Supplementary Results for Information Percolation in Segmented Markets

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Abstract: This supplement to "Information Percolation in Segmented Markets" houses Appendices D through L of the main paper, Duffie, Malamud, and Manso (2013).

This supplement of Duffie, Malamud, and Manso (2013) houses the following appendices of our paper, "Information Percolation in Segmented Markets." Appendices A through C are found in the main paper.

Supplementary Appendices

D. A proof of the existence and uniqueness of a strictly monotone equilibrium of the double auction.

E. An application of results from Appendix D to obtain approximations for the double-auction equilibrium for the case of large gains from trade.

F. An application of the results of Appendix E to derive approximations for the expected trading profits for the case of large gains from trade.

G. An application of the results of Appendix F to approximate the expected gains from information acquisition for the case of large gains from trade. From this, a derivation of general properties of equilibrium information acquisition. Appendices G.1 and G.2 apply these results to special cases of one and two classes of sellers, respectively.

H. The two-class model.

I. Results on endogenous investment in matching technology.

J. Proofs of results in Section I, on endogenous investment in matching technology.

K. Results for the case of dynamic information acquisition.

L. Proofs of results in Appendix K.

D Existence of Equilibrium for the Double Auction

Existence of equilibrium follows from Proposition 4.6 and the following general result.

Proposition D.1 *Fix a buyer class b and a seller class s such that*

$$\psi_b^H(x) \sim \mathrm{Exp}_{+\infty}(c, \gamma, -\alpha) \tag{1}$$

for some $c, \alpha > 0$ and some $\gamma \in \mathbb{R}$. If $\alpha < 1$, then there is no equilibrium associated with $V_0 = -\infty$. Suppose, however, that $\alpha > \alpha^*$ and that

$$\begin{aligned} -\gamma &< \frac{(\alpha+1)\log\alpha}{\log(\alpha+1)-\log\alpha}, && \text{if } \alpha \geq 2 \\ -\gamma &< \frac{\log(\alpha^2-\alpha)\,2^\alpha}{\log(\alpha+1)-\log\alpha}, && \text{if } \alpha < 2. \end{aligned}$$

Then, if the gain from trade $\overline{G}$ is sufficiently large, there exists a unique strictly monotone equilibrium with $V_0 = -\infty$. This equilibrium is in undominated strategies, and maximizes total welfare among all continuous nondecreasing equilibrium bidding policies.

In order to prove Proposition D.1, we apply the following auxiliary result.

Lemma D.2 *Suppose that $B, S : \mathbb{R} \to (v_b, v^H)$ are strictly increasing and that their inverses V_s and V_b satisfy*

$$v_b + \Delta_b P(V_s(z) + V_b(z)) = z.$$

Suppose further that $V_b'(z)$ solves (12) for all $z \in (v_b, v^H)$. Then $(B\,,\,S)$ is an equilibrium.

Proof. Recall that the seller maximizes

$$f_S(s) = \int_{V_b(s)}^{+\infty} (s - v_s - \Delta_s P(\theta+\phi))\ \Psi_b(P(\theta),\phi)\,d\phi. \tag{2}$$

To show that $S(\theta)$ is indeed optimal, it suffices to show that $f_S'(s) \geq 0$ for $s \leq S(\theta)$ and that $f_S'(s) \leq 0$ for $s \geq S(\theta)$. We prove only the first inequality. A proof of the second is analogous. So, let $s \leq S(\theta) \Leftrightarrow V_s(s) \leq \theta$. Then,

$$\begin{aligned} f_S'(s) &= V_b'(s)\,(-s + v_s + \Delta_s P(\theta + V_b(s)))\ \Psi_b(P(\theta), V_b(s)) + G_b(P(\theta), V_b(s)) \\ &= V_b'(s)\Psi_b(P(\theta), V_b(s))\left(-s + v_s + \Delta_s P(\theta+V_b(s)) + \frac{1}{V_b'(s) h_b(P(\theta), V_b(s))}\right). \end{aligned}$$

By Lemma 4.1, $h_b(p, V_b(s))$ is monotone decreasing in p. Therefore, by (26),

$$\frac{1}{V_b'(s)\, h_b(P(\theta), V_b(s))} \;\geq\; \frac{1}{V_b'(s)\, h_b(P(V_s(S)), V_b(s))} \;=\; s \,-\, v_s \,-\, \Delta_s\, P(V_s(s) + V_b(s)).$$

Hence,

$$\begin{aligned} f_S'(s) \quad &\geq \quad V_b'(s)\, \Psi_b(P(\theta), V_b(s)) \\ &\qquad \times (-s + v_s \,+ \Delta_s P(\theta + V_b(s)) \;+\; s \,-\, v_s \,-\, \Delta_s\, P(V_s(s) + V_b(s))) \;\geq\; 0, \end{aligned}$$

because $\theta \;\geq\; V_s(s)\,.$

For the buyer, it suffices to show that

$$f_B(b) \;=\; \max_b \int_{-\infty}^{V_s(b)} \big(v_b \,+\, \Delta_b P(\theta + \,\phi) \;-\; S(\theta)\big)\, \Psi_s(P(\phi), \theta)\, d\theta \tag{3}$$

satisfies $f_B'(b) \;\geq\; 0$ for $b \;\leq\; B(\phi)$, and satisfies $f_B'(b) \;\leq\; 0$ for $b \,\geq\, B(\phi)\,$. That is,

$$v_b \,+\, \Delta_b\, P(\phi \,+\, V_s(b)) \,-\, S(V_s(b)) \;=\; v_b \,+\, \Delta_b\, P(\phi \,+\, V_s(b)) \,-\, b \;\geq\; 0$$

for $b \;\leq B(\phi)$, and the reverse inequality for $b \geq B(\phi)$. For $b \;\leq B(\phi)$, we have $\phi \;\geq\; V_b(b)$ and therefore

$$v_b \,+\, \Delta_b\, P(\phi \,+\, V_s(b)) \,-\, b \;\geq\; v_b \,+\, \Delta_b\, P(V_b(b) \,+\, V_s(b)) \,-\, b \;=\; 0,$$

as claimed. The case of $b \geq B(\phi)$ is analogous. ■

Proof of Proposition D.1. It follows from Proposition 4.3 and Lemma D.2 that a strictly monotone equilibrium in undominated strategies exists if and only if there exists a solution $V_b(z)$ to (12) such that $V_b(v_b) \;=\; -\infty$ and

$$V_s(z) \;=\; \log \frac{z - v_b}{v^H - z} \;-\; V_b(z) \,-\, \log R$$

is monotone increasing in z and satisfies $V_s(v_b) \,=\, -\infty$, $V_s(v^H) \;=\, +\infty$. Furthermore, such an equilibrium is unique if the solution to the ODE (12) with $V_b(v_b) \;=\; -\infty$ is unique.

Fix a $t \leq T$ and denote for brevity $\gamma \;=\; \gamma_{it}$, $c \;=\; c_{it}$. Let also

$$g(z) \;=\; e^{(\alpha+1)\, V_b(z)}\,.$$

Then, a direct calculation shows that $V_b(z)$ solves (12) with $V_b(v_b) \;=\; -\infty$ if and only if $g(z)$ solves

$$\begin{aligned} &g'(z) \\ &= g(z)\, \frac{\alpha+1}{v_b - v_s} \left(\frac{z - v_b}{v^H - z}\, \frac{1}{h_b^H((\alpha+1)^{-1} \log g(z))} \,+\, \frac{1}{h_b^L((\alpha+1)^{-1} \log g(z))} \right), \end{aligned} \tag{4}$$

with $g(v_b) = 0$. By assumption and Lemma 4.1 ,

$$h_b^H(V) \sim c_i\,|V|^\gamma\, e^{(\alpha+1)V} \quad \text{and} \quad h_b^L(V) \sim c_i\,|V|^\gamma\, e^{\alpha V} \tag{5}$$

as $V \to -\infty$ because both $G_b^H(V)$ and $G_b^L(V)$ converge to 1. Hence, the right-hand side of (4) is continuous and the existence of a solution follows from the Euler theorem. Furthermore, when studying the asymptotic behavior of $g(z)$ as $z \downarrow v_b$, we can replace h_b^H and h_b^L by their respective asymptotics (5).

Indeed, let us consider

$$\begin{aligned}\tilde{g}'(z) \;=\; (\alpha+1)\,\tilde{g}(z)\,\frac{1}{v_b - v_s}\Bigg(&\frac{z-v_b}{v^H - z}\,\frac{1}{c\,((\alpha+1)^{-1}\log 1/\tilde{g})^\gamma\,\tilde{g}} \\ &+ \frac{1}{c((\alpha+1)^{-1}\log 1/\tilde{g})^\gamma\,\tilde{g}^{\alpha/(\alpha+1)}}\Bigg),\end{aligned} \tag{6}$$

with the initial condition $\tilde{g}(v_b) = 0$. We consider only values of z sufficiently close to v_b, so that $\log \tilde{g}(z) < 0$.

It follows from standard ODE comparison arguments and the results below that for any $\varepsilon > 0$ there exists a $\bar{z} > v_b$ such that

$$\left|\frac{g(z)}{\tilde{g}(z)} - 1\right| + \left|\frac{g'(z)}{\tilde{g}'(z)} - 1\right| \le \varepsilon \tag{7}$$

for all $z \in (v_b, \bar{z})$. The assumptions of the Proposition guarantee that the same asymptotics hold for the derivatives of the hazard rates, which implies that the estimates obtained in this manner are uniform.

First, we will consider the case of general (not necessarily large) $v_b - v_s$ and show that, when $\alpha < 1$, $g(z)$ decays so fast as $z \downarrow v_b$ that $V_s(z)$ cannot remain monotone increasing. A similar argument then implies that $V_s(z)$ cannot remain monotone increasing when $\overline{G}\alpha < 1$.

At points in the proof, we will define suitable positive constants denoted C_1, C_2, $C_3, \ldots$ without further mention.

Denote

$$\zeta = \frac{(\alpha+1)^{\gamma+1}}{c\,(v_b - v_s)}\,. \tag{8}$$

Then, we can rewrite (6) in the form

$$\tilde{g}'(z) = \frac{\zeta}{(\log 1/\tilde{g})^\gamma}\left(\frac{z - v_b}{v^H - z} + \tilde{g}^{1/(\alpha+1)}\right). \tag{9}$$

From this point, throughout the proof, without loss of generality, we assume that $v_b = 0$. Furthermore, after rescaling if necessary, we may assume that $v^H - v_b = 1$. Then, the same asymptotic considerations as above imply that, when studying the behavior of $\tilde{g}$ as $z \downarrow v_b$, we may replace $v^H - z \sim v^H - v_b$ in (6) by 1.

Let $A(z)$ be the solution to

$$z \; = \; \int_0^{A(z)} \zeta^{-1} \, (-\log x)^{\gamma} \, x^{-1/(\alpha+1)} \, dx \, .$$

A direct calculation shows that

$$B(z) \; \stackrel{def}{=} \; \int_0^z \zeta^{-1} \, (-\log x)^{\gamma} \, x^{-1/(\alpha+1)} \, dx \; \sim \; \zeta^{-1} \frac{\alpha+1}{\alpha} \, (-\log z)^{\gamma} \, z^{\alpha/(\alpha+1)} \, .$$

Conjecturing the asymptotics

$$A(z) \; \sim \; K \, (-\log z)^{\gamma(\alpha+1)/\alpha} \, z^{(\alpha+1)/\alpha} \tag{10}$$

and substituting these into $B(A(z)) = z$, we get

$$K \; = \; \zeta^{\frac{\alpha+1}{\alpha}} \left(\frac{\alpha}{\alpha+1} \right)^{\frac{(\gamma+1)(\alpha+1)}{\alpha}} \, .$$

Standard considerations imply that this is indeed the asymptotic behavior of $A(z)$. It is then easy to see that

$$A'(z) \; \sim \; K \, \frac{\alpha+1}{\alpha} \, (-\log z)^{\gamma(\alpha+1)/\alpha} \, z^{1/\alpha}. \tag{11}$$

By (9),

$$\tilde{g}'(z) \; \geq \; \frac{\zeta}{(\log 1/\tilde{g})^{\gamma}} \, \tilde{g}^{1/(\alpha+1)}.$$

Integrating this inequality, we get $\tilde{g}(z) \; \geq \; A(z)$. Now, the factor $(\log 1/\tilde{g})^{\gamma}$ is asymptotically negligible as $z \downarrow v_b$. Namely, for any $\varepsilon > 0$ there exists a $C_1 > 0$ such that

$$C_1 \, \tilde{g}^{1/(\alpha+\varepsilon+1)} \; \geq \; \frac{\zeta}{(\log 1/\tilde{g})^{\gamma}} \, \tilde{g}^{1/(\alpha+1)} \; \geq \; C_1^{-1} \, \tilde{g}^{1/(\alpha-\varepsilon+1)}.$$

Thus,

$$\left((\tilde{g})^{\frac{\alpha-\varepsilon}{1+\alpha-\varepsilon}} \right)' \; \geq \; C_2 \, .$$

Integrating this inequality, we get that

$$\tilde{g}(z) \; \geq \; C_3 \, (z - v_b)^{\frac{\alpha-\varepsilon+1}{\alpha-\varepsilon}} \, . \tag{12}$$

Let

$$l(z) \;=\; B(\tilde g(z)) \,-\, z\,.$$

Then, for small z, by (10),

$$\begin{aligned} l'(z) \;&=\; \tilde g'(z)\,\zeta^{-1}\,(-\log\tilde g)^{\gamma}\,\tilde g^{-1/(\alpha+1)} \;-\; 1 \\ &=\; \frac{\zeta}{(\log 1/\tilde g)^{\gamma}}\left(\frac{z}{v^H - z} \,+\, \tilde g^{1/(\alpha+1)}\right)\zeta^{-1}\,(-\log\tilde g)^{\gamma}\,\tilde g^{-1/(\alpha+1)} \;-\; 1 \\ &=\; \frac{z}{1-z}\,\frac{1}{\tilde g^{1/(\alpha+1)}} \qquad (13)\\ &=\; \frac{z}{1-z}\,\frac{1}{(A(l(z)+z))^{1/(\alpha+1)}} \\ &\le\; \frac{z}{1-z}\,\frac{1}{(A(l(z)))^{1/(\alpha+1)}}\,, \end{aligned}$$

where we have used the fact that $l(z) \ge 0$ because $h(0) = 0$ and $l'(z) \ge 0$. Integrating this inequality, we get that, for small z,

$$l(z) \;\le\; C_4\, z^{2(\alpha-\varepsilon)/(\alpha-\varepsilon+1)}.$$

Hence, for small z,

$$\tilde g(z) \;=\; A(l(z) \,+\, z) \;\le\; A((C_4+1)z^{2(\alpha-\varepsilon)/(\alpha-\varepsilon+1)}) \;\le\; C_5\, z^{2-\varepsilon}. \qquad (14)$$

Let $C(z)$ solve

$$\int_0^{C(z)} (-\log x)^{\gamma}\, dx \;=\; \zeta \int_0^z \frac{x}{1-x}\, dx\,.$$

A calculation similar to that for the function $A(z)$ implies that

$$C(z) \;\sim\; C_6\,(-\log z)^{\gamma}\, z^2 \qquad (15)$$

as $z \to 0$. Integrating the inequality

$$\tilde g'(z) \;\ge\; \frac{\zeta}{(-\log\tilde g)^{\gamma}}\,\frac{z}{1-z},$$

we get that

$$\tilde g(z) \;\ge\; C(z).$$

Let now $\alpha < 1$. Then, (14) immediately yields that the second term in the brackets in (6) is asymptotically negligible and, consequently,

$$\frac{\zeta}{(\log 1/\tilde g)^{\gamma}}\,\frac{z}{1-z} \;\le\; \tilde g'(z) \;\le\; \frac{(1+\varepsilon)\,\zeta}{(\log 1/\tilde g)^{\gamma}}\,\frac{z}{1-z} \qquad (16)$$

holds for sufficiently small z. Integrating this inequality implies that

$$C(z) \;\le\; \tilde{g}(z) \;\le\; (1+\varepsilon)\, C(z).$$

Now, (16) implies that

$$(1-\varepsilon)\, 2\, C(z) z^{-1} \;\le\; \tilde{g}'(z) \;\le\; 2\,(1+\varepsilon)\, C(z)\, z^{-1}$$

for sufficiently small[1] z.

Using the asymptotics (5) and repeating the same argument implies that $g(z)$ also satisfies these bounds. (The calculations for g are lengthier and omitted here.)

Now,

$$V_b'(z) \;=\; \frac{g'(z)}{(\alpha+1)\, g(z)} \;\ge\; (1-\varepsilon)\, \frac{2}{\alpha+1}\, z^{-1}.$$

Therefore,

$$V_s'(z) \;=\; \frac{1}{z\,(1-z)} \;-\; V_b'(z) \;<\; 0$$

for sufficiently small z. Thus, $V_s(z)$ cannot be monotone increasing and the equilibrium does not exist.

Let now $\alpha > 1$. We will now show that there exists a unique solution to (4) with $g(0) = 0$. Since the right-hand side loses Lipschitz continuity only at $z = 0$, it suffices to prove local uniqueness at $z = 0$. Hence, we need only consider the equation in a small neighborhood of $z = 0$. (It is recalled that we assume $v_b = 0$.)

As above, we prove the result directly for the ODE (6), and then explain how the argument extends directly to (4).

Suppose, to the contrary, that there exist two solutions $\tilde{g}_1$ and $\tilde{g}_2$ to (6). Define the corresponding functions l_1 and l_2 via $l_i \;=\; B(\tilde{g}_i) - z$. Both functions solve (13). Integrating over a small interval $[0, l]$, we get

$$|l_1(x) \;-\; l_2(x)| \;\le\; \int_0^x \frac{z}{1-z} \left| \frac{1}{(A(l_1(z)+z))^{1/(\alpha+1)}} \;-\; \frac{1}{(A(l_2(z)+z))^{1/(\alpha+1)}} \right| dz\,. \quad (17)$$

Now, we will use the following elementary inequality: There exists a constant $C_6 > 0$ such that

$$a^{1/\alpha} - b^{1/\alpha} \;\le\; \frac{C_6\,(a-b)}{a^{(\alpha-1)/\alpha} \,+\, b^{(\alpha-1)/\alpha}} \quad (18)$$

for $a > b > 0$. Indeed, let $x = b/a$ and $\beta = 1/\alpha$. Then, we need to show that

$$(1 + x^{1-\beta})\,(1 \;-\; x^{\beta}) \;\le\; C_6\,(1-x)$$

[1]We are using the same ε in all of these formulae. This can be achieved by shrinking if necessary the range of z under consideration.

for $x \in (0,1)$. That is, we must show that

$$x^{1-\beta} \; - \; x^{\beta} \;\leq\; (C_6 - 1)\,(1-x).$$

By continuity and compactness, it suffices to show that the limit

$$\lim_{x \to 1} \frac{x^{1-\beta} \; - \; x^{\beta}}{1-x}$$

is finite. This follows from L'Hôpital's rule.

By (10) and (11), we can replace the function $A(z)$ in (17) by its asymptotics (10) at the cost of getting a finite constant in front of the integral. Thus, for small z,

$$\begin{aligned} &|l_1(x) \; - \; l_2(x)| \\ &\leq \; C_7 \int_0^x z \left| \frac{((-\log(l_1+z))^{\gamma}\,(l_1+z))^{1/\alpha} - ((-\log(l_2+z))^{\gamma}\,(l_2+z))^{1/\alpha}}{((-\log(l_1+z))^{\gamma}\,(l_1+z))^{1/\alpha}((-\log(l_2+z))^{\gamma}\,(l_2+z))^{1/\alpha}} \right| dz\,. \end{aligned} \tag{19}$$

By (18),

$$\begin{aligned} &|((-\log(l_1+z))^{\gamma}\,(l_1+z))^{1/\alpha} - ((-\log(l_2+z))^{\gamma}\,(l_2+z))^{1/\alpha}| \\ &\leq \; C_6 \, \frac{|(-\log(l_1+z))^{\gamma}\,(l_1+z) \; - \; (-\log(l_2+z))^{\gamma}\,(l_2+z)|}{((-\log(l_1+z))^{\gamma}\,(l_1+z))^{(\alpha-1)/\alpha} + ((-\log(l_2+z))^{\gamma}\,(l_2+z))^{(\alpha-1)/\alpha}}\,. \end{aligned} \tag{20}$$

Now, consider some $\gamma > 0$. Then, for any sufficiently small $a > b > 0$, a direct calculation shows that

$$0 \;<\; (\log(1/a))^{\gamma}\,a \; - \; (\log(1/b))^{\gamma}\,b \;\leq\; ((\log(1/a))^{\gamma} \; + \; (\log(1/b))^{\gamma})\,(a-b).$$

If, instead, $\gamma \leq 0$, then the function $a \mapsto (\log(1/a))^{\gamma}\,a$ is continuously differentiable at $a = 0$, and hence

$$0 \;<\; (\log(1/a))^{\gamma}\,a \; - \; (\log(1/b))^{\gamma}\,b \;\leq\; C_8\,(a-b)\,.$$

Since $\alpha > 1$, the same calculation as that preceding (16) implies that, for sufficiently small z,

$$A(z) \;\leq\; \tilde{g}_i(z) \;=\; A(z + l_i(z)) \;\leq\; (1+\varepsilon)\,A(z)\;,\;\; i \;=\; 1,\,2.$$

Thus, for $z \in [0, \bar{\varepsilon}]$,

$$\begin{aligned} &\left| \frac{((-\log(l_1+z))^{\gamma}\,(l_1+z))^{1/\alpha} - ((-\log(l_2+z))^{\gamma}\,(l_2+z))^{1/\alpha}}{((-\log(l_1+z))^{\gamma}\,(l_1+z))^{1/\alpha}((-\log(l_2+z))^{\gamma}\,(l_2+z))^{1/\alpha}} \right| \\ &\quad\leq\; C_9\,|l_1(z) \; - \; l_2(z)|\,\frac{1}{z^{((\alpha+1)/\alpha)-\varepsilon}} \\ &\quad\leq\; C_9 \left(\sup_{z \in [0,\bar{\varepsilon}]} \; |l_1(z) \; - \; l_2(z)| \right) \frac{1}{z^{((\alpha+1)/\alpha)-\varepsilon}}. \end{aligned} \tag{21}$$

Thus, (19) implies that

$$
\begin{aligned}
|l_1(x) \;-\; l_2(x)| \le\; & C_{10}\left(\sup_{z\,\in\,[0,\bar\varepsilon]} |l_1(z) \;-\; l_2(z)|\right)\int_0^x z\,\frac{1}{z^{((\alpha+1)/\alpha)+\varepsilon}}\,dz \\
=\; & C_{11}\,(\bar\varepsilon)^{\frac{\alpha-1}{\alpha}-\varepsilon}\sup_{z\,\in\,[0,\bar\varepsilon]} |l_1(z) \;-\; l_2(z)|
\end{aligned}
\tag{22}
$$

for all $l \le \bar\varepsilon$. Taking the supremum over $l \in [0,\bar\varepsilon]$, we get

$$
\sup_{z\,\in\,[0,\bar\varepsilon]} |l_1(z) \;-\; l_2(z)| \;\le\; C_{11}\,(\bar\varepsilon)^{\frac{\alpha-1}{\alpha}-\varepsilon}\sup_{z\in[0,\bar\varepsilon]} |l_1(z) \;-\; l_2(z)|\,.
$$

Picking $\bar\varepsilon$ so small that $C_{11}\,(\bar\varepsilon)^{\frac{\alpha-1}{\alpha}-\varepsilon} < 1$ immediately yields that $l_1 = l_2$ on $[0,\bar\varepsilon]$ and hence, since the right-hand side of (6) is Lipschitz continuous for $z\,l \ne 0$, we have $l_1 = l_2$ for all z by a standard uniqueness result for ODEs.

The fact that the same result holds for the original equation (4) follows by the same arguments as above.

It remains to prove the last claim, namely the existence of equilibrium for sufficiently large $v_b - v_s$. By Proposition 4.3, it suffices to show that

$$
V_s'(z) \;=\; \frac{1}{z\,(1-z)} \;-\; V_b'(z) \;>\; 0 \tag{23}
$$

for all $z \in (0,1)$ provided that $v_b - v_s$ is sufficiently large.

It follows from the proof of Lemma 4.1 that

$$
G_L^{-1}\left((1-z)^{\frac{1}{(v_b-v_s)}}\right) \;\le\; V_b(z) \;\le G_H^{-1}\left((1-z)^{\frac{1}{(v_b-v_s)}}\right)\,.
$$

Thus, as $v_b - v_s \uparrow +\infty$, $V_b(z)$ converges to $-\infty$ uniformly on compact subsets of $[0,1)$. By assumption,

$$
\lim_{V\to+\infty} \frac{1}{h_b^H(V)} \;=\; \frac{1}{\alpha}\,, \qquad \lim_{V\to+\infty} \frac{1}{h_b^L(V)} \;=\; \frac{1}{\alpha+1}.
$$

Thus, as $z \uparrow 1$,

$$
V_b'(z) \;\sim\; \frac{1}{\alpha\,(v_b - v_s)}\,\frac{1}{1-z} \;<\; \frac{1}{z(1-z)}.
$$

Fixing a sufficiently small $\varepsilon > 0$, we will show below that there exists a threshold W such that (23) holds for all $v_b - v_s > W$ and all z such that $V_b(z) \le -\varepsilon^{-1}$. Since, by the assumptions made, $1/h_b^H(V)$ and $1/h_b^L(V)$ are uniformly bounded from above for $V \ge -\varepsilon^{-1}$, it will immediately follow from (12) that (23) holds for all z with $V_b(z) \ge -\varepsilon^{-1}$ as soon as $v_b - v_s$ is sufficiently large.

Thus, it remains to prove (23) when $V_b(z) \le -\varepsilon^{-1}$. We pick an ε so small that we can replace the ODE (4) by (6) when proving (23). That is, once we prove the claim for the "approximate" solution $\tilde{g}(z)$, the actual claim will follow from (7).

Let

$$\tilde{g}(z) \;=\; \frac{\zeta}{(-\log\zeta)^{\gamma}}\, f(z) \;\stackrel{def}{=}\; \varepsilon\, f(z)\;,\;\varepsilon \;=\; \frac{\zeta}{(-\log\zeta)^{\gamma}}\,.$$

Then, (4) is equivalent to the ODE

$$f'(z) \;=\; \left(\frac{\log(1/\zeta)}{\log(1/\zeta)\,+\,\log(1/f(z))}\right)^{\gamma}\left(\frac{z}{1-z}\,+\,\varepsilon^{\frac{1}{\alpha+1}}\, f(z)^{\frac{1}{\alpha+1}}\right)\,. \tag{24}$$

As $v_b - v_s \to +\infty$, we get that $\zeta, \varepsilon \to 0$. Let

$$f_0(z) \;\stackrel{def}{=}\; \int_0^z \frac{x}{1-x}\,dx \;=\; -\log(1-z)\,-\,z\,.$$

Using bounds analogous to that preceding (16), it is easy to see that

$$\lim_{v_b - v_s\to+\infty} f(z) \;=\; f_0(z)\;, \qquad \lim_{v_b - v_s\to+\infty} f'(z) \;=\; f_0'(z),$$

and that the convergence is uniform on compact subsets of $(0,1)$. Fixing a small $\varepsilon_1 > 0$, we have, for $z > \varepsilon_1$,

$$\begin{aligned}
\lim_{v_b - v_s\to\infty} V_b'(z) \;&=\; \lim_{v_b - v_s\to\infty} \frac{\tilde{g}'(z)}{(\alpha+1)\tilde{g}(z)}\\
&=\; \lim_{v_b - v_s\to\infty} \frac{f'(z)}{(\alpha+1)f(z)}\\
&=\; \frac{f_0'(z)}{(\alpha+1)f_0(z)}\\
&=\; \frac{z}{(\alpha+1)\,(1-z)\,(-\log(1-z)-z)}.
\end{aligned}$$

We then have

$$\frac{d^2}{dz^2}(-\log(1-z)) \;=\; \frac{1}{(1-z)^2} \;\ge\; 1.$$

Therefore, by Taylor's formula,

$$-\log(1-z) - z \;\ge\; \frac{1}{2}\,z^2\,.$$

Hence,

$$\frac{z}{(\alpha+1)\,(1-z)\,(-\log(1-z)-z)} \;\le\; \frac{2}{\alpha+1}\,\frac{1}{z(1-z)}.$$

Therefore (23) holds for large $v_b - v_s$ because $\alpha > 1$. This argument does not work as $z \to 0$ because $f(0) = f_0(0) = 0$. So, we need to find a way to get uniform upper bounds for $f'(z)/f(z)$ when z is small. By the comparison argument used above, and picking ε_1 sufficiently small, since our goal is to prove inequality (23), we can replace $1-z$ by 1 in (24).

In this part of the proof, it will be more convenient to deal with $\tilde{g}$ instead of f. By the above, we may replace $\tilde{g}$ by the function g_1 solving

$$g_1'(z) \;=\; \frac{\zeta}{(-\log(g_1))^\gamma}\left(z \,+\, g_1^{\frac{1}{\alpha+1}}\right).$$

Let

$$d(z) \;=\; \int_0^z \left(\log\left(\frac{1}{x}\right)\right)^\gamma dx\,,$$

$D(z) \;=\; d^{-1}(z)$, and $k(z) \;=\; D(g_1(z))$. Then, we can rewrite the ODE for g_1 as

$$k'(z) \;=\; \zeta\,\left(z \,+\, (D(k(z)))^{1/(\alpha+1)}\right)\,, \qquad k(0)=0.$$

Define $L(z)$ via

$$\int_0^{L(z)} (D(x))^{-1/(\alpha+1)}\,dx \;=\; z,$$

and let

$$\phi(z) \;=\; L(\zeta\,z) \,+\, \frac{1}{2}\,\zeta\,z^2 \;\geq\; L(\zeta z).$$

Then, by the monotonicity of $D(z)$,

$$\phi'(z) \;=\; \zeta\,L'(\zeta z) \,+\, \zeta\,z \;=\; \zeta\,\left(z \,+\, (D(L(\zeta z)))^{1/(\alpha+1)}\right) \;\leq\; \zeta\,(z \,+\, (D(\phi(\zeta z)))^{1/(\alpha+1)}).$$

By a comparison theorem for ODEs (for example, Hartman (1982), Theorem 4.1, p. 26),[2] we have

$$k(z) \;\geq\; \phi(z) \;\Leftrightarrow\; g_1(z) \;=\; D(k(z)) \;\geq\; D(\phi(z))\,. \tag{25}$$

Therefore, since the functions $x(-\log x)^\gamma$ and $x^{\alpha/(\alpha+1)}\,(-\log x)^\gamma$ are monotone increasing for small x, we have

$$\begin{aligned}(1+\alpha)\,V_b'(z) \;&=\; \frac{g'(z)}{g(z)}\\ &\leq\; (1+\varepsilon)\,\frac{g_1'(z)}{(\alpha+1)\,g_1(z)}\\ &=\; \frac{(1+\varepsilon)\zeta\,z}{g_1\,(-\log g_1)^\gamma} \,+\, \frac{(1+\varepsilon)\zeta}{g_1^{\alpha/(\alpha+1)}\,(-\log g_1)^\gamma}\\ &\leq\; \frac{(1+\varepsilon)\zeta\,z}{D(\phi(z))\,(-\log D(\phi(z)))^\gamma} \,+\, \frac{(1+\varepsilon)\zeta}{D(\phi(z))^{\alpha/(\alpha+1)}\,(-\log D(\phi(z)))^\gamma}.\end{aligned} \tag{26}$$

[2]Even though the right-hand side of the ODE in question is not Lipschitz continuous, the proof of this comparison theorem easily extends to our case because of the uniqueness of the solution, due to (22).

Thus, it suffices to show that

$$\frac{\zeta\, z^2}{D(\phi(z))\,(-\log D(\phi(z)))^{\gamma}} \;+\; \frac{\zeta\, z}{D(\phi(z))^{\alpha/(\alpha+1)}\,(-\log D(\phi(z)))^{\gamma}} \;<\; (1-\varepsilon)(1+\alpha)$$

for some $\varepsilon > 0$, and for all sufficiently small z and ζ. Now, a direct calculation similar to that for the functions $A(z)$ and $C(z)$ implies that

$$d(z) \;\sim\; z\,(-\log z)^{\gamma}$$

and therefore that

$$D(z) \;\sim\; z\,(-\log z)^{-\gamma}\,.$$

Thus, it suffices to show that

$$\begin{aligned} &\frac{\zeta\, z^2}{\phi(z)\,(-\log\phi)^{-\gamma}\,(-\log(\phi(z)\,(-\log\phi)^{-\gamma}))^{\gamma}} \\ &\qquad + \frac{\zeta\, z}{(\phi(z)\,(-\log\phi)^{-\gamma})^{\alpha/(\alpha+1)}\,(-\log(\phi(z)\,(-\log\phi)^{-\gamma}))^{\gamma}} \\ &\quad <\; (1-\varepsilon)(1+\alpha). \end{aligned} \tag{27}$$

Leaving the leading asymptotic term, we need to show that

$$\frac{\zeta\, z^2}{\phi(z)} \;+\; \frac{\zeta\, z}{(\phi(z))^{\alpha/(\alpha+1)}\,(-\log(\phi(z)))^{\gamma/(\alpha+1)}} \;<\; (1-\varepsilon)(1+\alpha)\,.$$

We have

$$\int_0^z (D(x))^{-1/(\alpha+1)}dx \;\sim\; \frac{\alpha+1}{\alpha}\, z^{\alpha/(\alpha+1)}\,(-\log z)^{\gamma/(\alpha+1)}.$$

Therefore

$$L(z) \;\sim\; \left(\frac{\alpha}{\alpha+1}\, z\right)^{(\alpha+1)/\alpha}\,(-\log z)^{-\gamma/\alpha}\,.$$

Hence, we can replace $\phi(z)$ by

$$\tilde\phi(z) \;\stackrel{def}{=}\; \left(\frac{\alpha}{\alpha+1}\,\zeta z\right)^{(\alpha+1)/\alpha}\,(-\log(\zeta z))^{-\gamma/\alpha} \;+\; \frac{1}{2}\,\zeta\, z^2\,.$$

Let

$$x \;=\; \frac{\zeta\, z^2}{(\zeta z)^{(\alpha+1)/\alpha}\,(-\log(\zeta z))^{-\gamma/\alpha}}\,.$$

Then,

$$\begin{aligned} &\frac{\zeta\, z^2}{\tilde\phi(z)} \;+\; \frac{\zeta\, z}{(\tilde\phi(z))^{\alpha/(\alpha+1)}\,(-\log(\tilde\phi(z)))^{\gamma/(\alpha+1)}} \\ &\quad=\; \frac{1}{\left(\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} + 0.5x\right)^{\alpha/(\alpha+1)}}\left(\frac{-\log(\zeta z)}{-\log\tilde\phi}\right)^{\gamma/(\alpha+1)} \;+\; \frac{x}{\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} + 0.5x}\,. \end{aligned}$$

We have

$$\begin{aligned}
\log(\tilde{\phi}) &= \log(\zeta\, z) + \log\left(\left(\frac{\alpha}{\alpha+1}\right)^{(\alpha+1)/\alpha} (\zeta z)^{1/\alpha} (-\log(\zeta z))^{-\gamma/\alpha} + 0.5\, z\right) \\
&\leq \log(\zeta z)
\end{aligned}$$

for small ζ, z. Furthermore, for any $\varepsilon > 0$ there exists a $\varepsilon > 0$ such that

$$\left(\frac{\alpha}{\alpha+1}\right)^{(\alpha+1)/\alpha} (\zeta z)^{1/\alpha} (-\log(\zeta z))^{-\gamma/\alpha} \geq (\zeta z)^{1/(\alpha-\varepsilon)}$$

for all $\zeta z \leq \varepsilon$. Hence,

$$\frac{\alpha-\varepsilon}{\alpha-\varepsilon+1} \leq \frac{-\log(\zeta z)}{-\log\tilde{\phi}} \leq 1$$

for all sufficiently small ζ, z. Consequently, to prove (26) it suffices to show that

$$\sup_{x>0} \chi(x) < 1+\alpha,$$

where

$$\chi(x) = \frac{1}{\left(\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} + 0.5x\right)^{\alpha/(\alpha+1)}} A_\alpha + \frac{x}{\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} + 0.5x},$$

with

$$A_\alpha = \max\left\{\left(\frac{\alpha}{\alpha+1}\right)^{\gamma/(\alpha+1)}, 1\right\}.$$

Let

$$K = \left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}}.$$

Then,

$$\chi'(x) = -\frac{0.5\, A_\alpha\, \alpha}{\alpha+1} \frac{1}{(K+0.5x)^{(2\alpha+1)/(\alpha+1)}} + \frac{K}{(K+0.5x)^2}.$$

Thus, $\chi'(x_*) = 0$ if and only if

$$K + 0.5x_* = \left(\frac{K}{\frac{0.5\, A_\alpha\, \alpha}{\alpha+1}}\right)^{\alpha+1},$$

which means that

$$x_* = 2\left(\left(\frac{2}{A_\alpha}\right)^{\alpha+1} - 1\right)\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}}.$$

Then,

$$
\begin{aligned}
&\chi(x_*) \\
&= \frac{1}{\left(\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} + 0.5x_*\right)^{\alpha/(\alpha+1)}} A_\alpha \; + \; \frac{x_*}{\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}} + 0.5x_*} \\
&= \frac{1}{\left((2/A_\alpha)^{\alpha+1}\,(\alpha/(\alpha+1))^{(\alpha+1)/\alpha}\right)^{\alpha/(\alpha+1)}} A_\alpha \; + \; \frac{2\left(\left(\frac{2}{A\,\alpha}\right)^{\alpha+1} - 1\right)\left(\frac{\alpha}{\alpha+1}\right)^{\frac{\alpha+1}{\alpha}}}{(2/A_\alpha)^{\alpha+1}\,(\alpha/(\alpha+1))^{(\alpha+1)/\alpha}} \qquad (28)\\
&= \left(\frac{A_\alpha}{2}\right)^{\alpha} \frac{\alpha+1}{\alpha} A_\alpha \; + \; 2 - \; 2\left(\frac{A_\alpha}{2}\right)^{\alpha+1} \; = \; 2 \; + \; \frac{A_\alpha^{\alpha+1}}{2^\alpha\,\alpha}\,.
\end{aligned}
$$

There are three candidates for x that achieve a maximum of χ, namely $x = 0$, $x = +\infty$, and $x = x_*$, which is positive if and only if $A_\alpha < 2$.

If $\gamma \geq 0$, then $A_\alpha = 1$, so $x = 0$ and $x = +\infty$ satisfy the required inequality as soon as $\alpha > 1$, whereas $\chi(x_*) < \alpha + 1$ if and only if $\alpha > \alpha_*$, where

$$\alpha_* \; = \; 1 \; + \; \frac{1}{\alpha_*\,2^{\alpha_*}}.$$

A calculation shows that $\alpha^* \in (1.30, 1.31)$.

If $\gamma < 0$, then

$$\chi(0) \; = \; \frac{(\alpha+1)\,A_\alpha}{\alpha}\,, \qquad \chi(+\infty) \; = \; 2,$$

and this gives the condition $A_\alpha < \alpha$. If $A_\alpha > 2$, that is, if

$$-\gamma \; > \; (\alpha+1)\,\frac{\log 2}{\log((\alpha+1)/\alpha)},$$

then we are done. Otherwise, we need the property

$$2 \; + \; \frac{A_\alpha^{\alpha+1}}{2^\alpha\,\alpha} \; < \; \alpha + 1 \; \Leftrightarrow \; -\gamma \; < \; \frac{\log\left((\alpha^2 - \alpha)\,2^\alpha\right)}{\log((\alpha+1)/\alpha)}\,.$$

■

E The Behavior of the Double Auction Equilibrium

Let

$$\zeta_{it} \; = \; \frac{(\alpha+1)}{c_{it}\,\overline{G}} \qquad (29)$$

and

$$\varepsilon_{it} \; = \; \frac{\zeta_{it}}{(|\log \zeta_{it}|/(\alpha+1))^{\gamma_{it}}}\,. \qquad (30)$$

Clearly, both ζ_{it} and ε_{it} are small when $\overline{G}$ is large.

Proposition E.1 *Let* $S_t = S_{i,j,t}$, $B_t = B_{i,j,t}$ *and* $\varepsilon_t = \varepsilon_{it}$. *We have, as* $\overline{G} \to \infty$,

$$S_t(\theta) \;\sim\; \mathcal{S}\left(\theta + \frac{1}{\alpha+1}\log\varepsilon_t\right),$$

where $\mathcal{S}(\theta)$ *is the inverse of the function in* z *defined by*

$$\log\frac{z-v_b}{v^H-z} - \frac{1}{\alpha+1}\log\left(\log\frac{1}{v^H-z} - (z-v_b)\right).$$

Similarly,

$$B_t(\theta) \;\sim\; \mathcal{B}\left(\theta - \frac{1}{\alpha+1}\log\varepsilon_t\right)$$

where $\mathcal{B}(z)$ *is the inverse of the function in* z *defined by*

$$\frac{1}{\alpha+1}\log\left(\log\frac{1}{v^H-z} - (z-v_b)\right).$$

Corollary E.2 *For any buyer-and-seller class pair* (i,j), $S_{i,j,t}(\theta)$ *is monotone decreasing in* t *and in any meeting probability* λ_i, *whereas* $B_{i,j,t}(\theta)$ *is monotone increasing in* t *and any* λ_i.

Proof. Without loss of generality, we assume for simplicity that $R = 1$. (This merely adds a constant to the inverse of the ask function, by Proposition 4.3.) We fix a time period $t \geq 0$ and omit the time index everywhere and write $V_b = V_{bt}$, $V_s = V_{st}$ for the inverses of the bid and ask functions. We also let $\gamma = \gamma_t$, $c = c_t$.

Let

$$\zeta \;=\; \zeta_t \;=\; \frac{(\alpha+1)^{\gamma+1}}{c\,\overline{G}}.$$

As in the proof of Proposition D.1, we define

$$g(z) \;=\; e^{(\alpha+1)V_b(z)} \;=\; \frac{\zeta}{(-\log\zeta)^{\gamma}}\,f(z) \;\stackrel{def}{=}\; \varepsilon\, f(z).$$

Then, as we have shown in the proof of Proposition D.1, we may assume that, for large $\overline{G}$,

$$f'(z) \;=\; \left(\frac{\log(1/\zeta)}{\log(1/\zeta) + \log(1/f(z))}\right)^{\gamma}\left(\frac{z-v_b}{v^H-z} + \varepsilon^{\frac{1}{\alpha+1}}\,f(z)^{\frac{1}{\alpha+1}}\right), \quad f(v_b) = 0. \quad (31)$$

See (24). Furthermore, as $\overline{G} \to \infty$, we have $\zeta, \varepsilon \to 0$,

$$\lim_{\overline{G}\to\infty} f(z) \;=\; f_0(z),$$

where

$$f_0(z) \;=\; (v^H - v_b)\,\log\frac{v^H - v_b}{v^H - z} \;-\; (z - v_b),$$

and the convergence is uniform on compact subsets of $[v_b, v^H)$.

From this point, for simplicity we take the case $\gamma = 0$. The general case follows by similar but lengthier arguments. Hence, we assume that f solves

$$f'(z) \;=\; \frac{z - v_b}{v^H - z} \;+\; \varepsilon^{1/\alpha+1}\, f^{1/(\alpha+1)}. \tag{32}$$

Since the solution $f(z)$ to (32) is uniformly bounded on compact subsets of $[v_b, v^H)$, by integrating (32) we find that

$$0 \;\le\; f(z) \;-\; f_0(z) \;=\; O(\varepsilon^{\frac{1}{\alpha+1}}\,(z \;-\; v_b)),$$

uniformly on compact subsets of $[v_b, v^H)$. Furthermore, $f_0(z) \;\le\; C_1\,(z - v_b)^2$, uniformly on compact subsets of $[v_b, v^H)$. Substituting these bounds into (32), we get

$$\begin{aligned} f(z) \;-\; f_0(z) \quad &\le\quad C_2\,\varepsilon^{\frac{1}{\alpha+1}} \int_{v_b}^{z} (\varepsilon^{1/\alpha+1}\,(z - v_b) \;+\; (z - v_b)^2)^{1/(\alpha+1)}\,dz \\ &\le\quad C_3\,\varepsilon^{\frac{1}{\alpha+1}}\,(z - v_b)\,(\varepsilon^{1/(\alpha+1)^2}\,(z - v_b)^{1/(\alpha+1)} \;+\; (z - v_b)^{2/(\alpha+1)}). \end{aligned}$$

Let now

$$l(z) \;=\; f(z)^{\alpha/(\alpha+1)} \;-\; \frac{\varepsilon^{1/\alpha+1}\alpha}{\alpha+1}\,(z - v_b)\,.$$

Then,

$$\begin{aligned} l'(z) \;&=\; \frac{\alpha}{\alpha+1}\,f'(z)\,f^{-1/(\alpha+1)} \;-\; \frac{\varepsilon^{1/\alpha+1}\alpha}{\alpha+1} \\ &=\; \frac{\alpha}{\alpha+1}\,\frac{z - v_b}{\left(\frac{\varepsilon^{1/\alpha+1}\alpha}{\alpha+1}\,(z - v_b) + l(z)\right)^{1/\alpha}} \\ &\le\; \frac{\alpha}{\alpha+1}\,\frac{z - v_b}{(l(z))^{1/\alpha}}\,. \end{aligned} \tag{33}$$

Integrating this inequality, we get

$$l(z) \;\le\; \frac{1}{2}(z - v_b)^2,$$

and therefore

$$f(z) \;\le\; C_4\,((z - v_b)^2 \;+\; \varepsilon^{1/\alpha}\,(z - v_b)^{(\alpha+1)/\alpha})\,. \tag{34}$$

Consequently,

$$e^{V_b(z)} \;=\; \varepsilon^{\frac{1}{\alpha+1}}\left(f_0(z) \;+\; o(\varepsilon^{\frac{1}{\alpha+1}}\,(z - v_b))\right)^{1/(\alpha+1)} \tag{35}$$

uniformly on compact subsets of $[v_b, v^H)$. Therefore,

$$\lim_{\varepsilon\to 0}\left(V_b(z) \;-\; \frac{1}{\alpha+1}\log\varepsilon\right) \;=\; \frac{1}{\alpha+1}\,\log f_0(z),$$

uniformly on compact subsets of (v_b, v^H).

Now, since $V_b \to -\infty$ uniformly on compact subsets of $[v_b, v^H)$,

$$V_s(z) \;=\; \log\frac{z-v_b}{v^H-z} \;-\; V_b(z)$$

converges to $+\infty$, uniformly on compact subsets of (v_b, v^H). Since $S(-\infty) = v_b$, standard arguments imply that $S(\theta)$ converges to v_b uniformly on compact subsets of $[-\infty, +\infty)$ (with $-\infty$ included). Furthermore,

$$\lim_{\varepsilon\to 0}\left(V_s(z) \;+\; \frac{1}{\alpha+1}\log\varepsilon\right) \;=\; \log\frac{z-v_b}{v^H-z} \;-\; \frac{1}{\alpha+1}\,\log f_0(z) \;\stackrel{def}{=}\; M(z),$$

uniformly on compact subsets of (v_b, v^H). Let $\mathcal{S}(z) \;=\; M^{-1}(z)$. We claim that

$$\lim_{\varepsilon\to 0} S\left(\theta \;-\; \frac{1}{\alpha+1}\log\varepsilon\right) \;=\; \mathcal{S}(\theta)\,, \tag{36}$$

uniformly on compact subsets of $\mathbb{R}$. Indeed, $S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)$ is the unique solution to the equation in y given by

$$\theta \;=\; V_s(y) \;+\; \frac{1}{\alpha+1}\log\varepsilon\,.$$

Since the right-hand side converges uniformly to the strictly monotone function $M(\,\cdot\,)$, this unique solution also converges uniformly to $\mathcal{S}(\theta)$. Furthermore, the equality

$$v_b \;+\; \Delta_b\,P(V_s(z) \;+\; V_b(z)) \;=\; z \;\Leftrightarrow v_b \;+\; \Delta_b\,P(\theta \;+\; V_b(S(\theta))) \;=\; S(\theta)$$

implies that

$$V_b\left(S\left(\theta \;-\; \frac{1}{\alpha+1}\log\varepsilon\right)\right) \;=\; \log\left(\frac{S-v_b}{v^H-S}\right) \;-\; \theta \;+\; \frac{1}{\alpha+1}\log\varepsilon$$

and therefore

$$V_b\left(S\left(\theta \;-\; \frac{1}{\alpha+1}\log\varepsilon\right)\right) \;-\; \frac{1}{\alpha+1}\log\varepsilon \;\to\; \log\left(\frac{\mathcal{S}(\theta)-v_b}{v^H-\mathcal{S}(\theta)}\right) \;-\; \theta.$$

We have

$$M(z) \;=\; \log\left(\frac{z-v_b}{(v^H-z)\left((v^H-v_b)\,\log\left(\frac{v^H-v_b}{v^H-z}\right) \;-\; (z-v_b)\right)^{1/(\alpha+1)}}\right).$$

Now, for $z \sim v_b$,

$$\log\left(\frac{v^H - v_b}{v^H - z}\right) \;=\; -\log\left(1 \;-\; \frac{z - v_b}{v^H - v_b}\right) \;\sim\; \frac{z - v_b}{v^H - v_b} \;+\; \frac{1}{2}\left(\frac{z - v_b}{v^H - v_b}\right)^2, \tag{37}$$

and therefore

$$M(z) \;\sim\; (1+\alpha)^{-1}\log(2(v^H - v_b)) \;+\; \frac{\alpha - 1}{\alpha + 1}\log\left(\frac{z - v_b}{v^H - v_b}\right) \tag{38}$$

as $z \to v_b$. Consequently, as $\theta \to -\infty$, we have

$$\mathcal{S}(\theta) \;\sim\; v_b \;+\; K\, e^{\frac{\alpha+1}{\alpha-1}\theta}$$

for some constant $K = K(\alpha)$. ■

F The Behavior of Some Important Integrals

For simplicity, many results in this section will be established under technical conditions on α. The general case can be handled similarly, but is significantly more messy. As above, we fix a pair $(i, j) = (b, s)$ and use S_t and B_t to denote the corresponding double auction equilibrium. Recall that ψ^H_{st} is the cross-sectional density of the information type of sellers at time t.

As previously, we consider the case of large $\overline{G}$ and use the notation $A \sim B$ to denote that $A/B \to 1$ when $\overline{G} \to \infty$.

Lemma F.1 *Let*

$$\frac{\alpha+1}{\alpha-1} > \alpha\,.$$

Then

$$\int_{\mathbb{R}} (v_b \;-\; S_\tau(y))\,\psi^H_{s\tau}(y)\,dy \;\sim\; c_{s\tau}\,\varepsilon^{\frac{\alpha}{\alpha+1}}\left|\frac{\log\varepsilon}{1+\alpha}\right|^{\gamma_{s\tau}}\int_{\mathbb{R}}(v_b - \mathcal{S}(y))\,e^{-\alpha y}\,dy$$

and

$$\int_{\mathbb{R}} (v_b \;-\; S_\tau(y))\,\psi^L_{s\tau}(y)\,dy \;=\; o(\varepsilon^{\frac{\alpha}{\alpha+1}})$$

as $\overline{G} \to \infty$.

Proof. In the following, we handle the cases of $\psi^L_{s\tau}$ and $\psi^H_{s\tau}$ simultaneously by using the notation "$\psi^{H,L}_{s\tau}$." Changing variables, we get

$$\begin{aligned}&\int_{\mathbb{R}} (v_b \;-\; S_\tau(y))\,\psi^{H,L}_{s\tau}(y)\,dy \\ &= \int_{\mathbb{R}} \psi^{H,L}_{s\tau}\left(y - \frac{1}{\alpha+1}\log\varepsilon\right)\left(v_b \;-\; S_\tau\left(y - \frac{1}{\alpha+1}\log\varepsilon\right)\right)\,dy\,.\end{aligned} \tag{39}$$

Furthermore, by Lemma G.6,

$$\lim_{\varepsilon\to 0} c_{s\tau}^{-1}\, \varepsilon^{-\{\alpha,\alpha+1\}/(\alpha+1)} \left|\frac{\log\varepsilon}{1+\alpha}\right|^{-\gamma_{s\tau}} \psi_{s\tau}^{H,L}\left(y - \frac{1}{\alpha+1}\log\varepsilon\right) \;=\; e^{-\{\alpha,\alpha+1\}\,y}.$$

By (36),

$$v_b \,-\, S_\tau\left(y - \frac{1}{\alpha+1}\log\varepsilon\right) \;\to\; v_b \,-\, \mathcal{S}(y)\,.$$

In order to conclude that

$$\begin{aligned} \lim_{\varepsilon\to 0} \varepsilon^{-\alpha/(\alpha+1)} \int_{\mathbb{R}} \psi_{s\tau}^{H}\left(y - \frac{1}{\alpha+1}\log\varepsilon\right) \left(v_b \,-\, S_\tau\left(y - \frac{1}{\alpha+1}\log\varepsilon\right)\right) dy \\ = \; c_{s\tau} \int_{\mathbb{R}} e^{-\alpha y}\,(v_b \,-\, \mathcal{S}(y))\, dy, \end{aligned} \tag{40}$$

and that

$$\int_{\mathbb{R}} \psi_{s\tau}^{L}\left(y - \frac{1}{\alpha+1}\log\varepsilon\right) \left(v_b \,-\, S_\tau\left(y - \frac{1}{\alpha+1}\log\varepsilon\right)\right) dy \;=\; o(\varepsilon^{\alpha/(\alpha+1)}),$$

we will show that the integrands

$$I(y) = \varepsilon^{-\alpha/(\alpha+1)}\, \psi_{s\tau}^{H}\left(y - \frac{1}{\alpha+1}\log\varepsilon\right) \left(v_b \,-\, S_\tau\left(y - \frac{1}{\alpha+1}\log\varepsilon\right)\right)$$

and

$$\varepsilon^{-\epsilon}\varepsilon^{-\alpha/(\alpha+1)}\, \psi_{s\tau}^{L}\left(y - \frac{1}{\alpha+1}\log\varepsilon\right) \left(v_b \,-\, S_\tau\left(y - \frac{1}{\alpha+1}\log\varepsilon\right)\right)$$

have an integrable majorant for some $\epsilon > 0$. Then, (40) will follow from the Lebesgue dominated convergence theorem.

We decompose the integral in question into three parts, as

$$\int_{-\infty}^{\frac{1}{1+\alpha}\log\varepsilon} I_1(y)\, dy \,+\, \int_{\frac{1}{1+\alpha}\log\varepsilon}^{A} I_2(y)\, dy \,+\, \int_{A}^{+\infty} I_3(y)\, dy,$$

and prove the required limit behavior for each integral separately. To this end, we will need to establish sharp bounds for $S(\theta)$ and $V_b(\theta)$.

Lemma F.2 *Let $\Omega \subset \mathbb{R}_+^2$ be a bounded open set and $\mathcal{L}(\theta,\varepsilon) \in C^b(\Omega)$ be a bounded, continuous function. Then we have*

$$S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right) \;\le\; v_b \,+\, C_1\,\mathcal{L}(\theta,\varepsilon) \tag{41}$$

for all $(\varepsilon, \theta) \in \Omega$ *if and only if*

$$\frac{1}{\alpha+1} \log f(v_b + \mathcal{L}(\theta, \varepsilon)) - \log(\mathcal{L}(\theta, \varepsilon)) \leq C_2 - \theta \, . \tag{42}$$

If (41) *holds, we have*

$$V_b\left(S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right) \leq \frac{\log\varepsilon}{1+\alpha} + C_3 + \log\mathcal{L}(\theta, \varepsilon) - \theta. \tag{43}$$

Proof. Applying V_s to both sides of (41) and using the fact that V_s is strictly increasing, we see that the desired inequality is equivalent to

$$\theta - \frac{1}{\alpha+1}\log\varepsilon \leq V_s(v_b + C_1\,\mathcal{L}) \, .$$

Now,

$$V_s(z) + \frac{1}{\alpha+1}\log\varepsilon = \log\frac{z-v_b}{v^H - z} - V_b(z) + \frac{1}{\alpha+1}\log\varepsilon = \log\frac{z-v_b}{v^H-z} - \frac{1}{\alpha+1}\log f(z) \, .$$

The claim follows because we are in the regime when $v^H - z$ is uniformly bounded away from zero.

Furthermore,

$$-\frac{\log\varepsilon}{1+\alpha} + V_b(S) = \log\left(\frac{S-v_b}{v^H - S}\right) - \theta - \log R \, . \tag{44}$$

If θ is bounded from above, then S is uniformly bounded away from v^H, and hence

$$\log\left(\frac{S-v_b}{v^H-S}\right) - \theta \leq C_4 + \log(S - v_b) - \theta.$$

The claim follows. ■

Lemma F.3 *Suppose that* $\varepsilon > 0$ *is sufficiently small. Fix an* $A > 0$*. Then, for*

$$\theta \in \left(\frac{1}{\alpha+1}\log\varepsilon, A\right) \tag{45}$$

we have

$$S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right) \leq v_b + C_5\, e^{\frac{\alpha+1}{\alpha-1}\theta} \, , \tag{46}$$

and for

$$\theta < \frac{1}{\alpha+1}\log\varepsilon, \tag{47}$$

we have that

$$S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right) \leq v_b + C_6\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\, e^{\frac{\alpha}{\alpha-1}\theta} \, . \tag{48}$$

Proof. By Lemma F.2, inequality (48) is equivalent to

$$\frac{1}{\alpha+1}\,\log f(v_b\,+\,C_6\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,e^{\frac{\alpha}{\alpha-1}\theta})\;-\;\log(C_6\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,e^{\frac{\alpha}{\alpha-1}\theta})\;\leq\;-\theta\,+\,C_7\,. \tag{49}$$

Under the condition (47),

$$\max\left\{(z-v_b)^2\,,\,\varepsilon^{1/\alpha}\,(z-v_b)^{(\alpha+1)/\alpha}\right\}\;=\;\varepsilon^{1/\alpha}\,(z-v_b)^{(\alpha+1)/\alpha} \tag{50}$$

for

$$z\;=\;C_8\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,e^{\frac{\alpha}{\alpha-1}\theta}\,.$$

Hence, by (34),

$$f(z)\;\leq\;C_9\,\varepsilon^{1/\alpha}\,(z-v_b)^{(\alpha+1)/\alpha}\,.$$

Consequently,

$$\begin{aligned}
&\frac{1}{\alpha+1}\,\log f(v_b\,+\,C_6\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,e^{\frac{\alpha}{\alpha-1}\theta})\;-\;\log\left(C_6\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,e^{\frac{\alpha}{\alpha-1}\theta}\right)\\
&\quad\leq\;C_{10}\;+\;\frac{1}{(\alpha+1)\,\alpha}\log\varepsilon\;+\;\frac{1}{\alpha}\left(\frac{\alpha}{\alpha-1}\theta\;+\;\frac{1}{(\alpha+1)(\alpha-1)}\log\varepsilon\right)\\
&\qquad-\;\left(\frac{\alpha}{\alpha-1}\theta\;+\;\frac{1}{(\alpha+1)(\alpha-1)}\log\varepsilon\right)\\
&\quad=\;-\theta\;+\;C_{10}\,,
\end{aligned} \tag{51}$$

and (48) follows.

Similarly, when θ satisfies (45), a direct calculation shows that

$$\max\left\{(z-v_b)^2\,,\,\varepsilon^{1/\alpha}\,(z-v_b)^{(\alpha+1)/\alpha}\right\}\;=\;(z-v_b)^2 \tag{52}$$

for

$$z\;=\;v_b\;+\;C_5\,e^{\frac{\alpha+1}{\alpha-1}\theta}\,.$$

Therefore, by (34),

$$\begin{aligned}
&\frac{1}{\alpha+1}\,\log f(v_b\,+\,C_5\,e^{\frac{\alpha+1}{\alpha-1}\theta})\;-\;\log(C_5\,e^{\frac{\alpha+1}{\alpha-1}\theta}\,)\\
&\qquad\qquad\leq\;C_{11}\;+\;\frac{2}{\alpha-1}\theta-\frac{\alpha\,+1}{\alpha-1}\theta\;=\;-\theta\;+\;C_{11},
\end{aligned} \tag{53}$$

and (46) follows. ■

As above, we recall that $\psi^H_{s\tau}$ is the cross-sectional density of the information type of sellers at time τ. As above, we handle the cases of $\psi^L_{s\tau}$ and $\psi^H_{s\tau}$ simultaneously by using the notation "$\psi^{H,L}_{s\tau}$."

Lemma F.4 *If*

$$\frac{\alpha+1}{\alpha-1} \;>\; \alpha,$$

then

$$\int_{-\infty}^{\frac{1}{\alpha+1}\,\log\varepsilon} \psi^{H,L}_{s\tau}\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\,\left(v_b \;-\; S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right)\,d\theta \;=\; o(\varepsilon^{\alpha/(\alpha+1)})\,.$$

Proof. By (47), since $\psi^{H,L}_{s\tau}$ is bounded, we get

$$\begin{aligned}
&\int_{-\infty}^{\frac{1}{\alpha+1}\,\log\varepsilon} \psi^{H,L}_{s\tau}\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\,\left(v_b \;-\; S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right)\,d\theta\\
&\le\; C_{12}\int_{-\infty}^{\frac{1}{\alpha+1}\,\log\varepsilon} \varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,e^{\frac{\alpha}{\alpha-1}\theta}\,d\theta\\
&=\; \varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}}\,\frac{\alpha-1}{\alpha}\,\varepsilon^{\frac{1}{(\alpha+1)(\alpha-1)}+\frac{\alpha}{(\alpha+1)(\alpha-1)}}\\
&=\; o(\varepsilon^{\alpha/(\alpha+1)})\,.
\end{aligned} \tag{54}$$

■

Lemma F.5 *If*

$$\frac{\alpha+1}{\alpha-1} \;>\; \alpha,$$

then

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\,\varepsilon^{-\frac{\alpha}{\alpha+1}}\int_{\frac{1}{\alpha+1}\,\log\varepsilon}^{A} \psi^{H}_{s\tau}\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\,\left(v_b \;-\; S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right)\,d\theta\\
&\qquad=\; c_{s\tau}\int_{-\infty}^{A}(v_b - \mathcal{S}(\theta))\,e^{-\alpha\,\theta}\,d\theta
\end{aligned} \tag{55}$$

and

$$\int_{\frac{1}{\alpha+1}\log\varepsilon}^{A} \psi^{L}_{s\tau}\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\,\left(v_b \;-\; S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right)\,d\theta \;=\; o(\varepsilon^{\alpha/(\alpha+1)}).$$

Proof. By assumption, as $x\to\infty$,

$$\psi^{H}_{s\tau}(x) \;\sim\; c_{s\tau}\,e^{-\alpha x}.$$

The claim follows from (36) and (45), which provides an integrable majorant. ■

The same argument implies the following result.

Lemma F.6 *We have*

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \varepsilon^{-\frac{\alpha}{\alpha+1}} \int_A^{+\infty} \psi^H_{s\tau}\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\left(v_b \,-\, S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right) d\theta \\
&\qquad = \; c_{s\tau} \int_A^{+\infty} (v_b - \mathcal{S}(\theta))\, e^{-\alpha\,\theta}\, d\theta
\end{aligned} \tag{56}$$

and

$$\int_A^{+\infty} \psi^L_{s\tau}\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\left(v_b \,-\, S\left(\theta - \frac{1}{\alpha+1}\log\varepsilon\right)\right) d\theta \;=\; o(\varepsilon^{\alpha/(\alpha+1)})\,.$$

■

We define, for $K \in \{H, L\}$,

$$\begin{aligned}
G^K_{\eta, q_{0,\tau-1}}(x) &= \int_x^{+\infty} (\eta^K \,*\, q^K_{0,\tau-1})(y)\, dy \\
F^K_{\eta, q_{0,\tau-1}}(x) &= 1 \;-\; G^K_{\eta, q_{0,\tau-1}}(x),
\end{aligned} \tag{57}$$

where $q_{0,\tau} = q_{i,0,\tau}$ is the density of increment to information type that an agent of class i will get during the time interval $[0, \tau]$ from trading with counterparties of class j. That is,

$$q_{i,0,0} \;=\; (1-\lambda_i)\,\delta_0 \;+\; \lambda_i\,\psi_{j\,0}.$$

and

$$q_{i,0,\tau+1} \;=\; (1-\lambda_i)\,q_{i,0,\tau} \;+\; \lambda_i \sum_j \kappa_{ij}\, q_{i,0,\tau} * \psi_{j\,\tau+1}.$$

Furthermore, everywhere in the sequel we assume that the density η of the type of an acquired signal packet satisfies $\eta^H \sim \mathrm{Exp}_{+\infty}(c_\eta, \gamma_\eta, -\alpha)$ for some $c_\eta,\ \gamma_\eta > 0$. This is without loss of generality by Condition 2 and Lemma 4.5 on p. 29, which together imply that any number of acquired signal packets satisfies this condition. That is, a convolution of densities satisfying the specified tail condition also satisfies the same condition. The same argument also implies that

$$q^H_{i,0,\tau} \sim \mathrm{Exp}_{+\infty}(c_{i,0,\tau}, \gamma_{i,0,\tau}, -\alpha)$$

for some $c_{i,0,\tau}, \gamma_{i,0,\tau} > 0$ and

$$\eta^H \,*\, q^H_{i,0,\tau} \;\sim\; \mathrm{Exp}_{+\infty}(C_{i,\eta,0,\tau}, \gamma_{i,0,\tau} + \gamma_\eta + 1, -\alpha)$$

for some $C_{i,\eta,0,\tau} > 0$.

Lemma F.7 *Suppose that*

$$\frac{(\alpha+1)^2}{\alpha-1} \; > \; 2\alpha+1\,.$$

Then,

$$\begin{aligned}&\int_{\mathbb{R}} \psi^H_{s\tau}(y)\,(v^H \;-\; S_\tau(y))\,F^H_{\eta,q_{0,\tau-1}}(V_{b\tau}(S_\tau(y)))\,dy \;\sim\; R^{-(\alpha+1)}\,c_{s\tau}\,\frac{c_{s,0,\tau-1}}{\alpha+1}\,C_{s,\eta,0,\tau-1}\\ &\times\varepsilon^{\frac{2\alpha+1}{\alpha+1}}\left|\frac{\log\varepsilon}{1+\alpha}\right|^{\gamma_{s\tau}+\gamma_{s,0,\tau-1}+\gamma_\eta+1}\int_{\mathbb{R}}(v^H \;-\; \mathcal{S}(y))\,\left(\frac{\mathcal{S}(y)-v_b}{v^H-\mathcal{S}(y)}\right)^{\alpha+1}e^{-y(2\alpha+1)}\,dy\end{aligned} \tag{58}$$

and

$$\begin{aligned}&\int_{\mathbb{R}} \psi^L_{s\tau}(y)\,(S_\tau(y)-v_b)\,F^L_{\eta,q_{0,\tau-1}}(V_{b\tau}(S_\tau(y)))\,dy \;\sim\; R^{-\alpha}\,c_{s\tau}\,\frac{c_{s,0,\tau-1}}{\alpha}\,C_{s,\eta,0,\tau-1}\\ &\times\varepsilon^{\frac{2\alpha+1}{\alpha+1}}\left|\frac{\log\varepsilon}{1+\alpha}\right|^{\gamma_{s\tau}+\gamma_{s,0,\tau-1}+\gamma_\eta+1}\int_{\mathbb{R}}(\mathcal{S}(y)-v_b)\,\left(\frac{\mathcal{S}(y)-v_b}{v^H-\mathcal{S}(y)}\right)^{\alpha}e^{-y(2\alpha+1)}\,dy\,.\end{aligned} \tag{59}$$

as $\overline{G}\to\infty$.

Proof. As $x\to-\infty$, we have

$$F^{H,L}_{\eta,q_{0,\tau-1}}(x) \;\sim\; \frac{c_{s,0,\tau-1}\,C_{s,\eta,0,\tau}}{\{\alpha+1,\alpha\}}\,e^{x\,\{\alpha+1,\alpha\}}\,|x|^{\gamma_{s,0,\tau-1}+\gamma_\eta+1}\,.$$

The claim follows by the arguments used in the proof of Lemma F.1. Special care is needed only because $(v^H-S)^{-1}$ blows up as $\theta\uparrow+\infty$.

By (44),

$$\begin{aligned}&F^H_{\eta,q_{0,\tau-1}}\left(V_b\left(S\left(\theta-\frac{1}{\alpha+1}\log\varepsilon\right)\right)\right)\\ &\le\; C_{13}\,\varepsilon\,\left(\frac{S-v_b}{v^H-S}\,e^{-\theta}\right)^{\alpha+1}\left|\log\left(\frac{S-v_b}{v^H-S}\,e^{-\theta}\varepsilon^{\frac{1}{\alpha+1}}\right)\right|^{\gamma_{s,0,\tau-1}+\gamma_\eta+1}\,.\end{aligned} \tag{60}$$

Thus, to get an integrable majorant in a neighborhood of $+\infty$, it would suffice to have a bound

$$v^H-S \;\ge\; C_{14}\,e^{-\beta\theta}$$

with some $\beta>0$ such that $\beta\alpha<2\alpha+1$, because this would guarantee that

$$\left(\frac{S-v_b}{v^H-S}\,e^{-\theta}\right)^{\alpha}\left|\log\left(\frac{S-v_b}{v^H-S}\,e^{-\theta}\varepsilon^{\frac{1}{\alpha+1}}\right)\right|^{\gamma_{s,0,\tau-1}+\gamma_\eta+1}e^{-\alpha\theta}\;\le\;\tilde{C}_{14}e^{-\bar{\varepsilon}\theta}$$

for some $\bar{\varepsilon}>0$. By the argument used in the proof of Lemma F.2, it suffices to show that for sufficiently large θ,

$$\frac{1}{\alpha+1}\,\log f(v_H\;-\;C_{14}\,e^{-\beta\theta})\;\le\;C_{15}\;+\;(\beta-1)\,\theta\,.$$

Now, it follows from (32) that

$$f'(z) \;\le\; f(z)^{1/(\alpha+1)} \,+\, \frac{v^H - v_b}{v^H - z}.$$

Since, for sufficiently small ε, $f(z)$ is uniformly bounded away from zero on compact subsets of $(v_b, v^H]$, we get

$$\frac{d}{dz}(f(z)^{\alpha/(\alpha+1)}) \;\le\; C_{16}\,(1 \,+\, (v^H - z)^{-1}),$$

for some $K > 0$ when z is close to v^H. Integrating this inequality, we get

$$f(z)^{\alpha/(\alpha+1)} \;\le\; C_{17}\,(1 \,-\, \log(v^H - z)).$$

Consequently,

$$\frac{1}{\alpha+1}\log f(v_H \,-\, C_{14}\,e^{-\beta\theta}) \;\le\; C_{18}\,\log\theta$$

if θ is sufficiently large. Hence, the required inequality holds for any $\beta > 1$ with a sufficiently large C_{14}, and the claim follows. ■

Lemma F.8 *Let*

$$\frac{\alpha+1}{\alpha-1} \;>\; \alpha\,.$$

Then

$$\begin{aligned}&\int_{\mathbb{R}} (S_\tau(y) - v_b)\,\times(\eta^H * q^H_{t,\tau-1})(y-\theta)\,dy\\ &\sim \frac{c_{b,t,\tau-1}}{\alpha+1}\,C_{b,\eta,0,\tau-1}\left|\frac{\log\varepsilon}{1+\alpha}\right|^{\gamma_{b,t,\tau-1}+\gamma_\eta+1}\varepsilon^{\frac{\alpha}{\alpha+1}}\int_{\mathbb{R}}(\mathcal{S}(y)-v_b)\,e^{-\alpha(y-\theta)}\,dy\end{aligned} \tag{61}$$

and

$$\int_{\mathbb{R}} (S_\tau(y) - v_b)\,\times(\eta^L * q^L_{t,\tau-1})(y-\theta)\,dy \;=\; o\left(\left|\frac{\log\varepsilon}{1+\alpha}\right|^{\gamma_{b,t,\tau-1}+\gamma_\eta+1}\varepsilon^{\frac{\alpha}{\alpha+1}}\right) \tag{62}$$

as $\overline{G}\to\infty$.

Lemma F.9 *Let*

$$\frac{(\alpha+1)\alpha}{\alpha-1} \;>\; \alpha\,.$$

Then we have, as $\overline{G} \to \infty$,

$$\begin{aligned}&\int_{\mathbb{R}} (S_\tau(y) - v_s)\, F^L_{b\tau}(V_{b\tau}(S_\tau(y)))\, (\eta^L * q^L_{t,\tau})(y-\theta)dy \sim\; c_{b,t,\tau-1}\, R^{-\alpha} C_{b,\eta,0,\tau-1} e^{(\alpha+1)\theta} \\ &\times \left|\frac{\log \varepsilon}{1+\alpha}\right|^{\gamma_{b,t,\tau-1}+\gamma_\eta+1} \varepsilon^{\frac{\alpha}{\alpha+1}}\, \frac{\alpha+1}{\alpha} \int_{\mathbb{R}} e^{-(2\alpha+1)y} \left(\frac{\mathcal{S}(y)-v_b}{v^H - \mathcal{S}(y)}\right)^\alpha dy\end{aligned} \tag{63}$$

and

$$\begin{aligned}&\int_{\mathbb{R}} (v^H - S_\tau(y))\, F^H_{b\tau}(V_{b\tau}(S_\tau(y)))\, (\eta^H * h^H_{t,\tau})(y-\theta)dy \\ &= \; o\left(\left|\frac{\log \varepsilon}{1+\alpha}\right|^{\gamma_{b,t,\tau-1}+\gamma_\eta+1} \varepsilon^{\frac{\alpha}{\alpha+1}}\right).\end{aligned} \tag{64}$$

G Proofs for Case of Initial Information Acquisition

For any given agent i, the expected utility $U_{i,t,\tau}$ from trading during the time interval $[t,\tau]$ is

$$U_{i,t,\tau}(\theta) \;=\; \sum_{r=t}^{\tau} u_{i,t,r}(\theta)\,,$$

where $u_{i,t,r}$ is the expected utility from trading at time r conditional on the agent's information at time t, evaluated at the information type outcome θ.

We denote further by $u_{i,t,r}(\theta;\eta)$ the expected utility from trading at time r conditional on the agent's information at time t after the agent has made the decision to acquire a signal packet with type density $\eta^{H,L}$, before the type of the acquired signal is observed. With this notation, $u_{i,t,r}(\theta) = u_{i,t,r}(\theta;\delta_0)$. The following lemma provides expressions for $u_{i,t,r}(\theta;\eta)$. These expressions follows directly from the definition of the double-auction trading mechanism.

Lemma G.1 *For a given buyer with posterior information type θ at time 0,*

$$\begin{aligned}u_{b,0,\tau}(\theta;\eta) \;&=\; P(\theta)\lambda \int_{\mathbb{R}} (v^H - S_\tau(y))\, G^H_{\eta,q_{0,\tau-1}}(V_{b\tau}(S_\tau(y)) - \theta)\, \psi^H_{s\tau}(y)\, dy \\ &+\; (1-P(\theta))\lambda \int_{\mathbb{R}} (v_b - S_\tau(y))\, G^L_{\eta,q_{0,\tau-1}}(V_{b\tau}(S_\tau(y)) - \theta)\, \psi^L_{s\tau}(y)\, dy,\end{aligned} \tag{65}$$

whereas a seller's utility is

$$\begin{aligned}u_{s,0,\tau}(\theta;\eta) \;&=\; P(\theta)\, \lambda \int_{\mathbb{R}} (S_\tau(y) - v^H)\, G^H_{b\tau}(V_{b\tau}(S_\tau(y)))\, (\eta^H * q^H_{0,\tau-1})(y-\theta)\, dy \\ &+\; (1-P(\theta))\, \lambda \int_{\mathbb{R}} (S_\tau(y) - v_s)\, G^L_{b\tau}(V_{b\tau}(S_\tau(y)))\, (\eta^L * q^L_{0,\tau-1})(y-\theta)\, dy.\end{aligned} \tag{66}$$

Here, by convention, we set $q^K_{t,t-1} \;=\; \delta_0$.

The next result provides approximate expressions for the gains from information acquisition when $\overline{G}$ is sufficiently large. Recall that the asymptotic behaviour for large $\overline{G}$ in the double auction between a class i of buyers and a class j of seller is determined by

$$\zeta_{it} \;=\; \frac{(\alpha+1)}{c_{it}\,\overline{G}}\,.$$

Lemma G.2 *Let b be a buyer of class i. Denote by $\mathbf{s}$ the set of seller classes with which buyers of class i trade and let*

$$\gamma_{s\tau} \;\equiv\; \max_{j\in\mathbf{s}} \gamma_{j\tau}.$$

Further, let

$$\mathbf{s}_m \;=\; \{j\in\mathbf{s} \;:\; \gamma_{j\tau}=\gamma_{s\tau}\}\,.$$

Let also $\gamma_\tau \;\equiv\; \gamma_{b\tau}$. Then

$$u_{b,0,\tau}(\theta;\eta)-u_{b,0,\tau}(\theta) \;\sim\; \frac{e^{-\alpha\theta}R^{-\alpha}}{1+Re^{\theta}}\, I_b^{\text{gain}}\,\lambda\int_{\mathbb{R}}(v^H \,-\, \mathcal{S}(y))\left(\frac{\mathcal{S}(y)-v_b}{v^H-\mathcal{S}(y)}\right)^{\alpha+1} e^{-y(2\alpha+1)}\,dy, \tag{67}$$

as $\overline{G}\to\infty$, where

$$\begin{aligned} I_b^{\text{gain}} \;&=\; \sum_{j\in\mathbf{s}_m} c_{j\tau}\,\frac{1}{\alpha(\alpha+1)}\left(\frac{\zeta_\tau}{(|\log\zeta_\tau|/(\alpha+1))^{\gamma_\tau}}\right)^{\frac{2\alpha+1}{\alpha+1}}\left|\frac{\log\zeta_\tau}{1+\alpha}\right|^{\gamma_{j\tau}} c_{s,0,\tau-1}\left|\frac{\log\zeta_\tau}{1+\alpha}\right|^{\gamma_{b,0,\tau-1}} \\ &\quad\times\left(C_{b,\eta,0,\tau}(\bar N_b,\bar N_s)\left|\frac{\log\zeta_\tau}{1+\alpha}\right|^{\gamma_\eta+1}-1\right) \\ &=\; \sum_{j\in\mathbf{s}_m} c_{j\tau}\,\frac{1}{\alpha(\alpha+1)}\,\zeta_\tau^{\frac{2\alpha+1}{\alpha+1}}\,c_{s,0,\tau-1}\left|\frac{\log\zeta_\tau}{1+\alpha}\right|^{\gamma_{b,0,\tau-1}-\frac{\alpha}{\alpha+1}\gamma_\tau+(\gamma_{j\tau}-\gamma_\tau)} \\ &\quad\times\left(C_{b,\eta,0,\tau}(\bar N_b,\bar N_s)\left|\frac{\log\zeta_\tau}{1+\alpha}\right|^{\gamma_\eta+1}-1\right)\,. \end{aligned} \tag{68}$$

Lemma G.3 *Let s be a seller of class i. Denote by $\mathbf{b}$ the set buyer classes with which seller of class i trade and let*

$$\gamma_\tau \;\equiv\; \max_{j\in\mathbf{b}} \gamma_{j\tau}.$$

Further, let

$$\mathbf{b}_m \;=\; \{j\in\mathbf{b} \;:\; \gamma_{j\tau}=\gamma_\tau\}\,.$$

Then

$$u_{s,0,\tau}(\theta;\eta) - u_{s,0,\tau}(\theta) \;\sim\; \frac{e^{(\alpha+1)\theta}\,R^{-\alpha}}{1+R\,e^{\theta}}\,I_s^{\text{gain}} \times G_s$$

as $\overline{G} \to \infty$, *where*

$$G_s = \lambda \int_{\mathbb{R}} \left((\mathcal{S}(y) - v_b) \;-\; \frac{\alpha+1}{\alpha}\, e^{-(\alpha+1)y} \left(\frac{\mathcal{S}(y) - v_b}{v^H - \mathcal{S}(y)} \right)^{\alpha} \right) e^{-\alpha y}\, dy$$

and

$$I_s^{\text{gain}} \;=\; \sum_{j \in \mathbf{b}_m} \zeta_{j\tau}^{\frac{\alpha}{\alpha+1}} \left(C_{b,\eta,0,\tau}(\bar{N}_b, \bar{N}_s) \left| \frac{\log \zeta_{j\tau}}{1+\alpha} \right|^{\gamma_\eta + 1} - 1 \right) c_{b,0,\tau-1} \left| \frac{\log \zeta_{j\tau}}{1+\alpha} \right|^{\gamma_{s,0,\tau-1} - \frac{\alpha}{\alpha+1}\gamma_\tau}.$$

Lemmas G.2 and G.3 follow directly from Lemmas F.1-F.9 above. The following result is then an immediate consequence.

Corollary G.4 *For buyers and sellers, the utility gain from acquiring information is convex in the number of signal packets acquired. Consequently, any optimal pure strategy is either to acquire the maximum number* $\bar{n}$ *of signal packets, or to acquire none.*

The next lemma is a direct consequence of Lemma 4.5 .

Lemma G.5 *Suppose that* $\lambda_{ij} \equiv \lambda_i \kappa_{ij} \neq 0$ *for all* i, j.[3] *Let* $\bar{N}_s$ *be the maximal number of signal packets for sellers, and* $\bar{N}_b$ *the maximal number of signals for buyers. Then, for any class* i,

$$\gamma_{i1} \;=\; \bar{N}_i + \mathbf{1}_{i \in s} \bar{N}_b + \mathbf{1}_{i \in b} \bar{N}_s,$$

and thus, for all $t \geq 2$,

$$\gamma_{it} \;=\; 2^{t-1}(\bar{N}_b + \bar{N}_s) \;-\; 1 \;+\; \bar{N}_i - \mathbf{1}_{i \in s} \bar{N}_s - \mathbf{1}_{i \in b} \bar{N}_b,$$

where we write $i \in b$ *if class* i *is a buyer class, and similarly for the sellers' classes. Furthermore,*

$$\gamma_{i,0,\tau-1} \;=\; \gamma_{i\tau} - \bar{N}_i\,.$$

Proof of Theorem 5.2 . It follows from Lemmas G.2-G.3 that it suffices to show that the exponents for $|\log \zeta|$ are monotone increasing in N if T is sufficiently large. For buyers, we have

$$\gamma_{b,0,\tau-1} - \frac{2\alpha+1}{\alpha+1}\gamma_\tau + \gamma_{j\tau} \;=\; -N_b - \frac{\alpha}{\alpha+1}\gamma_{b\tau} + \;\gamma_{j\tau}$$

[3] The case when some of the matching probabilities are zero can be studied by a limiting procedure.

whereas, for sellers, we need to show that

$$\gamma_{s,0,\tau-1} - \frac{\alpha}{\alpha+1}\gamma_\tau$$

is monotone increasing in the number of acquired signals. This follows directly from Lemma G.5. ■

Proof of Lemma C.3 . By the Perron-Frobenius Theorem (see Meyer (2000), chapter 8, page 668), we have

$$(\Lambda_s \Lambda_b)^{t-1} \sim r_s^{t-1} p_s q_s^T$$

where p_s and q_s are right and left Perron eigenvectors of $\Lambda_s \Lambda_b$ respectively, and r_s is the corresponding Perron eigenvalue. Similarly,

$$(\Lambda_b \Lambda_s)^{t-1} \sim r_b^{t-1} p_b q_b^T$$

where p_b and q_b are the right and left Perron eigenvectors of $\Lambda_b \Lambda_s$ respectively, and r_b is the corresponding Perron eigenvalue. Now, applying Λ_b to the identity $\Lambda_s \Lambda_b p_s = r_s p_s$, we get that $\Lambda_b p_s$ is a positive right eigenvector of $\Lambda_b \Lambda_s$ corresponding to a positive eigenvalue r_s. Uniqueness part of the Perron-Frobenius Theorem (see Meyer (2000), chapter 8, page 667) implies that $r_s = r_b$. To prove the last statement, we note that, by the Collatz-Wielandt formula (see Meyer (2000), chapter 8, page 667),

$$r_s = \max_x \min_i \frac{(\Lambda_s \Lambda_b x)_i}{x_i} = \min_i \frac{(\Lambda_s \Lambda_b p_s)_i}{p_{si}}$$

If we increase one of the elements of Λ_s or Λ_b, all coordinates of $\Lambda_s \Lambda_b p_s$ become strictly larger since $p_s > 0$, and hence the Collatz-Wielandt formula implies that r_s also strictly increases. ■

Proof of Proposition 5.5 . The claim of monotonicity in $N_{\min}$ and $\bar{n}$ follows directly from Lemmas F.1-F.9 and the proof of Theorem 5.2 . Furthermore, for large t, $c_{s_i,t}/c_{b_j,t}^{\alpha/(\alpha+1)}$ is monotone increasing in λ_{s_k,b_l} if and only if so does the principal eigenvalue r_s, and hence the claim follows from Lemma C.3 .

This completes the proof of the claim for seller and buyer classes from $\mathbf{s}$ and $\mathbf{b}$.

For a seller class $i \notin \mathbf{s}$, we have $c_{i,t} = \sum_j \lambda_{i,b_j} c_{j,t-1}^b$ by Lemma C.2 , and the claim follows. A similar argument applies for a buyer of class $i \notin \mathbf{b}$, with the only exception that λ_{i,s_j} appear in the denominator leading to within-class strategic substitutability of matching probabilities. The latter however is offset by the factor λ_i entering the expected gains from trade. ■

We will now study examples illustrating our general model. We will first treat the case of one class of sellers, and then consider the case of two classes of sellers.

G.1 One Class of Sellers

In order to calculate the equilibria, we will first need to determine the dependence of the cross-sectional type distributions on the model parameters. Suppose that buyers and sellers acquire N_b and N_s signal packets respectively. Then, let $\bar{N}_i = N_{\min} + N_i$ be the total number of signals packets that class i possesses. The maximum feasible number of signal packets is $N_{\max} = N_{\min} + \bar{n}$. Using Lemma 4.5 , we immediately get the following two technical lemmas.

Lemma G.6 *Suppose that at time* 0 *buyers and sellers acquire* $\bar{N}_b$ *and* $\bar{N}_s$ *signals respectively. Then,* $c_{bt} = c_{st} = c_t$ *and* $\gamma_{bt} = \gamma_{st} = \gamma_t$ *so that* ψ_{st} , $\psi_{bt} \sim \mathrm{Exp}_{-\infty}(c_t, \gamma_t, \alpha + 1)$ *for all* $t \geq 1$*, where* $\gamma_1 = \bar{N}_b + \bar{N}_s - 1$*. It follows that* $\gamma_t = 2\gamma_{t-1} + 1$ *for* $t \geq 2$*,*

$$c_1 \;=\; \lambda\, c_{s0} c_{b0} \frac{(\bar{N}_s - 1)!\,(\bar{N}_b - 1)!}{(\bar{N}_s + \bar{N}_b - 1)!}$$

and

$$c_{t+1} \;=\; \lambda\, c_t^2\, \frac{(\gamma_t!)^2}{\gamma_{t+1}!}\,.$$

In particular,

$$\gamma_t \;=\; 2^{t-1}(\bar{N}_b + \bar{N}_s) \;-\; 1$$

and

$$c_t \;=\; D_{\bar{N}_b, \bar{N}_s}(t)\, c_0^{2^{t-1}(\bar{N}_s + \bar{N}_b)}\, \lambda^{2^t - 1},$$

for a model-independent combinatorial function $D_{\bar{N}_b, \bar{N}_s}(t)$*.*

Lemma G.7 *For* $i = b$ *or* $i = s$*, we have* $q^H_{i,0,\tau} \sim \mathrm{Exp}_{-\infty}(c_{i,0,\tau}, \gamma_{i,0,\tau}, \alpha + 1)$*, where*

$$\gamma_{i,0,\tau} \;=\; (2^\tau \;-\; 1)\,(\bar{N}_b + \bar{N}_s) \;-\; 1 \;+\; \bar{N}_j$$

and

$$c_{i,0,\tau} \;=\; D_{i,\bar{N}_b,\bar{N}_s}(0,\tau)\, c_0 \lambda^{2^{\tau+1}-1},$$

for model-independent combinatorial function $D_{i,\bar{N}_b,\bar{N}_s}(0,\tau)$ *.*

We will also need the following auxiliary lemma, whose proof is straightforward.

Lemma G.8 *For* $i \in \{b, s\}$*, let* $\mathrm{Gain}_i(\bar{N}_s, \bar{N}_b)$ *denote the utility gain from acquiring the maximum number* $\bar{n} = N_{\max} - N_{\min}$ *of signal packets, for a market in which all other buyers and sellers have* $\bar{N}_b$ *and* $\bar{N}_s$ *signal packets, respectively. Let*

$$\begin{aligned} &\pi_1 \equiv \mathrm{Gain}_s(N_{\max}, N_{\min}), \quad \pi_2 \equiv \mathrm{Gain}_s(N_{\min}, N_{\min}), \\ &\pi_3 \equiv \mathrm{Gain}_b(N_{\max}, N_{\max}), \quad \pi_4 \equiv \mathrm{Gain}(N_{\max}, N_{\min}). \end{aligned} \tag{69}$$

Then:

- $(N_{\max}, N_{\min})$ *is an equilibrium if and only if* $\pi \in [\pi_4, \pi_1]$.
- $(N_{\max}, N_{\max})$ *is an equilibrium if and only if* $\pi \leq \pi_3$.
- $(N_{\min}, N_{\min})$ *is an equilibrium if and only if* $\pi \geq \pi_2$.

Lemma G.9 *Let* $\widetilde{T} \equiv \log_2(\alpha+1)+1$. *Then, the following are true:*

- *If* $T = 0$ *then* $\pi_1 = \pi_2 > \pi_4 > \pi_3$. *Thus, an equilibrium exists if and only if* $\pi \notin (\pi_3, \pi_4)$.
- *If* $0 < T < \widetilde{T}$ *then* $\pi_1 > \pi_2 > \pi_4 > \pi_3$, *and an equilibrium exists if and only if* $\pi \notin (\pi_3, \pi_4)$.
- *If* $t > \widetilde{T}$ *then* $\pi_1 > \pi_2 > \pi_3 > \pi_4$, *and an equilibrium always exists.*
- *For all* i, π_i *is increasing in* $N_{\min}$ *and in* $\bar{n}$.

Proof. For small values of ε, the constants π_1, π_2, π_3, and π_4 satisfy

$$\pi_k \;\sim\; \mathfrak{A}_i(0,\bar{N}_s,\bar{N}_b)\, Z_i(0,\bar{N}_s,\bar{N}_b),$$

for corresponding pairs of $\bar{N}_b, \bar{N}_s$. Here,

$$\mathfrak{A}_i(0,\bar{N}_s,\bar{N}_b) \;=\; (N_{\max}-N_{\min})^{-1}\left(C_{j,\eta_{N_{\max}},0,T}\left|\frac{\log\zeta}{1+\alpha}\right|^{N_{\max}} - C_{j,\eta_{N_{\min}},0,T}\left|\frac{\log\zeta}{1+\alpha}\right|^{N_{\min}}\right), \tag{70}$$

where $j = s$ when $i = b$, and where $j = b$ when $i = s$. Furthermore,

$$\begin{aligned} Z_b(0,\bar{N}_s,\bar{N}_b) \;&\sim\; \lambda\,\frac{R^{-\alpha}}{1+R}\,\mathfrak{D}_b(0,c_0,\bar{N}_b,\bar{N}_s,\alpha)\lambda^{2^T-1}\left(\frac{1}{\lambda^{2^T-1}\,\overline{G}}\right)^{\frac{2\alpha+1}{\alpha+1}} \\ &\qquad\times\;\lambda^{2^T-1}\,\left|\log(\overline{G})\right|^{(2^{T-1}\;-\;1)\,(\bar{N}_b+\bar{N}_s)\;-\;1+\bar{N}_s-\frac{\alpha}{\alpha+1}(2^{T-1}(\bar{N}_b+\bar{N}_s)-1)} \\ &=\;\frac{R^{-\alpha}}{1+R}\,\mathfrak{D}_b(0,c_0,\bar{N}_b,\bar{N}_s,\alpha)\,\lambda^{\frac{2^T-(\alpha+1)+2\alpha+1}{\alpha+1}}\,(\overline{G})^{-\frac{2\alpha+1}{\alpha+1}} \\ &\qquad\times\;|\log\overline{G}|^{\frac{2^{T-1}}{\alpha+1}(\bar{N}_b+\bar{N}_s)-\bar{N}_b-\frac{1}{\alpha+1}}, \end{aligned} \tag{71}$$

for some function $\mathfrak{D}_b(0,c_0,\bar{N}_b,\bar{N}_s,\alpha)$. Similarly,

$$\begin{aligned} Z_s(0,\bar{N}_s,\bar{N}_b) \;&\sim\; \frac{R^{-\alpha}}{1+R}\,\mathfrak{D}_s(0,c_0,\bar{N}_b,\bar{N}_s,\alpha)\,\lambda^{\frac{2^T-(\alpha+1)+2\alpha+1}{\alpha+1}}\,(\overline{G})^{-\frac{\alpha}{\alpha+1}} \\ &\qquad\times\;|\log\overline{G}|^{\frac{2^{T-1}}{\alpha+1}(\bar{N}_b+\bar{N}_s)-\bar{N}_b-\frac{1}{\alpha+1}}, \end{aligned} \tag{72}$$

for some function $\mathfrak{D}_s(c_0, \bar{N}_b, \bar{N}_s, \alpha)$. For $T = 0$, there is only one trading round and therefore

$$Z_s(0, \bar{N}_s, \bar{N}_b) \;=\; \frac{R^{-\alpha}}{1+R}\mathfrak{D}_b(c_0, \bar{N}_b, \bar{N}_s, \alpha)\,\lambda\,(\overline{G})^{-\frac{\alpha}{\alpha+1}}|\log(\overline{G})|^{-(\bar{N}_b-1)\alpha/(\alpha+1)}$$

and

$$Z_b(0, \bar{N}_s, \bar{N}_b) \;=\; \frac{R^{-\alpha}}{1+R}\mathfrak{D}_s(c_0, \bar{N}_b, \bar{N}_s, \alpha)\lambda(\overline{G})^{-\frac{2\alpha+1}{\alpha+1}}\,|\log \overline{G}|^{-(\bar{N}_b-1)\alpha/(\alpha+1)+(\bar{N}_s-\bar{N}_b)}.$$

When $\overline{G}$ is sufficiently large, $Z_s > Z_b$ and the impact of $\mathfrak{D}_i$ and $C_{i,\eta,0,\tau}(\bar{N}_b, \bar{N}_s)$ is small and does not affect the monotonicity results. The claim follows by direct calculation. ■

G.2 Two Classes of Sellers

As above, we denote by $\bar{N}_i = N_{\min} + N_i$ the total number of signal packets held by agents of class i. We have the following results.

Let $\bar{N}_s = \max\{\bar{N}_1, \bar{N}_2\}$ and let $m \in \{1, 2\}$ be the corresponding seller class that acquired more information and $-m$ be the other seller class. Then,

$$\lambda \;=\; \begin{cases} 0.5\lambda_m\ , & \bar{N}_1 \neq \bar{N}_2 \\ 0.5(\lambda_1 + \lambda_2)\ , & \bar{N}_1 = \bar{N}_2\,. \end{cases}$$

Lemma G.10 *We have $\psi_{l,t} \sim\ \mathrm{Exp}_{-\infty}(c_{lt}, \gamma_{lt}, \alpha + 1)$ for $l \in \{s_1, s_2, b\}$ for all $t \geq 1$, where $\gamma_{s_k,1} = \bar{N}_k + \bar{N}_b - 1$ and $\gamma_{b1} \;=\; \bar{N}_s + \bar{N}_b - 1$, and where, for $t \geq 2$,*

$$\gamma_{s_k,t} \;=\; \gamma_{s_k,t-1} + \gamma_{b,t-1} + 1 \tag{73}$$

$$\gamma_{b,t} \;=\; \gamma_{b,t-1} + \gamma_{s_m,t-1} + 1 \tag{74}$$

and where further

$$c_{b1} \;=\; \lambda\, c_{s0}c_{b0}\frac{(\bar{N}_s-1)!\,(\bar{N}_b-1)!}{(\bar{N}_s+\bar{N}_b-1)!}\ ,\ c_{s_k,1} \;=\; \lambda_k\, c_{s_k,0}c_{b0}\frac{(\bar{N}_k-1)!\,(\bar{N}_b-1)!}{(\bar{N}_k+\bar{N}_b-1)!},$$

$$c_{b,t+1} \;=\; c_{bt}\,\frac{\gamma_{bt}!\,\gamma_{s_m,t}!}{\gamma_{b,t+1}!}\begin{cases}\lambda\, c_{s_m,t}, & \bar{N}_1 \neq \bar{N}_2 \\ 0.5\,(\lambda_1\, c_{s_1,t} + \lambda_2\, c_{s_2,t})\ , & \bar{N}_1 = \bar{N}_2,\end{cases}$$

and

$$c_{s_k,t+1} \;=\; c_{bt}\,\frac{\gamma_{bt}!\,\gamma_{s_m,t}!}{\gamma_{b,t+1}!}\,\lambda_k\, c_{s_k,t}.$$

Consequently,

$$\gamma_{bt} \;=\; \gamma_{s_m,t} \;=\; 2^{t-1}(\bar{N}_b + \bar{N}_s) \;-\; 1$$

and, for $t \geq 2$,

$$\gamma_{s_{-m},t} \;=\; 2^{t-1}(\bar{N}_b + \bar{N}_s) \;-\; 1 \;+\; \bar{N}_{-m} - \bar{N}_s.$$

Thus, for $\bar{N}_1 \neq \bar{N}_2$,

$$c_{bt} \;=\; D_{\bar{N}_b,\bar{N}_s}(t)\, c_0^{2^{t-1}(\bar{N}_s+\bar{N}_b)}\,(0.5\lambda_m)^{2^t-1} \;,\; c_{s_k,t} \;=\; D_{\bar{N}_b,\bar{N}_1,\bar{N}_2}(t)\,\lambda_k^t\,(0.5\lambda_m)^{2^t-t-1},$$

for some combinatorial functions $D_{\bar{N}_b,\bar{N}_s}(t), D_{k,\bar{N}_b,\bar{N}_1,\bar{N}_2}(t)$.

However, when $\bar{N}_1 = \bar{N}_2$, *we get*

$$c_{b,t} \;=\; d_{b,\bar{N}_b,\bar{N}_s}(t)\,(\lambda_1^t + \lambda_2^t)\prod_{r=1}^{t-1}(\lambda_1^r + \lambda_2^r)^{2^{t-r-1}}$$

and

$$c_{s_k,t} \;=\; d_{s,\bar{N}_b,\bar{N}_s}(t)\,\lambda_k^t\prod_{r=1}^{t-1}(\lambda_1^r + \lambda_2^r)^{2^{t-r-1}},$$

for some combinatorial functions $d_{b,\bar{N}_b,\bar{N}_s}(t)$ *and* $d_{s,\bar{N}_b,\bar{N}_s}(t)$.

Now, we need to calculate $\gamma_{t,\tau}$.

Lemma G.11 *We have* $h^H_{l,t,\tau} \sim \mathrm{Exp}_{-\infty}(c_{l,t,\tau}, \gamma_{t,\tau}, \alpha+1)$, *where*

$$c_{s_k,t,t} \;=\; \lambda_k\, c_{bt}, \qquad \gamma_{t,t} \;=\; \gamma_{bt},$$

and

$$c_{b,t,t} \;=\; \begin{cases} 0.5\lambda_m\, c_{s_m,t}\ , & \bar{N}_1 \neq \bar{N}_2 \\ 0.5(\lambda_1 c_{s_1,t} + \lambda_2\, c_{s_2,t})\ , & \bar{N}_1 = \bar{N}_2\,. \end{cases}$$

Then we define inductively

$$c_{s_k,t,\tau+1} \;=\; \lambda_k\, c_{s_k,t,\tau}\, c_{b,\tau+1}\,\frac{\gamma_{s_k,t,\tau}!\,\gamma_{b,\tau+1}!}{\gamma_{s,0,\tau+1}!},\;\; \gamma_{s,0,\tau+1} \;=\; \gamma_{s,0,\tau} + \gamma_{b,\tau+1} + 1$$

and

$$c_{b,t,\tau+1} \;=\; c_{b,t,\tau}\,\frac{\gamma_{b,t,\tau}!\,\gamma_{s_m,\tau+1}!}{\gamma_{b,t,\tau+1}!}\begin{cases} 0.5\lambda_m c_{s_m,\tau+1}\ , & \bar{N}_1 \neq \bar{N}_2 \\ 0.5\,(\lambda_1 c_{s_1,\tau+1} \;+\; \lambda_2\, c_{s_2,\tau+1})\ , & \bar{N}_1 = \bar{N}_2\,, \end{cases}$$

and

$$\gamma_{b,t,\tau+1} \;=\; \gamma_{b,t,\tau} + \gamma_{s_m,\tau+1} + 1.$$

In particular, for $t > 0$,

$$\gamma_{l,t,\tau} \;=\; (2^\tau \;-\; 2^{t-1})\,(\bar{N}_b + \bar{N}_s) \;-\; 1\ ,\; l \in \{s_1, s_2, b\},$$

For $t = 0$,

$$\gamma_{s,0,\tau} \;=\; (2^\tau \;-\; 1)\,(\bar{N}_b + \bar{N}_s) \;-\; 1 \;+\; \bar{N}_b \;,\;\; \gamma_{b,0,\tau} \;=\; (2^\tau \;-\; 1)\,(\bar{N}_b + \bar{N}_s) \;-\; 1 \;+\; \bar{N}_s \,.$$

If $\bar{N}_1 \neq \bar{N}_2$ *then*

$$c_{b,t,\tau} \;=\; D_{b,\bar{N}_b,\bar{N}_s}(t,\tau,c_0)\,\lambda_m^{2^{\tau+1}-2^t}, \qquad c_{s_m,t,\tau} \;=\; D_{s,\bar{N}_b,\bar{N}_s}(t,\tau,c_0)\,\lambda_m^{2^{\tau+1}-2^t},$$

and

$$c_{s_{-m},t,\tau} \;=\; \left(\frac{\lambda_{-m}}{\lambda_m}\right)^{\tau-t+1} c_{s_m,t,\tau}$$

for all $t \geq 0$, *for some combinatorial functions* $D_{\bar{N}_b,\bar{N}_s}(t,\tau)$ *and* $D_{l,\bar{N}_b,\bar{N}_s}(0,\tau)$.

When $\bar{N}_1 = \bar{N}_2$,

$$c_{s_k,t,\tau} \;=\; d_{s,\bar{N}_b,\bar{N}_s}(t,\tau)\lambda_k^{\tau-t+1}\left(\prod_{r=1}^{t-1}(\lambda_1^r+\lambda_2^r)^{2^{\tau-r}-2^{t-r-1}}\right)\prod_{r=t}^{\tau}(\lambda_1^r+\lambda_2^r)^{2^{\tau-r}}$$

and

$$c_{b,t,\tau} \;=\; d_{b,\bar{N}_b,\bar{N}_s}(t,\tau)\frac{\lambda_1^{\tau+1}+\lambda_2^{\tau+1}}{\lambda_1^t+\lambda_2^t}\left(\prod_{r=1}^{t-1}(\lambda_1^r+\lambda_2^r)^{2^{\tau-r}-2^{t-r-1}}\right)\prod_{r=t}^{\tau}(\lambda_1^r+\lambda_2^r)^{2^{\tau-r}}$$

for all $t \geq 0$, *for some combinatorial functions* $d_{k,\bar{N}_b,\bar{N}_s}(t,\tau)$ *and* $d_{b,\bar{N}_b,\bar{N}_s}(t,\tau)$.

Proposition G.12 *Suppose that* $T > \widetilde{T}$. *Let* $\lambda_1 \leq \lambda_2$. *In equilibrium, we always have* $\bar{N}_b \leq \bar{N}_1 \leq \bar{N}_2$. *Furthermore, there exist constants* $\pi_1 > \pi_2 > \pi_3 > \pi_4 > \pi_5 > \pi_6$ *such that the following are true:*

1. *If* $\pi > \pi_1$ *then the unique equilibrium is* $(\bar{N}_b, \bar{N}_1, \bar{N}_2) = (N_{\min}, N_{\min}, N_{\min})$.

2. *If* $\pi_1 > \pi > \pi_2$ *then there are two equilibria:*

   - $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\min}, N_{\min})$
   - $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\max}, N_{\max})$.

3. *If* $\pi_2 > \pi > \pi_3$ *then there are three equilibria:*

   - $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\min}, N_{\min})$
   - $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\max}, N_{\max})$
   - $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\min}, N_{\max})$.

4. *If* $\pi_3 > \pi > \pi_4$ *then there are two equilibria:*

   - $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\min}, N_{\min})$
   - $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\max}, N_{\max})$.

5. *If* $\pi_4 > \pi > \pi_5$ *then there is a unique equilibrium*

$$(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\max}, N_{\max}).$$

6. *If* $\pi_5 > \pi > \pi_6$ *there are two equilibria:*

   - $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\max}, N_{\max}, N_{\max})$
   - $(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\min}, N_{\max}, N_{\max})$.

7. *If* $\pi_6 > \pi$ *then there is a unique equilibrium*

$$(\bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; (N_{\max}, N_{\max}, N_{\max}).$$

Proof. Denote by $\text{Gain}_i(\bar{N}_b, \bar{N}_1, \bar{N}_2)$ the gains from acquiring the maximal number of signals for an agent of class i, conditional on the numbers of signals packets acquired by all other agents. As in Lemma G.8, we define

$$\begin{aligned}
\pi_1 \;&\equiv\; \text{Gain}_1(N_{\min}, N_{\max}, N_{\max}), \quad \pi_2 \;\equiv\; \text{Gain}_2(N_{\min}, N_{\min}, N_{\max}) \\
\pi_3 \;&\equiv\; \text{Gain}_1(N_{\min}, N_{\min}, N_{\max}), \quad \pi_4 \;\equiv\; \text{Gain}_2(N_{\min}, N_{\min}, N_{\min}) \\
\pi_5 \;&\equiv\; \text{Gain}_b(N_{\max}, N_{\max}, N_{\max}), \quad \pi_6 \;\equiv\; \text{Gain}_b(N_{\min}, N_{\max}, N_{\max})\,.
\end{aligned} \tag{75}$$

Then, it suffices to prove that π_i are monotone decreasing in i. As in the proof of Lemma G.9, we have

$$\pi_i \;\sim\; \mathfrak{A}_i\, Z_i,$$

and it remains to study the asymptotic behavior of Z_i. We have

$$Z_b(0, \bar{N}_b, \bar{N}_1, \bar{N}_2) \;=\; Z_b^{s_1}(0, \bar{N}_b, \bar{N}_1, \bar{N}_2) \;+\; Z_b^{s_2}(0, \bar{N}_b, \bar{N}_1, \bar{N}_2),$$

where

$$\begin{aligned}
Z_b^{s_k}(0, \bar{N}_b, \bar{N}_1, \bar{N}_2) \;\sim\; &0.5\lambda_k \frac{R^{-\alpha}}{1+R}\, \mathfrak{D}_b(0, c_0, \bar{N}_b, \bar{N}_s, \alpha)\, c_{bT} \left(\frac{1}{c_{bT}\,\overline{G}}\right)^{\frac{2\alpha+1}{\alpha+1}} \\
&\times\; c_{b,0,\tau-1}\,|\log \overline{G}|^{\gamma_{b,0,T-1}-\frac{\alpha}{\alpha+1}\gamma_T}
\end{aligned} \tag{76}$$

for some function $\mathfrak{D}_b(0, c_0, \bar{N}_b, \bar{N}_s, \alpha)$. Similarly,

$$Z_{s_k}(0, \bar{N}_b, \bar{N}_1, \bar{N}_2) \;\sim\; \frac{R^{-\alpha}}{1+R}\, \mathfrak{D}_s(0, c_0, \bar{N}_b, \bar{N}_s, \alpha)\, \lambda_k\, c_{s_k,0,T-1}\, (c_{bT}\, \overline{G})^{-\frac{\alpha}{\alpha+1}} \times |\log \overline{G}|^{\gamma_{s,0,T-1} - \frac{\alpha}{\alpha+1}\gamma_T}\,. \tag{77}$$

We first study equilibria with $\bar{N}_1 = N_{\min} < \bar{N}_2 = N_{\max}$. Since, for both seller classes, the surpluses from acquiring information are of comparable magnitude and are much larger than those of the buyers, we ought to have $\bar{N}_b = N_{\min}$. This will be an equilibrium if

$$\pi > (Z_b^{s_1}(0, N_{\min}, N_{\min}, N_{\max}) \;+\; Z_b^{s_2}(0, N_{\min}, N_{\min}, N_{\max}))\, \mathfrak{A}_b,$$

but this automatically follows from

$$\pi_3 \;\sim\; \mathfrak{A} Z_{s_1}(0, N_{\min}, N_{\min}, N_{\max}) \;<\; \pi \;<\; \mathfrak{A} Z_{s_2}(0, N_{\min}, N_{\min}, N_{\max}) \;\sim\; \pi_2.$$

Since $Z_{s_1}/Z_{s_2} = (\lambda_1/\lambda_2)^{\tau+1}$, this is only possible if $\lambda_1 < \lambda_2$. Furthermore,

$$Z_{s_k}(0, N_{\min}, N_{\min}, N_{\max}) \;\sim\; \frac{R^{-\alpha}}{1+R}\, \tilde{\mathfrak{D}}_s\, \lambda_k \left(\frac{\lambda_k}{\lambda_2}\right)^T \lambda_2^{\frac{1}{\alpha+1}(2^T-1)} \overline{G}^{-\frac{\alpha}{\alpha+1}} \times |\log \overline{G}|^{(2^{T-1}\;-\;1)\,(N_{\min}+N_{\max})\;-\;1 + N_{\min} - \frac{\alpha}{\alpha+1}(2^{T-1}(N_{\min}+N_{\max})-1)}. \tag{78}$$

Now, $\bar{N}_1 = \bar{N}_2 = N_{\max},\; \bar{N}_b = N_{\min}$ forms an equilibrium if and only if

$$\pi > \pi_6 \;\sim\; (Z_b^{s_1}(0, N_{\min}, N_{\max}, N_{\max}) \;+\; Z_b^{s_2}(0, N_{\min}, N_{\max}, N_{\max}))\, \mathfrak{A}_b$$

and

$$\begin{aligned} \pi < \pi_1 \;&\sim\; Z_{s_1}(0, N_{\min}, N_{\max}, N_{\max}) \\ &\sim\; \frac{R^{-\alpha}}{1+R}\, \tilde{\mathfrak{D}}_s\, \lambda_1 \frac{\lambda_1^T}{(\lambda_1^T + \lambda_2^T)^{\alpha/(\alpha+1)}} \prod_{r=1}^{T-1} (\lambda_1^r + \lambda_2^r)^{\frac{1}{\alpha+1}2^{T-r}} \\ &\times \overline{G}^{-\frac{\alpha}{\alpha+1}} |\log \overline{G}|^{(2^{T-1}\;-\;1)\,(N_{\min}+N_{\max})\;-\;1 + N_{\min} - \frac{\alpha}{\alpha+1}(2^{T-1}(N_{\min}+N_{\max})-1)}. \end{aligned} \tag{79}$$

Next, $\bar{N}_b = \bar{N}_1 = \bar{N}_2 = N_{\min}$ is an equilibrium if and only if

$$\begin{aligned} \pi > \pi_4 \;&\sim\; Z_{s_2}(0, N_{\min}, N_{\min}, N_{\min}) \\ &\sim\; \frac{R^{-\alpha}}{1+R}\, \tilde{\mathfrak{D}}_s\, \lambda_2 \frac{\lambda_2^T}{(\lambda_1^T + \lambda_2^T)^{\alpha/(\alpha+1)}} \\ &\times \prod_{r=1}^{T-1} (\lambda_1^r + \lambda_2^r)^{\frac{1}{\alpha+1}2^{T-r}} \overline{G}^{-\frac{\alpha}{\alpha+1}} |\log \overline{G}|^{(2^{T-1}\;-\;1)\,(2N_{\min})\;-\;1 + N_{\min} - \frac{\alpha}{\alpha+1}(2^{T-1}(2N_{\min})-1)}\,. \end{aligned} \tag{80}$$

Finally, $\bar{N}_b = \bar{N}_1 = \bar{N}_2 = N_{\max}$ is an equilibrium if and only if

$$\pi < \pi_5 \;\sim\; (Z_b^{s_1}(0, N_{\max}, N_{\max}, N_{\max}) \;+\; Z_b^{s_2}(0, N_{\max}, N_{\max}, N_{\max}))\,\mathfrak{A}_b\,. \tag{81}$$

The fact that π_i decreases with i follows directly from their asymptotic expressions. ■

Lemma G.13 *There exists a unique solution $\hat{T} > \max\{2, \widetilde{T}\}$ to the equation $(\alpha+1)\hat{T} = 2^{\hat{T}} - 1$, and a unique solution $\bar{T}$ to the equation $(2\alpha+1)\bar{T} = 2^{\bar{T}} - 1$. Furthermore,*

$$\frac{\prod_{r=0}^{T-1}(\lambda_1^r + \lambda_2^r)^{2^{T-1-r}}}{(\lambda_1^T + \lambda_2^T)^\alpha}$$

- *is monotone decreasing in λ_2 for all $\lambda_2 \geq \lambda_1$ if $T \leq \hat{T}$.*
- *is monotone increasing in λ_2 for all $\lambda_2 \geq \lambda_1$ if $T \geq \bar{T}$.*

Proof. The fact that $\hat{T}$ exists and is unique follows directly from the convexity of the function 2^T. To prove that $\widetilde{T} < \hat{T}$, we need to show that $(\alpha+1)\widetilde{T} > 2^{\widetilde{T}} - 1$. Substituting $\widetilde{T} = \log_2(\alpha+1) + 1$, we get

$$2^{\widetilde{T}} - 1 - (\alpha+1)\widetilde{T} = 2(\alpha+1) - 1 - (\alpha+1)(\log_2(\alpha+1)+1) = \alpha - (\alpha+1)\log_2(\alpha+1) < 0,$$

because $\alpha + 1 > 2$ implies that $\log_2(\alpha+1) > 1$.

Let now $x = \lambda_2/\lambda_1 \geq 1$. Then, by homogeneity, it suffices to show that

$$\frac{\prod_{r=0}^{T-1}(1 + x^r)^{2^{T-1-r}}}{(1 + x^T)^\alpha}$$

is monotone decreasing in x. Differentiating, we see that we need to show that

$$\sum_{r=1}^{T-1} 2^{T-1-r} r \,\frac{x^r}{1+x^r} \;\leq\; \alpha\, T\, \frac{x^T}{1+x^T}\,.$$

Since $x \geq 1$, we have

$$\frac{x^r}{1+x^r} \leq \frac{x^T}{1+x^T}.$$

Therefore, using the simple identity

$$\sum_{r=1}^{T-1} 2^{T-1-r} r \;=\; 2^T - 1 - T,$$

we get

$$\sum_{r=1}^{T-1} 2^{T-1-r} r \frac{x^r}{1+x^r} \leq (2^T - 1 - T) \frac{x^T}{1+x^T} \leq \alpha\, T \frac{x^T}{1+x^T}$$

for all $T \leq \hat{T}$. Similarly, since

$$\frac{x^r}{1+x^r} \geq \frac{1}{2} \geq \frac{1}{2}\frac{x^T}{1+x^T},$$

we get that

$$\sum_{r=1}^{T-1} 2^{T-1-r} r \frac{x^r}{1+x^r} \geq (2^T - 1 - T) \frac{1}{2}\frac{x^T}{1+x^T} \geq \alpha\, T \frac{x^T}{1+x^T}$$

for all $T \geq \bar{T}$. ■

The next proposition gives the partial-equilibrium impact on the information gathering incentives of class-1 sellers of increasing the contact probability λ_2 of the more active sellers.

Proposition G.14 *Suppose Condition 2 holds and $\lambda_1 \leq \lambda_2$. Fixing the numbers N_1, N_2, and N_b of signal packets gathered by all agents, consider the utility $u_{1n} - u_{1N_1}$ of a particular class-1 seller for gathering n signal packets. There exist integers $\bar{T}$ and $\hat{T}$, larger than the time $\widetilde{T}$ of Proposition 5.6 such that, for any $n > N_1$, the utility gain $u_{1n} - u_{1N_1}$ of acquiring additional signal packets is decreasing in λ_2 for $0 < T < \hat{T}$ and is increasing in λ_2 for $T > \bar{T}$.*

Proofs of Propositions G.14 and 5.7 . Monotonicity of the gains Gain_1 follows from Lemma G.13 and the expressions for this gain, provided in the proof of Proposition G.12. Proposition 5.7 follows from Lemma G.13 if we set $\mathcal{K} = \pi_1$. ■

H Two-Class Case

This appendix focuses more closely on information acquisition externalities by specializing to the case in which all investors have the same contact probability λ. In this case, there are only two classes of investors, buyers b and sellers s. For a small time horizon T, the lack of complementarity suggested by Proposition 5.7 implies that symmetric equilibria may fail to exist. For larger T, symmetric equilibria always exist and are generally non-unique.

Definition H.1 *An asymmetric rational expectations equilibrium is: for each class* $i \in \{b, s\}$*, the masses* $p_{in}, n = 0, \ldots, \bar{n}$*, where* p_{in} *is the mass of the sub-group of group* i *that acquires exactly* n *packets; for each time* t *and seller-buyer pair* (i, j)*, a pair* (S_{ijt}, B_{ijt}) *of bid and ask functions; and for each class* i *and time* t*, a cross-sectional type distribution* ψ_{it} *such that:*

(1) The cross-sectional type distribution ψ_{it} *is initially* $\psi_{i0} = \sum_{n=0}^{\bar{n}} p_{in}\, \overline{\psi}^{*(N_{\min}+n)}$ *and satisfies the evolution equation (6).*

(2) The bid and ask functions (S_{ijt}, B_{ijt}) *form the equilibrium uniquely defined by Theorem 4.7.*

(3) Each $n \in \{0, \ldots, \bar{n}\}$ *with* $p_{in} > 0$ *solves* $\max_{n \in \{0,\ldots,\bar{n}\}} u_{in}$*, for each class* i*.*

It turns out that, in all asymmetric equilibria, agents in each sub-group either do not acquire information at all or acquire the maximal number $\bar{n}$ of signal packets. We will denote the corresponding strategy $(\bar{n}, p_i)$, meaning that a group of mass p_i of agents of class i acquires the maximal number $\bar{n}$ of packets and the other sub-group (of mass $1 - p_i$) does not acquire any information.

Proposition H.2 *There exist thresholds* $\overline{\pi} > \hat{\pi} > \underline{\pi}$ *such that the following are true.*

1. *If* $T < \widetilde{T}$ *then:*
   - *A symmetric equilibrium exists if and only if* $\pi \notin (\underline{\pi}, \hat{\pi})$*.*
   - *An asymmetric equilibrium exists if and only if* $\pi \geq \underline{\pi}$*.*
2. *If* $T > \widetilde{T}$*, then:*
   - *A symmetric equilibrium always exists.*
   - *Asymmetric equilibria exist if and only if* $\pi \leq \overline{\pi}$*.*

Furthermore, there is always at most one equilibrium in which different sub-groups of sellers acquire different amounts of information, and at most one equilibrium in which different sub-groups of buyers acquire different amounts of information.

In order to determine how the equilibrium mass of those agents who acquire information depends on the model parameters, we need to study the behavior of the gain from acquiring information. The next proposition studies externalities from information acquisition by other agents on the information acquisition incentives of any given agent in an out-of-equilibrium setting.

Proposition H.3 *For all i, the gain*

$$\text{Gain}_i \;=\; \max_{n>0}\{(u_{in}-u_{i0})/n\}$$

from information acquisition is increasing in $N_{\min}, \bar{n}$, and λ. Fix an i and suppose that only a subgroup of mass p_i of class-i agents acquire information. Let us also fix the information acquisition policy of the other class.

1. *If $T > \widetilde{T}$, then Gain_i/p_i is monotone increasing in p_i.*
2. *If $T < \widetilde{T}$, then Gain_i/p_i is monotone decreasing in p_i.*

Proof of Proposition H.3. Let $\overline{\pi} \equiv \pi_1 > \pi_2 \equiv \underline{\pi}$. Suppose that a mass p of buyers acquire $\bar{n}$ packets and the rest (mass $1-p$) do acquire no packets. For our asymptotic formulae, this is equivalent to simply multiplying c_0 by $p^{1/N_{\max}}$ for the initial density of the buyers' type distribution. Furthermore, the same recursive calculation as above implies that $c_{i,\tau}$ is proportional to $p^{2^{\tau-1}}$ for $\tau > 0$ whereas $c_{i,0,\tau}$ is proportional to $p^{2^\tau-1}$. By the same argument as above, sellers always acquire more information and therefore we ought to have that $\bar{N}_s = N_{\max}$. The equilibrium condition is just the indifference condition for a buyer,

$$p\pi \;=\; \text{Gain}_b,$$

because then a seller will always acquire information since the gain from doing so is always higher for him. Substituting the asymptotic expressions for the gains of information acquisition, we get the asymptotic relation

$$p\pi \;\sim\; p^{\min\{1,2^{T-1}+1\}}\,p^{\max\{2^{T-1}-1,0\}}\,p^{-\frac{2\alpha+1}{\alpha+1}\min\{1,2^{T-1}\}}\,\pi_3.$$

For $T < \widetilde{T}$, this gives a unique equilibrium value of p for any $\pi \geq \pi_3$. For $T > \widetilde{T}$, this gives a unique value of p for all $\pi \leq \pi_3$.

Similarly, for the case when different groups of sellers acquire different amounts of information, the equilibrium condition is

$$p\pi \;\sim\; p^{\max\{2^{T-1},1\}}\,p^{-\frac{\alpha}{\alpha+1}\min\{1,2^{T-1}\}}\,\pi_1$$

For $T < \widetilde{T}$, this gives a unique equilibrium value for p for any $\pi \geq \pi_1$. For $T > \widetilde{T}$, this gives a unique value for p for all $\pi \leq \pi_1$.

The fact that there are no equilibria in which both buyers and sellers acquire information asymmetrically follows from the expressions for the asymptotic size of the gains of information acquisition. ■

The intuition behind Proposition H.3 is similar to that behind Proposition G.14. An increase in the mass p gives rise to both a learning effect and a pricing effect. The learning effect dominates the pricing effect if and only if there are sufficiently many trading rounds, that is, when $T > \widetilde{T}$.

Now, the equilibrium indifference condition, determining the mass p_i is given by

$$\pi \; = \; p_i^{-1} \, \mathrm{Gain}_i(p_i, \lambda, N_{\min}, \bar{n}). \tag{82}$$

Proposition H.3 immediately yields the following result.

Proposition H.4 *The following are true:*[4]

- *If $T > \widetilde{T}$ then equilibrium masses p_b and p_s are decreasing in $\lambda, N_{\min}, \bar{n}$;*
- *If $T < \widetilde{T}$ then equilibrium masses p_b and p_s are increasing in $\lambda, N_{\min}, \bar{n}$.*

We note that a stark difference between the monotonicity results of Propositions G.14 and H.3. By Proposition H.3, in the two-class model, gains from information acquisition *are always increasing in the "market liquidity" parameter* λ. By contrast, Proposition G.14 shows that, with more than two classes, this is not true anymore. *Gains may decrease with liquidity.* The effect of this monotonicity of gains differs, however, between symmetric and asymmetric equilibria. In symmetric equilibiria, the effect goes in the intuitive direction: Since gains increase with λ, so does the equilibrium amount of information acquisition. By contrast, equation (82) shows that, for asymmetric equilibria, the effect goes in the opposite direction: Since the gains increase in both λ and the mass p of agents that acquire information (when $T > \widetilde{T}$), this mass must go down in equilibrium in order to make the agents indifferent between acquiring and not acquiring information.

From this result, we can also consider the effect of "education policies" such as the following.

- Educating agents before they enter the market by increasing the number $N_{\min}$ of endowed signal packets.
- Increasing the number $\overline{n}$ of signal packets that can be acquired.

[4] Recall that, by Proposition H.2, equilibrium masses p_b and p_s are always unique (if they exist).

Proposition H.4 implies that, in a dynamic model with sufficiently many trading rounds, both policies improve market efficiency. By contrast, a static model that does not account for the effects of information percolation would lead to the opposite policy implications.

I Endogenous Investment in Matching Technology

In this section, we take initial information endowments as given and instead focus on endogenous investment in matching technologies. In particular, the initial type densities are characterized by a fixed vector $N = (N_1, \ldots, N_M)$ of initially acquired signal packets. Before the initial signals are revealed to each agent, agents in class i individually choose an amount $\chi_{ij} \in K \equiv \{\underline{\chi}, \overline{\chi}\}$ to invest in a technology for meeting investors of class j, for some minimum investment $\underline{\chi} > 0$ and maximum investment $\overline{\chi} > \underline{\chi}$. We examine the case of symmetric choices within classes, so that agents of class i commonly choose the investment χ_{ij}. Given these choices, in each period the probability with which an agent of class i meets some agent in class j is $f_{ij}(\chi_{ij}, \chi_{ji})$, for a given function $f_{ij} : K \times K \to (0, 1)$. By the exact law of large numbers, this technology satisfies

$$m_i f_{ij}(\chi_{ij}, \chi_{ji}) \ = \ m_j f_{ji}(\chi_{ji}, \chi_{ij}).$$

We always make the non-satiation assumption that

$$\sum_{j \neq i} f_{ij}(\overline{\chi}, \overline{\chi}) < 1.$$

Given the $M \times (M - 1)$ matching-technology investments $\chi = (\chi_{ij})$, the cross-sectional type density ψ_{it} of the class-i agents satisfies the evolution equation

$$\psi_{i,t+1} \ = \ \left(1 - \sum_{j \neq i} f_{ij}(\chi_{ij}, \chi_{ji})\right) \psi_{i,t} \ + \ \sum_{j \neq i} f_{ij}(\chi_{ij}, \chi_{ji}) \, \psi_{i,t} * \psi_{j,t}. \tag{83}$$

Similarly, given χ, a particular agent of class i who makes the technology choice $c \in K^{M-1}$ has a Markov type process whose probability density $\psi_t^{c,\chi}$ at time t satisfies the Kolmogorov forward equation

$$\psi_{t+1}^{c,\chi} \ = \ \left(1 - \sum_{j \neq i} f_{ij}(c_j, \chi_{ji})\right) \psi_t^{c,\chi} \ + \ \sum_{j \neq i} f_{ij}(c_j, \chi_{ji}) \, \psi_t^{c,\chi} * \psi_{j,t}. \tag{84}$$

We will be applying the following technical assumption.

Condition 3 . *For any integer $T > 1$ and any pair (i, j) of agent classes, the function $c \mapsto (f_{ij}(\overline{\chi}, c))^T - (f_{ij}(\underline{\chi}, c))^T$ is nonnegative and monotone increasing in c.*

This assumption guarantees that the increase in matching probabilities associated with investing in a more effective matching technology is increasing in the investments in matching technology by other agents. The complementarity property holds, for example, for the constant-returns-to-scale technology of Duffie, Malamud and Manso (2009), by which $f_{ij}(\chi_{ij}, \chi_{ji}) = k_{ij}\chi_{ij}\chi_{ji}$ for some constant k_{ij}. The idea is natural: the greater the efforts of other agents at being matched, the more easily are they found by improving one's own search technology.

Definition I.1 *A (symmetric) rational expectations equilibrium consists of matching technology investments $\chi = (\chi_{ij})$; for each time t and each seller-buyer pair (i, j), a pair (S_{ijt}, B_{ijt}) of bid and ask functions; and for each class i and time t, a cross-sectional type density ψ_{it} such that:*

1. *The cross-sectional type density ψ_{it} satisfies the evolution equation (83).*

2. *The bid and ask functions (S_{ijt}, B_{ijt}) are the equilibrium bidding strategies uniquely defined by Theorem 4.7.*

3. *The matching-technology investments $\chi_i = (\chi_{i1}, \ldots, \chi_{iM})$ of class i maximize, for any agent of class i, the expected total trading gains net of matching-technology costs. That is, χ_i solves*

$$\sup_{c \in K^{M-1}} \quad U_i(c, \chi),$$

where

$$U_i(c, \chi) = E\left(\sum_{t=1}^{T}\sum_{j} f_{ij}(c_j, \chi_{ji})\, v_{ijt}(\Theta_t^{c,\chi}; B_{ijt}, S_{ijt})\right) - (\chi_{i1} + \cdots + \chi_{iM}), \quad (85)$$

where the agent's type process $\Theta_t^{c,\chi}$ has probability density $\psi_t^{c,\chi}$ satisfying (84) and the expected gain v_{ijt} associated with a given sort of trading encounter is as defined by (9) or (11), depending on whether class-i agents are sellers or buyers, respectively.

We say that *search is a strategic complement* if, for any agent class i and any matching technology investments $\chi = (\chi_{ij})$, the utility gain $U_i(c', \chi) - U_i(c, \chi)$ associated with increasing the matching technology investments from c to $c' \geq c$ is increasing in χ_{-i}, the matching-technology investments of classes $j \neq i$. The main result of this section is the following theorem.

Theorem I.2 *Suppose Conditions 2 and 3 hold. Let $\bar{T}$ and $\bar{g}$ be as in Proposition 5.5. Then, for any proportional gain from trade $\overline{G} > \overline{g}$ and market duration $T > \bar{T}$, search is a strategic complement.*

The intuition for this result is analogous to that of Proposition 5.5. If other agents are assumed to have increased their ability to find counterparties, and thereby collect more information from trading encounters, then under the stated conditions a given agent is encouraged to do the same in order to mitigate adverse selection in trade with better informed counterparties.

This complementarity can be responsible for the existence or non-existence of equilibria, depending on the duration T of markets, just as in the previous section. The Tarski (1955) fixed point theorem implies the following analogue of Corollary 5.3.

Corollary I.3 *Suppose Conditions 2 and 3 hold. For any proportional gain from trade $\overline{G} > \overline{g}$ and market duration $T > \bar{T}$, there exists a symmetric equilibrium. Furthermore, the set of equilibria investments in matching technology is a lattice with respect to the natural partial order on $K^{M \times (M-1)}$.*

Just as with the discussion following Corollary 5.3, a maximal and a minimal element of the set of equilibria can be selected by the same standard iteration procedure. We also have the following comparative-statics variant of Proposition 5.5 .

Proposition I.4 *Under Conditions 2 and 3, there exist some $\overline{g}$ and $\bar{T}$ such that for any proportional gain from trade $\overline{G} > \overline{g}$ and market duration $T > \bar{T}$, equilibrium investment in matching technology is increasing in the initial vector N of acquired signal packets.*

The intuition behind this result is analogous to that behind Theorem 5.2. If traders are initially better informed and T is large, then the learning effect dominates, giving agents an incentive to invest more in search technologies.

This result also illustrates the role of cross-class externalities. Even if agents in class j do not trade with those in class i, an increase in the initial information endowment of class i increases the search incentives of class j. This is a "pure" learning externality in that, if class i is better informed, this information will eventually percolate to the trading counterparties of class j. This encourages class j to have a better search technology.[5]

We also consider the incentive effects for the formation of "trading networks." (Because our model is based on a continuum agents, the network effect is with respect to agent classes, not individual agents.) With respect to information gathering incentives, agents prefer to trade with better informed agents. This incentive can even overcome the associated direct impact of adverse selection.

Condition 4 (Symmetry). Classes are symmetric in the sense that they have equal masses $m_i = m_j$ and f_{ij} does not depend on (i, j).

Theorem I.5 *Suppose that class-i agents are initially better informed than those of class j, that is, $N_i > N_j$. Under Conditions 2, 3, and 4, there exists some $\overline{g}$ and some $\bar{T}$ such that for any proportional gain from trade $\overline{G} > \overline{g}$ and market duration $T > \bar{T}$, in any equilibrium:*

1. *There is more investment in matching with class i than with class j. That is, $\chi_{ki} \geq \chi_{kj}$ for all k.*
2. *Class-i agents invest more in matching technology than do class-j agents. That is, $\chi_i \geq \chi_j$.*

The incentive effects associated with this result naturally support the existence of "hub-spoke trading networks," with better informed agents situated in the "center," and with other agents trading more with central agents by virtue of establishing trading relationships, meaning investment in the associated matching technologies. As a result, one expects a positive correlation between the frequency of trade of a class of agents and its information quality. While this effect accounts for learning opportunities, pricing effects, and adverse selection, we do not capture some other important effects, such as those associated with size variation in trades and risk aversion.

Finally, we note that if the market duration is moderate, meaning that $T \in (\widetilde{T}, \bar{T})$, the learning effect may not be strong enough to create the complementarity effects that we have described. Indeed, there are counterexamples for the 3-class model of the previous section.

[5] As before, this is based on the assumption that there is an ordered path of classes connecting class i with class j.

J Proofs: Endogenous Matching Technology

Proof of Theorem I.2. To prove the theorem, we need to show that, for any $k \neq i$, the utility gain for an agent of class i from searching more for agents of class k is monotone increasing in the search efforts of all other agents. This utility gain is given by

$$\begin{aligned}
&\sum_{t=0}^{T} \sum_{j\neq i,k} f_{ij}(\chi_{ij},\chi_{ji}) \int_{\mathbb{R}} 0.5(\pi^H_{i,j,t}(\theta) h^H_{i,t+1}(\chi_i^{k,+},\theta) + \pi^L_{i,j,t}(\theta) h^L_{i,t+1}(\chi_i^{+,k},\theta))\, d\theta \\
&+ \sum_{t=0}^{T} f_{ik}(\overline{\chi},\chi_{ki}) \int_{\mathbb{R}} 0.5(\pi^H_{i,k,t}(\theta) h^H_{i,t+1}(\chi_i^{k,+},\theta) + \pi^L_{i,j,t}(\theta) h^L_{i,t+1}(\chi_i^{k,+},\theta))\, d\theta \\
&- \sum_{t=0}^{T} \sum_{j\neq i,k} f_{ij}(\chi_{ij},\chi_{ji}) \int_{\mathbb{R}} 0.5(\pi^H_{i,j,t}(\theta) h^H_{i,t+1}(\chi_i^{k,-},\theta) + \pi^L_{i,j,t}(\theta) h^L_{i,t+1}(\chi_i^{k,-},\theta))\, d\theta \\
&- \sum_{t=0}^{T} f_{ik}(\underline{\chi},\chi_{ki}) \int_{\mathbb{R}} 0.5(\pi^H_{i,k,t}(\theta) h^H_{i,t+1}(\chi_i^{k,-},\theta) + \pi^L_{i,j,t}(\theta) h^L_{i,t+1}(\chi_i^{k,-}\theta))\, d\theta,
\end{aligned} \tag{86}$$

where $\chi_i^{k,\pm}$ coincides with χ_i, but with χ_{ik} replaced by $\underline{\chi}$ and $\overline{\chi}$, respectively. Lemmas G.2-G.3 imply that the leading asymptotic term of this gains from search can be written as

$$\sum_{t=0}^{T} (f_{ik}(\overline{\chi},\chi))^t - (f_{ik}(\underline{\chi},\chi))^t) K_{ijt},$$

for some nonnegative coefficients K_{ijt} that do not depend on χ_{ik}. Furthermore, a slight modification of the proof of Proposition 5.5 implies that these coefficients K_{ijt} are monotone increasing in the matching-technology investments of other classes whenever T is sufficiently large. The claim follows. ■

Proof of Proposition I.4. The proof follows directly from the arguments in the proof of Theorem I.2 because the coefficients K_{ijt} are monotone in the initial amount of information. ■

Proof of Theorem I.5. It follows from Lemmas G.2-G.3 that the expected profit from trading with better informed classes is always larger when T is sufficiently large, and that these profits are larger for initially better informed agents. The claim follows. ■

K Dynamic Information Acquisition

For settings in which side investment in information gathering can be done dynamically, based on learning over time, we are able to get analytical results only with a sufficiently low cost of information acquisition, corresponding to a per-packet cost of signals of $\pi_4 > \pi$, the case considered in the previous appendix.

In this case, we can show that there is a "threshold equilibrium," characterized by thresholds $\underline{X}_{it} < \overline{X}_{it}$, such that agents of type i acquire additional information only when their log-likelihood is in the interval $(\underline{X}_{it}, \overline{X}_{it})$.

The timing of the game is as follows. At the beginning of each period t, an agent may acquire information. Trading then takes place after an agent meets a counterparty with probability λ. Without loss of generality, when they acquire information, they choose between $N_{\min}$ and $N_{\max}$ packets. Otherwise, there will be multiple thresholds for each intermediate level number of signals. This is feasible to model, but much more complicated.

We let $\psi_{i,t}$ denote the cross-sectional density of types after information acquisition, and before trading takes place, and let $\chi_{i,t}$ denote the cross-sectional density of types after the auctions take place. Thus,

$$\psi_{i,t+1} \; = \; (\chi_{i,t}\, I_{(\underline{X}_{i,t+1}, \overline{X}_{i,t+1})}) * \eta_{N_{\max}} \; + \; (\chi_{i,t}\, I_{\mathbb{R}\setminus(\underline{X}_{i,t+1}, \overline{X}_{i,t+1})}) * \eta_{N_{\min}}$$

is the density that determines the bid and ask functions, and

$$\chi_{i,t+1} \; = \; (1-\lambda)\, \psi_{i,t+1} \; + \; \lambda\, \psi_{b,t+1} * \psi_{s,t+1}, \; i = b, s,$$

is the cross-sectional density of types after the auctions took place.

We now denote by $Q_{i,t,\tau}(\theta, x)$ the cross-sectional type density at time τ right before the auctions take place of an agent of class i conditional on his type being θ at time t *after the information has been acquired.* Then, conditional on his type being θ at time t before information has been acquired, depending on whether the agent acquires $N_i \in \{N_{\max}, N_{\min}\}$ signals, his type density at time τ is

$$R^{N_i}_{i,t,\tau}(\theta, x) \; = \; \int_{\mathbb{R}} \eta_{N_i}(z-\theta)\, Q_{i,t,\tau}(z, x)\, dz\,.$$

Furthermore, $Q_{i,t,\tau}(z, x)$ satisfies the recursion

$$Q_{i,t,\tau+1}(\theta, x) \; = \; \Big(q_{i,t,\tau}(\theta, \cdot) I_{[\underline{X}_{i,\tau+1}, \overline{X}_{i,\tau+1}]}\Big) * \eta_{N_{\max}} \; + \; \Big(q_{i,t,\tau}(\theta, \cdot) I_{[\underline{X}_{i,\tau+1}, \overline{X}_{i,\tau+1}]}\Big) * \eta_{N_{\min}}$$

where

$$q_{i,t,\tau}(\theta,\cdot) \;=\; \lambda\, Q_{i,t,\tau}(\theta,\cdot) * \psi_{j,\tau} \;+\; (1-\lambda)\, Q_{i,t,\tau}\,.$$

We will also need the following additional technical condition.

Condition 3. Suppose there exist $K, \epsilon > 0$ such that

$$|\overline{\psi}^H(-x)e^{(\alpha+1)x} - c_0| + |\overline{\psi}^H(x)e^{\alpha x} - c_0| \;\leq\; K\, e^{-\epsilon\, x} \tag{87}$$

for all $x > 0$.

Theorem K.1 *There exist $A, g > 0$ such that, for all $\overline{G} > g$ and all $\pi < e^{-A\overline{G}}$, there exists a threshold equilibrium.*

We let $M_{it}^{H,L}$ note the mass of agents of class i who acquire information at time t, indicating with a superscript the corresponding outcome of Y, H or L.

Theorem K.2 *There exists a critical time t^* such that the following hold in any threshold equilibrium under the conditions of Theorem K.1.*

- *Sellers:*
  - *For both H and L, the mass $M_{st}^{H,L}$ is monotone decreasing with t, and increasing in $\overline{G}^{-1}, T, N_{\max}$.*
  - *$M_{st}^{H,L}$ is monotone increasing in λ for $t < t^*$ and is monotone decreasing in λ for $t \geq t^*$.*
- *Buyers:*
  - *The mass $M_{bt}^{H,L}$ is monotone decreasing in t and is increasing in $T, N_{\max}$.*
  - *M_{bt}^{H} is monotone increasing in $\overline{G}^{-1}$.*
  - *M_{bt}^{H} is monotone increasing in λ for $t < t^*$ and is monotone decreasing in λ for $t \geq t^*$.*
  - *M_{bt}^{L} is monotone decreasing in $\overline{G}^{-1}$.*

L Proofs: Dynamic Information Acquisition

In this section we study asymptotic equilibrium behavior when $\overline{G}$ and π^{-1} become large. Furthermore, we will assume that π^{-1} is significantly larger than $\overline{G}$, so that $\pi^{-1}/\overline{G}^A$ is large for a sufficiently large $A > 0$. Throughout the proof, we will constantly use the notation $X >> Y$ if, asymptotically, $X - Y \to +\infty$.

L.1 Exponential Tails

Note that, by Lemma 4.5 ,

$$\chi_{i0}^H(x) \ = \ (1-\lambda)\,\psi_{i,0}^H \ + \ \lambda\,\psi_{b,0}^H * \psi_{s,0}^H \ \sim \ c_0\,e^{-\alpha x}\,|x|^{2\,N_{\max}-1} \ = \ c_0\,e^{-\alpha x}\,|x|^{\gamma_0}\,.$$

Furthermore, as we show below, in any equilibrium we always have

$$\underline{X}_{bt} << \underline{X}_{b,t+1} << \underline{X}_{st} << \underline{X}_{s,t+1} \tag{88}$$

and

$$\overline{X}_{b,t+1} << \overline{X}_{bt} << \overline{X}_{s,t+1} << \overline{X}_{st}. \tag{89}$$

Lemma L.1 *Suppose that $x \to +\infty$ and $\overline{X}_{it} \ \to \ +\infty$ in such a way that*

$$\overline{X}_{b,t+1} << \overline{X}_{bt} << \overline{X}_{s,t+1} << \overline{X}_{st}$$

for all t and such that, for any fixed i, t, the difference $x - \overline{X}_{i,t}$ either stays bounded or converges to $+\infty$ or converges to $-\infty$. Let $\mathbf{1}_L$ be the indicator of the L state. Then,

$$\psi_{it}(x) \ \sim \ C_{it}^{\psi}\,e^{-(\alpha+\mathbf{1}_L)x}\,x^{\gamma_t^{\psi}}$$

and

$$\chi_{it}(x) \ \sim \ C_{it}^{\chi}\,e^{-(\alpha+\mathbf{1}_L)x}\,x^{\gamma_t^{\chi}},$$

where

$$\gamma_t^{\psi} \ = \ N_{\max} + \gamma_{t-1}^{\chi}$$

and

$$\gamma_t^{\chi} \ = \ 2\gamma_t^{\psi} + 1\,.$$

The powers m_t^{ψ}, m_t^{χ} with which λ enters $C_{it}^{\psi,\chi}$ satisfy

$$m_t^{\chi} = 2m_t^{\psi} + 1\ ,\ m_t^{\psi} = m_{t-1}^{\chi}\,.$$

Furthermore, there exists a constant $\mathfrak{K}_1$ *such that*

$$|\psi_{it}(x)| \ \leq \ \mathfrak{K}_1 \, e^{-(\alpha+\mathbf{1}_L)x} \, x^{\gamma_t^\psi}$$

and

$$|\chi_{it}(x)| \ \leq \ \mathfrak{K}_1 \, e^{-(\alpha+\mathbf{1}_L)x} \, x^{\gamma_t^\chi} \, .$$

In addition, there exists a $\delta_{it} > 0$ *such that*

$$|\psi_{it}(x) e^{\alpha x} \, x^{-\gamma_t^\psi} - C_{it}^\psi| \ < \ \mathfrak{K}_1 e^{-\delta_{it} x}$$

for all $x > A$.

Proof. The proof is by induction. Recall that

$$\eta_N \ = \ \overline{\psi}^{*N}$$

Fix a sufficiently large $A > 0$. Then,

$$\begin{aligned}
\chi_{i,t+1} \quad &= \quad \int_{\mathbb{R}} \psi_{bt}(x-y)\, \psi_{st}(y)\, dy \ = \ \int_{-\infty}^{x} \psi_{bt}(x-y) \psi_{st}(y)\, dy \\
&\quad + \int_{x}^{+\infty} \psi_{bt}(x-y)\, \psi_{st}(y)\, dy \\
&= \quad \int_0^{+\infty} \psi_{bt}(y)\psi_{st}(x-y) dy \ + \ \int_{-\infty}^{0} \psi_{bt}(y)\psi_{st}(x-y)\, dy \\
&= \quad \int_0^{A} \psi_{bt}(y)\psi_{st}(x-y) dy \ + \ \int_{A}^{+\infty} \psi_{bt}(y)\psi_{st}(x-y)\, dy \\
&\quad + \int_{-\infty}^{0} \psi_{bt}(y)\psi_{st}(x-y)\, dy \\
&\equiv \quad I_1 \ + \ I_2 \ + \ I_3 \, .
\end{aligned}$$

Pick an A so large that ψ_{bt} can be replaced by its asymptotic from the induction hypothesis. Note that we can only take the "relevant" asymptotic coming from the values of y satisfying $y < \overline{X}_{b,T}$ because the tail behavior coming from "further away" regimes are asymptotically negligible. Then,

$$\begin{aligned}
I_2 \ &= \ \int_A^{+\infty} \psi_{bt}(y)\psi_{st}(x-y)\, dy \\
&\sim \ \int_A^{+\infty} C\, e^{-(\alpha+\mathbf{1}_L)y}\, y^{\gamma_t^\psi}\, \psi_{st}(x-y)\, dy \\
&= \ C\, e^{-(\alpha+\mathbf{1}_L)x}\, x^{\gamma_t^\psi} \int_{-\infty}^{x-A} e^{(\alpha+\mathbf{1}_L)y}\, |1-y/x|^{\gamma_t^\psi}\, \psi_{st}(y)\, dy \, . \qquad (90)
\end{aligned}$$

Now, applying l'Hôpital's rule and using the induction hypothesis, we get that

$$\frac{\int_{-\infty}^{x-A} e^{(\alpha+\mathbf{1}_L)y}\,\psi_{st}(y)dy}{(\gamma_t^{\psi}+1)^{-1}x^{\gamma_t^{\psi}+1}} \;\sim\; C_{st}.$$

Thus, we have proved the required asymptotic for the term I_2.

To bound the term I_1, we again use the induction hypothesis and get

$$\int_0^A \psi_{bt}(y)\psi_{st}(x-y)\,dy \;\le\; \mathfrak{K}_1\int_0^A \psi_{bt}(y)e^{-(\alpha+\mathbf{1}_L)(x-y)}\,|x-y|^{\gamma_t^{\psi}}\,dy \;\sim\; e^{-(\alpha+\mathbf{1}_L)x}|x|^{\gamma_t^{\psi}}\tilde{C}_2,$$

for some constant $\tilde{C}_2$, so the term I_1 is asymptotically negligible relative to I_2.

Finally, for the term I_3, we have

$$\int_x^{+\infty} \psi_{bt}(x-y)\psi_{st}(y)\,dy \;=\; \int_{-\infty}^0 \psi_{bt}(y)\psi_{st}(x-y)\,dy\,. \tag{91}$$

Now, picking a sufficiently large $A>0$ and using the same argument as above, we can replace the integral by $\int_{-A}^0$ and then use the induction hypothesis to replace $\psi_{st}(x-y)$ by $C_{st}e^{-(\alpha+\mathbf{1}_L)(x-y)}|x-y|^{\gamma_t^{\psi}}$. Therefore,

$$\begin{aligned}\int_{-\infty}^0 \psi_{bt}(y)\psi_{st}(x-y)\,dy \;&\sim\; \int_{-\infty}^0 \psi_{bt}(y)C_{st}e^{-(\alpha+\mathbf{1}_L)(x-y)}\,|x-y|^{\gamma_t^{\psi}}\,dy\\ &\sim\; C_{st}e^{-(\alpha+\mathbf{1}_L)x}\,x^{\gamma_t^{\psi}}\int_{-\infty}^0 \psi_{bt}(y)e^{(\alpha+\mathbf{1}_L)y}\,dy\,,\end{aligned} \tag{92}$$

which is negligible relative to I_2. Thus, we have completed the proof of the induction step for χ_{it}. It remains to prove it for ψ_{it}. We have

$$\begin{aligned}\psi_{it}(x) \;&=\; \int_{\underline{X}_{it}}^{\overline{X}_{it}} \chi_{i,t-1}(y)\eta_{N_{\max}}(x-y)\,dy \;+\; \int_{-\infty}^{\underline{X}_{it}} \chi_{i,t-1}(y)\eta_{N_{\min}}(x-y)\,dy\\ &\quad+\;\int_{\overline{X}_{it}}^{+\infty} \chi_{i,t-1}^H(y)\eta_{N_{\min}}^H(x-y)\,dy\\ &=\int_{-\infty}^{\overline{X}_{it}} \chi_{i,t-1}(y)\eta_{N_{\max}}(x-y)\,dy \;-\; \int_{-\infty}^{\underline{X}_{it}} \chi_{i,t-1}(y)\eta_{N_{\max}}^H(x-y)\,dy\\ &\quad+\;\int_{-\infty}^{\underline{X}_{it}} \chi_{i,t-1}^H(y)\eta_{N_{\min}}(x-y)\,dy + \int_{\overline{X}_{it}}^{+\infty} \chi_{i,t-1}(y)\eta_{N_{\min}}(x-y)\,dy\,.\end{aligned} \tag{93}$$

Since $\underline{X}_{i1}\to-\infty$, the induction hypothesis implies that

$$\begin{aligned}&\int_{-\infty}^{\underline{X}_{it}} \chi_{i,t-1}(y)\eta_N(x-y)\,dy \;\sim\; \int_{-\infty}^{\underline{X}_{it}} C_{i,t-1}e^{(\alpha+\mathbf{1}_H)y}|y|^{\gamma_{i,t-1}^{\chi}}\,c_0^N\,e^{-(\alpha+\mathbf{1}_L)(x-y)}\,|x-y|^{N-1}\,dy\\ &=\; C_{i,t-1}\,c_0^N\,e^{-(\alpha+\mathbf{1}_L)x}\,x^{N-1}\int_{-\infty}^0 e^{(2\alpha+1)(y+\underline{X}_{it})}|y+\underline{X}_{it}|^{\gamma_{i,t-1}^{\chi}}\,|1-(y+\underline{X}_{it})/x|^{N-1}\,dy\\ &=\; o\left(e^{-(\alpha+\mathbf{1}_L)x}\,x^{N-1}\,e^{(2\alpha+1)\underline{X}_{it}}|\underline{X}_{it}|^{\gamma_{i,t-1}^{\chi}}\right)\,.\end{aligned} \tag{94}$$

The same argument as above (the induction step for χ_{it}) implies that

$$\int_{\mathbb{R}} \chi_{i,t-1}(y)\eta_N(x-y)\,dy \;\sim\; C\,e^{-(\alpha+\mathbf{1}_L)x}\,x^{\gamma^{\chi}_{i,t-1}+N}\,.$$

Now, we will have to consider two different cases. If $x - \overline{X}_{it} \to +\infty$, we can replace $\eta_N^H(x-y)$ in the integral below by $c_0^N\,|x-y|^{N-1}\,e^{-\alpha(x-y)}$ and get

$$\begin{aligned}&\int_{-\infty}^{\overline{X}_{i1}} \chi_{i,t-1}(y)\eta_N(x-y)dy\\ \sim\;& c_0^N e^{-(\alpha+\mathbf{1}_L)x}\,|x|^{N-1}\int_{-\infty}^{\overline{X}_{i1}} \chi_{i,t-1}(y)\,e^{(\alpha+\mathbf{1}_L)y}\,|1-y/x|^{N-1}\,dy\end{aligned} \tag{95}$$

Using l'Hopital's rule and the induction hypothesis, we get

$$\int_{-\infty}^{\overline{X}_{it}} \chi_{i,t-1}(y)\,e^{(\alpha+\mathbf{1}_L)y}\,dy \;\sim\; C\,(\overline{X}_{it})^{\gamma^{\chi}_{i,t-1}+1}.$$

It remains to consider the case when $x - \overline{X}_{it}$ stays bounded from above. In this case,

$$\int_{-\infty}^{\overline{X}_{it}} \chi_{i,t-1}(y)\eta_N(x-y)\,dy \;=\; \int_{x-\overline{X}_{it}}^{+\infty} \chi_{i,t-1}(x-z)\eta_N(z)\,dy\,. \tag{96}$$

Now, the same argument as in (94) implies that $\int_{-\infty}^{x-\overline{X}_{i1}} \chi_{i,t-1}(x-z)\eta_N(z)\,dy$ is asymptotically negligible relative to $\int_{x-\overline{X}_{i1}}^{+\infty} \chi_{i,t-1}(x-z)\eta_N(z)\,dy$ because $x-\overline{X}_{i1}$ is bounded from above. Therefore,

$$\int_{x-\overline{X}_{it}}^{+\infty} \chi^H_{i,t-1}(x-z)\eta^H_N(z)\,dy \;\sim\; \int_{-\infty}^{+\infty} \chi^H_{i,t-1}(x-z)\eta^H_N(z)\,dy \;\sim\; C\,e^{-\alpha x}\,x^{\gamma^{\chi}_{i,t-1}+N}$$

for the H state, and similarly for the state L. The induction step follows now from (93). The proof of the upper bounds for the densities is analogous. ■

The arguments in the proof of Lemma L.1 also imply the following result.

Lemma L.2 *Under the hypothesis of Lemma L.1, we have that, when $\theta \to +\infty$ so that $\theta - x \to +\infty$,*

$$\begin{aligned} q_{i,t,\tau}(\theta,x) &\sim C^q_{i,t,\tau}\,e^{(\alpha+\mathbf{1}_H)(x-\theta)}\,|x-\theta|^{\gamma^q_{t,\tau}}\\ Q_{i,t,\tau}(\theta,x) &\sim C^Q_{i,t,\tau}\,e^{(\alpha+\mathbf{1}_H)(x-\theta)}\,|x-\theta|^{\gamma^Q_{t,\tau}}\\ R^{N_i}_{i,t,\tau}(\theta,x) &\sim C^{R,N_i}_{i,t,\tau}\,e^{(\alpha+\mathbf{1}_H)(x-\theta)}\,|x-\theta|^{\gamma^Q_{t,\tau}+N_i}, \end{aligned} \tag{97}$$

where

$$\begin{aligned} \gamma^q_{t,\tau} &= \gamma^Q_{t,\tau} + \gamma^{\psi}_{\tau} + 1\\ \gamma^Q_{t,\tau} &= \gamma^q_{t,\tau-1} + N_{\max}\,. \end{aligned} \tag{98}$$

Lemmas L.1 and L.2 immediately yield the next result.

Lemma L.3 *We have:*

$$\begin{aligned} \gamma_t^{\psi} &= (2^{t+1}-1)N_{\max}-1 \\ \gamma_t^{\chi} &= (2^{t+2}-2)N_{\max}-1 \\ \gamma_{t,\tau}^{q} &= (2^{\tau+2}-2^{t+1}-1)N_{\max}-1 \\ \gamma_{t,\tau}^{Q} &= (2^{\tau+1}-2^{t+1})N_{\max}-1\,. \end{aligned} \tag{99}$$

Furthermore, the powers $m_{t,\tau}$ of λ with which λ enters the corresponding coefficients c_t and $C_{t,\tau}$ are given by:

$$\begin{aligned} m_t^{\psi} &= 2^{t-1}-1 \\ m_t^{\chi} &= 2^{t}-1 \\ m_{t,\tau}^{q} &= 2^{\tau+1}-2^{t} \\ m_{t,\tau}^{Q} &= 2^{\tau}-2^{t}\,. \end{aligned} \tag{100}$$

L.2 Gains from Information Acquisition

For any given agent i, the expected utility $U_{i,t,\tau}$ from trading during the time interval from t to τ immediately before information is acquired can be represented as

$$U_{i,t,\tau}(\theta) \;=\; \sum_{r=t}^{\tau} u_{i,t,\tau}(\theta)\,.$$

Suppose that, at time t, an agent of type i with posterior θ makes a decision to acquire information with type density η. Then, the agent knows that his type at time τ, at the moment when the next auction takes place, his posterior will be distributed as $\delta_\theta * \eta * g^K_{i,t,\tau-1}$.

We will use the following notation:

$$G_{t,\tau}^{K,R,N}(\theta,x) \;=\; \int_x^{+\infty} R_{t,\tau}^{K,N}(\theta,y)\,dy\;,\;\; F_{t,\tau}^{K,R,N}(\theta,x) \;=\; 1\;-\;G_{t,\tau}^{K,R,N}(\theta,x),$$

for $K\in\{H,L\}$.

The following analog of Lemma G.1 holds.

Proposition L.4 *For a given buyer with posterior θ at time t, before the time-t auction takes place,*

$$\begin{aligned} u_{b,t,\tau}(N,\theta) \;&=\; P(\theta)\lambda\int_{\mathbb{R}}(v^H-S_\tau(y))\,G_{t,\tau}^{H,R,N}(\theta,V_{b\tau}(S_\tau(y)))\,\psi_{s\tau}^{H}(y)\,dy \\ &\quad+\;(1-P(\theta))\lambda\int_{\mathbb{R}}(v_b-S_\tau(y))\,G_{t,\tau}^{L,R,N}(\theta,V_{b\tau}(S_\tau(y)))\,\psi_{s\tau}^{L}(y)\,dy, \end{aligned} \tag{101}$$

whereas for a seller,

$$u_{s,t,\tau}(N,\theta) \;=\; P(\theta)\,\lambda \int_{\mathbb{R}} (S_\tau(y) - v^H)\, G^H_{b\tau}(V_{b\tau}(S_\tau(y)))\, R^{H,N}_{t,\tau}(\theta,y)\,dy \\ +\; (1-P(\theta))\,\lambda \int_{\mathbb{R}} (S_\tau(y) - v_s)\, G^L_{b\tau}(V_{b\tau}(S_\tau(y)))\, R^{L,N}_{t,\tau}(\theta,y)\,dy. \tag{102}$$

Thus, the gain from acquiring additional information is given by

$$\sum_{\tau>t} (u_{i,t,\tau}(N_{\max},\theta) \;-\; u_{i,t,\tau}(N_{\min},\theta))\,.$$

The following lemma provides asymptotic behavior of these gains for extreme type values.

Lemma L.5 *We have*

- *For a buyer:*
  - *As $\theta \to +\infty$,*

$$u_{b,t,\tau}(N_{\max},\theta) - u_{b,t,\tau}(N_{\min},\theta) \;\sim\; C^{R,N_{\max}}_{b,t,\tau} e^{-(\alpha+1)\theta} |\theta|^{\gamma^Q_{t,\tau}+N_{\max}} \\ \times \int_{\mathbb{R}} (v^H \;-\; S_\tau(y)) \left(\frac{S_\tau(y)-v_b}{v^H - S_\tau(y)}\right)^{\alpha+1} e^{-(\alpha+1)y}\, \psi^H_{s\tau}(y)\,dy. \tag{103}$$

  - *As $\theta \to -\infty$,*

$$u_{b,t,\tau}(N_{\max},\theta) - u_{b,t,\tau}(N_{\min},\theta) \;\sim\; C^{R,N_{\max}}_{b,t,\tau}\, Re^{(\alpha+1)\theta} |\theta|^{\gamma^Q_{t,\tau}+N_{\max}} \\ \times \int_{\mathbb{R}} (v^H \;-\; S_\tau(y)) \left(\frac{S_\tau(y)-v_b}{v^H - S_\tau(y)}\right)^{-\alpha} e^{\alpha y}\, \psi^H_{s\tau}(y)\,dy. \tag{104}$$

- *For a seller, as $\theta \to -\infty$,*[6]

$$u_{s,0,\tau}(N_{\max},\theta) - u_{s,0,\tau}(N_{\min},\theta) \;\sim\; C^{R,N_{\max}}_{s,0,\tau}\, Re^{(\alpha+1)\theta} |\theta|^{\gamma^Q_{t,\tau}+N_{\max}} \\ \times \int_{\mathbb{R}} \Bigg(\Big((S_\tau(y) - v_b) + (v^H - S_\tau(y))\, F^H_{b\tau}(V_{b\tau}(S_\tau(y))) \Big) e^y \\ + \Big((S_\tau(y) - v_b) + (v_s - S_\tau(y))\, F^L_{b\tau}(V_{b\tau}(S_\tau(y))) \Big) \Bigg)\, e^{-(\alpha+1)y}\,dy. \tag{105}$$

Furthermore, the derivatives of $u_{i,t,\tau}(N_{\max},\theta) \,-\, u_{i,t,\tau}(N_{\min},\theta)$ with respect to θ have the same asymptotic behavior, but with all constants on the right-hand sides multiplied by $\alpha+1$ when $\theta \to -\infty$ and by $-(\alpha+1)$ when $\theta \to +\infty$.

[6]The case $\theta \to +\infty$ will be considered separately below.

Proof. Throughout the proof, we will often interchange limit and integration without showing the formal justification, which is based the same arguments as in the case of initial information acquisition considered above. However, the calculations are lengthy and omitted for the reader's convenience.

We have

$$\int_{\mathbb{R}} (v^H - S_\tau(y))\, G_{t,\tau}^{H,R,N}(\theta, V_{b\tau}(S_\tau(y)))\, \psi_{s\tau}^H(y)\, dy \;=\; (v^H - v_b)$$
$$+ \int_{\mathbb{R}} (v_b - S_\tau(y))\, \psi_{s\tau}^H(y)\, dy \;-\; \int_{\mathbb{R}} (v^H - S_\tau(y))\, F_{t,\tau}^{H,R,N}(\theta, V_{b\tau}(S_\tau(y)))\, \psi_{s\tau}^H(y)\, dy \tag{106}$$

and

$$\int_{\mathbb{R}} (v_b - S_\tau(y))\, G_{t,\tau}^{L,R,N}(\theta, V_{b\tau}(S_\tau(y)))\, \psi_{s\tau}^L(y)\, dy$$
$$= \int_{\mathbb{R}} (v_b - S_\tau(y))\, \psi_{s\tau}^L(y)\, dy \;-\; \int_{\mathbb{R}} (v_b - S_\tau(y))\, F_{t,\tau}^{L,R,N}(\theta, V_{b\tau}(S_\tau(y)))\, \psi_{s\tau}^L(y)\, dy. \tag{107}$$

By Lemma L.2, for a fixed x, we have

$$F_{t,\tau}^{R,N}(\theta, V_{b\tau}(S_\tau(y))) \;\sim\; C_{b,t,\tau}^{F,R,N}\, e^{(\alpha+\mathbf{1}_H)(V_{b\tau}(S_\tau(y))-\theta)}\, |\theta|^{\gamma_{t,\tau}^Q+N},$$

and the first claim follows from the identity

$$V_{b\tau}(S_\tau(y)) \;=\; \log\frac{S_\tau(y)-v_b}{v^H - S_\tau(y)} - y.$$

The case of the limit $\theta \to -\infty$ is completely analogous.

It remains to consider the case of a seller. The term corresponding to state H gives

$$\int_{\mathbb{R}} (S_\tau(y) - v^H)\, G_{b\tau}^H(V_{b\tau}(S_\tau(y)))\, R_{t,\tau}^{L,N}(\theta, y)\, dy$$
$$= (v_b - v^H) \;+\; \int_{\mathbb{R}} \Big((S_\tau(y) - v_b) + (v^H - S_\tau(y))\, F_{b\tau}^H(V_{b\tau}(S_\tau(y)))\Big)\, R_{t,\tau}^{L,N}(\theta, y)\, dy. \tag{108}$$

In the limit as $\theta \to -\infty$,

$$\begin{aligned}
&\int_{\mathbb{R}} \Big((S_\tau(y) - v_b) + (v^H - S_\tau(y))\, F_{b\tau}^H(V_{b\tau}(S_\tau(y)))\Big)\, R_{t,\tau}^{L,N}(\theta, y)\, dy \\
&\sim\; C_{s,0,\tau}^{R,N} e^{\alpha\theta}\, |\theta|^{\gamma_{t,\tau}^Q+N} \int_{\mathbb{R}} \Big((S_\tau(y) - v_b) + (v^H - S_\tau(y))\, F_{b\tau}^H(V_{b\tau}(S_\tau(y)))\Big) e^{-\alpha y}\, dy.
\end{aligned} \tag{109}$$

The term corresponding to state L gives

$$\begin{aligned}&\int_{\mathbb{R}} (S_\tau(y) - v_s)\, G^L_{b\tau}(V_{b\tau}(S_\tau(y)))\, R^{L,N}_{t,\tau}(\theta, y)\, dy \\ &= (v_b - v_s) \;+\; \int_{\mathbb{R}} \Big((S_\tau(y) - v_b) + (v_s - S_\tau(y))\, F^L_{b\tau}(V_{b\tau}(S_\tau(y))) \Big)\, R^{L,N}_{t,\tau}(\theta, y)\, dy.\end{aligned} \tag{110}$$

In the limit as $\theta \to -\infty$,

$$\begin{aligned}&\int_{\mathbb{R}} \Big((S_\tau(y) - v_b) + (v_s - S_\tau(y))\, F^L_{b\tau}(V_{b\tau}(S_\tau(y))) \Big)\, R^{L,N}_{t,\tau}(\theta, y)\, dy \\ &\sim\; C^{R,N}_{s,0,\tau} e^{(\alpha+1)\theta}\, |\theta|^{\gamma^Q_{t,\tau}+N} \int_{\mathbb{R}} \Big((S_\tau(y) - v_b) + (v_s - S_\tau(y))\, F^L_{b\tau}(V_{b\tau}(S_\tau(y))) \Big) e^{-(\alpha+1)y}\, dy.\end{aligned} \tag{111}$$

This completes the proof.

The claim concerning the derivatives with respect to θ is proved analogously. ■

The arguments of the proof of Lemmas F.1-F.9 imply the following result.

Lemma L.6 *Let*

$$\frac{(\alpha+1)^2}{\alpha - 1} > \alpha\,.$$

Then

$$\begin{aligned}&\int_{\mathbb{R}} (v^H \;-\; S_\tau(y))\, \left(\frac{S_\tau(y) - v_b}{v^H - S_\tau(y)} \right)^{\alpha+1} e^{-(\alpha+1)y}\, \psi^H_{s\tau}(y)\, dy \\ &\sim\; c_{s\tau}\, \varepsilon^{\frac{2\alpha+1}{\alpha+1}} \left| \frac{\log \varepsilon}{\alpha + 1} \right|^{\gamma_\tau} \int_{\mathbb{R}} (v^H \;-\; \mathcal{S}(y))\, \left(\frac{\mathcal{S}(y) - v_b}{v^H - \mathcal{S}(y)} \right)^{\alpha+1} e^{-(2\alpha+1)y}\, dy.\end{aligned} \tag{112}$$

Similarly, we have the following result.

Lemma L.7 *Let*

$$\frac{\alpha+1}{\alpha - 1} \;>\; \alpha\,.$$

Then

$$\begin{aligned}&\int_{\mathbb{R}} \Bigg(\Big((S_\tau(y) - v_b) + (v^H - S_\tau(y))\, F^H_{b\tau}(V_{b\tau}(S_\tau(y))) \Big) e^{y} \\ &+ \Big((S_\tau(y) - v_b) + (v_s - S_\tau(y))\, F^L_{b\tau}(V_{b\tau}(S_\tau(y))) \Big) \Bigg)\, e^{-(\alpha+1)y}\, dy \\ &\sim\; \varepsilon^{\frac{\alpha}{\alpha+1}} \int_{\mathbb{R}} \left((\mathcal{S}(y) - v_b) e^{-\alpha y} - \frac{\alpha+1}{\alpha}\, e^{-(2\alpha+1)y} \left(\frac{\mathcal{S}(y) - v_b}{v^H - \mathcal{S}(y)} \right)^{\alpha} \right)\, dy.\end{aligned} \tag{113}$$

In order to prove the next asymptotic result, we will need the following auxiliary lemma.

Lemma L.8 *Let $f(z)$ solve*

$$f'(z) \;=\; \left(\frac{\log(1/\zeta)}{\log(1/\zeta)\;+\;\log(1/f(z))}\right)^{\gamma}\left(z\;+\;\varepsilon^{\frac{1}{\alpha+1}}\,f(z)^{\frac{1}{\alpha+1}}\right), \tag{114}$$

with $f(0)=0$. Then, $r_\varepsilon(y) \;=\; f(\varepsilon^{\frac{1}{\alpha-1}}y)\varepsilon^{-2/(\alpha-1)}$ converges to the function $r(y)$ that is the unique solution to

$$r'(y) \;=\; y \;+\; (r(y))^{1/(\alpha+1)}\,, \qquad r(0)=0$$

as $\varepsilon\to 0$.

Proof. We have

$$\begin{aligned}
r_\varepsilon'(y) \;&=\; \varepsilon^{-\frac{1}{\alpha-1}}\,f'(\varepsilon^{\frac{1}{\alpha-1}}y)\\
&=\; \varepsilon^{-\frac{1}{\alpha-1}}\left(\frac{\log(1/\zeta)}{\log(1/\zeta)\;+\;\log(1/f(\varepsilon^{\frac{1}{\alpha-1}}y))}\right)^{\gamma}\left(\varepsilon^{\frac{1}{\alpha-1}}y\;+\;\varepsilon^{\frac{1}{\alpha+1}}\,f(\varepsilon^{\frac{1}{\alpha-1}}y)^{\frac{1}{\alpha+1}}\right)\\
&=\; \left(\frac{\log(1/\zeta)}{\log(1/\zeta)\;+\;\log(\varepsilon^{-2/(\alpha-1)}/r_\varepsilon(y))}\right)^{\gamma}\left(y\;+\;(r_\varepsilon(y))^{\frac{1}{\alpha+1}}\right).
\end{aligned} \tag{115}$$

The right-hand side of this equation converges to $y\;+\;(r_\varepsilon(y))^{1/(\alpha+1)}$. The fact that $r_\varepsilon(y)$ converges to $r(y)$ follows from the uniqueness part of the proof of Proposition D.1 and standard continuity arguments. ■

Lemma L.9 *We have*

$$\begin{aligned}
&\int_{\mathbb{R}}(v^H\;-\;S_\tau(y))\left(\frac{S_\tau(y)-v_b}{v^H-S_\tau(y)}\right)^{-\alpha}e^{\alpha y}\,\psi_{s\tau}^H(y)\,dy\\
&\qquad\sim\;\varepsilon^{-\alpha/(\alpha-1)}\int_0^\infty y^{-\alpha-1}\,\phi_{s\tau}\left(y^{-1}\,r(y)^{1/(\alpha+1)}\right)\,dy,
\end{aligned} \tag{116}$$

where

$$\phi_{s\tau}(y) \;=\; y^{-\alpha}\psi_{s\tau}^H(-\log y)\,.$$

Proof. For simplicity, we make the normalization $v^H=1,\; v_b=0$.

We make the change of variable $S_\tau(y)=z,\; y\;=\;V_{s\tau}(z)\,,\; dy\;=\;V_{s\tau}'(z)\,dz$. Using the identity $V_{s\tau}(z)\;=\;\log\frac{z}{1-z}\;-\;V_{b\tau}(z)$, we get

$$V_{s\tau}'(z)\;=\;\frac{1}{z(1-z)}-V_{b\tau}(z)\,.$$

We will also use the notation $g(z) = e^{(\alpha+1)V_{b\tau}(z)}$ from the proof of Proposition D.1. Then, we have

$$\begin{aligned}
&\int_{\mathbb{R}} (v^H - S_\tau(y)) \left(\frac{S_\tau(y) - v_b}{v^H - S_\tau(y)}\right)^{-\alpha} e^{\alpha y}\, \psi^H_{s\tau}(y)\, dy \\
&= \int_0^1 (1-z) \left(\frac{1-z}{z}\right)^{\alpha} e^{\alpha\, V_{s\tau}(z)}\, \psi^H_{s\tau}(V_{s\tau}(z))\, V'_{s\tau}(z)\, dz \\
&= \int_0^1 (1-z)\, e^{-\alpha V_{b\tau}(z)}\, \psi^H_{s\tau}\left(\log\frac{z}{1-z} - V_{b\tau}(z)\right)\left(\frac{1}{z(1-z)} - V'_{b\tau}(z)\right) dz \\
&= \int_0^1 (1-z)\, g(z)^{-\frac{\alpha}{\alpha+1}}\, \psi^H_{s\tau}\left(\log\frac{z}{1-z} - \frac{\log g(z)}{\alpha+1}\right)\left(\frac{1}{z(1-z)} - \frac{g'(z)}{(\alpha+1)g(z)}\right) dz.
\end{aligned} \tag{117}$$

As we have shown in the proof of Proposition D.1, $g(z)/\varepsilon$ converges to a limit $f_0(z)$ when $\varepsilon \to 0$. A direct calculation based on dominated convergence theorem and the bounds for $f(z)$ established in the proof of Proposition D.1 implies that the limit

$$\lim_{\varepsilon\to 0} \varepsilon^{\frac{\alpha}{\alpha+1}} \int_r^1 (1-z)\, g(z)^{-\frac{\alpha}{\alpha+1}}\, \psi^H_{s\tau}\left(\log\frac{z}{1-z} - \frac{\log g(z)}{\alpha+1}\right)\left(\frac{1}{z(1-z)} - \frac{g'(z)}{(\alpha+1)g(z)}\right) dz$$

exists and is finite for any $r > 0$. By contrast, as we will show below,

$$\varepsilon^{\frac{\alpha}{\alpha+1}} \int_0^r (1-z)\, g(z)^{-\frac{\alpha}{\alpha+1}}\, \psi^H_{s\tau}\left(\log\frac{z}{1-z} - \frac{\log g(z)}{\alpha+1}\right)\left(\frac{1}{z(1-z)} - \frac{g'(z)}{(\alpha+1)g(z)}\right) dz$$

blows up to $+\infty$ as $\varepsilon \to 0$. Therefore, the part $\int_r^1$ of the integral is asymptotically negligible and we will in the sequel only consider the integral $\int_0^r$ with a sufficiently small $r > 0$. Then, it follows from the proof of Proposition D.1 that we may assume that $g(z) = \varepsilon f(z)$ where $f(z)$ solves the ODE (114). For the same reason, we may replace $1 - z$ by 1. It also follows from the proof of Proposition D.1 that

$$K_2\, D(q(z)) \;\le\; g(z) \;\le\; K_1\, D(q(z)) \tag{118}$$

for some $K_1 > K_2 > 0$, where

$$D(x) \;=\; x\,(-\log x)^{-\gamma},$$

with $\gamma = \gamma_\tau$, and

$$q(z) \;=\; \zeta^{1+1/\alpha}\, C\, z^{(\alpha+1)/\alpha}\, (-\log(\zeta z))^{-\gamma/\alpha} \;+\; \frac{1}{2}\,\zeta\, z^2$$

for some constant $C > 0$.

Denote

$$\phi(e^{-x}) \;=\; e^{\alpha x}\,\psi^H_{s\tau}(x)\,.$$

We have $\psi^H_{s\tau}(x) \sim\; c_{s\tau}e^{-\alpha\, x}|x|^{\gamma_\tau}$ when $x \to +\infty$ and $\psi^H_{s\tau}(x) \sim\; c_{s\tau}e^{(\alpha+1)x}|x|^{\gamma_\tau}$ when $x \to -\infty$. Therefore,

$$\phi(y) \;=\; y^{-\alpha}\,\psi^H_{s\tau}(-\log y) \;\sim\; c_{s\tau}y^{-\alpha}\,e^{-\alpha(-\log y)}|\log y|^{\gamma_\tau} \;=\; c_{s\tau}|\log y|^{\gamma_\tau}$$

when $y \to 0$ and, similarly,

$$\phi(y) \;\sim\; c_{s\tau}y^{-2\alpha-1}|\log y|^{\gamma_\tau}$$

as $y \to +\infty$.

With this notation, we have

$$\begin{aligned}&\int_0^r (1-z)\,g(z)^{-\frac{\alpha}{\alpha+1}}\,\psi^H_{s\tau}\left(\log z - \frac{\log g(z)}{\alpha+1}\right)\left(\frac{1}{z(1-z)} - \frac{g'(z)}{(\alpha+1)g(z)}\right)dz\\ &\quad= \int_0^r z^{-\alpha}\,\phi\left(z^{-1}\,g^{1/(\alpha+1)}\right)\left(\frac{1}{z(1-z)} - \frac{g'(z)}{(\alpha+1)g(z)}\right)dz\,.\end{aligned} \tag{119}$$

By (118), for some $K_3 > 0$,

$$\begin{aligned}\frac{g'(z)}{g(z)} \;\le\; K_3\,&\frac{\zeta^{1/\alpha}Cz^{1/\alpha}\,(-\log(\zeta z))^{-\gamma/\alpha}\,\left(\frac{\alpha+1}{\alpha} \;+\; \frac{\gamma}{\alpha}\,(-\log(\zeta z))^{-1}\right) +\; z}{\zeta^{1/\alpha}\,C\,z^{(\alpha+1)/\alpha}\,(-\log(\zeta z))^{-\gamma/\alpha} \;+\; \frac{1}{2}\,z^2}\\ &\times(-\log q(z))^{-\gamma}\,(1 \;+\; \gamma\,(-\log q(z))^{-1}).\end{aligned} \tag{120}$$

Since we are in the regime when both z and ζ are small, $1 \;+\; \gamma\,(-\log q(z))^{-1} \;\sim\; 1$, so we can ignore this factor when we determine the asymptotic behavior. Furthermore, for the same reason,

$$\frac{\alpha+1}{\alpha} \;\le\; z\,\frac{\zeta^{1/\alpha}Cz^{1/\alpha}\,(-\log(\zeta z))^{-\gamma/\alpha}\,\left(\frac{\alpha+1}{\alpha} \;+\; \frac{\gamma}{\alpha}\,(-\log(\zeta z))^{-1}\right) +\; z}{\zeta^{1/\alpha}\,C\,z^{(\alpha+1)/\alpha}\,(-\log(\zeta z))^{-\gamma/\alpha} \;+\; \frac{1}{2}\,z^2} \;\le\; 2$$

for small ζ, z. Therefore, since for small ζ, z $(-\log q(z))^{-\gamma}$ is sufficiently small, we have

$$\frac{1}{z} \;-\; \frac{g'(z)}{g(z)} \;\sim\; \frac{1}{z} \tag{121}$$

for small z, ζ.

Making the transformation $z \;=\; \zeta^{1/(\alpha-1)}\,(-\log\zeta)^{-\gamma/(\alpha-1)}y$, standard dominated convergence arguments together with Lemma L.8 imply that

$$\begin{aligned}&\int_0^r z^{-\alpha-1}\,\phi\left(z^{-1}\,g^{1/(\alpha+1)}\right)\,dz\\ &= \left(\frac{\zeta}{(-\log\zeta)^\gamma}\right)^{-\alpha/(\alpha-1)}\int_0^{r\left(\frac{\zeta}{(-\log\zeta)^\gamma}\right)^{-1/(\alpha-1)}} y^{-\alpha-1}\,\phi\left(y^{-1}\,(r_\varepsilon(y))^{1/(\alpha+1)}\right)\,dy\\ &\sim\; \left(\frac{\zeta}{(-\log\zeta)^\gamma}\right)^{-\alpha/(\alpha-1)}\int_0^\infty y^{-\alpha-1}\,\phi\left(y^{-1}\,r(y)^{1/(\alpha+1)}\right)\,dy,\end{aligned} \tag{122}$$

completing the proof. ■

Lemma L.10 *We have*

$$V_{b\tau}(z) \;\approx\; \frac{1}{\overline{G}\alpha}\log\frac{1}{1-z} \;+\; K(\varepsilon)$$

as $z\uparrow 1$, *for some constant* $K(\varepsilon)$.

Proof. As above, we will everywhere use the normalization $v^H = 1,\ v_b = 0$. For brevity, let $h^{H,L} = h^{H,L}_{b\tau}$. We have

$$V'_{b\tau}(z) \;=\; (\overline{G})^{-1}\left(\frac{z}{1-z}\,\frac{1}{h^H(V_{b\tau}(z))} \;+\; \frac{1}{h^L(V_{b\tau}(z))}\right),$$

and therefore

$$V_{b\tau}(z) \;=\; V_{b\tau}(z_0) \;+\; (\overline{G})^{-1}\int_{z_0}^{z}\left(\frac{y}{1-y}\,\frac{1}{h^H(V_{b\tau}(y))} \;+\; \frac{1}{h^L(V_{b\tau}(y))}\right)dy$$

for any $z_0\in(0,1)$. A direct application of l'Hopital's rule implies that

$$\frac{1}{h^H(x)} \;=\; \frac{G^H(x)}{\psi^H(x)} \;\to\; \alpha^{-1}$$

as $x\to+\infty$. Using the identity

$$\frac{G^H(x)}{\psi^H(x)} \;-\; \alpha^{-1} \;=\; e^{\alpha x}\,\frac{\int_x^{+\infty} e^{-\alpha y}((y/x)^\gamma e^{\alpha y}y^{-\gamma}\psi^H(y) - e^{\alpha x}x^{-\gamma}\psi^H(x))\,dy}{e^{\alpha x}x^{-\gamma}\,\psi^H(x)},$$

it is possible to show that this will converge to zero at least as fast as $x^{-\gamma}$. Indeed, condition (87) implies that we can replace $e^{\alpha y}y^{-\gamma}\psi^H(y)$ by its limit value c_τ as the difference will be asymptotically negligible. Thus, it remains to consider

$$e^{\alpha x}\int_x^{+\infty} e^{-\alpha y}((y/x)^\gamma - 1)\,dy \;=\; \int_0^\infty e^{-\alpha y}((1+y/x)^\gamma - 1)\,dy \;\le\; x^{-\gamma}\int_0^\infty e^{-\alpha y}y^\gamma\,dy.$$

Therefore, we can write

$$\begin{aligned} V_{b\tau}(z) \;&=V_{b\tau}(z_0) \;+\; (\overline{G})^{-1}\int_{z_0}^{z}\left(\frac{y}{1-y}\,\frac{1}{h^H(V_{b\tau}(y))} \;+\; \frac{1}{h^L(V_{b\tau}(y))}\right)dy \\ &=\; V_{b\tau}(z_0) \;+\; \frac{1}{\overline{G}\alpha}(-z \;-\; \log(1-z) - (-z_0 - \log(1-z_0))) \\ &\quad+\; \frac{1}{\overline{G}}\int_{z_0}^{z}\left(\frac{y}{1-y}\left(\frac{1}{h^H(V_{b\tau}(y))} - \frac{1}{\alpha}\right) \;+\; \frac{1}{h^L(V_{b\tau}(y))}\right)dy. \end{aligned} \tag{123}$$

Consequently, when $z \uparrow 1$,

$$V_{b\tau}(z) \;\sim\; \frac{1}{\overline{G}\alpha} \log \frac{1}{1-z} \;+\; K(\varepsilon),$$

where

$$\begin{aligned} K(\varepsilon) \;&=\; V_{b\tau}(z_0) \;+\; \frac{1}{\overline{G}\alpha}(-1 + z_0 + \log(1 - z_0)) \\ &\quad+\; \frac{1}{\overline{G}} \int_{z_0}^{1} \left(\frac{y}{1-y} \left(\frac{1}{h^H(V_{b\tau}(y))} - \frac{1}{\alpha} \right) \;+\; \frac{1}{h^L(V_{b\tau}(y))} \right) dy, \end{aligned} \tag{124}$$

and the claim follows. ■

Lemma L.11 *When $\overline{G}$ becomes large, $K(\varepsilon)$ converges to*

$$K \;=\; A - \int_{-\infty}^{A} \alpha^{-1}\, h^H(x) dx \;+\; \int_{A}^{+\infty} (1 - h^H(x)/\alpha)\, dx\,.$$

Proof. Based on the change of variables

$$V_{b\tau}(y) = x\ ,\ \ dy = B_\tau'(x)\, dx = \overline{G} \left(\frac{B_\tau(x)}{1 - B_\tau(x)} \frac{1}{h^H(x)} \;+\; \frac{1}{h^L(x)} \right)^{-1},$$

we have

$$\begin{aligned} &\frac{1}{\overline{G}} \int_{z_0}^{1} \left(\frac{y}{1-y} \left(\frac{1}{h^H(V_{b\tau}(y))} - \frac{1}{\alpha} \right) \;+\; \frac{1}{h^L(V_{b\tau}(y))} \right) dy \\ &= \int_{V_{b\tau}(z_0)}^{+\infty} \frac{\frac{B_\tau(x)}{1-B_\tau(x)} \left(\frac{1}{h^H(x)} - \frac{1}{\alpha} \right) \;+\; \frac{1}{h^L(x)}}{\frac{B_\tau(x)}{1-B_\tau(x)} \frac{1}{h^H(x)} \;+\; \frac{1}{h^L(x)}}\, dx. \end{aligned} \tag{125}$$

When $\overline{G} \to \infty,\ B_\tau(x) \to v^H = 1$. Hence the leading asymptotic of the integrand is given by $1 - h^H(x)/\alpha$. Therefore, for any $A > 0$,

$$\begin{aligned} &\int_{V_{b\tau}(z_0)}^{+\infty} \frac{\frac{B_\tau(x)}{1-B_\tau(x)} \left(\frac{1}{h^H(x)} - \frac{1}{\alpha} \right) \;+\; \frac{1}{h^L(x)}}{\frac{B_\tau(x)}{1-B_\tau(x)} \frac{1}{h^H(x)} \;+\; \frac{1}{h^L(x)}}\, dx \\ &\approx\; \int_{V_{b\tau}(z_0)}^{+\infty} (1 - h^H(x)/\alpha)\, dx \\ &=\; A - V_{b\tau}(z_0) \;-\; \int_{V_{b\tau}(z_0)}^{A} \alpha^{-1}\, h^H(x) dx \;+\; \int_{A}^{+\infty} (1 - h^H(x)/\alpha)\, dx \\ &\approx\; A - V_{b\tau}(z_0) \;-\; \int_{-\infty}^{A} \alpha^{-1}\, h^H(x) dx \;+\; \int_{A}^{+\infty} (1 - h^H(x)/\alpha)\, dx, \end{aligned} \tag{126}$$

and the claim follows. ■

Lemma L.12 *When $\theta \to +\infty$ and $\overline{G} \to \infty$ in such a way that $\theta - \log \varepsilon/(\alpha+1) \to +\infty$, we have*

$$u_{s,0,\tau}(N_{\max}, \theta) - u_{s,0,\tau}(N_{\min}, \theta) \;\sim\; e^{-(\alpha+1)\frac{\theta}{\overline{G}\alpha - 1}}\, |\theta|^{\gamma_\tau}\, Z$$

and

$$\frac{\partial}{\partial \theta}\, \left(u_{s,0,\tau}(N_{\max}, \theta) - u_{s,0,\tau}(N_{\min}, \theta)\right) \;\sim\; -\frac{\alpha+1}{\overline{G}\alpha - 1} e^{-(\alpha+1)\frac{\theta}{\overline{G}\alpha - 1}}\, |\theta|^{\gamma_\tau}\, Z$$

for some constant $Z > 0$.

Proof. When $y \to \infty$ we have $S_\tau(y) \to 1$. Thus,

$$\begin{aligned} y \;&=\; V_{s\tau}(S_\tau(y)) \;=\; \log \frac{S_\tau(y)}{1 - S_\tau(y)} \;-\; V_{b\tau}(S_\tau(y)) \\ &\sim\; \left(1 - \frac{1}{\overline{G}\alpha}\right) \log \frac{1}{1 - S_\tau(y)} \;-\; K(\varepsilon)\,. \end{aligned} \tag{127}$$

Therefore,

$$1 \;-\; S_\tau(y+\theta) \;\sim\; e^{-(y+\theta+K(\varepsilon))/\left(1 - \frac{1}{\overline{G}\alpha}\right)}$$

when $\theta \to \infty$ and

$$V_{b\tau}(S_\tau(y+\theta)) \;\sim\; \frac{y+\theta+K(\varepsilon)}{1 - (\overline{G}\alpha)^{-1}} - (y+\theta).$$

Hence

$$G^H_{b\tau}(V_{b\tau}(S_\tau(y+\theta))) \;\sim\; \frac{c_\tau}{\alpha}\, |\theta|^{\gamma_\tau}\, e^{-\alpha \frac{y+\theta+\overline{G}\alpha K}{\overline{G}\alpha - 1}}.$$

Therefore, in the high state, we get

$$\begin{aligned} &\int_{\mathbb{R}} (S_\tau(y+\theta) - 1)\, G^H_{b\tau}(V_{b\tau}(S_\tau(y+\theta)))\, R^{H,N}_{t,\tau}(\theta, y+\theta)\, dy \\ &\sim\; -\frac{c_\tau}{\alpha}\, |\theta|^{\gamma_\tau} \int_{\mathbb{R}} e^{-(y+\theta+K(\varepsilon))/\left(1 - \frac{1}{\overline{G}\alpha}\right)}\, e^{-\alpha \frac{y+\theta+\overline{G}\alpha K}{\overline{G}\alpha - 1}}\, R^{H,N}_{t,\tau}(\theta, y+\theta)\, dy \\ &=\; -|\theta|^{\gamma_\tau} e^{-(\alpha+1)\frac{\overline{G}\alpha}{\overline{G}\alpha - 1} K(\varepsilon)}\, e^{-\theta\left(\frac{\alpha(\overline{G}+1)}{\alpha\overline{G} - 1}\right)}\, \frac{c_\tau}{\alpha} \int_{\mathbb{R}} e^{-(\alpha+1)\frac{y}{\overline{G}\alpha - 1}} e^{-y}\, R^{H,N}_{t,\tau}(\theta, y+\theta)\, dy. \end{aligned} \tag{128}$$

In the low state, using $S_\tau(y+\theta) - v_s \sim \overline{G}$, we get

$$\begin{aligned} &\int_{\mathbb{R}} (S_\tau(y+\theta) - v_s)\, G^L_{b\tau}(V_{b\tau}(S_\tau(y+\theta)))\, R^{L,N}_{t,\tau}(\theta, y+\theta)\, dy \\ &\quad\sim\; \overline{G}\, |\theta|^{\gamma_\tau}\, \frac{c_\tau}{\alpha+1} \int_{\mathbb{R}} e^{-(\alpha+1)\frac{y+\theta+\overline{G}\alpha K}{\overline{G}\alpha - 1}}\, R^{L,N}_{t,\tau}(\theta, y+\theta)\, dy \\ &\quad\sim\; \overline{G}\, e^{-(\alpha+1)\frac{\theta+\overline{G}\alpha K}{\overline{G}\alpha - 1}}\, |\theta|^{\gamma_\tau}\, \frac{c_\tau}{\alpha+1} \int_{\mathbb{R}} e^{-(\alpha+1)\frac{y}{\overline{G}\alpha - 1}} R^{L,N}_{t,\tau}(\theta, y+\theta)\, dy. \end{aligned} \tag{129}$$

Thus, in the limit as $\overline{G} \to 0$, the gain in the low state from acquiring information satisfies

$$\begin{aligned} & e^{-(\alpha+1)\frac{\theta+\overline{G}\alpha K}{\overline{G}\alpha-1}} \, |\theta|^{\gamma_\tau} \, \frac{\overline{G} c_\tau}{\alpha+1} \int_{\mathbb{R}} e^{-(\alpha+1)\frac{y}{\overline{G}\alpha-1}} \, (R_{t,\tau}^{L,N_{\max}}(\theta, y+\theta) - R_{t,\tau}^{L,N_{\min}}(\theta, y+\theta)) \, dy \\ \sim \; & \frac{1}{\alpha} e^{-(\alpha+1)K} \, e^{-(\alpha+1)\frac{\theta}{\overline{G}\alpha-1}} \, |\theta|^{\gamma_\tau} \, c_\tau \int_{\mathbb{R}} (-y)(\eta_{N_{\max}}^L - \eta_{N_{\min}}^L) * h_{t,\tau}^L(y) \, dy, \end{aligned} \tag{130}$$

whereas the loss in the H state is asymptotically negligible because the additional factor $\overline{G}$ is missing. ■

The following lemma completes the proof of Theorem K.1.

Lemma L.13 *There exist $g, A > 0$ such that a threshold equilibrium exists whenever $\overline{G} > g$ and $\pi < e^{-A\overline{G}}$. In any such equilibrium, conditions (88) and (89) hold as $\pi^{-1}, \overline{G} \to \infty$.*

Proof. Fix a threshold acquisition policy $\{\overline{X}_{it}, \underline{X}_{it}\}_{i=b,s,t\geq 1}$ of all the agents in the market. It follows from the above (Lemmas L.1, L.3, L.5 and L.12) that there exist constants $a, g, B > 0$ such that the gains from information acquisition are monotone decreasing in $|\theta|$ when $|\theta| > B$, $\overline{G} > g$ and

$$\min\{\min_{i,t} |\overline{X}_{it}|, \min_{i,t} |\underline{X}_{it}|\} > a\overline{G}\,.$$

Therefore, the optimal acquisition policy for any agent is also of threshold type, given by $\{\tilde{\overline{X}}_{it}, \tilde{\underline{X}}_{it}\}$, whenever π is sufficiently small. It follows from the proofs of Lemmas L.5 and L.12 that, in fact, there exists an $A > 0$ such that $\pi < e^{-A\overline{G}}$ is sufficient for this. Clearly, choosing $A > 0$ sufficiently big, we can achieve that

$$\min\{\min_{i,t} |\tilde{\overline{X}}_{it}|, \min_{i,t} |\tilde{\underline{X}}_{it}|\} > a\overline{G}\,.$$

Making the change of variables $\theta \to Re^\theta/(1 + Re^\theta)$, we immediately get that the mapping from $\{\overline{X}_{it}, \underline{X}_{it}\}_{i=b,s,t\geq 1}$ to $\{\tilde{\overline{X}}_{it}, \tilde{\underline{X}}_{it}\}_{i=b,s,t\geq 1}$ maps bounded convex set into itself. Therefore, existence of a threshold equilibrium follows by the Brower fixed point Theorem. The fact that any equilibrium satisfies (88) and (89) follows by a careful examination of alternative cases, is very lengthy and is therefore omitted. ■

We can now calculate approximations for the optimal acquisition thresholds. Though we cannot prove that an equilibrium is unique, the next result implies that the equilibrium is *asymptotically unique*, in the sense that the asymptotic behavior of the equilibrium thresholds is the same for any equilibrium.

Lemma L.14 *For any equilibrium, in the limit when $\overline{G} \to \infty$ and $\pi \to 0$ in such a way that $\pi < e^{-A\overline{G}}$, the optimal information acquisition thresholds satisfy*

1.

$$(\alpha+1)\overline{X}_{bt} \approx \overline{K}_{b,t,\tau} + \left(m^{Q}_{t,T} - \frac{\alpha}{\alpha+1} m^{\psi}_{T}\right)\log\lambda + \log(\pi^{-1}) - \frac{2\alpha+1}{\alpha+1}\log\overline{G} \\ + \ (\gamma^{Q}_{t,T} + N_{\max})\log(\log(\pi^{-1}\overline{G}^{-\frac{2\alpha+1}{\alpha+1}})) - \frac{\alpha}{\alpha+1}\gamma^{\psi}_{T}\log\log\overline{G}. \tag{131}$$

2.

$$-(\alpha+1)\underline{X}_{bt} \ \approx \ \underline{K}_{b,t,\tau} \ + \log R \ + \ (\log(\pi^{-1}) + (\gamma^{Q}_{t,T} + N_{\max})\log(\log(\pi^{-1}))) \\ + \ \frac{\alpha}{\alpha-1}\log\overline{G} \ + \ \left(\gamma^{Q}_{t,T} + N_{\max} + \frac{\alpha}{\alpha-1}\gamma^{\psi}_{T}\right)\log\log\overline{G}. \tag{132}$$

3.

$$(\alpha+1)\overline{X}_{st} \ \approx \ (\overline{G}\alpha - 1) \\ \left(\log(\pi^{-1}) + \overline{K}_{s,0,T} + \gamma_T \log\left(\frac{\overline{G}\alpha - 1}{\alpha+1}(\log(\pi^{-1}) + \overline{K}_{s,0,T})\right)\right). \tag{133}$$

4.

$$-(\alpha+1)\underline{X}_{st} \ \approx \ \underline{K}_{s,0,\tau} \ + \ \left(m^{Q}_{t,T} - \frac{\alpha}{\alpha+1} m^{\psi}_{T}\right)\log\lambda \ + \ \log(\pi^{-1}) \\ - \ \frac{\alpha}{\alpha+1}\log\overline{G} \ + \ (\gamma^{Q}_{t,T} + N_{\max})\log(\log(\pi^{-1}\overline{G}^{-\frac{\alpha}{\alpha+1}})) \\ - \frac{\alpha}{\alpha+1}\gamma^{\psi}_{T}\log\log\overline{G}, \tag{134}$$

where $\gamma^{\psi}_{T} = (2^{T+1} - 1)N_{\max} - 1$ and $\gamma^{Q}_{t,T} = (2^{T+1} - 2^{t+1})N_{\max} - 1$.

Proof. The proof follows directly from Lemma L.5 and Lemmas L.6-L.12. ■

Proof of Theorem K.2. This theorem follows from substituting the asymptotic expressions of Lemma L.14 into the asymptotic formulae of Lemma L.1 for the tail behaviour of the densities of type distributions. ■

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