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BLACK'S CONSOL RATE CONJECTURE

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This paper confirms a version of a conjecture by Fischer Black regarding consol rate models for the term structure of interest rates. A consol rate model is one in which the stochastic behavior of the short rate is influenced by the consol rate. Since the consol rate is itself determined, via the usual discounted present value formula, by the short rate, such models have an inherent fixed point aspect. Under an equivalent martingale measure, purely technical regularity conditions are given for the stochastic differential equation defining the short rate and the consol rate to be consistent with the definition of the consol rate as the yield on a perpetual annuity. The results are based on an extension of the theory for the forward–backward stochastic differential equations to infinite-horizon settings. Under additional compatibility conditions, we also show that the consol rate is uniquely determined and given as a function of the short rate.

1. Introduction. The main objective of this paper is to confirm and explore a conjecture by Fischer Black regarding consol rate models for the term structure of interest rates. A consol rate model is one in which the stochastic behavior of the short rate is influenced by the consol rate. Since the consol rate is itself determined, via the usual discounted present value formula, by the short rate, such models have an inherent fixed-point aspect. Under purely technical conditions, we show whether or not a stochastic differential equation defining the short rate and the consol rate is consistent with the definition of the consol rate as the yield on a perpetual annuity. The results are based on an extension of the theory for the forward–backward stochastic differential equations to infinite-horizon settings.

We fix a filtered probability space $(\Omega, \mathscr{F}, P; \{\mathscr{F}_t\}_{t \geq 0})$, satisfying the usual conditions. [See, e.g., Protter (1990) for background technical definitions]. A consol is a perpetual annuity, that is, a security paying dividends continually and in perpetuity at a constant rate, which can be taken without loss of generality to be 1. A short rate process is a nonnegative progressively measurable process. We are interested in models of a short rate process r and

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a price process Y for the consol that satisfy the expected discounted value formula

$$(1.1)\qquad Y_t = E\left[\int_t^\infty \exp\left(-\int_t^s r_u\,du\right) ds \,\middle|\, \mathscr{F}_t\right], \qquad t \geq 0,$$

where E denotes expectation under P. Without getting into the associated definitions and related notions of arbitrage, (1.1) is consistent with the role of P as an "equivalent martingale measure," in the sense of Harrison and Kreps (1979), which can be consulted for the associated theory. It is not unusual in applications to work from the beginning under such a probability measure, and we do so. Since the yield on a consol is the reciprocal of its price, and since reciprocation is a smooth mapping from $(0,\infty)$ to $(0,\infty)$ with a smooth inverse, it makes no difference whether we work in terms of the price process Y or its yield process $l = Y^{-1}$, also known as the consol rate process.

Since the work of Brennan and Schwartz (1979), there has been interest in term-structure models based on a stochastic differential equation for the short rate r and the consol rate l. Since $Y = l^{-1}$, we can equally well work in terms of a stochastic differential equation for (r, Y) of the form

$$(1.2)\qquad dr_t = \mu(r_t, Y_t)\,dt + \alpha(r_t, Y_t)\,dW_t,$$

$$(1.3)\qquad dY_t = (r_t Y_t - 1)\,dt + A(r_t, Y_t)\,dW_t,$$

where W is a standard Brownian motion in $\mathbb{R}^2$ and where μ, α, and A are measurable functions on $(0,\infty) \times (0,\infty)$ into $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^2$, respectively, satisfying technical conditions. The drift $r_t Y_t - 1$ shown for Y is implied directly by (1.1) and Itô's formula, and is interpreted as a statement that the expected rate of return on the consol (under the measure P) is the short (riskless) rate r.

Our first objective is to show how to determine whether the diffusion function A on the consol price process is consistent with the characterization (1.1) of the consol price. It is clear that not any choice for A will work. For example, we cannot have $\alpha(r, y)^{\mathrm{T}} A(r, y) = 0$ for all (r, y), unless both α and A are identically zero, since the increments of r and Y must be correlated in a very particular fashion, given the definition (1.1) of Y. Indeed, we want to resolve whether any A can be chosen in a manner consistent with (1.1). Since Y_t depends on the conditional distribution of $\{r_s\colon s \geq t\}$, which in turn depends at least on μ and α (not to mention the dependence of r, through Y, on A itself), we should expect that the diffusion function A for Y depends in a particular way on μ and α.

In a private communication, Black has conjectured that, under at most technical conditions, for any (μ, α) there is always a choice for A that works. We confirm that conjecture. We also provide additional technical conditions under which (1.2) and (1.3) are consistent with (1.1) if and only if $Y_t = \varphi(r_t)$ and (consequently) $A(r_t, Y_t) = \varphi'(r_t)\alpha(r_t, \varphi(r_t))$, where φ is a C^2 solution of the ordinary differential equation

$$\varphi'(x)\mu(x,\varphi(x)) \\ - x\varphi(x) + \tfrac{1}{2}\varphi''(x)\|\alpha(x,\varphi(x))\|^2 + 1 = 0, \qquad x \in (0,\infty). \tag{1.4}$$

Under the same technical conditions, there is a unique solution to this ordinary differential equation (ODE). This result, connecting the ODE (1.4) with solutions to the model (1.1)–(1.3) of short rate and consol rate, is based on recent developments in the theory of forward–backward stochastic differential equations reported in Ma and Yong (1993) and Ma, Protter and Yong (1993).

The conclusion that the consol price Y_t is, under technical regularity conditions, necessarily of the form $\varphi(r_t)$ is somewhat surprising, in that one of the main objectives of the Brenann–Schwartz model is to provide two state variables for the term structure: the short rate r_t and the consol rate l_t. From the above, we have $l_t = 1/\varphi(r_t)$, and the single state variable r_t is therefore sufficient. It may be that the technical regularity conditions that we impose rules out some interesting cases.

An easy illustration is the following mild extension of an example due to Fischer Black, from a private communication:

$$dr_t = \left(k_1 r_t + \frac{k_2}{Y_t}\right) dt + r_t v\, dW_t, \tag{1.5}$$

$$dY_t = (r_t Y_t - 1)\, dt + A(r_t, Y_t)\, dW_t, \tag{1.6}$$

where k_1 and k_2 are constants, W is a standard Brownian in $\mathbb{R}^2$ and $v \in \mathbb{R}^2$. We can think of the function A as unknown and to be determined in terms of the other information. With $\|v\|^2 = k_1 + k_2$, we conjecture that it is possible to take $Y_t = c/r_t \equiv \varphi(r_t)$, for some constant c. Plugging this conjecture for φ into (1.4) uniquely yields $c = 1$ and $A(r, y) = A(r, \varphi(r)) = -v/r$. Is there some other choice for A that works and that perhaps generates a model in which we cannot treat Y_t as $\varphi(r_t)$ for some function φ? We do not know the answer to this question, since (1.5) does not satisfy the regularity conditions for uniqueness of solutions that we provide in this paper. (For example, our regularity implies the property that the short rate r stays in some interval $[\underline{r}, \bar{r}]$, for positive constants $\underline{r}$ and $\bar{r}$.)

More generally, we replace the short rate r in (1.2) with a "state process" X in $\mathbb{R}^n$, and assume that the short rate is given by $r_t = h(X_t)$ for some well behaved function h. We give conditions under which there is a solution for the consol price in the form $Y_t = \theta(X_t)$, where θ solves a partial differential equation analogous to (1.4). In this general case, however, the conditions ensuring the uniqueness of solutions for (X, Y) are given implicitly (see Section 4). For the case $n = 1$, Remark 4.8 treats the case of invertible h, so that (1.1)–(1.3) can be recovered.

Hogan (1993) has shown that special cases of a stochastic differential equation proposed by Brennan and Schwartz (1979) for (r, l) fail to have a finite-valued solution. Our results indicate caution in proposing any particular stochastic differential equation for the short rate and consol rate, whether or not it has a finite solution, if the consol rate is intended to represent the yield on a consol in a manner consistent with the proposed model. The diffusion of the consol must be chosen consistently with the solution of a nontrivial fixed point problem involving the drift and diffusion of the short rate.

One could also apply our results to a general class of financial market valuation problems. Suppose, for instance, that one wants to develop a model in which the stochastic behavior of the state process is influenced in a particular way by certain asset prices, which are in turn determined by the usual discounted expected valuation approach, with future state prices given in terms of future state variables. We show how this can be done consistently, and give a differential equation for the asset price in terms of the state variables. In the infinite-horizon setting, uniqueness is shown under somewhat narrower sufficient conditions.

2. Forward–backward stochastic differential equations, nodal solutions and asset valuation. We first recall some results in Ma and Yong (1993) and Ma, Protter and Yong (1993). Let $(\Omega, \mathscr{F}, P; \{\mathscr{F}_t\})$ be a fixed filtered probability space satisfying the usual conditions. Let W be a d-dimensional Brownian motion defined on this space. We further assume that $\mathscr{F}_t$ is $\sigma\{W_s: 0 \le s \le t\}$, augmented by the P-null sets in $\mathscr{F}$. Consider the following forward–backward stochastic differential equation (SDE) in a finite horizon:

$$
\begin{aligned}
&dX_t = b(X_t, Y_t)\,dt + \sigma(X_t, Y_t)\,dW_t, \qquad t \in [0, T],\\
&dY_t = -\hat{b}(X_t, Y_t)\,dt - Z_t\,dW_t, \qquad t \in [0, T],\\
&X_0 = x, \qquad Y_T = g(X_T),
\end{aligned}
\tag{2.1}
$$

where b, σ, $\hat{b}$ and g are some given functions satisfying certain conditions and $T > 0$. The unknown processes X, Y and Z take values in $\mathbb{R}^n$, $\mathbb{R}^m$ and $\mathbb{R}^{m \times d}$, respectively, and we will always assume that they are $\{\mathscr{F}_t\}$-adapted (see Definition 2.1 below).

We can think of X in (2.1) as a state variable for the problem, which has a given initial condition x and whose dynamics are influenced by a variable Y, which has a terminal value given in terms of X_T. For this reason, one can call X the "forward" variable and Y the "backward" variable, hence the term "forward–backward stochastic differential equation." Provided Z satisfies the usual integrability condition $E[\int_0^T \|Z_t\|^2\,dt] < \infty$, we can also write

$$
Y_t = E\left[g(X_T) + \int_t^T \hat{b}(X_s, Y_s)\,ds \,\middle|\, \mathscr{F}_t\right], \qquad t \in [0, T]. \tag{2.2}
$$

In all of our applications in this paper, Y will turn out to be a vector of prices of certain financial securities, while X is a state variable that affects both the dynamics and the final value of Y. Determining a solution for (X, Y) may be thought of as a fixed point problem. In other applications, such as models of recursive utility, we may think of Y_t as the vector of current utilities of the various economic agents in a given Pareto problem, as in Epstein (1987) (deterministic case) or Duffie, Geoffard and Skiadas (1994) (stochastic case). In that problem, we may think of X as determined by the aggregate consumption process and the vector of utility weights. In general, we use the following definition.

DEFINITION 2.1. A process (X, Y, Z) is called an *adapted solution* of (2.1) if it is $\{\mathscr{F}_t\}$-adapted, square-integrable and satisfies

$$(2.3)\qquad \begin{aligned} X_t &= x + \int_0^t b(X_s, Y_s)\,ds + \int_0^t \sigma(X_s, Y_s)\,dW_s, \\ Y_t &= g(X_T) + \int_t^T \hat{b}(X_s, Y_s)\,ds + \int_t^T Z_s\,dW_s, \qquad t \in [0, T]. \end{aligned}$$

Moreover, if there exists a function θ: $\mathbb{R}^n \times [0, T] \to \mathbb{R}^m$, which is C^1 in t and C^2 in x, such that the adapted solution (X, Y, Z) satisfies

$$(2.4)\quad Y_t = \theta(X_t, t), \qquad Z_t = -\sigma(X_t, \theta(X_t, t))^T \theta_x(X_t, t), \qquad t \geq 0,$$

then (X, Y, Z) is called a *nodal solution* of (2.1).

In Section 3 we give conditions on $(\sigma, b, \hat{b}, g)$ under which there is indeed a nodal solution (X, Y, Z).

We use the term "nodal solution" because θ in some cases turns out to be the "nodal surface" of the viscosity solution to a certain Hamilton–Jacobi–Bellman equation [see Ma and Yong (1993) for details]. From Ma, Protter, and Yong (1993), we know that under certain conditions, any adapted solution of (2.1) must be a nodal solution. Moreover, the nodal solution is unique. As a matter of fact, this nodal solution can be constructed in the following way: First, find the unique classical solution θ of the parabolic system:

$$(2.5)\qquad \begin{aligned} &\theta_t^k + \tfrac{1}{2}\operatorname{tr}\big(\theta_{xx}^k \sigma(x, \theta)\sigma(x, \theta)^T\big) + \big\langle b(x, \theta), \theta_x^k \big\rangle + \hat{b}^k(x, \theta) = 0, \\ &\qquad\qquad\qquad\qquad (t, x) \in (0, T) \times \mathbb{R}^n, 1 \leq k \leq m, \\ &\theta(T, x) = g(x), \qquad x \in \mathbb{R}^n. \end{aligned}$$

Second, solve the forward SDE:

$$(2.6)\qquad \begin{aligned} dX_t &= b(X_t, \theta(X_t, t))\,dt + \sigma(X_t, \theta(X_t, t))\,dW_t, \qquad t \in [0, T], \\ X_0 &= x. \end{aligned}$$

Finally, set Y_t and Z_t as in (2.4). This will give the nodal solution of (2.1). Such a method was called a four step scheme in Ma, Protter and Yong (1993), where more general cases were studied.

In what follows, we restrict ourselves to the case $m = 1$ and

$$(2.7)\qquad \hat{b}(x, y) = 1 - h(x)y, \qquad (x, y) \in \mathbb{R}^n \times \mathbb{R},$$

for some function $h\colon \mathbb{R}^n \to (0,\infty)$. Then, (2.1) becomes

$$\begin{aligned} dX_t &= b(X_t,Y_t)\,dt + \sigma(X_t,Y_t)\,dW_t, \qquad t\in[0,T],\\ dY_t &= (h(X_t)Y_t - 1)\,dt - \langle Z_t, dW_t\rangle, \qquad t\in[0,T],\\ X_0 &= x, \qquad Y_T = g(X_T), \end{aligned} \tag{2.8}$$

By the usual variation of constants formula, we have

$$\begin{aligned} g(X_T) = Y_T &= \exp\left(\int_t^T h(X_u)\,du\right)Y_t - \int_t^T \exp\left(\int_s^T h(X_u)\,du\right)ds \\ &\quad - \int_t^T \exp\left(\int_s^T h(X_u)\,du\right)\langle Z_s, dW_s\rangle, \qquad t\in[0,T]. \end{aligned} \tag{2.9}$$

This implies that

$$\begin{aligned} Y_t &= \exp\left(-\int_t^T h(X_u)\,du\right)g(X_T) + \int_t^T \exp\left(-\int_t^s h(X_u)\,du\right)ds \\ &\quad + \int_t^T \exp\left(-\int_t^s h(X_u)\,du\right)\langle Z_s, dW_s\rangle, \qquad t\in[0,T]. \end{aligned} \tag{2.10}$$

Taking conditional expectations and noting that $\int\langle Z, dW\rangle$ is a martingale (as is the case under the square-integrability condition on Z), we obtain

$$\begin{aligned} Y_t = E\bigg[&\exp\left(-\int_t^T h(X_u)\,du\right)g(X_T) \\ &+ \int_t^T \exp\left(-\int_t^s h(X_u)\,du\right)ds\,\bigg|\,\mathscr{F}_t\bigg], \qquad t\in[0,T]. \end{aligned} \tag{2.11}$$

Hence, it is expected that (2.8) should be very closely related to the following problem, which we call the *finite-horizon valuation* (*FHV*) *problem*:

PROBLEM FHV. Find an adapted process (X,Y) such that

$$\begin{aligned} dX_t &= b(X_t,Y_t)\,dt + \sigma(X_t,Y_t)\,dW_t, \qquad t\in[0,T],\ X_0 = x,\\ Y_t &= E\bigg[\exp\left(-\int_t^T h(X_u)\,du\right)g(X_T) \\ &\qquad + \int_t^T \exp\left(-\int_t^s h(X_u)\,du\right)ds\,\bigg|\,\mathscr{F}_t\bigg], \qquad t\in[0,T]. \end{aligned} \tag{2.12}$$

In the FHV problem (2.12), treating $r_t = h(X_t)$ as the short rate of interest in a finance setting, we may think of Y as the price of a security claiming a continual constant unit dividend until time T, at which time the security is valued at $g(X_T)$. The unusual aspect of this formulation, relative to the typical model, is that the state process X has dynamics that depend on the price process Y, while Y_t itself depends on the conditional distribution of X_s for $s \geq t$. If we take $g \equiv 0$, we have a finite horizon annuity valuation problem in which the annuity price influences the short rate. The consol rate problem is the infinite-horizon version of this problem. We will show technical conditions (Proposition 3.1) under which $Y_t = \theta(X_t, t)$ for some unique function θ, so that there is no role for Y as a separate state variable. Later, under additional regularity, we will show that this is also true in the infinite-horizon setting (Section 4) for some time-independent θ, and that the finite-horizon solutions will converge to the infinite-horizon solution as the horizon $T \to \infty$ (Section 5).

Any adapted process (X, Y) satisfying (2.12) is called an adapted solution of Problem FHV. Moreover, we have the following notion.

DEFINITION 2.2. An adapted solution (X, Y) of Problem FHV is called a nodal solution of Problem FHV if there exists a function $\theta: \mathbb{R}^n \times [0, T] \to \mathbb{R}$, which is C^1 in t and C^2 in x, such that

$$Y_t = \theta(X_t, t), \qquad t \in [0, T]. \tag{2.13}$$

We will call (2.8) the forward–backward SDE associated with Problem FHV.

The main purpose of this paper is to study the following problem, which is called the *infinite-horizon consol rate (IHCR) problem* in the sequel.

PROBLEM IHCR. Find an adapted, locally square-integrable process (X, Y), such that

$$\begin{aligned} dX_t &= b(X_t, Y_t)\,dt + \sigma(X_t, Y_t)\,dW_t, \qquad t \in [0, \infty),\ X_0 = x, \\ Y_t &= E\left[\int_t^\infty \exp\left(-\int_t^s h(X_u)\,du\right) ds \,\middle|\, \mathscr{F}_t\right], \qquad t \in [0, \infty). \end{aligned} \tag{2.14}$$

Any adapted process (X, Y) satisfying (2.14) is called an *adapted solution* of Problem IHCR. Moreover, we have the following definition similar to Definition 2.2.

DEFINITION 2.3. An adapted solution (X, Y) of Problem IHCR is called a nodal solution of Problem IHCR if there exists a bounded C^2 function θ with θ_x being bounded, such that

$$Y_t = \theta(X_t), \qquad t \in [0, \infty). \tag{2.15}$$

We note that in Definition 2.3, the function θ is time-independent because of the infinite horizon. Formally, the forward–backward SDE associated with Problem IHCR is

$$\begin{aligned} dX_t &= b(X_t, Y_t)\,dt + \sigma(X_t, Y_t)\,dW_t, \qquad t \in [0, \infty), \\ dY_t &= (h(X_t)Y_t - 1)\,dt - \langle Z_t, dW_t \rangle, \qquad t \in [0, \infty), \\ X_0 &= x, \qquad Y_t \text{ is bounded a.s., uniformly in } t \in [0, \infty). \end{aligned} \tag{2.16}$$

We shall verify this in the next section. Also, we should note that, in general, the asymptotic behavior of Y_t at $t = \infty$ is not known. We therefore only impose the boundedness of Y instead. As before, we may introduce the following definition.

DEFINITION 2.4. A process (X, Y, Z) is called an adapted solution of (2.16) if it is $\{\mathscr{F}_t\}$-adapted, locally square-integrable and if

$$(2.17)\quad \begin{aligned} X_t &= x + \int_0^t b(X_s, Y_s)\, ds + \int_0^t \sigma(X_s, Y_s)\, dW_s, \quad t \in [0, \infty), \\ Y_t &= Y_u + \int_u^t [h(X_s)Y_s - 1]\, ds - \int_u^t \langle Z_s, dW_s \rangle, \quad 0 \le u \le t < \infty. \end{aligned}$$

Moreover, if there exists a bounded C^2 function θ with θ_x being bounded, such that the adapted solution (X, Y, Z) satisfies the relations

$$(2.18)\quad Y_t = \theta(X_t), \qquad Z_t = -\sigma(X_t, \theta(X_t))^T \theta_x(X_t), \qquad t \in [0, \infty),$$

then we call (X, Y, Z) a nodal solution of (2.16).

3. Existence of nodal solutions. In this section we study the existence of nodal solutions to both Problem FHV and Problem IHCR. We shall also establish the relationship between these problems and the associated forward–backward SDEs, and some properties of the nodal solutions. Let us first make some *standing assumptions*.

(H1) The functions σ, b and h are C^1 with bounded partial derivatives and there exist constants λ, $\mu > 0$ and some continuous increasing function ν: $[0, \infty) \to [0, \infty)$, such that

$$(3.1)\quad \lambda I \le \sigma(x, y)\sigma(x, y)^T \le \mu I, \qquad (x, y) \in \mathbb{R}^n \times \mathbb{R},$$

$$(3.2)\quad |b(x, y)| \le \nu(|y|), \qquad (x, y) \in \mathbb{R}^n \times \mathbb{R},$$

$$(3.3)\quad \inf_{x \in \mathbb{R}^n} h(x) \equiv \delta > 0, \qquad \sup_{x \in \mathbb{R}^n} h(x) \equiv \gamma < \infty.$$

(H2) The function g is bounded in $C^{2+\alpha}(\mathbb{R}^n)$, for some $\alpha > 0$.

Let us begin with Problem FHV.

PROPOSITION 3.1. *Let $(H1)$–$(H2)$ hold. Then, Problem FHV admits a unique adapted solution (X, Y). Moreover, it is in fact a nodal solution.*

PROOF. First, from the previous section, we see that if (X, Y, Z) is an adapted solution of (2.8), then (X, Y) is an adapted solution of Problem FHV. Conversely, let (X, Y) be any adapted solution of Problem FHV. We shall prove that there exists an adapted $\mathbb{R}^d$-valued process Z such that (X, Y, Z) is an adapted solution to (2.8).

To this end, let us define a measurable process

$$(3.4)\quad U_t \triangleq \exp\left(-\int_t^T h(X_u)\,du\right)g(X_T) + \int_t^T \exp\left(-\int_t^s h(X_u)\,du\right)ds.$$

Clearly, for each $t \in [0,T]$, U_t is $\mathscr{F}_T$-measurable. Let $Y_t \triangleq E[U_t \mid \mathscr{F}_t]$, $t \in [0,T]$. Then note that the Brownian filtration $\{\mathscr{F}_t\}$ is actually continuous [cf. Karatzas and Shreve (1988)] and Y is continuous and is indistinguishable from the optional (or predictable) projection of U. Furthermore, it holds that

$$(3.5)\qquad E\left[\int_t^T H_s U_s\,ds \middle| \mathscr{F}_t\right] = E\left[\int_t^T H_s Y_s\,ds \middle| \mathscr{F}_t\right],$$

for any $t \in [0,T]$, P-a.s., where H is any bounded progressively measurable process [see, for example, Dellacherie and Meyer (1982), Section 6].

On the other hand, for each fixed $\omega \in \Omega$, one can check by direct computation that U satisfies the ODE:

$$(3.6)\qquad U_t(\omega) = g(X_T(\omega)) + \int_t^T (h(X_s(\omega))U_s(\omega) - 1)\,ds.$$

Taking conditional expectation on both sides of (3.6) and applying (3.5), we see that Y_t satisfies

$$(3.7)\qquad Y_t = E\left[g(X_T) + \int_t^T (h(X_s)Y_s - 1)\,ds \middle| \mathscr{F}_t\right],\qquad P\text{-a.s.}$$

Now by applying the martingale representation theorem and following an argument like that in Ma, Protter and Yong [(1993), Section 5] (note the boundedness of g, h and the adaptedness of Y), we see that there exists an adapted $\mathbb{R}^d$-valued square-integrable process $\{Z_t\colon 0 \le t \le T\}$, such that

$$(3.8)\qquad Y_t = g(X_T) + \int_t^T (h(X_s)Y_s - 1)\,ds - \int_t^T \langle Z_s, dW_s\rangle.$$

In other words, (X,Y,Z) solves (2.8). (In some finance applications, the Brownian motion W does not generate the given filtration $\{\mathscr{F}_t\}$, as assumed in this paper, but rather is obtained as the martingale part of a Brownian motion under a different measure, via Girsanov's theorem. Even in this more general setting, it can be seen that W generates all martingales as stochastic integrals in the above sense).

Finally, since the process Y is one dimensional, the results in Ma, Protter and Yong (1993) show that (2.8) possesses a unique adapted solution which can be constructed via the four step scheme; namely, any adapted solution of (2.8) must be the nodal solution, proving the proposition. □

To study the Problem IHCR, we need the following lemma. The proof of the lemma is quite standard, but we nevertheless sketch it for the benefit of the reader.

LEMMA 3.2. *Let* $(H1)$ *hold. Then the following equation admits a classical solution* $\theta \in C^{2+\alpha}(\mathbb{R}^n)$:

$$\frac{1}{2}\operatorname{tr}\big(\theta_{xx}\sigma(x,\theta)\sigma^T(x,\theta)\big) + \langle b(x,\theta), \theta_x\rangle - h(x)\theta + 1 = 0, \qquad x \in \mathbb{R}^n. \tag{3.9}$$

such that

$$\frac{1}{\gamma} \le \theta(x) \le \frac{1}{\delta}, \qquad x \in \mathbb{R}^n. \tag{3.10}$$

SKETCH OF THE PROOF. Let $B_R(0)$ be the ball of radius $R > 0$ centered at the origin. We consider (3.9) in $B_R(0)$ with the homogeneous Dirichlet boundary condition. By Gilbarg and Trudinger [(1977), Theorem 14.10] there exists a solution $\theta^R \in C^{2+\alpha}(B_R(0))$ for some $\alpha > 0$. By the maximum principle, we have

$$0 \le \theta^R(x) \le \frac{1}{\delta}, \qquad x \in B_R(0). \tag{3.11}$$

Next, for any fixed $x_0 \in \mathbb{R}^n$ and $R > |x_0| + 2$, by Gilbarg and Trudinger [(1977), Theorem 14.6], we have

$$\big|\theta_x^R(x)\big| \le C, \qquad x \in B_1(x_0), \tag{3.12}$$

where the constant C is independent of $R > |x_0| + 2$. This, together with the boundedness of σ and the first partial derivatives of σ, b, h, implies that as a linear equation in θ [regarding $\sigma(x,\theta(x))$ and $b(x,\theta(x))$ as known functions], the coefficients are bounded in C^1. Hence, by Schauder's interior estimates, we obtain that

$$\|\theta^R\|_{C^{2+\alpha}(B_1(x_0))} \le C, \qquad R > |x_0| + 2. \tag{3.13}$$

Then, we can let $R \to \infty$ along some sequence to get a limit function $\theta(x)$. By the standard diagonalization argument, we may assume that θ is defined in the whole of $\mathbb{R}^n$. Clearly, $\theta \in C^{2+\alpha}(\mathbb{R}^n)$ and is a classical solution of (3.9). Finally, by the maximum principle again, we obtain (3.10). □

Now, we come up with the following existence result for the Problem IHCR.

THEOREM 3.3. *Let* $(H1)$ *hold. Then there exists at least one nodal solution* (X,Y) *of Problem IHCR.*

PROOF. By Lemma 3.2, we can find a classical solution $\theta \in C^{2+\alpha}(\mathbb{R}^n)$ of (3.9). Now, we consider the (forward) SDE

$$\begin{aligned} &dX_t = b(X_t, \theta(X_t))\,dt + \sigma(X_t, \theta(X_t))\,dW_t, \qquad t > 0,\\ &X_0 = x. \end{aligned} \tag{3.14}$$

Since θ_x is bounded and b and σ are uniformly Lipschitz, (3.14) admits a unique strong solution X_t, $t \in [0, \infty)$. Next, we define

$$Y_t = \theta(X_t), \qquad t \in [0, \infty). \tag{3.15}$$

We are going to show that (X, Y) is an adapted solution of Problem IHCR. In fact, by using Itô's formula, we have

$$\begin{aligned} dY_t &= \big[\langle \theta_x(X_t), b(X_t, \theta(X_t))\rangle \\ &\qquad + \tfrac{1}{2}\operatorname{tr}\big(\theta_{xx}(X_t)\sigma(X_t, \theta(X_t))\sigma^T(X_t, \theta(X_t))\big)\big]\,dt \\ &\qquad + \langle \theta_x(X_t), \sigma(X_t, \theta(X_t))\,dW_t\rangle \\ &= (h(X_t)\theta(X_t) - 1)\,dt + \langle \sigma^T(X_t, \theta(X_t))\theta_x(X_t), dW_t\rangle \\ &= (h(X_t)Y_t - 1)\,dt - \langle Z_t, dW_t\rangle, \end{aligned} \tag{3.16}$$

where $Z_t = -\sigma(X_t, \theta(X_t))^T\theta_x(X_t)$ and (3.9) has been used. Clearly, (3.16) can be regarded as a linear nonhomogeneous SDE in Y_t. Thus, the usual variation of constants formula gives

$$\begin{aligned} Y_T &= \exp\left(\int_t^T h(X_u)\,du\right)Y_t - \int_t^T \exp\left(\int_s^T h(X_u)\,du\right)ds \\ &\qquad + \int_t^T \left\langle \exp\left(\int_s^T h(X_u)\,du\right)Z_s, dW_s\right\rangle, \qquad 0 \le t \le T < \infty. \end{aligned} \tag{3.17}$$

The above can also be written as

$$\begin{aligned} Y_t &= \exp\left(-\int_t^T h(X_u)\,du\right)Y_T + \int_t^T \exp\left(-\int_t^s h(X_u)\,du\right)ds \\ &\qquad - \int_t^T \left\langle \exp\left(-\int_t^s h(X_u)\,du\right)Z_s, dW_s\right\rangle, \qquad 0 \le t \le T < \infty. \end{aligned} \tag{3.18}$$

Next, we take conditional expectations, using the local square integrability of Z implied by the properties of σ and θ to obtain

$$\begin{aligned} Y_t = E\Bigg[&\exp\left(-\int_t^T h(X_u)\,du\right)Y_T \\ &+ \int_t^T \exp\left(-\int_t^s h(X_u)\,du\right)ds \Bigg| \mathscr{F}_t\Bigg], \qquad 0 \le t \le T < \infty. \end{aligned} \tag{3.19}$$

Since θ is bounded, so is Y [see (3.15)]. Consequently, by condition (3.3),

$$\left|\exp\left(-\int_t^T h(X_r)\,dr\right)Y_T\right| \le \frac{1}{\delta}e^{-\delta(T-t)} \to 0, \qquad T \to \infty. \tag{3.20}$$

On the other hand, for any $T \le T'$,

$$\begin{aligned} &\left|\int_T^{T'} \exp\left(-\int_t^s h(X_u)\,du\right)ds\right| \\ &\quad \le \int_T^{T'} e^{-\delta(s-t)}\,ds = e^{\delta t}\frac{e^{-\delta T} - e^{-\delta T'}}{\delta} \to 0, \qquad T, T' \to \infty. \end{aligned} \tag{3.21}$$

Namely, for each fixed (t, ω), the integral above converges as $T \to \infty$. Hence, by the dominated convergence theorem, we can send $T \to \infty$ in (3.19) to get

$$Y_t = E\left[\int_t^\infty \exp\left(-\int_t^s h(X_u)\,du\right) ds \,\middle|\, \mathscr{F}_t\right], \qquad t \in [0, \infty). \tag{3.22}$$

This shows that (X, Y) is an adapted solution of Problem IHCR. By (3.15), this solution is nodal. □

Next, we discuss the possibility of representing the solution of Problem IHCR as the adapted solution of forward–backward SDE (2.16). As was pointed out in the Introduction, in the general infinite-horizon case, we do not have a unique solution of the forward–backward SDE if the value at infinity cannot be specified. Nevertheless, we shall show below and in the next section that uniqueness in a certain class of solutions will still hold, which will be sufficient for our purpose.

To begin with, let us establish a result concerning the nonautonomous ODE with infinite duration. Let H: $[0, \infty) \to [\delta, \infty)$ be a continuous function, where $\delta > 0$ is given, and consider the ODE:

$$\frac{dU_t}{dt} = H_t U_t - 1. \tag{3.23}$$

We have the following lemma.

LEMMA 3.4. *There exists a unique bounded solution of* (3.23) *defined on* $[0, \infty)$. *Moreover, such a solution has an explicit expression*:

$$U_t = \int_t^\infty \exp\left(-\int_t^s H_u\,du\right) ds. \tag{3.24}$$

PROOF. The existence follows from a direct verification that the function U defined by (3.24) is a bounded solution of (3.23). To see the uniqueness, it suffices to show that any bounded solution of (3.23) must be of the form (3.24). Indeed, let U be any bounded solution to (3.23) defined on $[0, \infty)$. For any $0 \le t \le T$, we can apply the variation of constants formula to get

$$U_t = \exp\left(-\int_t^T H_u\,du\right) U_T + \int_t^T \exp\left(-\int_t^s H_u\,du\right) ds. \tag{3.25}$$

Since U_T is bounded (for all $T > 0$), we have

$$\left|\exp\left(-\int_t^T H_r\,dr\right) U_T\right| \le Ce^{-\delta(T-t)} \to 0 \quad \text{as } T \to 0. \tag{3.26}$$

Hence a similar argument proving (3.22) shows that (3.24) holds. This proves the lemma. □

PROPOSITION 3.5. *If* (X, Y) *is an adapted solution to Problem IHCR, then there exists an adapted* $\mathbb{R}^d$*-valued locally square-integrable process* Z, *such that* (X, Y, Z) *is an adapted solution of* (2.16).

PROOF. Suppose that (X, Y) is an adapted solution to Problem IHCR. Define

$$U_t = \int_t^\infty \exp\left(-\int_t^s h(X_u)\, du \right) ds. \tag{3.27}$$

By the assumption on the function h, U_t is well defined for each $t \geq 0$. Clearly, Y is the optional projection of U; to wit, $Y_t = E(U_t \mid \mathscr{F}_t)$, $t \geq 0$.

Now for each fixed $\omega \in \Omega$, define $H_t(\omega) = h(X_t(\omega))$, $t \geq 0$. Then $t \mapsto H_t(\omega)$ is continuous and bounded below by $\delta > 0$. Therefore, by Lemma 3.4, we see that U is the unique bounded solution of the ODE (with random coefficients):

$$\frac{dU_t}{dt} = h(X_t)U_t - 1. \tag{3.28}$$

Now, whenever $0 \leq t < T < \infty$, we have

$$U_t = U_T - \int_t^T [h(X_s)U_s - 1]\, ds, \tag{3.29}$$

whence

$$\begin{aligned} Y_t &= E(U_t \mid \mathscr{F}_t) \\ &= E\left[U_T - \int_t^T [h(X_s)U_s - 1]\, ds \middle| \mathscr{F}_t \right] \\ &= E\left[Y_T - \int_t^T [h(X_s)Y_s - 1]\, ds \middle| \mathscr{F}_t \right], \end{aligned} \tag{3.30}$$

where we have used the fact that Y is the optional projection of U. Thus, a by now standard argument using the martingale representation theorem, as in the finite horizon case, leads to the existence of an adapted square-integrable process $Z^{(T)}$ defined on $[0, T]$, such that

$$Y_t = Y_T - \int_t^T [h(X_s)Y_s - 1]\, ds + \int_t^T \langle Z_s^{(T)}, dW_s \rangle, \qquad t \in [0, T]. \tag{3.31}$$

It remains to show that there exists an adapted locally integrable process Z taking values in $\mathbb{R}^d$ such that

$$\int_t^T \langle Z_s, dW_s \rangle = \int_t^T \langle Z_s^{(T)}, dW_s \rangle, \qquad 0 \leq t \leq T < \infty. \tag{3.32}$$

To see this, note that (3.31) holds for any $T > 0$, so if $0 \leq T_1 < T_2 < \infty$, then for $t \in [0, T_1]$, we have

$$\begin{aligned} Y_t &= Y_{T_1} - \int_t^{T_1} [h(X_s)Y_s - 1]\, ds + \int_t^{T_1} \langle Z_s^{(T_1)}, dW_s \rangle \\ &= Y_{T_2} - \int_t^{T_2} [h(X_s)Y_s - 1]\, ds + \int_t^{T_2} \langle Z_s^{(T_2)}, dW_s \rangle. \end{aligned} \tag{3.33}$$

Setting $t = T_1$, we get

$$Y_{T_1} = Y_{T_2} - \int_{T_1}^{T_2}[h(X_s)Y_s - 1]\,ds + \int_{T_1}^{T_2}\langle Z_s^{(T_2)}, dW_s\rangle. \tag{3.34}$$

Plugging (3.34) into (3.33), we obtain that

$$\int_t^{T_1}\langle Z_s^{(T_2)} - Z_s^{(T_1)}, dW_s\rangle = 0, \quad \text{for all } t \in [0, T_1]. \tag{3.35}$$

This leads to the property

$$E\left(\int_0^{T_1}\left[Z_s^{(T_2)} - Z_s^{(T_1)}\right]^2 ds\right) = 0. \tag{3.36}$$

In other words, $Z^{(T_1)} = Z^{(T_2)}$, $dt \otimes dP$-almost surely on $[0, T_1] \times \Omega$. Therefore, modulo a $dt \otimes dP$-null set, we can define a process Z by $Z_t = Z_t^{(N)}$, if $t \in [0, N]$, where $N = 1, 2, \dots,$ and it is fairly easy to check as before that (3.32) holds. Therefore, (3.31) can be rewritten as

$$Y_t = Y_T - \int_t^T[h(X_s)Y_s - 1]\,ds + \int_t^T\langle Z_s, dW_s\rangle, \tag{3.37}$$

for all $T > 0$, or equivalently, one has

$$dY_t = [h(X_t)Y_t - 1]\,dt - \langle Z_t, dW_t\rangle, \qquad t \in [0, \infty). \tag{3.38}$$

Finally, the boundedness of Y follows easily from the definition of Y and the fact that $U_t \le 1/\delta$, $\forall t \ge 0$, P-a.s., proving the proposition. □

It is worth noting that although the bounded solution U of the random ODE (3.23) over the infinite horizon is unique, the uniqueness of the adapted solution to the forward–backward SDE (2.16) over an infinite duration is still unknown. Theorem 3.3 and Proposition 3.5 suggested two ways to construct the adapted solutions to such forward–backward SDEs. In the next section we shall prove that if both X and Y are one dimensional, then the adapted solution to the forward–backward SDE over an infinite horizon is unique, under some explicit compatibility conditions, and such adapted solutions must be nodal (see Theorem 4.1). In the higher-dimensional case, such a result is also proved (Theorem 4.5), but the condition that we have to impose is implicit, and the general uniqueness result is far from obvious. Nevertheless, one would expect that the uniqueness should hold at least among the nodal solutions. The next result and the remark following it explore its possibility. Recall from Definition 2.4 that a nodal solution can be given by an arbitrary bounded C^2 function θ with bounded gradient.

PROPOSITION 3.6. *Let* $(H1)$ *hold. Suppose that the forward–backward SDE* (2.16) *has a nodal solution* (X, Y, Z); *namely,* (2.18) *holds for some* *bounded C^2 function θ with bounded gradient. Then θ must satisfy the ODE* (3.9).

PROOF. Let (2.18) hold for some bounded C^2 function θ with bounded gradient. Since θ is C^2, we can apply Itô's formula to $Y_t = \theta(X_t)$. This leads to

$$\begin{aligned} dY_t = {} & \big[\langle b(X_t, \theta(X_t)), \theta_x(X_t)\rangle \\ & + \tfrac{1}{2}\operatorname{tr}\big(\theta_{xx}(X_t)\sigma\sigma^T(X_t, \theta(X_t))\big)\big]\, dt \\ & + \langle \theta_x(X_t), \sigma(X_t, \theta(X_t))\, dW_t\rangle. \end{aligned} \tag{3.39}$$

Comparing (3.39) with (2.16) and noting that $Y_t = \theta(X_t)$, we obtain that

$$\begin{aligned} & \langle b(X_t, \theta(X_t)), \theta_x(X_t)\rangle \\ & \quad + \tfrac{1}{2}\operatorname{tr}\big(\theta_{xx}(X_t)\sigma\sigma^T(X_t, \theta(X_t))\big) = h(X_t)\theta(X_t) - 1, \end{aligned} \tag{3.40}$$

for all $t \geq 0$, P-almost surely. Define a continuous function $F\colon \mathbb{R}^n \to \mathbb{R}$ by

$$\begin{aligned} F(x) \triangleq {} & \langle b(x, \theta(x)), \theta_x(x)\rangle \\ & + \tfrac{1}{2}\operatorname{tr}\big(\theta_{xx}(x)\sigma\sigma^T(x, \theta(x))\big) - h(x)\theta(x) + 1. \end{aligned} \tag{3.41}$$

We shall prove that $F \equiv 0$. In fact, note that in this case, X actually satisfies the forward SDE

$$\begin{aligned} dX_t &= \tilde{b}(X_t)\, dt + \tilde{\sigma}(X_t)\, dW_t, \quad t \geq 0, \\ X_0 &= x, \end{aligned} \tag{3.42}$$

where $\tilde{b}(x) \triangleq b(x, \theta(x))$ and $\tilde{\sigma}(x) \triangleq \sigma(x, \theta(x))$. Therefore, X is a time-homogeneous Markov process with some transition probability density $p(t, x, y)$. Since both $\tilde{b}$ and $\tilde{\sigma}$ are bounded and satisfy a Lipschitz condition and since $\sigma\sigma^{\mathrm{T}}$ is uniformly positive definite, it is well known [see, e.g., Friedman (1964, 1975)] that for each $y \in \mathbb{R}^n$, $p(\cdot, \cdot, y)$ is the fundamental solution of the parabolic partial differential equation (PDE)

$$\frac{1}{2}\sum_{i,j=1}^{n} \tilde{a}^{ij}(x)\frac{\partial^2 p}{\partial x_i\, \partial x_j} + \sum_{i=1}^{n} \tilde{b}^i(x)\frac{\partial p}{\partial x_i} - \frac{\partial p}{\partial t} = 0 \tag{3.43}$$

and it is positive everywhere. Now by (3.40), we have that $F(X_t) = 0$ for all $t \geq 0$, P-a.s., whence $E_{0,x}[F^2(X_t)] = 0$, for all $t > 0$. Since

$$E_{0,x}\big[F^2(X_t)\big] = \int_{\mathbb{R}^n} p(t, x, y)F^2(y)\, dy, \qquad t > 0, \tag{3.44}$$

and $p(t, x, y)$ is positive everywhere, we have $F(y) = 0$ almost everywhere under the Lebesgue measure in $\mathbb{R}^n$. The result then follows from the continuity of F. $\square$

REMARK 3.7. The essence of Proposition 3.6 is that the only possible nodal solution for the infinite-horizon forward–backward SDE (2.16) is the one constructed using the solution of ODE (3.9). Therefore, if (3.9) has multiple solutions, we do not have uniqueness of the nodal solution and the number of the nodal solutions will be exactly the same as that of the solutions to (3.9). However, if the solution of (3.9) is unique, then the nodal solution of (2.16) (or equivalently, Problem IHCR) will be unique as well.

4. Uniqueness of the nodal solution. In this section we study the uniqueness of the nodal solution of Problem IHCR. We first consider the one-dimensional case, that is, when X and Y are both one-dimensional processes. For simplicity, we denote

$$a(x,y) = \tfrac{1}{2}\|\sigma(x,y)\|^2, \qquad (x,y) \in \mathbb{R}^2. \tag{4.1}$$

Let us make some further assumptions:

(H3) Let $m = n = 1$ and the functions a, b, h satisfy the following:

$$h \text{ is strictly increasing,} \tag{4.2}$$

$$\begin{aligned} &a(x,y)h(x) - (h(x)y-1)\int_0^1 a_y(x,(1-\beta)y+\beta\hat{y})\,d\beta \geq \eta > 0, \\ &\int_0^1 \big[a(x,y)b_y(x,(1-\beta)y+\beta\hat{y}) \\ &\qquad - a_y(x,(1-\beta)y+\beta\hat{y})b(x,y)\big]\,d\beta \geq 0, \\ &\qquad\qquad y,\hat{y} \in \left[\frac{1}{\gamma},\frac{1}{\delta}\right], \qquad x \in \mathbb{R}. \end{aligned} \tag{4.3}$$

Condition (4.3) essentially says that the coefficients b, σ and h should be somewhat "compatible." Although a little complicated, it is still quite explicit and easily verifiable. For example, a sufficient conditions for (4.3) are

$$\begin{aligned} &a(x,y)h(x) - (h(x)y-1)a_y(x,w) \geq \eta > 0, \\ &a(x,y)b_y(x,w) - a_y(x,w)b(x,y) \geq 0, \\ &\qquad\qquad y,w \in \left[\frac{1}{\gamma},\frac{1}{\delta}\right], \ x \in \mathbb{R}. \end{aligned} \tag{4.4}$$

It is readily seen that the following equations will guarantee (4.4):

$$a_y(x,y) = 0, \qquad b_y(x,y) \geq 0, \qquad (x,y) \in \mathbb{R} \times \left[\frac{1}{\gamma},\frac{1}{\delta}\right]. \tag{4.5}$$

In particular, if both a and b are independent of y, then (4.3) holds automatically.

Our main result of this section is the following uniqueness theorem.

THEOREM 4.1. *Let* $(H1)$ *and* $(H3)$ *hold. Then Problem IHCR has a unique adapted solution. Moreover, this solution is nodal.*

To prove the above result, we need the following lemmas.

LEMMA 4.2. *Let h be strictly increasing and let θ solve*

$$a(x,\theta)\theta_{xx} + b(x,\theta)\theta_x - h(x)\theta + 1 = 0, \qquad x \in \mathbb{R}. \tag{4.6}$$

Suppose x_M is a local maximum of θ and x_m is a local minimum of θ with $\theta(x_m) \le \theta(x_M)$. Then, $x_m > x_M$.

PROOF. Since h is strictly increasing, from (4.6) we see that θ is not identically constant in any interval. Therefore $x_m \neq x_M$. Now let us look at x_M. It is clear that $\theta_x(x_M) = 0$ and $\theta_{xx}(x_M) \le 0$. Thus, from (4.6) we obtain that

$$\theta(x_M) \le \frac{1}{h(x_M)}. \tag{4.7}$$

Similarly, we have

$$\theta(x_m) \ge \frac{1}{h(x_m)}. \tag{4.8}$$

Since $\theta(x_m) \le \theta(x_M)$, we have

$$\frac{1}{h(x_m)} \le \frac{1}{h(x_M)}, \tag{4.9}$$

whence $x_M < x_m$ because h is strictly increasing. □

LEMMA 4.3. *Under the conditions of Theorem* 4.1, (4.6) *admits a unique solution θ.*

PROOF. From Lemma 4.2, we see that if x_m is a global minimum, then there will be no local maximum on (x_m, ∞). Thus, θ is strictly monotone increasing on (x_m, ∞) since h is so. By the boundedness of θ, θ_x and θ_{xx}, we see that

$$\lim_{x\to\infty} \theta_x(x) = \lim_{x\to\infty} \theta_{xx}(x) = 0 \tag{4.10}$$

and $\lim_{x\to\infty} \theta(x)$ exists. Thus, by (4.4), we have

$$\lim_{x\to\infty} \theta(x) = \frac{1}{h(+\infty)}. \tag{4.11}$$

On the other hand, we have seen that

$$\lim_{x\to\infty} \theta(x) > \theta(x_m) \ge \frac{1}{h(x_m)} > \frac{1}{h(+\infty)}, \tag{4.12}$$

which contradicts (4.11). This means that θ has no global minimum. From Lemma 4.2, we see that θ can have at most one global maximum point x_M.

Thus, on $(-\infty, x_M)$, θ is strictly monotone increasing. Hence, we have [similar to (4.10) and (4.11)]

$$\lim_{x\to-\infty}\theta_x(x) = \lim_{x\to-\infty}\theta_{xx}(x) = 0, \tag{4.13}$$
$$\lim_{x\to-\infty}\theta(x) = \frac{1}{h(-\infty)},$$

but from Lemma 4.2,

$$\lim_{x\to-\infty}\theta(x) < \theta(x_M) \le \frac{1}{h(x_M)} \le \frac{1}{h(-\infty)}, \tag{4.14}$$

which contradicts (4.13). Hence, θ cannot have any maximum points either. Consequently, θ is monotone on $\mathbb{R}$. Finally, since

$$\theta(-\infty) = \frac{1}{h(-\infty)} > \frac{1}{h(+\infty)} = \theta(+\infty), \tag{4.15}$$

it is necessary that θ is monotone decreasing.

Next, let θ and $\hat\theta$ be two solutions of (4.6). Then, $w \equiv \hat\theta - \theta$ satisfies

$$\begin{aligned}
0 &= a(x,\hat\theta)w_{xx} + b(x,\hat\theta)w_x \\
&\quad - \left(h(x) - \int_0^1 \left[a_y(x,\theta+\beta w)\theta_{xx} + b_y(x,\theta+\beta w)\theta_x\right] d\beta\right) w \\
&= a(x,\hat\theta)w_{xx} + b(x,\hat\theta)w_x \\
&\quad - \left(h(x) - \int_0^1 \left[a_y(x,\theta+\beta w)\frac{h(x)\theta - 1 - b(x,\theta)\theta_x}{a(x,\theta)} \right.\right. \\
&\qquad\qquad \left.\left. + b_y(x,\theta+\beta w)\theta_x\right] d\beta\right) w \\
&= a(x,\hat\theta)w_{xx} + b(x,\hat\theta)w_x \\
&\quad - \frac{1}{a(x,\theta)}\left(a(x,\theta)h(x) - (h(x)\theta - 1)\int_0^1 a_y\big(x,(1-\beta)\theta + \beta\hat\theta\big)\, d\beta \right. \\
&\qquad + |\theta_x| \int_0^1 \left[a(x,\theta) b_y\big(x,(1-\beta)\theta + \beta\hat\theta\big) \right. \\
&\qquad\qquad \left.\left. - a_y\big(x,(1-\beta)\theta + \beta\hat\theta\big) b(x,\theta)\right] d\beta\right) w \\
&\equiv \tilde a(x)w_{xx} + \tilde b(x)w_x - c(x)w.
\end{aligned} \tag{4.16}$$

Here, we used the fact that $\theta_x(x) = -|\theta_x(x)|$ (since θ is decreasing). By (H3), we see that $c(x) \ge \eta/\mu > 0$ for all $x \in \mathbb{R}$ (note that by (3.10), $\theta, \hat\theta \in$

$[1/\gamma, 1/\delta]$). From (H1), we also see that $\tilde{a}$ and $\tilde{b}$ are bounded. Thus, by the lemma that will be proved below, we obtain $w = 0$, proving the uniqueness. □

LEMMA 4.4. *Let w be a bounded classical solution of*

$$\tilde{a}(x)w_{xx} + \tilde{b}(x)w_x - c(x)w = 0, \qquad x \in \mathbb{R}, \tag{4.17}$$

with $c(x) \geq c_0 > 0$, $\tilde{a}(x) \geq 0$, $x \in \mathbb{R}^n$, and with $\tilde{a}$ and $\tilde{b}$ bounded. Then, $w(x) \equiv 0$.

PROOF. For any $\alpha > 0$, let us consider $\Phi(x) = w(x) - \alpha|x|^2$. Since w is bounded, there exists some x_0 at which Φ attains its global maximum. Thus, $\Phi'(x_0) = 0$ and $\Phi''(x_0) \leq 0$, and which means that

$$w_x(x_0) = 2\alpha x_0, \qquad w_{xx}(x_0) \leq 2\alpha. \tag{4.18}$$

Now, by (4.17),

$$\begin{aligned} c_0 w(x_0) &= \tilde{a}(x_0)w_{xx}(x_0) + \tilde{b}(x_0)w_x(x_0) \\ &\leq 2\alpha\big(\tilde{a}(x_0) + \tilde{b}(x_0)x_0\big). \end{aligned} \tag{4.19}$$

For any $x \in \mathbb{R}$, by the definition of x_0, we have (note the boundedness of $\tilde{a}$ and $\tilde{b}$)

$$\begin{aligned} w(x) - \alpha|x|^2 &\leq w(x_0) - \alpha|x_0|^2 \\ &\leq \alpha\big(2\tilde{a}(x_0) + 2\tilde{b}(x_0)x_0 - |x_0|^2\big) \leq C\alpha. \end{aligned} \tag{4.20}$$

Sending $\alpha \to 0$, we obtain $w(x) \leq 0$. Similarly, we can show that $w(x) \geq 0$. Thus $w(x) \equiv 0$. □

This lemma is not new. Since the proof is simple, we have provided it for completeness. Also, it is not hard to see that similar results hold for higher-dimensional cases. Actually, much more general comparison results can be found in the literature [see Crandall, Ishii, and Lions (1992)].

PROOF OF THEOREM 4.1. Let (X, Y) be any adapted solution of Problem IHCR. Then, by Proposition 3.4, there exists an adapted process Z such that (X, Y, Z) is an adapted solution of (2.16). Now, under (H1) and (H3), (4.6) admits a unique classical solution θ with $\theta_x \leq 0$. We set

$$\tilde{Y}_t = \theta(X_t), \qquad \tilde{Z}_t = -\sigma(X_t, \theta(X_t))^T \theta_x(X_t), \qquad t \in [0, \infty). \tag{4.21}$$

By Itô's formula, we have [note (4.1)]

$$\begin{aligned} d\tilde{Y}_t &= [\theta_x(X_t)b(X_t, Y_t) + \theta_{xx}(X_t)a(X_t, Y_t)]\, dt \\ &\quad + \big\langle \sigma(X_t, Y_t)^T \theta_x(X_t), dW_t \big\rangle. \end{aligned} \tag{4.22}$$

Hence, with (2.16), we obtain [note (4.6)] that for any $0 \le u < t < \infty$,

$$
\begin{aligned}
E\big(\tilde{Y}_u - Y_u\big)^2 &= E\big(\tilde{Y}_t - Y_t\big)^2 \\
&\quad - E\int_u^t \Big(2\big(\tilde{Y}_s - Y_s\big)\big[\theta_x(X_s)b(X_s, Y_s) \\
&\qquad\qquad + \theta_{xx}(X_s)a(X_s, Y_s) - h(X_s)Y_s + 1\big] \\
&\qquad\qquad + \|\sigma(X_s, Y_s)\theta_x(X_s) + Z_s\|^2\Big)\, ds \\
&= E\big(\tilde{Y}_t - Y_t\big)^2 \\
&\quad - E\int_u^t \Big[2\big(\tilde{Y}_s - Y_s\big)\Big[\theta_x(X_s)\big(b(X_s, Y_s) - b\big(X_s, \tilde{Y}_s\big)\big) \\
&\qquad\qquad + \theta_{xx}(X_s)\big(a(X_s, Y_s) - a\big(X_s, \tilde{Y}_s\big)\big) \\
&\qquad\qquad - h(X_s)\big(Y_s - \tilde{Y}_s\big)\Big] + \|\tilde{Z}_s - Z_s\|^2\Big]\, ds \\
&\le E\big(\tilde{Y}_t - Y_t\big)^2 \\
&\quad - 2\int_u^t E\Big[\big(\tilde{Y}_s - Y_s\big)^2\Big(h(X_s) + |\theta_x(X_s)| \\
&\qquad\qquad \times \int_0^1 b_y\big(X_s, Y_s + \beta\big(\tilde{Y}_s - Y_s\big)\big)\, d\beta \\
&\qquad\qquad - \theta_{xx}(X_s)\int_0^1 a_y\big(Z_s, Y_s + \beta\big(\tilde{Y}_s - Y_s\big)\big)\, d\beta\Big)\Big]\, ds \\
&= E\big(\tilde{Y}_t - Y_t\big)^2 \\
&\quad - 2\int_u^t E\Big[\big(\tilde{Y}_s - Y_s\big)^2\Big(h(X_s) + |\theta_x(X_s)| \\
&\qquad\qquad \times \int_0^1 b_y\big(X_s, \tilde{Y}_s + \beta\big(Y_s - \tilde{Y}_s\big)\big)\, d\beta \\
&\qquad\qquad + \frac{b\big(X_s, \tilde{Y}_s\big)\theta_x(X_s) - h(X_s)\tilde{Y}_s + 1}{a\big(X_s, \tilde{Y}_s\big)} \\
&\qquad\qquad \times \int_0^1 a_y\big(Z_s, \tilde{Y}_s + \beta\big(Y_s - \tilde{Y}_s\big)\big)\Big)\, d\beta\Big]\, ds \qquad (4.23) \\
&= E\big(\tilde{Y}_t - Y_t\big)^2 \\
&\quad - 2\int_u^t E\Bigg[\frac{\big(\tilde{Y}_s - Y_s\big)^2}{a(X_s, Y_s)}\Big(a(X_s, Y_s)h(X_s)
\end{aligned}
$$

$$
\begin{aligned}
&-\big(h(X_s)\tilde{Y}_s - 1\big)\int_0^1 a_y\big(X_s, \tilde{Y}_s + \beta(Y_s - \tilde{Y}_s)\big)\, d\beta \\
&+|\theta_x(X_s)|\int_0^1\Big[a\big(X_s, \tilde{Y}_s\big)b_y\big(X_s, \tilde{Y}_s + \beta\big(Y_s - \tilde{Y}_s\big)\big) \\
&\qquad -b\big(X_s, \tilde{Y}_s\big)a_y\big(X_s, \tilde{Y}_s + \beta\big(Y_s - \tilde{Y}_s\big)\big)\Big)\, d\beta\Bigg]\, ds \\
\le{}& E\big(\tilde{Y}_t - Y_t\big)^2 - \frac{2\eta}{\mu}\int_u^t E\big(\tilde{Y}_s - Y_s\big)^2\, ds.
\end{aligned}
$$

Define $\varphi(t) = E(\tilde{Y}_t - Y_t)^2$ and $\alpha = 2\eta/\mu > 0$. Then (4.23) can be written as

$$
\varphi(u) \le \varphi(t) - \alpha\int_u^t \varphi(s)\, ds, \qquad 0 \le u < t < \infty. \tag{4.24}
$$

Thus,

$$
\begin{aligned}
\frac{d}{dt}\left(e^{-\alpha t}\int_u^t \varphi(s)\, ds\right) &= e^{-\alpha t}\left(\varphi(t) - \alpha\int_u^t \varphi(s)\, ds\right) \\
&\ge e^{-\alpha t}\varphi(u), \qquad t \in [u, \infty).
\end{aligned} \tag{4.25}
$$

Integrating over $[u, T]$, we obtain (note Y and $\tilde{Y}$ are bounded, and so is φ)

$$
\frac{e^{-\alpha u} - e^{-\alpha T}}{\alpha}\varphi(u) \le e^{-\alpha T}\int_0^T \varphi(s)\, ds \le CTe^{-\alpha T}, \qquad T > 0. \tag{4.26}
$$

Therefore, it is necessary that $\varphi(u) = 0$. This implies that

$$
Y_u = \tilde{Y}_u \equiv \theta(X_u), \qquad u \in [0, \infty), \text{ a.s. } \omega \in \Omega. \tag{4.27}
$$

Hence, (X, Y) is a nodal solution. Finally, suppose (X, Y) and $(\hat{X}, \hat{Y})$ are any adapted solutions of Problem IHCR. Then, by the above proof, we must have

$$
Y_t = \theta(X_t), \qquad \hat{Y}_t = \theta\big(\hat{X}_t\big), \qquad t \in [0, \infty). \tag{4.28}
$$

Thus, by (2.16), we see that X_t and $\hat{X}_t$ satisfy the same forward SDE with the same initial condition. Thus, by the uniqueness of the strong solution to such an SDE, $X = \hat{X}$. Consequently, $Y = \hat{Y}$. This proves the theorem. □

Let us indicate an obvious extension of Theorem 4.1 to higher dimensions.

THEOREM 4.5. *Let* $(H1)$ *hold and suppose there exists a solution* θ *to* (3.9) *satisfying*

$$
\begin{aligned}
h(x) - \int_0^1\Bigg[&\sum_{i,j=1}^n a_y^{ij}\big(x, (1-\beta)\theta(x) + \beta\hat{\theta}\big)\theta_{x_i x_j}(x) \\
&- \sum_{i=1}^n b_y^i\big(x, (1-\beta)\theta(x) + \beta\hat{\theta}\big)\theta_{x_i}(x)\Bigg]\, d\beta \ge \eta > 0, \\
&\qquad x \in \mathbb{R}^n,\ \hat{\theta} \in \left[\frac{1}{\gamma}, \frac{1}{\delta}\right].
\end{aligned} \tag{4.29}
$$

Then Problem IHCR has a unique adapted solution. Moreover, this solution is nodal and is determined by the given solution θ.

SKETCH OF THE PROOF. First of all, by an estimate similar to (4.16), we can prove that (3.9) has no other solution except $\theta(x)$. Then, by a proof similar to that of Theorem 4.1, we obtain the conclusion here. □

COROLLARY 4.6. *Let $(H1)$ hold and both a and b be independent of y. Then Problem IHCR has a unique adapted solution which is nodal.*

PROOF. In the present case, condition (4.29) trivially holds. Thus, Theorem 4.5 applies. □

REMARK 4.7. We note that for the case in which a and b are independent of y, the forward equation for X is decoupled from Y. This special case has been studied by Duffie and Lions (1993) in the context of recursive utility models (under weaker regularity conditions). In other words, we can treat the infinite-horizon recursive utility model as a special case.

REMARK 4.8. The fact that h is strictly increasing implies that it has an inverse h^{-1}, so the unique solution (X, Y) to Problem IHCR can be treated as the unique solution (r, Y) to the consol rate problem described in the Introduction, by taking $X = h^{-1}(r)$. Moreover, if h is C^2, Itô's formula implies that we have the unique solution (r, Y) to (1.2) and (1.3), with

$$\begin{aligned} \mu(r,y) &= h'(h^{-1}(r))b(h^{-1}(r),y) \\ &\quad + \tfrac{1}{2}h''(h^{-1}(r))\|\sigma(h^{-1}(r),y)\|^2, \\ \alpha(r,y) &= h'(h^{-1}(r))\sigma(h^{-1}(r),y). \end{aligned} \tag{4.30}$$

Indeed, this unique solution is obtained at $Y = \theta(h^{-1}(r))$, where θ is the unique solution of (4.6). The use of X rather than r as the "forward" state variable for this problem is due simply to the fact that it is easier to state regularity conditions in terms of (b, σ, h) than directly in terms of (μ, α).

REMARK 4.9. As a point of reference, we note that Itô's formula implies that (r, Y) solves (1.2) and (1.3) if and only if $(r, l = Y^{-1})$ solves the SDE

$$\begin{aligned} dr_t &= \hat{\mu}(r_t, l_t)\,dt + \hat{\sigma}(r_t, l_t)\,dW_t, \\ dl_t &= \left(l_t^2 - r_t l_t + \frac{l_t^3}{2}\|A(r_t, l_t)\|^2\right)dt + \hat{A}\,(r_t, l_t)\,dW_t, \end{aligned} \tag{4.31}$$

where

$$\begin{aligned} \hat{\mu}(r,l) &= \mu(r, l^{-1}), \\ \hat{\alpha}(r,l) &= \alpha(r, l^{-1}), \\ \hat{A}\,(r,l) &= -l^2 A(r, l^{-1}). \end{aligned} \tag{4.32}$$

Thus the same characterization given above can equally well be given in terms of ($\hat{\mu}$, $\hat{\alpha}$, $\hat{A}$).

5. The limit of Problem FHV. In Section 2 we posed the consol rate problems in both finite- and infinite-horizon cases. Practically, it would be nice to know whether the limit of the Problem FHV is the Problem IHCR. The purpose of this section is to show that this is indeed the case, under certain conditions.

We first prove the following lemma.

LEMMA 5.1. *Let w be the classical solution of the equation*

$$(5.1)\qquad \begin{gathered} w_t - \sum_{ij=1}^{n} a^{ij}(x,t) w_{x_i x_j} - \sum_{i=1}^{n} b^i(x,t) w_{x_i} + c(x,t) w = 0, \\ (x,t) \in \mathbb{R}^n \times [0,\infty), \qquad w|_{t=0} = w_0(x). \end{gathered}$$

Suppose that

$$(5.2)\qquad \begin{gathered} \lambda I \le \big(a^{ij}(x,t)\big) \le \mu I, \\ |b^i(x,t)| \le C, \qquad 1 \le i \le n, \\ c(x,t) \ge \eta > 0, \qquad (x,t) \in \mathbb{R}^n \times [0,\infty), \\ |w_0(x)| \le M, \end{gathered}$$

for some positive constants λ, μ, η, C and M. Then

$$(5.3)\qquad |w(x,t)| \le M e^{-\eta t}, \qquad (x,t) \in \mathbb{R}^n \times [0,\infty).$$

PROOF. First, let $R > 0$ and consider the following initial-boundary value problem:

$$(5.4)\qquad \begin{gathered} w_t^R - \sum_{i,j=1}^{n} a^{ij}(x,t) w_{x_i x_j}^R - \sum_{i=1}^{n} b^i(x,t) w_{x_i}^R + c(x,t) w^R = 0, \\ (x,t) \in B_R \times [0,\infty), \qquad w^R|_{\partial B_R} = 0, \qquad w^R|_{t=0} = w_0(x)\chi^R(x), \end{gathered}$$

where B_R is the ball of radius $R > 0$ centered at 0 and χ^R is some "cutoff" function. Then, we know that (5.4) admits a unique classical solution $w^R \in C^{2+\alpha, 1+\alpha/2}(B_R \times [0,\infty))$ for some $\alpha > 0$, where $C^{2+\alpha, 1+\alpha/2}$ is the space of all functions $v(x,t)$ which are C^2 in x and C^1 in t with Hölder continuous $v_{x_i x_j}$ and v_t of exponent α and $\alpha/2$, respectively. Moreover, we have

$$(5.5)\qquad |w^R(x,t)| \le M, \qquad (x,t) \in B_R \times [0,\infty),$$

and for any $x_0 \in \mathbb{R}^n$ and $T > 0$ $(0 < \alpha' < \alpha)$,

$$w^R \to w \text{ in } C^{2+\alpha', 1+\alpha'/2}(B_1(x_0) \times [0,T]) \text{ as } R \to \infty, \tag{5.6}$$

where w is the solution of (5.1). Now, we let $\psi(x,t) = Me^{-(\eta-\varepsilon)t}$ $(\varepsilon > 0)$. Then

$$\begin{aligned} &\psi_t - \sum_{i,j=1}^n a^{ij}(x,t)\psi_{x_i x_j} - \sum_{i=1}^n b^i(x,t)\psi_{x_i} + c(x,t)\psi \\ &\qquad = (c(x,t) - \eta + \varepsilon)M^{-(\eta-\varepsilon)t} \geq \varepsilon M^{-(\eta-\varepsilon)t} > 0, \\ &\psi|_{\partial B_R} > 0 = w^R|_{\partial B_R}, \qquad \psi|_{t=0} = M \geq w_0(x) = w^R|_{t=0}. \end{aligned} \tag{5.7}$$

Thus, by Friedman [(1964), Chapter 2, Theorem 16], we have

$$w^R(x,t) \leq \psi(x,t) = M^{-(\eta-\varepsilon)t}, \qquad (x,t) \in B_R \times [0,\infty). \tag{5.8}$$

Similarly, we can prove that

$$w^R(x,t) \geq -Me^{-(\eta-\varepsilon)t}, \qquad (x,t) \in B_R \times [0,\infty). \tag{5.9}$$

Since the right-hand sides of (5.8) and (5.9) are independent of R, we see that

$$|w(x,t)| \leq Me^{-(\eta-\varepsilon)t}, \qquad (x,t) \in \mathbb{R}^n \times [0,\infty). \tag{5.10}$$

Hence, (5.3) follows by sending $\varepsilon \to 0$. □

Our main result of this section is the following theorem.

THEOREM 5.2. *Let (H1) and (H2) hold and let θ be a solution of* (3.9) *with the property* (4.29). *Let (X^K, Y^K) be the nodal solution of Problem FHV in $[0, K]$ and let (X, Y) be the nodal solution of Problem IHCR determined by θ. Then,*

$$\lim_{K \to \infty} E|Y_t^K - Y_t|^2 + E|X_t^K - X_t|^2 = 0, \tag{5.11}$$

uniformly in t on compacts.

PROOF. By Proposition 3.1, we see that (X_t^K, Y_t^K) satisfies

$$Y_t^K = \theta^K(X_t^K, t), \qquad t \in [0, K], \text{a.s. } \omega \in \Omega, \tag{5.12}$$

where θ^K is the solution of the parabolic equation

$$\begin{aligned} &\theta_t^K + \sum_{i,j=1}^n a^{ij}(x, \theta^K)\theta_{x_i x_j}^K + \sum_{i=1}^n b^i(x, \theta^K)\theta_{x_i}^K - h(x)\theta^K + 1 = 0, \\ &\qquad (x,t) \in \mathbb{R}^n \times [0,T), \qquad \theta^K|_{t=T} = g(x). \end{aligned} \tag{5.13}$$

Next, we define φ to be the solution of

$$\begin{aligned} &\varphi_t - \sum_{i,j=1}^n a^{ij}(x, \varphi)\varphi_{x_i x_j} - \sum_{i=1}^n b^i(x, \varphi)\varphi_{x_i} + h(x)\varphi - 1 = 0, \\ &\qquad (x,t) \in \mathbb{R}^n \times (0,\infty), \qquad \varphi|_{t=0} = g(x). \end{aligned} \tag{5.14}$$

Clearly, we have

$$\theta^K(x,t) = \varphi(x, K-t), \qquad (x,t) \in \mathbb{R}^n \times [0, K]. \tag{5.15}$$

Now, we let $w(x,t) = \varphi(x,t) - \theta(x)$. Then

$$\begin{aligned} w_t - \sum_{i,j=1}^n a^{ij}(x,\varphi) w_{x_i x_j} - \sum_{i=1}^n b(x,\varphi) w_{x_i} \\ - \Bigg[h(x) - \int_0^1 \Bigg(\sum_{i,j=1}^n a_y^{ij}(x, \theta + \beta w)\theta_{x_i x_j} \\ + \sum_{i=1}^n b_y^i(x, \theta+\beta w)\theta_{x_i}\Bigg) d\beta \Bigg] w = 0, \\ w|_{t=0} = g(x) - \theta(x). \end{aligned} \tag{5.16}$$

We note that both $\varphi(x,t)$ and $\theta(x)$ lie in $[1/\gamma, 1/\delta]$. Thus, by condition (4.29) and Lemma 5.1, we see that

$$\begin{aligned} |\theta^K(x,t) - \theta(x)| &= |\varphi(x, K-t) - \theta(x)| \\ &\le \frac{1}{\delta} e^{-\eta(K-t)}, \qquad (x,t) \in \mathbb{R}^n \times [0,K], K > 0. \end{aligned} \tag{5.17}$$

Now, we look at the following forward SDEs:

$$\begin{aligned} dX_t^K &= b\big(X_t^K, \theta^K(X_t^K, t)\big)\,dt + \sigma\big(X_t^K, \theta^K(X_t^K,t)\big)\,dW_t, \\ X_0^K &= x; \end{aligned} \tag{5.18}$$

$$\begin{aligned} dX_t &= b(X_t, \theta(X_t))\,dt + \sigma(X_t, \theta(X_t))\,dW_t, \\ X_0 &= x. \end{aligned} \tag{5.19}$$

By Itô's formula, we have

$$\begin{aligned} E|X_t^K - X_t|^2 &= E\int_0^t \Big[2\big\langle X_s^K - X_s, b\big(X_s^K, \theta^K(X_s^K, s)\big) - b(X_s, \theta(X_s))\big\rangle \\ &\qquad + \mathrm{tr}\Big(\big[\sigma\big(X_s^K, \theta^K(X_s^K, s)\big) - \sigma(X_s, \theta(X_s))\big] \\ &\qquad \times \big[\sigma\big(X_s^K, \theta^K(X_s^K,s)\big) - \sigma(X_s, \theta(X_s))\big]^T\Big)\Big]\,ds \\ &\le CE\int_0^t \Big[|X_s^K - X_s| \big(|X_s^K - X_s| + \big|\theta^K(X_s^K, s) - \theta(X_s^K)\big|\big) \\ &\qquad + \big(|X_s^K - X_s| + \big|\theta^K(X_s^K,s) - \theta(X_s^K)\big|\big)^2\Big]\,ds \\ &\le C\int_0^t \big[E|X_s^K - X_s|^2 + \exp(-2\eta(K-s))\big]\,ds \\ &\le C\int_0^t |X_s^K - X_s|^2\,ds + C\exp(-2\eta(K-t)). \end{aligned} \tag{5.20}$$

Applying Gronwall's inequality, we obtain that

$$E\left(|X_t^K - X_t|^2\right) \le Ce^{-2\eta(K-t)}, \qquad t \in [0, K], K > 0. \tag{5.21}$$

Furthermore,

$$\begin{aligned} E\left(|Y_t^K - Y_t|^2\right) &= E\left(\left|\theta^K(X_t^K, t) - \theta(X_t)\right|^2\right) \\ &\le 2E\left(\left|\theta^K(X_t^K, t) - \theta(X_t^K)\right|^2\right) + 2E\left(\left|\theta(X_t^K) - \theta(X_t)\right|^2\right) \\ &\le Ce^{-2\eta(K-t)} + CE\left(|X_t^K - X_t|^2\right) \le Ce^{-2\eta(K-t)}, \\ &\qquad t \in [0, K], K > 0. \end{aligned} \tag{5.22}$$

Finally, let $K \to \infty$, the conclusion follows. □

REMARK 5.3. From Section 4, we see that for the one-dimensional case, under (H3), the solution θ of (3.9) satisfies (4.29). Thus, the result of Theorem 5.2 holds under (H1)–(H3).

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REFERENCES

BLACK, F. (1993). Private communication.

BRENNAN, M. and SCHWARTZ, E. (1979). A continuous time approach to the pricing of bonds. *J. Banking Finance* **3** 133–155.

CRANDALL, M. G., ISHII, H. and LIONS, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.* (*NS*) **27** 1–67.

DELLACHERIE, C. and MEYER, P. A. (1982). *Probabilities and Potentials B*. North-Holland, Amsterdam.

DUFFIE, D., GEOFFARD, P.-Y. and SKIADAS, C. (1994). Efficient and equilibrium allocations with stochastic differential utility. *J. Math. Econom.* **23** 133–146.

DUFFIE, D. and LIONS, P.-L. (1993). PDE solutions of stochastic differential utility. *J. Math. Econom.* **21** 577–606.

EPSTEIN, L. (1987). The global stability of efficient intertemporal allocations. *Econometrica* **55** 329–355.

FRIEDMAN, A. (1964). *Partial Differential Equations of Parabolic Type*. Prentice-Hall, Englewood Cliffs, NJ.

FRIEDMAN, A. (1975). *Stochastic Differential Equations and Applications* **1**. Academic Press, New York.

GILBARG, D. and TRUDINGER, N. S. (1977). *Elliptic Partial Differential Equations of Second Order*. Springer, Berlin.

HARRISON, M. and KREPS, D. (1979). Martingales and arbitrage in multiperiod securities markets. *J. Econom. Theory* **20** 381–408.

HOGAN, M. (1993). Problems in certain two factor term structure models. *Ann. Appl. Probab.* **3** 576–581.

KARATZAS, I. and SHREVE, S. E. (1988). *Brownian Motion and Stochastic Calculus*. Springer, Berlin.

MA, J. and YONG, J. (1993). Solvability of forward–backward SDE's and the nodal set of Hamilton–Jacobi–Bellman equations. Preprint.

MA, J., PROTTER, P. and YONG, J. (1993). Solving forward–backward stochastic differential equations explicitly—A four step scheme. Preprint.

PROTTER, P. (1990). *Stochastic Integration and Differential Equations. A New Approach*. Springer, Berlin.

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