may not have the same internal cardinality as the set of type-l agents to be matched with type-k agents. Such sets are modified in the second step so that a matching for those agents becomes possible. The third step is to randomly match the agents in the divided groups accordingly. The random matching as constructed is shown to be an independent directed random matching with the given parameters in the final step.

Step 1: For each $k \in S$, let $\eta_k = 1 - \sum_{r \in S} q_{kr}$ (the no-matching probability for a type-k agent), and $I_k = \{i \in I : \alpha^0(i) = k, \pi^0(i) = i\}$ (the set of type-k agents who are initially unmatched). For each agent $i \in I_k$, define a probability ζ_i on $S \cup \{J\}$ such that $\zeta_i(l) = q_{kl}$ for $l \in S$ and $\zeta_i(J) = \eta_k$. For each agent $i \in I$ such that $\pi^0(i) \neq i$, define a probability ζ_i on $S \cup \{J\}$ such that $\zeta_i(l) = \delta_J(l)$ for $l \in S \cup \{J\}$, where $\delta_J(l)$ is 1 if $l = J$ and zero otherwise. Let $\Omega_0 = (S \cup \{J\})^I$ be the internal set of all the internal functions from I to $S \cup \{J\}$, and μ_0 the internal product probability measure $\Pi_{i \in I} \zeta_i$ on $(\Omega_0, \mathcal{A}_0)$, where $\mathcal{A}_0$ is the internal power set of Ω_0. For each fixed sample realization $\omega_0 \in \Omega_0$, $k, l \in S$, the agents in the set $\bar{A}^{\omega_0}_{kl} = \{i \in I_k : \omega_0(i) = l\}$ are intended to be matched with agents in the set $\bar{A}^{\omega_0}_{lk}$.

Step 2: The point is that one may not be able to produce an internal partial matching so that agents in $\bar{A}^{\omega_0}_{kl}$ are matched with agents in $\bar{A}^{\omega_0}_{lk}$, since $\bar{A}^{\omega_0}_{kl}$ and $\bar{A}^{\omega_0}_{lk}$ may not have the same internal cardinality when $k \neq l$, and $\bar{A}^{\omega_0}_{kk}$ may not have even internal cardinality to allow an internal full matching on $\bar{A}^{\omega_0}_{kk}$. Thus, we need to modify those sets. For $k, l \in S$ with $k \neq l$, let

$$C^{\omega_0}_{kl} = \{A_{kl} : A_{kl} \subseteq \bar{A}^{\omega_0}_{kl}, A_{kl} \text{ is internal and } |A_{kl}| = \min\{|\bar{A}^{\omega_0}_{kl}|, |\bar{A}^{\omega_0}_{lk}|\}\}.$$

It is clear that $C^{\omega_0}_{kl} \neq \emptyset$. For any $k \in S$, let $C^{\omega_0}_{kk}$ be the set of all those sets in the form $\bar{A}^{\omega_0}_{kk} \backslash \{i\}$ for $i \in \bar{A}^{\omega_0}_{kk}$ if $|\bar{A}^{\omega_0}_{kk}|$ is odd, and $C^{\omega_0}_{kk}$ the set with one element $\bar{A}^{\omega_0}_{kk}$ if $|\bar{A}^{\omega_0}_{kk}|$ is even. Denote the product space $\prod_{k,l \in S} C^{\omega_0}_{kl}$ by C^{ω_0}. Define an internal probability measure μ^{ω_0} on C^{ω_0} with its internal power set by letting[48] $\mu^{\omega_0}(\mathbf{A}) = \frac{1}{|C^{\omega_0}|}$ for $\mathbf{A} \in C^{\omega_0}$. The purpose of introducing the space C^{ω_0} and the internal probability measure μ^{ω_0} is to randomly remove some agents from the sets $\bar{A}^{\omega_0}_{kl}$ to obtain the modified sets $A^{\omega_0}_{kl}$ with the desired properties for an internal matching. Let

$$\Omega_1 = \{(A_{kl})_{k,l \in S} : A_{kl} \subseteq I \text{ and } A_{kl} \text{ is internal, where } k, l \in S\}.$$

The probability measure μ^{ω_0} can be trivially extended to the common sample space Ω_1 with its internal power set by letting $\mu^{\omega_0}(\mathbf{A}) = 0$ for $\mathbf{A} \in \Omega_1 \setminus C^{\omega_0}$.

Given the hyperfinite internal probability space $(\Omega_0, \mathcal{A}_0, \mu_0)$ and internal transition probability $\mu^{\omega_0}, \omega_0 \in \Omega_0$, we can define internal probability measure μ_1 on $\Omega_0 \times \Omega_1$ with its internal power set by letting $\mu_1(\omega_0, \mathbf{A}) = \mu_0(\omega_0) \times \mu^{\omega_0}(\mathbf{A})$ for any $\omega_0 \in \Omega_0$ and $\mathbf{A} \in \Omega_1$.

Step 3: For any fixed $\omega_0 \in \Omega_0$ and $\mathbf{A}^{\omega_0} = (A_{kl})_{k,l \in S} \in C^{\omega_0}$, we consider internal partial matchings on I that match agents from A_{kl} to A_{lk}. We only need to consider those sets A_{kl} which are nonempty. Let $B^{\omega_0}_k = I_k \backslash (\bigcup_{l \in S} A^{\omega_0}_{kl})$, which is the set of initially unmatched agents who remain unmatched. Let $\bar{B}^{\omega_0}_k$ denote the set $\{i \in I_k : \omega_0(i) = J\}$; then it is clear that $B^{\omega_0}_k = \bar{B}^{\omega_0}_k \cup \bigcup_{l \in S} \left(\bar{A}^{\omega_0}_{kl} \setminus A^{\omega_0}_{kl}\right)$. Let $B^{\omega_0} = \cup_{k=1}^{K} B^{\omega_0}_k$. For each $k \in S$, let $\Omega^{\omega_0, \mathbf{A}^{\omega_0}}_{kk}$ be the internal set of all the internal full matchings on $A^{\omega_0}_{kk}$. Let $\mu^{\omega_0, \mathbf{A}^{\omega_0}}_{kk}$ be the internal counting probability measure on $\Omega^{\omega_0, \mathbf{A}^{\omega_0}}_{kk}$. For $k, l \in S$ with $k < l$, let $\Omega^{\omega_0, A^{\omega_0}}_{kl}$ be the internal set of all the internal bijections from $A^{\omega_0}_{kl}$ to $A^{\omega_0}_{lk}$. Let $\mu^{\omega_0, \mathbf{A}^{\omega_0}}_{kl}$ be the internal counting probability on $A^{\omega_0}_{kl}$. Let Ω_2 be the internal set of all the internal partial matchings from I to I. Define $\Omega^{\omega_0, \mathbf{A}^{\omega_0}}_2$ to be the set of $\phi \in \Omega_2$, with

(i) the restriction $\phi|_H = \pi^0|_H$, where H is the set $\{i : \pi^0(i) \neq i\}$ of initially matched agents.

(ii) $\{i \in I_k : \phi(i) = i\} = B^{\omega_0}_k$ for each $k \in S$.

(iii) the restriction $\phi|_{A^{\omega_0}_{kk}} \in \Omega^{\omega_0, \mathbf{A}^{\omega_0}}_{kk}$ for $k \in S$.

[48] We shall also use an element d of a set D to represent the singleton set $\{d\}$. Here $\mu^{\omega_0}(\mathbf{A})$ actually means $\mu^{\omega_0}(\{\mathbf{A}\})$.

(iv) for $k,l \in S$ with $k < l$, $\phi|_{A_{kl}^{\omega_0}} \in \Omega_{kl}^{\omega_0,\mathbf{A}^{\omega_0}}$.

Property (i) means that initially matched agents remain matched with the same partners. The rest is clear.

Define an internal probability measure $\mu_2^{\omega_0,\mathbf{A}^{\omega_0}}$ on Ω_2 such that

(i) for $\phi \in \Omega_2^{\omega_0,\mathbf{A}^{\omega_0}}$,

$$\mu_2^{\omega_0,\mathbf{A}^{\omega_0}}(\phi) = \prod_{1 \le k \le l \le K, A_{kl}^{\omega_0} \neq \emptyset} \mu_{kl}^{\omega_0,\mathbf{A}^{\omega_0}}(\phi|_{A_{kl}^{\omega_0}}).$$

(ii) $\phi \notin \Omega_2^{\omega_0,\mathbf{A}^{\omega_0}}$, $\mu_2^{\omega_0,\mathbf{A}^{\omega_0}}(\phi) = 0$.

The purpose of introducing the space $\Omega_2^{\omega_0,\mathbf{A}^{\omega_0}}$ and the internal probability measure $\mu_2^{\omega_0,\mathbf{A}^{\omega_0}}$ is to match the agents in A_{kl} to the agents A_{lk} randomly. The probability measure $\mu_2^{\omega_0,\mathbf{A}^{\omega_0}}$ is trivially extended to the common sample space Ω_2.

Define an internal probability measure P_0 on $\Omega = \Omega_0 \times \Omega_1 \times \Omega_2$ with the internal power set $\mathcal{F}_0$ by letting

$$P_0\left((\omega_0, \mathbf{A}, \omega_2)\right) = \begin{cases} \mu_1(\omega_0, \mathbf{A}) \times \mu_2^{\omega_0,\mathbf{A}}(\omega_2) & \text{if } \mathbf{A} \in C^{\omega_0} \\ 0 & \text{otherwise.} \end{cases}$$

For $(i,\omega) \in I \times \Omega$, let $\pi(i,(\omega_0,\mathbf{A},\omega_2)) = \omega_2(i)$ and

$$g(i,\omega) = \begin{cases} \alpha^0(\pi(i,\omega)) & \text{if } \pi(i,\omega) \neq i \\ J & \text{if } \pi(i,\omega) = i. \end{cases}$$

Denote the corresponding Loeb probability spaces of the internal probability spaces $(\Omega, \mathcal{F}_0, P_0)$ and $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes P_0)$ respectively by $(\Omega, \mathcal{F}, P)$ and $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Since π is an internal function from $I \times \Omega$ to I, it is $\mathcal{I} \boxtimes \mathcal{F}$-measurable.

Denote the internal set $\left\{(\omega_0, \mathbf{A}, \omega_2) \in \Omega : \omega_0 \in \Omega_0, \mathbf{A} \in C^{\omega_0},\ \omega_2 \in \Omega_2^{\omega_0,\mathbf{A}}\right\}$ by $\hat{\Omega}$. By the construction of P_0, it is clear that $P_0\left(\hat{\Omega}\right) = 1$. By its construction, it is clear that π is an internal random matching and satisfies part (i) of the lemma.

Step 4: It remains to prove parts (ii) and (iii) of the lemma. Define an internal process f from $I \times \Omega$ to $S \cup \{J\}$ such that for any $(i,\omega) \in I \times \Omega$,

$$f(i,\omega) = \begin{cases} \omega_0(i) & \text{if } \pi^0(i) = i \\ \alpha^0(\pi^0(i)) & \text{if } \pi^0(i) \neq i. \end{cases}$$

It is clear that if $\alpha^0(i) = k$ and $\pi^0(i) = i$, then

$$P(f_i = l) \simeq P_0(f_i = l) = \mu_0(\omega_0(i) = l) = \zeta_i(l) = q_{kl},$$

which means[49] that $P(f_i = l) = {}^\circ q_{kl}$. Similarly, we have $P(f_i = J) = {}^\circ\eta_k$. It is also obvious that for $i \neq j$ in I, f_i and f_j are independent random variables on the sample space $(\Omega, \mathcal{F}, P)$. The exact law of large number as in Lemma 1 implies that for P-almost all $\omega = (\omega_0, \mathbf{A}, \omega_2) \in \Omega$, $\lambda(\{\alpha^0(i) = k, \pi^0(i) = i, \omega_0(i) = l\}) = {}^\circ\hat{\rho}_{kJ} \cdot {}^\circ q_{kl}$ holds for any $k, l \in S$, and $\lambda(\{\alpha^0(i) = k, \pi^0(i) = i, \omega_0(i) = J\}) = {}^\circ\hat{\rho}_{kJ} \cdot {}^\circ\eta_k$, which means that

$$\frac{|\bar{A}^{\omega_0}_{kl}|}{\hat{M}} \simeq \hat{\rho}_{kJ} q_{kl} \simeq \hat{\rho}_{lJ} q_{lk} \simeq \frac{|\bar{A}^{\omega_0}_{lk}|}{\hat{M}} \text{ and } \frac{|\bar{B}^{\omega_0}_{k}|}{\hat{M}} \simeq \hat{\rho}_{kJ}\eta_k. \tag{44}$$

Let $\tilde{\Omega}$ be the set of $\omega = (\omega_0, \mathbf{A}, \omega_2) \in \Omega$ such that Equation (44) holds. Then, $P\left(\tilde{\Omega}\right) = 1$, and hence $P\left(\hat{\Omega} \cap \tilde{\Omega}\right) = 1$.

Fix any $\omega = (\omega_0, \mathbf{A}, \omega_2) \in \hat{\Omega} \cap \tilde{\Omega}$; then $\mathbf{A} = \mathbf{A}^{\omega_0}$ for some $\mathbf{A}^{\omega_0} \in C^{\omega_0}$ and $\omega_2 \in \Omega_2^{\omega_0, \mathbf{A}^{\omega_0}}$. For any $k \neq l \in S$, we have

$$\frac{|A^{\omega_0}_{kl}|}{\hat{M}} = \min\left(\frac{|\bar{A}^{\omega_0}_{kl}|}{\hat{M}}, \frac{|\bar{A}^{\omega_0}_{lk}|}{\hat{M}}\right) \simeq \hat{\rho}_{kJ} q_{kl} \simeq \frac{|\bar{A}^{\omega_0}_{kl}|}{\hat{M}} \text{ and } \frac{|A^{\omega_0}_{kk}|}{\hat{M}} \simeq \frac{|\bar{A}^{\omega_0}_{kk}|}{\hat{M}} \simeq \hat{\rho}_{kJ} q_{kk}, \tag{45}$$

which also implies that

$$\frac{|B^{\omega_0}_{k}|}{\hat{M}} \simeq \hat{\rho}_{kJ}\eta_k \simeq \frac{|\bar{B}^{\omega_0}_{k}|}{\hat{M}}.$$

For any $i \in I_k$, $i \in A^{\omega_0}_{kl}$ if and only if $\pi(\omega_0, \mathbf{A}^{\omega_0}, \omega_2) = \omega_2(i) \in A^{\omega_0}_{lk}$; and $i \in B^{\omega_0}_k$ if and only if $\pi(\omega_0, \mathbf{A}^{\omega_0}, \omega_2) = \omega_2(i) = J$. Hence, for the fixed $\omega = (\omega_0, \mathbf{A}^{\omega_0}, \omega_2)$, and for any $k, l \in S$, we can obtain that if $i \in A^{\omega_0}_{kl} \subseteq \bar{A}^{\omega_0}_{kl}$, $f(i, \omega) = \omega_0(i) = l = \alpha^0(\omega_2(i)) = g(i, \omega)$; if $i \in \bar{B}^{\omega_0}_k \subseteq B^{\omega_0}_k$, $f(i, \omega) = \omega_0(i) = J = \alpha^0(\omega_2(i)) = g(i, \omega)$. For any $i \in I \backslash (\cup_{k \in S} I_k)$ which means $\pi^0(i) \neq i$, we can obtain that $f(i, \omega) = \alpha^0(\pi^0(i)) = \alpha^0(\pi(i, \omega)) = g(i, \omega)$. It is clear that the set $\{i \in I : f(i, \omega) \neq g(i, \omega)\}$ is a subset of $\bigcup_{l \in S}\left(\bar{A}^{\omega_0}_{kl} \setminus A^{\omega_0}_{kl}\right)$, which has λ-measure zero by (45).

By the fact that $P\left(\hat{\Omega} \cap \tilde{\Omega}\right) = 1$, we know that for P-almost all $\omega \in \Omega$,

$$\lambda\left(i \in I : f(i, \omega) = g(i, \omega)\right) = 1.$$

Since the Loeb product space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a Fubini extension as discussed in Subsection D.4, the Fubini property implies that for λ-almost all $i \in I$, $g(i, \omega)$ is equal to $f(i, \omega)$ for P-almost all $\omega \in \Omega$. Hence g satisfies part (ii) of the lemma. Let $\tilde{I}$ be an $\mathcal{I}$-measurable set with $\lambda(\tilde{I}) = 1$ such that for any $i \in \tilde{I}$, $g_i(\omega) = f_i(\omega)$ for P-almost all $\omega \in \Omega$. Therefore, by the construction of f, we know that the collection of random variables $\{g_i\}_{i \in \tilde{I}}$ is mutually independent in the sense that any finitely many random variables from that collection are mutually independent. This also implies part (iii) of the lemma. ■

Proof of Theorem 1: We follow Lemma 7. Let α^0 be an internal type function from I to S such that[50] $\lambda_0\left(\{\alpha^0(i) = k\}\right) \simeq p_k$ for any $k \in S$. Let $\pi^0(i) = i$ for any $i \in I$. Given that

[49] Recall that for a bounded hyperreal number $x \in {}^*\mathbb{R}$, ${}^\circ x$ is its standard part.

[50] For any given $p \in \Delta$, the atomless property of λ_0 implies the existence of such an α^0.

matching probability function q from $S \times S$ to $\mathbb{R}_+$ with $\sum_{r \in S} q_{kr} \leq 1$ and $p_k q_{kl} = p_l q_{lk}$ for all $k, l \in S$, the condition $\hat{\rho}_{kJ} q_{kl} \simeq \hat{\rho}_{lJ} q_{lk}$ in the statement of Lemma 7 is obviously satisfied. It is clear that the random matching π and the probability measure P constructed in Lemma 7 satisfies all the conditions in Theorem 1. Let α be α^0. Then α and π, which are defined on a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, are a type function and an independent directed random matching with respective parameters p and q. For each $\omega \in \Omega$, since π_ω is an internal bijection on I and λ_0 is the hyperfinite counting probability measure on $\mathcal{I}_0$, it is obvious that π_ω is measure-preserving from the Loeb space $(I, \mathcal{I}, \lambda)$ to itself.

E.2 Proof of Theorem 5

What we need to do is to construct sequences of internal transition probabilities, internal type functions, and internal random matchings. Since we need to consider random mutation, random matching and random type changing with break-up at each time period, three internal measurable spaces with internal transition probabilities will be constructed at each time period. After the construction, we need to check the satisfiability of Markov conditional independence for each step.

In contrast to the settings in Duffie and Sun (2007), the matched agents considered in this paper may form an enduring partnership after the matching step. Matched agents will not participate in the search process until they break up. During their partnership, a matched agent and her partner may change their types with correlation. Therefore, one needs to keep track of those matched agents and their partners at each step, which means to work with extended-type processes that incorporate the types of the agents and their partners in this paper rather than simply the type processes of the agents in Duffie and Sun (2007). This brings substantial difficulties in the construction of the dynamical system and the proof of the property of Markov conditional independence for the extended-type processes.

Let T_0 be the hyperfinite discrete time line $\{n\}_{n=0}^{M}$ and $(I, \mathcal{I}_0, \lambda_0)$ be the agent space, where $I = \{1, \ldots, \hat{M}\}$, $\mathcal{I}_0$ is the internal power set on I, λ_0 is the internal counting probability measure on $\mathcal{I}_0$, M and $\hat{M}$ are unlimited hyperfinite numbers in ${}^*\mathbb{N}_\infty$. We transfer the sequences of numbers $b^n, \theta^n, \sigma^n, \varsigma^n, n \in \mathbb{N}$ to the nonstandard universe to obtain $b^n, \theta^n, \sigma^n, \varsigma^n, n \in {}^*\mathbb{N}$. The transfer of the sequence of functions $q^n, n \in \mathbb{N}$ to the nonstandard universe is denoted by ${}^*q^n, n \in {}^*\mathbb{N}$. Then, for any $k, l \in S$, ${}^*q^n_{kl}$ is an internal function from ${}^*\hat{\Delta}$ to ${}^*[0,1]$. Let $\hat{q}^n_{kl}(\hat{\rho}) = ({}^*q^n_{kl})(\hat{\rho})$ and $\hat{\eta}^n_k = 1 - \sum_{l \in S} \hat{q}^n_{kl}(\hat{\rho})$ for any $k, l \in S$ and $\hat{\rho} \in {}^*\hat{\Delta}$. Note that an object with an upper left star means the transfer of a standard object to the nonstandard universe.

We shall first consider the case of an initial condition Π^0 that is deterministic. Let $\{A_{kl}\}_{(k,l) \in \hat{S}}$ be an internal partition of I such that $\frac{|A_{kl}|}{\hat{M}} \simeq \ddot{p}_{kl}$ for any $k \in S$ and $l \in S \cup \{J\}$, and $|A_{kl}| = |A_{lk}|$ and $|A_{kk}|$ are even for any $k, l \in S$. Let α^0 be an internal function from $(I, \mathcal{I}_0, \lambda_0)$ to S such that $\alpha^0(i) = k$ if $i \in \bigcup_{l \in S \cup \{J\}} A_{kl}$. Let π^0 be an internal partial matching from I to I such that $\pi^0(i) = i$ on $\bigcup_{k \in S} A_{kJ}$, and the restriction $\pi^0|_{A_{kl}}$ is an internal bijection from A_{kl} to A_{lk} for any $k, l \in S$. Let

$$g^0(i) = \begin{cases} \alpha^0(\pi^0(i)) & \text{if } \pi^0(i) \neq i \\ J & \text{if } \pi^0(i) = i. \end{cases}$$

It is clear that $\lambda_0(\{i : \alpha^0(i) = k, g^0(i) = l\}) \simeq \check{p}^0_{kl}$ for any $k \in S$ and $l \in S \cup \{J\}$.

Suppose that the dynamical system $\mathbb{D}$ has been constructed up to time period $n - 1 \in {}^*\mathbb{N}$. That is, $\{(\Omega_m, \mathcal{F}_m, Q_m)\}_{m=1}^{3n-3}$ and $\{\alpha^l, \pi^l\}_{l=0}^{n-1}$ have been constructed, where each Ω_m is a hyperfinite internal set with its internal power set $\mathcal{F}_m$, Q_m an internal transition probability from Ω^{m-1} to $(\Omega_m, \mathcal{F}_m)$, α^l an internal type function from $I \times \Omega^{3l-1}$ to the type space S, and π^l an internal random matching[51] from $I \times \Omega^{3l}$ to I. Here, $\Omega^m = \prod_{j=1}^m \Omega_j$, and $\{\omega_j\}_{j=1}^m$ will also be denoted by ω^m when there is no confusion. Denote the internal product transition probability $Q_1 \otimes Q_2 \otimes \cdots \otimes Q_m$ by Q^m, and $\otimes_{j=1}^m \mathcal{F}_j$ by $\mathcal{F}^m$ (which is simply the internal power set on Ω^m). Then, Q^m is the internal product of the internal transition probability Q_m with the internal probability measure Q^{m-1}.

We shall now consider the constructions for time n. We first work with the random mutation step. Let $\Omega_{3n-2} = S^I$ (the space of all internal functions from I to S) with its internal power set $\mathcal{F}_{3n-2}$. For each $i \in I$, $\omega^{3n-3} \in \Omega^{3n-3}$, if $\alpha^{n-1}(i, \omega^{3n-3}) = k$, define a probability measure $\gamma_i^{\omega^{3n-3}}$ on S by letting $\gamma_i^{\omega^{3n-3}}(l) = b^n_{kl}$ for each $l \in S$. Define an internal probability measure $Q_{3n-2}^{\omega^{3n-3}}$ on $(S^I, \mathcal{F}_{3n-2})$ to be the internal product measure $\prod_{i \in I} \gamma_i^{\omega^{3n-3}}$. Let $\bar{\alpha}^n : \left(I \times \prod_{m=1}^{3n-2} \Omega_m\right) \to S$ be such that $\bar{\alpha}^n\left(i, \omega^{3n-2}\right) = \omega_{3n-2}(i)$. Let $\bar{g}^n : \left(I \times \prod_{m=1}^{3n-2} \Omega_m\right) \to S \cup \{J\}$ be such that

$$\bar{g}^n\left(i, \omega^{3n-2}\right) = \begin{cases} \bar{\alpha}^n(\pi^{n-1}(i, \omega^{3n-3}), \omega^{3n-2}) & \text{if } \pi^{n-1}(i, \omega^{3n-3}) \neq i \\ J & \text{if } \pi^{n-1}(i, \omega^{3n-3}) = i. \end{cases}$$

Let $\check{\rho}^n_{\omega^{3n-2}} = \lambda_0\left(\bar{\alpha}^n_{\omega^{3n-2}}, \bar{g}^n_{\omega^{3n-2}}\right)^{-1}$ be the internal cross-sectional extended type distribution after random mutation.

Next, we consider the step of directed random matching. Let $(\Omega_{3n-1}, \mathcal{F}_{3n-1}) = (\bar{\Omega}, \bar{\mathcal{F}})$, where $(\bar{\Omega}, \bar{\mathcal{F}})$ is the measurable space constructed in the proof of Lemma 7. For any given $\omega^{3n-2} \in \Omega^{3n-2}$, the type function is $\bar{\alpha}^n_{\omega^{3n-2}}(\,\cdot\,)$ while the partial matching function is $\pi^{n-1}_{\omega^{3n-3}}(\,\cdot\,)$. We can construct an internal probability measure $Q_{3n-1}^{\omega^{3n-2}} = P_{\bar{\alpha}^n_{\omega^{3n-2}}, \pi^{n-1}_{\omega^{3n-3}}, \hat{q}^n(\check{\rho}^n_{\omega^{3n-2}})}$ and a directed random matching $\pi_{\bar{\alpha}^n_{\omega^{3n-2}}, \pi^{n-1}_{\omega^{3n-3}}, \hat{q}^n(\check{\rho}^n_{\omega^{3n-2}})}$ by Lemma 7. Let $\bar{\pi}^n : \left(I \times \prod_{m=1}^{3n-1} \Omega_m\right) \to I$

[51] To handle the deterministic case at the initial step with $l = 0$ ($3l-1 = -1$ and $3l = 0$), one can let $\Omega^0 = \Omega^{-1}$ be a singleton set.

be such that

$$\bar{\pi}^n\left(i, \omega^{3n-1}\right) = \pi_{\bar{\alpha}^n_{\omega^{3n-2}}, \pi^{n-1}_{\omega^{3n-3}}, \hat{q}^n(\check{\rho}^n_{\omega^{3n-2}})}(i, \omega_{3n-1}),$$

$$\bar{g}^n(i, \omega^{3n-1}) = \begin{cases} \bar{\alpha}^n(\bar{\pi}^n(i, \omega^{3n-1}), \omega^{3n-2}) & \text{if } \bar{\pi}^n(i, \omega^{3n-1}) \neq i \\ J & \text{if } \bar{\pi}^n(i, \omega^{3n-1}) = i. \end{cases}$$

Now, we consider the final step of random type changing with break-up for matched agents. Let $\Omega_{3n} = (S \times \{0,1\})^I$ with its internal power set $\mathcal{F}_{3n}$, where 0 represents "unmatched" and 1 represents "paired"; each point $\omega_{3n} = (\omega^1_{3n}, \omega^2_{3n}) \in \Omega_{3n}$ is an internal function from I to $S \times \{0,1\}$. Define a new type function $\alpha^n : (I \times \Omega^{3n}) \to S$ by letting $\alpha^n(i, \omega^{3n}) = \omega^1_{3n}(i)$. Fix $\omega^{3n-1} \in \Omega^{3n-1}$. For each $i \in I$, (1) if $\bar{\pi}^n(i, \omega^{3n-1}) = i$ (i is not paired after the matching step at time n), let $\tau_i^{\omega^{3n-1}}$ be the probability measure on the type space $S \times \{0,1\}$ that gives probability one to the type $\left(\bar{\alpha}^n(i, \omega^{3n-2}), 0\right)$ and zero for the rest; (2) if $\bar{\pi}^n(i, \omega^{3n-1}) \neq i$ (i is paired after the matching step at time n), $\bar{\alpha}^n(i, \omega^{3n-2}) = k$, $\bar{\pi}^n(i, \omega^{3n-1}) = j$ and $\bar{\alpha}^n(j, \omega^{3n-2}) = l$, define a probability measure $\tau_{ij}^{\omega^{3n-1}}$ on $(S \times \{0,1\}) \times (S \times \{0,1\})$ such that $\tau_{ij}^{\omega^{3n-1}}((k',1),(l',1)) = (1-\theta^n_{kl})\sigma^n_{kl}(k',l')$ and $\tau_{ij}^{\omega^{3n-1}}((k',0),(l',0)) = \theta^n_{kl}\varsigma^n_{kl}(k')\varsigma^n_{lk}(l')$ for $k', l' \in S$, and zero for the rest. Let $A^n_{\omega^{3n-1}} = \{(i,j) \in I \times I : i < j,\ \bar{\pi}^n(i, \omega^{3n-1}) = j\}$ and $B^n_{\omega^{3n-1}} = \{i \in I : \bar{\pi}^n(i, \omega^{3n-1}) = i\}$. Define an internal probability measure $Q_{3n}^{\omega^{3n-1}}$ on $(S \times \{0,1\})^I$ to be the internal product measure

$$\prod_{i \in B^n_{\omega^{3n-1}}} \tau_i^{\omega^{3n-1}} \otimes \prod_{(i,j) \in A^n_{\omega^{3n-1}}} \tau_{ij}^{\omega^{3n-1}}.$$

Let

$$\pi^n(i, \omega^{3n}) = \begin{cases} J & \text{if } \bar{\pi}^n(i, \omega^{3n-1}) = J \text{ or } \omega^2_{3n}(i) = 0 \text{ or } \omega^2_{3n}(\bar{\pi}^n(i, \omega^{3n-1})) = 0 \\ \bar{\pi}^n(i, \omega^{3n-1}) & \text{otherwise.} \end{cases}$$

and

$$g^n(i, \omega^{3n}) = \begin{cases} \alpha^n(\pi^n(i, \omega^{3n}), \omega^{3n}) & \text{if } \pi^n(i, \omega^{3n}) \neq i \\ J & \text{if } \pi^n(i, \omega^{3n}) = i. \end{cases}$$

It is clear that π^n is a random matching and Equation (19) holds.

Keep repeating the construction. We can then construct a hyperfinite sequence of internal transition probabilities $\{(\Omega_m, \mathcal{F}_m, Q_m)\}_{m=1}^{3M}$ and a hyperfinite sequence of internal type functions and internal random matchings $\{(\alpha^n, \pi^n)\}_{n=0}^{M}$.

Let $(I \times \Omega^{3M}, \mathcal{I}_0 \otimes \mathcal{F}^{3M}, \lambda_0 \otimes Q^{3M})$ be the internal product probability space of $(I, \mathcal{I}_0, \lambda_0)$ and $(\Omega^{3M}, \mathcal{F}^{3M}, Q^{3M})$. Denote the Loeb spaces of $(\Omega^{3M}, \mathcal{F}^{3M}, Q^{3M})$ and the internal product $(I \times \Omega^{3M}, \mathcal{I}_0 \otimes \mathcal{F}^{3M}, \lambda_0 \otimes Q^{3M})$ by $(\Omega^{3M}, \mathcal{F}, P)$ and $(I \times \Omega^{3M}, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ respectively. For simplicity, let Ω^{3M} be denoted by Ω, Q^{3M} be denoted by P_0. As discussed in Subsection D.4, the Loeb product space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a Fubini extension.

In the following, we will often work with functions or sets that are measurable in $(\Omega^m, \mathcal{F}^m, Q^m)$ or its Loeb space for some $m \le 3M$, which may be viewed as functions or sets based on $(\Omega^{3M}, \mathcal{F}^{3M}, Q^{3M})$ or its Loeb space by allowing for dummy components for the tail part. We can thus continue to use P to denote the Loeb measure generated by Q^m for convenience. Since all the type functions, random matchings and the partners' type functions are internal in the relevant hyperfinite settings, they are all $\mathcal{I} \boxtimes \mathcal{F}$-measurable when viewed as functions on $I \times \Omega$.

For $n = 0$, the initial independence condition in the definition of Markov conditional independence in Subsection A.2 is trivially satisfied. Suppose that the Markov conditional independence are satisfied up to period $n-1 \in \mathbb{N}$. It remains to check the Markov conditional independence for each step of random mutation, random matching, and match-induced type changes with break-up in period n.

For the mutation step in period n, fix any $(a_1, r_1), (a_2, r_2)$ and $(k_1^t, l_1^t), (k_2^t, l_2^t)$, $t = 1, \ldots, n-1$ in $\hat{S}$. For any agents i and j with $i \neq j$, we can obtain that

$$
\begin{aligned}
P & \left(\bar{\beta}_i^n = (a_1, r_1), \bar{\beta}_j^n = (a_2, r_2), \beta_i^t = (k_1^t, l_1^t), \beta_j^t = (k_2^t, l_2^t), t = 1, \ldots, n-1\right) \\
& \simeq \int_{D_{ij}^{3n-3}} Q_{3n-2}^{\omega^{3n-3}} \left(\bar{\beta}^n(i, \omega^{3n-2}) = (a_1, r_1), \bar{\beta}^n(j, \omega^{3n-2}) = (a_2, r_2)\right) dQ^{3n-3}(\omega^{3n-3}) \\
& = \int_{\underline{D}_{ij}^{3n-3}} Q_{3n-2}^{\omega^{3n-3}} \left(\bar{\beta}^n(i, \omega^{3n-2}) = (a_1, r_1), \bar{\beta}^n(j, \omega^{3n-2}) = (a_2, r_2)\right) dQ^{3n-3}(\omega^{3n-3}) \\
& \quad + \int_{\overline{D}_{ij}^{3n-3}} Q_{3n-2}^{\omega^{3n-3}} \left(\bar{\beta}^n(i, \omega^{3n-2}) = (a_1, r_1), \bar{\beta}^n(j, \omega^{3n-2}) = (a_2, r_2)\right) dQ^{3n-3}(\omega^{3n-3}),
\end{aligned}
$$

where

$$
D_{ij}^{3n-3} = \{\omega^{3n-3} : \beta^t(i, \omega^{3t}) = (k_1^t, l_1^t), \beta^t(j, \omega^{3t}) = (k_2^t, l_2^t), t = 1, \ldots, n-1\}
$$

$$
\underline{D}_{ij}^{3n-3} = \{\omega^{3n-3} : \pi^{n-1}(i, \omega^{3n-3}) \neq j, \beta^t(i, \omega^{3t}) = (k_1^t, l_1^t), \beta^t(j, \omega^{3t}) = (k_2^t, l_2^t), t = 1, \ldots, n-1\}
$$

$$
\overline{D}_{ij}^{3n-3} = \{\omega^{3n-3} : \pi^{n-1}(i, \omega^{3n-3}) = j, \beta^t(i, \omega^{3t}) = (k_1^t, l_1^t), \beta^t(j, \omega^{3t}) = (k_2^t, l_2^t), t = 1, \ldots, n-1\}.
$$

Fix any agent $i \in I$. It is clear that $\overline{D}_{ij}^{3n-3} \cap \overline{D}_{ij'}^{3n-3} = \emptyset$ for different j and j'. Then there are at most countably many $j \in I$ such that $P(\overline{D}_{ij}^{3n-3}) > 0$. Let $F_i^{3n-3} = \{j \in I : j \neq i, P(\overline{D}_{ij}^{3n-3}) = 0\}$; then $\lambda(F_i^{3n-3}) = 1$. Fix any $j \in F_i^{3n-3}$. The probability for agents i and j to be partners is zero at the end of period $n-1$. When agents i and j are not partners, their random extended types will be independent by the construction of $Q^{\omega^{3n-3}}_{3n-2}$. Hence, we can obtain that

$$
\begin{aligned}
P & \left(\bar\beta^n_i = (a_1, r_1), \bar\beta^n_j = (a_2, r_2), \beta^t_i = (k^t_1, l^t_1), \beta^t_j = (k^t_2, l^t_2), t = 1, \ldots, n-1\right) \\
& \simeq \int_{\underline{D}^{3n-3}_{ij}} Q^{\omega^{3n-3}}_{3n-2} \left(\bar\beta^n(i, \omega^{3n-2}) = (a_1, r_1), \bar\beta^n(j, \omega^{3n-2}) = (a_2, r_2)\right) dQ^{3n-3}(\omega^{3n-3}) \\
& = \int_{\underline{D}^{3n-3}_{ij}} Q^{\omega^{3n-3}}_{3n-2} \left(\bar\beta^n(i, \omega^{3n-2}) = (a_1, r_1)\right) Q^{\omega^{3n-3}}_{3n-2} \left(\bar\beta^n(j, \omega^{3n-2}) = (a_2, r_2)\right) dQ^{3n-3}(\omega^{3n-3}) \\
& = \int_{\underline{D}^{3n-3}_{ij}} B^{3n-2}_{k^{n-1}_1 l^{n-1}_1}(a_1, r_1) B^{3n-2}_{k^{n-1}_2 l^{n-1}_2}(a_2, r_2) dQ^{3n-3}(\omega^{3n-3}) \\
& \simeq P(D^{3n-3}_{ij}) B^{3n-2}_{k^{n-1}_1 l^{n-1}_1}(a_1, r_1) B^{3n-2}_{k^{n-1}_2 l^{n-1}_2}(a_2, r_2),
\end{aligned}
$$

where

$$
B^{3n-2}_{kl}(r, s) = \begin{cases} b^n_{kr} b^n_{ls} & \text{if } l, s \in S \\ b^n_{kr} & \text{if } l = s = J \\ 0 & \text{otherwise.} \end{cases}
$$

Thus, for λ-almost all agent $j \in I$,

$$
\begin{aligned}
P & \left(\bar\beta^n_i = (a_1, r_1), \bar\beta^n_j = (a_2, r_2) \,|\, \beta^t_i = (k^t_1, l^t_1), \beta^t_j = (k^t_2, l^t_2), t = 1, \ldots, n-1\right) \\
& = B^{3n-2}_{k^{n-1}_1 l^{n-1}_1}(a_1, r_1) B^{3n-2}_{k^{n-1}_2 l^{n-1}_2}(a_2, r_2).
\end{aligned} \tag{46}
$$

Note that for any $i \in I$,

$$
\begin{aligned}
P(\bar\beta^n_i = (a_1, r_1), \beta^{n-1}_i = (k^{n-1}_1, l^{n-1}_1)) & \simeq \int_{E^{3n-3}_i} Q^{\omega^{3n-3}}_{3n-2} \left(\bar\beta^n(i, \omega^{3n-2}) = (a_1, r_1)\right) dQ^{3n-3}(\omega^{3n-3}) \\
= \int_{E^{3n-3}_i} B^{3n-2}_{k^{n-1}_1 l^{n-1}_1}(a_1, r_1) dQ^{3n-3}(\omega^{3n-3}) & \simeq P(E^{3n-3}_i) B^{3n-2}_{k^{n-1}_1 l^{n-1}_1}(a_1, r_1),
\end{aligned}
$$

where $E^{3n-3}_i = \{\omega^{3n-3} : \beta^{n-1}(i, \omega^{3n-3}) = (k^{n-1}_1, l^{n-1}_1)\}$. Then, we have

$$
P(\bar\beta^n_i = (a_1, r_1) \,|\, \beta^{n-1}_i = (k^{n-1}_1, l^{n-1}_1)) = B^{3n-2}_{k^{n-1}_1 l^{n-1}_1}(a_1, r_1). \tag{47}
$$

Hence, Equations (13) and (14) in the definition of dynamical system are satisfied. By Equation (46), we can obtain for each $i \in I$, and for λ-almost all $j \in I$,

$$
\begin{aligned}
P & \left(\bar\beta^n_i = (a_1, r_1), \bar\beta^n_j = (a_2, r_2) \,|\, \beta^t_i = (k^t_1, l^t_1), \beta^t_j = (k^t_2, l^t_2), t = 1, \ldots, n-1\right) \\
& = P\left(\bar\beta^n_i = (a_1, r_1) \,|\, \beta^{n-1}_i = (k^{n-1}_1, l^{n-1}_1)\right) P\left(\bar\beta^n_j = (a_2, r_2) \,|\, \beta^{n-1}_j = (k^{n-1}_2, l^{n-1}_2)\right).
\end{aligned} \tag{48}
$$

Hence, Equation (23) in the definition of Markov conditional independence is satisfied.

For the random matching step in period n, fix any $(a_1, r_1), (a_2, r_2)$ in $S \times S$ and any $(k^t_1, l^t_1), (k^t_2, l^t_2)$ in $\hat S$ for $t = 1, \ldots, n-1$. Fix any $\omega^{3n-2} \in \Omega^{3n-2}$. Let $A^{\omega^{3n-3}} = \{i \in I : \pi^{n-1}_{\omega^{3n-3}}(i) \neq i\}$. By Lemma 1 (i), we know that

$$
Q^{\omega^{3n-2}}_{3n-1} \left(\omega_{3n-1} \in \Omega_{3n-1} : \bar\pi^n\left(i, (\omega^{3n-2}, \omega_{3n-1})\right) = \pi^{n-1}\left(i, \omega^{3n-3}\right) \text{ for any } i \in A^{\omega^{3n-3}}\right) = 1,
$$

which implies that Equation (15) holds.

Lemma 2 and Equation (48) imply that the extended type process $\bar{\beta}^n$ is essentially pairwise independent. It follows from the exact law of large numbers in Lemma 1 that for P-almost all $\omega^{3n-2} \in \Omega^{3n-2}$,

$$\check{\rho}^n(\omega^{3n-2}) \simeq \check{p}^n(\omega^{3n-2}) = \lambda \left(\bar{\beta}^n_{\omega^{3n-2}}\right)^{-1} = \mathbb{E}\left(\check{p}^n\left(\omega^{3n-2}\right)\right) = \tilde{p}^n \simeq \mathbb{E}\left(\check{\rho}^n\right). \tag{49}$$

Then Equation (17) is equivalent to

$$P(\bar{\bar{g}}^n_i = l \,|\, \bar{\alpha}^n_i = k, \bar{g}^n_i = J) = q^n_{kl}\left(\tilde{p}^n\right).$$

Since paired agents do not match in this step, their extended types will not change. Thus, to verify Equation (24), we only need to prove, for the event

$$A_n = \{\bar{\beta}^n_i = (a_1, J), \bar{\beta}^n_j = (a_2, J),\, \beta^t_i = (k^t_1, l^t_1),\, \beta^t_j = (k^t_2, l^t_2), t = 1, \ldots, n-1\},$$

that

$$\begin{aligned} &P\Big(\bar{\bar{\beta}}^n_i = (a_1, r_1), \bar{\bar{\beta}}^n_j = (a_2, r_2) \,|\, A_n\Big) \\ &\quad = P\left(\bar{\bar{\beta}}^n_i = (a_1, r_1) \,|\, \bar{\beta}^n_i = (a_1, J)\right) P\left(\bar{\bar{\beta}}^n_j = (a_2, r_2) \,|\, \bar{\beta}^n_j = (a_2, J)\right). \end{aligned}$$

Fix any $k \in S$. If $\tilde{p}^n_{kJ} = \int_I P(\bar{\beta}^n_i = (k, J)) d\lambda(i) = 0$, then $P(\bar{\beta}^n_i = (k, J)) = 0$ for λ-almost all agent $i \in I$, which means that Equation (24) automatically holds. It follows from the continuity requirement above Equation (49) that

$$\check{\rho}^n_{a_1 J} \hat{q}^n_{a_1 r_1}(\check{\rho}^n) \simeq \tilde{p}^n_{a_1 J} q^n_{a_1 r_1}\left(\tilde{p}^n\right)$$

for P-almost all $\omega^{3n-2} \in \Omega^{3n-2}$. Suppose $\tilde{p}^n_{a_1 J} > 0$ and $\tilde{p}^n_{a_2 J} > 0$. Hence, we can obtain that for P-almost all $\omega^{3n-2} \in \Omega^{3n-2}$, $\hat{q}^n_{a_1 r_1}(\check{\rho}^n) \simeq q^n_{a_1 r_1}\left(\tilde{p}^n\right)$ and $\hat{q}^n_{a_2 r_2}(\check{\rho}^n) \simeq q^n_{a_2 r_2}\left(\tilde{p}^n\right)$.

We can now derive

$$
\begin{aligned}
&\int_I \int_I \Big| P\Big(\bar{\bar{\beta}}_i^n = (a_1, r_1), \bar{\bar{\beta}}_j^n = (a_2, r_2), \bar{\beta}_i^n = (a_1, J), \bar{\beta}_j^n = (a_2, J), \\
&\qquad\qquad \beta_i^t = (k_1^t, l_1^t), \beta_j^t = (k_2^t, l_2^t), t = 1, \dots, n-1\Big) \\
&\quad - q_{a_1 r_1}^n\left(\tilde{p}^n\right) q_{a_2 r_2}^n\left(\tilde{p}^n\right) P(D_{ij}^{3n-2}) \ \Big| \ d\lambda(j) d\lambda(i) \\
&\simeq \int_I \int_I \Big| \int_{D_{ij}^{3n-2}} \Big(Q_{3n-1}^{\omega^{3n-2}}\left(\bar{\bar{g}}^n(i, \omega^{3n-1}) = r_1, \bar{\bar{g}}^n(j, \omega^{3n-1}) = r_2\right) \\
&\quad - \hat{q}_{a_1 r_1}^n\left(\check{\rho}^n(\omega^{3n-2})\right) \hat{q}_{a_2 r_2}^n\left(\check{\rho}^n(\omega^{3n-2})\right) \Big) dQ^{3n-2}(\omega^{3n-2}) \ \Big| \ d\lambda_0(j) d\lambda_0(i) \\
&\le \int_I \int_I \int_{\Omega^{3n-2}} \mathbf{1}_{D_{ij}^{3n-2}}(\omega^{3n-2}) \Big| Q_{3n-1}^{\omega^{3n-2}}\left(\bar{\bar{g}}^n(i, \omega^{3n-1}) = r_1, \bar{\bar{g}}^n(j, \omega^{3n-1}) = r_2\right) \\
&\quad - \hat{q}_{a_1 r_1}^n\left(\check{\rho}^n(\omega^{3n-2})\right) \hat{q}_{a_2 r_2}^n\left(\check{\rho}^n(\omega^{3n-2})\right) \ \Big| \ dQ^{3n-2}(\omega^{3n-2}) \ d\lambda_0(j) \ d\lambda_0(i) \\
&= \int_{\Omega^{3n-2}} \int_I \int_I \mathbf{1}_{D_{ij}^{3n-2}}(\omega^{3n-2}) \Big| Q_{3n-1}^{\omega^{3n-2}}\left(\bar{\bar{g}}^n(i, \omega^{3n-1}) = r_1, \bar{\bar{g}}^n(j, \omega^{3n-1}) = r_2\right) \\
&\quad - \hat{q}_{a_1 r_1}^n\left(\check{\rho}^n(\omega^{3n-2})\right) \hat{q}_{a_2 r_2}^n\left(\check{\rho}^n(\omega^{3n-2})\right) \ \Big| \ d\lambda_0(j)\, d\lambda_0(i)\, dQ^{3n-2}(\omega^{3n-2}),
\end{aligned}
\tag{50}
$$

where

$$
\begin{aligned}
D^{3n-2} = \{(\omega^{3n-2}, i, j) : \quad & \bar{\beta}^n(i, \omega^{3n-2}) = (a_1, J), \bar{\beta}^n(j, \omega^{3n-2}) = (a_2, J), \\
& \beta^t(i, \omega^{3t}) = (k_1^t, l_1^t), \beta^t(j, \omega^{3t}) = (k_2^t, l_2^t), t = 1, \dots, n-1\},
\end{aligned}
$$

D_{ij}^{3n-2} is the (i,j)-section of D^{3n-2}, and $\mathbf{1}_{D_{ij}^{3n-2}}$ is the indicator function of the set $\mathbf{1}_{D_{ij}^{3n-2}}$ in Ω^{3n-2}. By Lemma 7 (iii), it is clear that for λ-almost all $i \in I$, for λ-almost all $j \in I$, and for any $\omega^{3n-2} \in D_{ij}^{3n-2}$, we have

$$
Q_{3n-1}^{\omega^{3n-2}}\left(\bar{\bar{g}}^n(i, \omega^{3n-1}) = r_1, \bar{\bar{g}}^n(j, \omega^{3n-1}) = r_2\right) \simeq \hat{q}_{a_1 r_1}^n\left(\check{\rho}^n(\omega^{3n-2})\right) \hat{q}_{a_2 r_2}^n\left(\check{\rho}^n(\omega^{3n-2})\right).
$$

Hence, the last term of Equation (50) is equal to an infinitesimal. Therefore, the first term of Equation (50) is equal to zero, which implies that for λ-almost all $i \in I$

$$
P\Big(\bar{\bar{\beta}}_i^n = (a_1, r_1), \bar{\bar{\beta}}_j^n = (a_2, r_2) \,|\, A_n\Big) = q_{a_1 r_1}^n\left(\tilde{p}^n\right) q_{a_2 r_2}^n\left(\tilde{p}^n\right), \tag{51}
$$

for λ-almost all $j \in I$.

For $i \in I$, let $E_i^{3n-2} = \{\omega^{3n-2} : \bar{\beta}^n(i, \omega^{3n-2}) = (a_1, J)\}$. We can obtain that for λ-almost all $i \in I$, and for any $\omega^{3n-2} \in E_i^{3n-2}$,

$$
P\left(\bar{\bar{\beta}}_i^n = (a_1, r_1), \bar{\beta}_i^n = (a_1, J)\right) \simeq \int_{E_i^{3n-2}} Q_{3n-1}^{\omega^{3n-2}}\left(\bar{\bar{g}}^n(i, \omega^{3n-1}) = r_1\right) dQ^{3n-2}(\omega^{3n-2}),
$$

and $Q_{3n-1}^{\omega^{3n-2}}\left(\bar{\bar{g}}^n(i, \omega^{3n-1}) = r_1\right) \simeq \hat{q}_{a_1 r_1}^n\left(\check{\rho}^n(\omega^{3n-2})\right)$. Hence, we can obtain that for λ-almost all

$i \in I$,

$$
\begin{aligned}
& P\left(\bar{\bar{\beta}}_i^n = (a_1, r_1), \bar{\beta}_i^n = (a_1, J)\right) \\
\simeq\ & \int_{E_i^{3n-2}} Q_{3n-1}^{\omega^{3n-2}}\left(\bar{\bar{g}}^n(i, \omega^{3n-1}) = r_1\right) dQ^{3n-2}(\omega^{3n-2}) \\
\simeq\ & \int_{E_i^{3n-2}} \hat{q}_{a_1 r_1}^n\left(\check{\rho}^n(\omega^{3n-2})\right) dQ^{3n-2}(\omega^{3n-2}) \simeq P(E_i^{3n-2}) q_{a_1 r_1}^n(\tilde{p}^n).
\end{aligned}
$$

Therefore, we have for λ-almost all $i \in I$,

$$
P\left(\bar{\bar{\beta}}_i^n = (a_1, r_1) \,|\, \bar{\beta}_i^n = (a_1, J)\right) = q_{a_1 r_1}^n(\tilde{p}^n). \tag{52}
$$

Since $\check{p}^n(\omega^{3n-2}) \simeq \tilde{p}^n$ for P-almost all $\omega^{3n-2} \in \Omega^{3n-2}$, Equation (52) implies Equation (17). Combining Equations (51) and (52) together, we have

$$
\begin{aligned}
P\Big(\bar{\bar{\beta}}_i^n = (a_1, r_1), \bar{\bar{\beta}}_j^n = (a_2, r_2) \,|\, A_n\Big) &= q_{a_1 r_1}^n(\tilde{p}^n)\, q_{a_2 r_2}^n(\tilde{p}^n) \\
&= P\left(\bar{\bar{\beta}}_i^n = (a_1, r_1) \,|\, \bar{\beta}_i^n = (a_1, J)\right) P\left(\bar{\bar{\beta}}_j^n = (a_2, r_2) \,|\, \bar{\beta}_j^n = (a_2, J)\right).
\end{aligned}
$$

Hence, Equation (24) in the definition of Markov conditional independence is satisfied.

For the step of type changing with break-up in period n, fix any $(a_1, r_1), (a_2, r_2), (x_1, y_1), (x_2, y_2)$, and $(k_1^t, l_1^t), (k_2^t, l_2^t)$, $t = 1, \ldots, n-1$ in $\hat{S}$. For any agents i and j with $i \neq j$, we can obtain that

$$
\begin{aligned}
& P\Big(\beta_i^n = (a_1, r_1), \beta_j^n = (a_2, r_2), \bar{\bar{\beta}}_i^n = (x_1, y_1), \bar{\bar{\beta}}_j^n = (x_2, y_2), \\
& \qquad \beta_i^t = (k_1^t, l_1^t), \beta_j^t = (k_2^t, l_2^t), t = 1, \ldots, n-1\Big) \\
& \simeq \int_{D_{ij}^{3n-1}} Q_{3n}^{\omega^{3n-1}}\left(\beta^n(i, \omega^{3n}) = (a_1, r_1), \beta^n(j, \omega^{3n}) = (a_2, r_2)\right) dQ^{3n-1}(\omega^{3n-1}) \\
& = \int_{\underline{D}_{ij}^{3n-1}} Q_{3n}^{\omega^{3n-1}}\left(\beta^n(i, \omega^{3n}) = (a_1, r_1), \beta^n(j, \omega^{3n}) = (a_2, r_2)\right) dQ^{3n-1}(\omega^{3n-1}) \\
& \quad + \int_{\overline{D}_{ij}^{3n-1}} Q_{3n}^{\omega^{3n-1}}\left(\beta^n(i, \omega^{3n}) = (a_1, r_1), \beta^n(j, \omega^{3n}) = (a_2, r_2)\right) dQ^{3n-1}(\omega^{3n-1}),
\end{aligned}
$$

where

$$
\begin{aligned}
D_{ij}^{3n-1} = \{\omega^{3n-1} : \quad & \bar{\bar{\beta}}_i^n = (x_1, y_1), \bar{\bar{\beta}}_j^n = (x_2, y_2), \\
& \beta^t(i, \omega^{3t}) = (k_1^t, l_1^t), \beta^t(j, \omega^{3t}) = (k_2^t, l_2^t), t = 1, \ldots, n-1\},
\end{aligned}
$$

$$
\begin{aligned}
\underline{D}_{ij}^{3n-1} = \{\omega^{3n-1} : \bar{\pi}^n(i, \omega^{3n-1}) \neq j, \quad & \bar{\bar{\beta}}_i^n = (x_1, y_1), \bar{\bar{\beta}}_j^n = (x_2, y_2), \\
& \beta^t(i, \omega^{3t}) = (k_1^t, l_1^t), \beta^t(j, \omega^{3t}) = (k_2^t, l_2^t), t = 1, \ldots, n-1\},
\end{aligned}
$$

$$\overline{D}_{ij}^{3n-1} = \{\omega^{3n-1} : \bar{\pi}^n(i,\omega^{3n-1}) = j, \quad \bar{\bar{\beta}}_i^n = (x_1,y_1), \bar{\bar{\beta}}_j^n = (x_2,y_2),$$
$$\beta^t(i,\omega^{3t}) = (k_1^t,l_1^t), \beta^t(j,\omega^{3t}) = (k_2^t,l_2^t), t = 1,\ldots,n-1\}.$$

Fix any agent $i \in I$. It is clear that $\overline{D}_{ij}^{3n-1} \cap \overline{D}_{ij'}^{3n-1} = \emptyset$ for different j and j'. Then there are at most countably many $j \in I$ such that $P(\overline{D}_{ij}^{3n-1}) > 0$. Let $F_i^{3n-1} = \{j \in I : j \neq i, P(\overline{D}_{ij}^{3n-1}) = 0\}$; then $\lambda(F_i^{3n-1}) = 1$. Next, fix any $j \in F_i^{3n-1}$. The probability for agents i and j to be partners is zero at the matching step in period n. When agents i and j are not partners, their random extended types will be independent by the construction of $Q_{3n}^{\omega^{3n-1}}$. Hence, we can obtain that

$$\begin{aligned}
&P\Big(\beta_i^n = (a_1,r_1), \beta_j^n = (a_2,r_2), \bar{\bar{\beta}}_i^n = (x_1,y_1), \bar{\bar{\beta}}_j^n = (x_2,y_2),\\
&\qquad\qquad \beta_i^t = (k_1^t,l_1^t), \beta_j^t = (k_2^t,l_2^t), t = 1,\ldots,n-1\Big)\\
&\quad \simeq \int_{\underline{D}_{ij}^{3n-1}} Q_{3n}^{\omega^{3n-1}} \left(\beta^n(i,\omega^{3n}) = (a_1,r_1), \beta^n(j,\omega^{3n}) = (a_2,r_2)\right) dQ^{3n-1}(\omega^{3n-1})\\
&\quad = \int_{\underline{D}_{ij}^{3n-1}} Q_{3n}^{\omega^{3n-1}} \left(\beta^n(i,\omega^{3n}) = (a_1,r_1)\right) Q_{3n}^{\omega^{3n-1}} \left(\beta^n(j,\omega^{3n}) = (a_2,r_2)\right) dQ^{3n-1}(\omega^{3n-1})\\
&\quad = \int_{\underline{D}_{ij}^{3n-1}} B_{x_1y_1}^{3n}(a_1,r_1) B_{x_2y_2}^{3n}(a_2,r_2)\, dQ^{3n-1}(\omega^{3n-1})\\
&\quad \simeq P(D_{ij}^{3n-1}) B_{x_1y_1}^{3n}(a_1,r_1) B_{x_2y_2}^{3n}(a_2,r_2),
\end{aligned}$$

where

$$B_{kl}^{3n}(r,s) = \begin{cases} (1-\theta_{kl}^n)\sigma_{kl}^n(r,s) & \text{if } l,s \in S\\ \theta_{kl}^n \varsigma_{kl}^n(r) & \text{if } l \in S \text{ and } s = J\\ \delta_k(r)\delta_J(s) & \text{if } l = J. \end{cases}$$

Therefore, for any $i \in I$, and for λ-almost all $j \in I$,

$$\begin{aligned}
&P\Big(\beta_i^n = (a_1,r_1), \beta_j^n = (a_2,r_2) \,|\, \bar{\bar{\beta}}_i^n = (x_1,y_1), \bar{\bar{\beta}}_j^n = (x_2,y_2), &(53)\\
&\qquad \beta_i^t = (k_1^t,l_1^t), \beta_j^t = (k_2^t,l_2^t), t = 1,\ldots,n-1\Big)\\
&= B_{x_1y_1}^{3n}(a_1,r_1) B_{x_2y_2}^{3n}(a_2,r_2). &(54)
\end{aligned}$$

Note that for any agent $i \in I$,

$$\begin{aligned}
P(\beta_i^n = (a_1,r_1), \bar{\bar{\beta}}_i^n = (x_1,y_1)) &\simeq \int_{E_i^{3n-1}} Q_{3n}^{\omega^{3n-1}} \left(\beta^n(i,\omega^{3n}) = (a_1,r_1)\right) dQ^{3n-1}(\omega^{3n-1})\\
= \int_{E_i^{3n-1}} B_{x_1y_1}^{3n}(a_1,r_1)\, dQ^{3n-3}(\omega^{3n-3}) &\simeq P(E_j^{3n-1}) B_{x_1y_1}^{3n}(a_1,r_1),
\end{aligned}$$

where $E_i^{3n-1} = \{\omega^{3n-1} : \bar{\bar{\beta}}^n(i,\omega^{3n-1}) = (x_1,y_1)\}$. This implies that

$$P(\beta_i^n = (a_1,r_1) \,|\, \bar{\bar{\beta}}_i^n = (x_1,y_1)) = B_{x_1y_1}^{3n}(a_1,r_1).$$

Hence, Equations (20), (21) and (22) in the definition of the dynamical system $\mathbb{D}$ are satisfied. By Equation (53), we can obtain for each $i \in I$, and for λ-almost all $j \in I$,

$$\begin{aligned}
&P\Big(\beta_i^n = (a_1, r_1), \beta_j^n = (a_2, r_2) \,|\, \bar{\bar{\beta}}_i^n = (x_1, y_1), \bar{\bar{\beta}}_j^n = (x_2, y_2), \\
&\qquad \beta_i^t = (k_1^t, l_1^t), \beta_j^t = (k_2^t, l_2^t), t = 1, \dots, n-1\Big) \\
&= P\left(\beta_i^n = (a_1, r_1) \,|\, \bar{\bar{\beta}}_i^n = (x_1, y_1)\right) P\left(\beta_j^n = (a_2, r_2) \,|\, \bar{\bar{\beta}}_j^n = (x_2, y_2)\right).
\end{aligned}$$

Hence, Equation (25) in the definition of Markov conditional independence is satisfied.

In summary, we have shown the validity of Equations (13) to (22), and (23) to (25). Hence $\mathbb{D}$ is a dynamical system with the Markov conditional independence property, where the initial condition Π^0 is deterministic. Note that, for each $n \in \mathbb{N}$ and $\omega \in \Omega$, since π_ω^n and $\bar{\pi}_\omega^n$ are internal bijections on I, it is obvious that π_ω^n and $\bar{\pi}_\omega^n$ are measure-preserving from the Loeb space $(I, \mathcal{I}, \lambda)$ to itself.

Finally, we consider the case that the initial extended type process β^0 is *i.i.d.* across agents. We shall use the construction for the case of deterministic initial condition. We choose $n = -1$ to be the initial period so that we can have some flexibility in choosing the parameters in period 0. Assume that at $n = -1$, all agents have type 1, and no agents are matched. Namely, the initial type function is $\alpha^{-1} \equiv 1$ while the initial matching is $\pi^{-1} \equiv i$.

Denote $\sum_{r \in S \cup \{J\}} \ddot{p}_{kr}^0$ by $\ddot{p}_k^0$. For the parameters in period 0, let

$$b_{kr}^0 = \begin{cases} \ddot{p}_r^0 & \text{if } k = 1 \\ \delta_k(r) & \text{if } k \neq 1, \end{cases}$$

$$q_{kl}^0(\hat{p}) = \begin{cases} \frac{1}{\hat{p}_{kJ}} \min\left(\frac{\ddot{p}_{kl}^0 \hat{p}_{kJ}}{\ddot{p}_k^0}, \frac{\ddot{p}_{kl}^0 \hat{p}_{lJ}}{\ddot{p}_l^0}\right) & \text{if } \hat{p}_{kJ} \neq 0, \ddot{p}_k^0 \neq 0 \text{ and } \ddot{p}_l^0 \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$\sigma_{kl}^0(k', l') = \delta_k(k')\delta_l(l')$, $\varsigma_{kl}^0(k') = \delta_k(k')$, and $\theta_{kl}^0 = 0$ for any $k, k', l, l' \in S$. Following the construction for the case of deterministic initial condition, there exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ on which is defined a dynamical system $\mathbb{D} = (\Pi^n)_{n=-1}^\infty$ that is Markov conditionally independent with the parameters $(b^n, q^n, \sigma^n, \theta^n)_{n=0}^\infty$.

By Lemma 4, $\tilde{p}_{kl}^0 = \delta_J(l)\ddot{p}_k^0$. It follows from part (2) of Theorem 4 that,

$$z_{(1J)(kl)}^0 = \ddot{p}_k^0 \frac{\ddot{p}_{kl}^0}{\ddot{p}_k^0} = \ddot{p}_{kl}^0,$$

$$z_{(1J)(kJ)}^0 = 1 - \sum_{l \in S} z_{(1J)(kl)}^0 = 1 - \sum_{l \in S} \ddot{p}_{kl}^0 = \ddot{p}_{kJ}^0.$$

Therefore, for λ-almost all $i \in I$,

$$P(\beta_i^0 = (k, l)) = P(\beta_i^0 = (k, l) \,|\, \beta_i^{-1} = (1, J)) P(\beta_i^{-1} = (1, J)) = z_{(1J)(kl)}^0 = \ddot{p}_{kl}^0$$

for any $k \in S$, $l \in S \cup \{J\}$. Part (3) of Theorem 4 implies the essential pairwise independence of β^0. Thus, we can simply start the dynamical system $\mathbb{D}$ from time zero instead of time -1 so that we can have an *i.i.d.* initial extended type process β^0.

E.3 Proofs of Propositions 2 and 5

In this subsection, the unit interval $[0,1]$ will have a different notation in a different context. Recall that $(L, \mathcal{L}, \chi)$ is the Lebesgue unit interval, where χ is the Lebesgue measure defined on the Lebesgue σ-algebra $\mathcal{L}$. We shall prove Proposition 5 first. The proof of Proposition 2 then follows easily.

Note that the agent space used in the proof of Theorem 5 is a hyperfinite Loeb counting probability space. As shown in Proposition 6, a hyperfinite index set of agents has the external cardinality of the continuum. The purpose of Proposition 5 is to show that one can find some extension of the Lebesgue unit interval as the agent space so that the associated version of Theorem 5 still holds.

Fix a Fubini extension $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$ as constructed in the proof of Theorem 5. Following Appendix A of Sun and Zhang (2009) and Appendix B in Duffie and Sun (2012), we can state the following lemma.[52]

Lemma 8 *There exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ such that:*

(1) The agent space $(I, \mathcal{I}, \lambda)$ is an extension of the Lebesgue unit interval $(L, \mathcal{L}, \chi)$.

(2) There exists a surjective mapping φ from I to $\hat{I}$ such that $\varphi^{-1}(\hat{i})$ has the cardinality of the continuum for any $\hat{i} \in \hat{I}$ and φ is measure preserving, in the sense that for any $A \in \hat{\mathcal{I}}$, $\varphi^{-1}(A)$ is measurable in $\mathcal{I}$ with $\lambda[\varphi^{-1}(A)] = \hat{\lambda}(A)$.

(3) Let Φ be the mapping $(\varphi, \mathrm{Id}_\Omega)$ from $I \times \Omega$ to $\hat{I} \times \Omega$, that is, $\Phi(i,\omega) = (\varphi, \mathrm{Id}_\Omega)(i,\omega) = (\varphi(i), \omega)$ for any $(i,\omega) \in I \times \Omega$. Then Φ is measure preserving from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$ in the sense that for any $V \in \hat{\mathcal{I}} \boxtimes \mathcal{F}$, $\Phi^{-1}(V)$ is measurable in $\mathcal{I} \boxtimes \mathcal{F}$ with $(\lambda \boxtimes P)[\Phi^{-1}(V)] = (\hat{\lambda} \boxtimes P)(V)$.

Denote the MCI dynamical system with parameters $(b, q, \sigma, \varsigma, \theta)$ and a deterministic initial condition, as constructed in proof of Theorem 5 by $\hat{\mathbb{D}}$. For that dynamical system, we add a hat to the relevant type processes, matching functions, and partners' type processes. We shall follow the proof of Theorem 4 in Duffie and Sun (2012).

Proof of Proposition 5: Based on the dynamical system $\hat{\mathbb{D}}$ on the Fubini extension $(\hat{I} \times$

[52] Parts (2) and (3) of Lemma 8 are taken from Lemma 11 in Duffie and Sun (2012).

$\Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$, we shall now define, inductively, a new dynamical system $\mathbb{D}$ on the Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

For any $\hat{i}, \hat{i}' \in \hat{I}$ with $\hat{i} \neq \hat{i}'$, let $\Theta^{\hat{i},\hat{i}'}$ be a bijection from $\varphi^{-1}(\hat{i})$ to $\varphi^{-1}(\hat{i}')$, and $\Theta^{\hat{i}',\hat{i}}$ be the inverse mapping of $\Theta^{\hat{i},\hat{i}'}$. This is possible since both $\varphi^{-1}(\hat{i})$ and $\varphi^{-1}(\hat{i}')$ have cardinality of the continuum.

Let α^0 be the mapping $\hat{\alpha}^0(\varphi)$ from I to S,

$$\pi^0(i) = \begin{cases} i & \text{if } \hat{\pi}^0(\varphi(i)) = \varphi(i) \\ \Theta^{\varphi(i),\, \hat{\pi}^0(\varphi(i))}(i) & \text{if } \hat{\pi}^0(\varphi(i)) \neq \varphi(i), \end{cases}$$

and $g^0(i) = \alpha^0\left(\pi^0(i)\right) = \hat{g}^0(\varphi(i))$. By the measure preserving property of φ in Lemma 8, we know that $\beta^0 = (\alpha^0, g^0)$ is $\mathcal{I}$-measurable type function with distribution $\hat{p}^0$ on $S \times (S \cup \{J\})$.

For each time period $n \geq 1$, let $\bar{\alpha}^n$ and α^n be the respective mappings $\hat{\bar{\alpha}}^n(\Phi)$ and $\hat{\alpha}^n(\Phi)$ from $I \times \Omega$ to S. Define mappings $\bar{\pi}^n$, and π^n from $I \times \Omega$ to I such that for each $(i, \omega) \in I \times \Omega$,

$$\bar{\pi}^n(i, \omega) = \begin{cases} i & \text{if } \hat{\bar{\pi}}^n_\omega(\varphi(i)) = \varphi(i) \\ \Theta^{\varphi(i),\, \hat{\bar{\pi}}^n_\omega(\varphi(i))}(i) & \text{if } \hat{\bar{\pi}}^n_\omega(\varphi(i)) \neq \varphi(i) \end{cases}$$

$$\pi^n(i, \omega) = \begin{cases} i & \text{if } \hat{\pi}^n_\omega(\varphi(i)) = \varphi(i) \\ \Theta^{\varphi(i),\, \hat{\pi}^n_\omega(\varphi(i))}(i) & \text{if } \hat{\pi}^n_\omega(\varphi(i)) \neq \varphi(i). \end{cases}$$

When $\pi^n_\omega(\varphi(i)) \neq \varphi(i)$, π^n_ω defines a full matching on $\varphi^{-1}(\hat{H}^n_\omega)$, where $\hat{H}^n_\omega = \hat{I} - \{i : \hat{\pi}_\omega(i)^n = i\}$, which implies that $\pi^n_\omega(i) \neq i$. Hence, π^n is a well-defined mapping from $I \times \Omega$ to I. For the same reason, $\bar{\pi}^n$ is well defined.

Since Φ is measure-preserving and $\hat{\bar{\alpha}}^n$ and $\hat{\alpha}^n$ are measurable mappings from $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$ to S. By the definitions of $\bar{\alpha}^n$ and α^n, it is obvious that for each $i \in I$,

$$\bar{\alpha}^n_i = \hat{\bar{\alpha}}^n_{\varphi(i)} \text{ and } \alpha^n_i = \hat{\alpha}^n_{\varphi(i)}. \tag{55}$$

Next, we consider the property of $\bar{\pi}^n$ and π^n. Fix any $\omega \in \Omega$. Let $H^n_\omega = I - \{i : \pi^n_i = i\}$; then $H^n_\omega = \varphi^{-1}(\hat{H}^n_\omega)$. Pick any $i \in H^n_\omega$ and denote $\pi^n_\omega(i)$ by j. Then, $\varphi(i) \in \hat{H}^n_\omega$. The definition of π^n implies that $j = \Theta^{\varphi(i),\, \hat{\pi}^n_\omega(\varphi(i))}(i)$. Since $\Theta^{\varphi(i),\, \hat{\pi}^n_\omega(\varphi(i))}$ is a bijection between $C_{\varphi(i)}$ and $C_{\hat{\pi}^n_\omega(\varphi(i))}$, it follows that $\varphi(j) = \varphi(\pi^n_\omega(i)) = \hat{\pi}^n_\omega(\varphi(i))$ by the definition of φ. Thus, $j = \Theta^{\varphi(i),\, \varphi(j)}(i)$. Since the inverse of $\Theta^{\varphi(i),\, \varphi(j)}$ is $\Theta^{\varphi(j),\, \varphi(i)}$, we know that $\Theta^{\varphi(j),\, \varphi(i)}(j) = i$. By the full matching property of $\hat{\pi}^n_\omega$, $\varphi(j) \neq \varphi(i)$, $\varphi(j) \in \hat{H}^n_\omega$ and $\hat{\pi}^n_\omega(\varphi(j)) = \varphi(i)$. Hence, we have $j \neq i$, and

$$\pi^n_\omega(j) = \Theta^{\varphi(j),\, \hat{\pi}^n_\omega(\varphi(j))}(j) = \Theta^{\varphi(j),\, \varphi(i)}(j) = i.$$

This means that the composition of π^n_ω with itself on H^n_ω is the identity mapping on H^n_ω, which also implies that π^n_ω is a bijection on H^n_ω. Therefore π^n_ω is a full matching on $H^n_\omega = I - \{i : \pi^n_i = i\}$.

We define $g^n : I \times \Omega \to S \cup \{J\}$ by

$$g^n(i,\omega) = \begin{cases} \alpha^n(\pi^n(i,\omega)) & \text{if } \pi^n(i,\omega) \neq i \\ J & \text{if } \pi^n(i,\omega) = i. \end{cases}$$

As noted in the above paragraph, for any fixed $\omega \in \Omega$, $\varphi(\pi^n_\omega(i)) = \hat{\pi}^n_\omega(\varphi(i))$ for $i \in H^n_\omega$. When $i \notin H^n_\omega$, we have $\varphi(i) \notin \hat{H}^n_\omega$, and $\pi^n_\omega(i) = i$, $\hat{\pi}^n_\omega(\varphi(i)) = \varphi(i)$. Therefore, $\varphi(\pi^n_\omega(i)) = \hat{\pi}^n_\omega(\varphi(i))$ for any $i \in I$. Then, for any (i,ω) such that $\pi^n_i \neq i$,

$$g^n(i,\omega) = \hat{\alpha}^n(\varphi(\pi^n(i,\omega)),\omega) = \hat{\alpha}^n(\hat{\pi}^n(\varphi(i),\omega),\omega) = \hat{g}^n(\varphi(i),\omega) = \hat{g}^n(\Phi)(i,\omega).$$

For any (i,ω) such that $\pi^n_i = i$,

$$g^n(i,\omega) = J = \hat{g}^n(\varphi(i),\omega) = \hat{g}^n(\Phi)(i,\omega).$$

We can prove that $\bar{g}^n(i,\omega) = \hat{\bar{g}}^n(\Phi)(i,\omega)$ and $\bar{\bar{g}}^n(i,\omega) = \hat{\bar{\bar{g}}}^n(\Phi)(i,\omega)$ in the same way. Hence, the measure-preserving property of Φ implies that g^n is $\mathcal{I} \boxtimes \mathcal{F}$-measurable. The previous three identities on the partners' processes also mean that for any $i \in I$,

$$g^n_i(\,\cdot\,) = \hat{g}^n_{\varphi(i)}(\,\cdot\,),\ \bar{g}^n_i(\,\cdot\,) = \hat{\bar{g}}^n_{\varphi(i)}(\,\cdot\,),\ \bar{\bar{g}}^n_i(\,\cdot\,) = \hat{\bar{\bar{g}}}^n_{\varphi(i)}(\,\cdot\,).$$

Since $\bar{\alpha}^n = \hat{\bar{\alpha}}^n(\Phi)$ and $\bar{g}^n = \hat{\bar{g}}^n(\Phi)$, Equation (13) implies that for λ-almost all $i \in I$,

$$\begin{aligned} P(\bar{\alpha}^n_i = k_2, \bar{g}^n_i = l_2 \,|\, \alpha^{n-1}_i = k_1, g^{n-1}_i = l_1) &= P(\hat{\bar{\alpha}}^n_{\varphi(i)} = k_2, \hat{\bar{g}}^n_{\varphi(i)} = l_2 |\, \hat{\alpha}^{n-1}_{\varphi(i)} = k_1, \hat{g}^{n-1}_{\varphi(i)} = l_1) \\ &= b^n_{k_1k_2} b^n_{l_1l_2}. \end{aligned}$$

Similarly, we can obtain that for λ-almost all $i \in I$,

$$P(\bar{\alpha}^n_i = k_2, \bar{g}^n_i = r \,|\, \alpha^{n-1}_i = k_1, g^{n-1}_i = J) = b^n_{k_1k_2}\delta_J(r),$$

$$P(\bar{\bar{g}}^n_i = l \,|\, \bar{\alpha}^n_i = k, \bar{g}^n_i = J, \check{p}^n) = q^n_{kl}(\check{p}^n(\omega)),$$

$$P(\alpha^n_i = l_1, g^n_i = r \,|\, \bar{\alpha}^n_i = k_1, \bar{\bar{g}}^n_i = J) = \delta_{k_1}(l_1)\,\delta_J(r),$$

$$P(\alpha^n_i = l_1, g^n_i = J \,|\, \bar{\alpha}^n_i = k_1, \bar{\bar{g}}^n_i = k_2) = \theta^n_{k_1k_2}\varsigma^n_{k_1k_2}(l_1),$$

$$P(\alpha^n_i = l_1, g^n_i = l_2 \,|\, \bar{\alpha}^n_i = k_1, \bar{\bar{g}}^n_i = k_2) = (1-\theta^n_{k_1k_2})\sigma^n_{k_1k_2}(l_1,l_2).$$

Therefore, $\mathbb{D}$ is a dynamical system with random mutation, directed random matching and type changing with break-up and with the parameters $(p^0, b, q, \sigma, \varsigma, \theta)$.

It remains to check the Markov conditional independence for $\mathbb{D}$. Since the dynamical system $\hat{\mathbb{D}}$ is Markov conditionally independent, for each $n \geq 1$, there is a set $\hat{I}' \in \hat{\mathcal{I}}$ with $\hat{\lambda}(\hat{I}') = 1$, and for each $\hat{i} \in \hat{I}'$, there exists a set $\hat{E}_{\hat{i}} \in \hat{\mathcal{I}}$ with $\hat{\lambda}(\hat{E}_{\hat{i}}) = 1$, with Equations (23) to (25) being satisfied for any $\hat{i} \in \hat{I}'$ and any $\hat{j} \in \hat{E}_{\hat{i}}$. Let $I' = \varphi^{-1}(\hat{I}')$. For any $i \in I'$, let $E_i = \varphi^{-1}(\hat{E}_{\varphi(i)})$. Since φ is measure-preserving, $\lambda(I') = \lambda(E_i) = 1$. Fix any $i \in I'$, and any $j \in E_i$. Denote $\varphi(i)$ by $\hat{i}$ and $\varphi(j)$ by $\hat{j}$. Then, it is obvious that $\hat{i} \in \hat{I}'$ and $\hat{j} \in \hat{E}_{\hat{i}}$. Therefore Equations (23) to (25) are satisfied for any $i' \in I'$ and any $j' \in E_{i'}$. Therefore the dynamical system $\mathbb{D}$ is Markov conditionally independent.

Now, we check the measure-preserving property as stated in Footnote 34.[53] Let $\mathcal{I}' \boxtimes \mathcal{F} = \{B' \subseteq I \times \Omega : B' = \Phi^{-1}(B) \text{ for some } B \in \hat{\mathcal{I}} \boxtimes \mathcal{F}\}$. By Lemma 8, $\mathcal{I}'$ and $\mathcal{I}' \boxtimes \mathcal{F}$ are sub σ-algebras of $\mathcal{I}$ and $\mathcal{I} \boxtimes \mathcal{F}$ respectively. Note that α^n, $\bar{\alpha}^n$, g^n, $\bar{g}^n$ and $\bar{g}^n$ are still measurable on $(I \times \Omega, \mathcal{I}' \boxtimes \mathcal{F}, \lambda \boxtimes P)$, then the dynamical system $\mathbb{D}$ is also Markov conditionally independent on $(I \times \Omega, \mathcal{I}' \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Note that, for each $n \in \mathbb{N}$ and $\omega \in \Omega$, $\hat{\pi}^n_\omega$ and $\hat{\bar{\pi}}^n_\omega$ are measure-preserving from the Loeb space $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$ to itself. Therefore, for any $A \in \mathcal{I}'$,

$$\lambda((\pi^n_\omega)^{-1}(A)) = \hat{\lambda}(\phi((\pi^n_\omega)^{-1}(A))) = \hat{\lambda}((\hat{\pi}^n_\omega)^{-1}(\phi(A))) = \hat{\lambda}(\phi(A)) = \lambda(A),$$

which implies that π^n is measure preserving. We can prove that $\bar{\pi}^n$ is measure preserving on $(I \times \Omega, \mathcal{I}' \boxtimes \mathcal{F}, \lambda \boxtimes P)$ in the same way.

By using exactly the same proof as that given at the end of the proof of Theorem 5, we can have an *i.i.d.* (instead of deterministic) initial extended type process β^0 in the statement of this proposition. ∎

Proof of Proposition 2: In the proof of Proposition 5, take the initial extended type distribution $\hat{p}^0_{kl} = p_k \delta_J(l)$. Assume that there is no genuine random mutation. Then, it is clear that $\tilde{p}^0_{kl} = p_k \delta_J(l)$ for any $k \in S$. Consider the random matching π^1 in period one.

Fix an agent i with $\alpha^0(i) = k$. We have $P(\bar{\alpha}^1_i = k) = 1$, $P\left(\bar{\bar{g}}^1_i = l\right) = q_{kl}$ and $P\left(\bar{\bar{g}}^1_i = J\right) = \eta_k$. Similarly, Equation (24) implies that the process $\bar{\bar{g}}^1$ is essentially pairwise independent. By taking the type function α to be α^0, the matching function π to be $\bar{\pi}^1$, and the associated process g to be $\bar{\bar{g}}^1$, the proposition holds. ∎

[53]The measure-preserving property as stated in Footnotes 16 and 19 can be checked by using the same idea.