Floating–Fixed Credit Spreads

Darrell Duffie and Jun Liu

We examine the term structure of yield spreads between floating-rate and fixed-rate notes of the same credit quality and maturity. Floating–fixed spreads are theoretically characterized in some practical cases and quantified in a simple model in terms of maturity, credit quality, yield volatility, yield-spread volatility, correlation between changes in yield spreads and default-free yields, and other determining variables. We show that if the issuer's default risk is risk-neutrally independent of interest rates, the sign of floating–fixed spreads is determined by the term structure of the risk-free forward rate.

Intuitively, if the term structure is upward sloping, investors anticipate that floating-rate coupons will increase with time. Default risk for a given issuer increases with time because the issuer cannot survive to time t unless it also survives to each time $s < t$. Because the higher anticipated coupon payments of later dates are also more likely to be lost to default, investors must be compensated by a floating-rate spread that is slightly larger than the fixed-rate spread.

In terms of magnitude, however, in most practical cases, floating–fixed spreads are small, typically a few basis points at most, as will be shown by our examples.[1] Our persistent queries to market practitioners generated almost no examples in which market participants make a distinction between par floating-rate spreads and par fixed-rate spreads. The exception was certain cases in which one of these forms of debt was viewed as "more liquid" than the other, an issue that we do not pursue.

For example, consider an issuer whose credit quality implies a fixed-rate spread on five-year par-coupon debt of 100 basis points (bps) over the rate on default-free five-year par-coupon fixed-rate debt. Suppose changes in credit quality are not correlated with state prices (in a sense to be made precise). In a typical upward-sloping term-structure environment, based on the steady-state behavior of a two-factor CIR (Cox–Ingersoll–Ross 1985) model fitted to LIBOR swap rates recorded during the 1990s, floating-rate debt of the same credit quality and maturity would be issued at a spread of roughly 101 bps.[2] This is, of course, not to say that the issuer should prefer to issue fixed over floating debt but, rather, that a slightly higher credit spread is required to compensate investors paying par for floating-rate debt.

As suggested by this example, the magnitude of the floating–fixed spread associated with default risk is sufficiently small that one could safely attribute any nontrivial differences that may exist in actual fixed and floating rates of the same credit quality to institutional differences between the fixed- and floating-rate note markets.

For our model, the floating–fixed spread is roughly linear in the issuer's fixed-rate credit spread, roughly linear in the slope of the yield curve, roughly linear in the level of the yield curve, and roughly linear in the correlation between changes in default-free yields and fixed-rate yield spreads. The floating–fixed spread is nonlinear in maturity. There is essentially no dependence in the level of the yield curve when the slope is held constant. The floating–fixed spread is greatest at high yield-spread volatility and high correlation between yield spread and default-free yields.

Our methodology for valuing defaultable debt is that of Duffie and Singleton (1999). Our numerical examples are based on three-factor term-structure models. Two of the three state variables determine a LIBOR swap term-structure model estimated from LIBOR swap data; the expected default-loss-rate process is based on all three factors, which allows for correlation between default risk and LIBOR swap rates. For purposes of studying the effects of correlation between yields and credit spreads, we move from the multifactor CIR setting to a "quadratic-Gaussian" credit-spread model.

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Getting Started

For simplicity, we begin in a discrete time setting: Let $\Gamma_{m,n}$ denote the time m price of a default-free zero-coupon bond maturing at time $n > m$.

The one-period default-free floating-rate coupon at time n, $c(n)$, is

$$c(n) = (\Gamma_{n-1,n})^{-1} - 1. \tag{1}$$

The coupon rate at time 0 for fixed-rate par-valued default-free debt maturing at time N, $C(N)$, is determined by

$$C(N) = \frac{1 - \Gamma_{0,N}}{\sum_{n=1}^{N} \Gamma_{0,n}}. \tag{2}$$

The in-n-for-1 forward rate for maturity n is defined by

$$\begin{aligned} f(n) &= \frac{\Gamma_{0,n-1} - \Gamma_{0,n}}{\Gamma_{0,n}} \\ &= \frac{\Gamma_{0,n-1}}{\Gamma_{0,n}} - 1. \end{aligned} \tag{3}$$

We will later use the relationship

$$\sum_{n=1}^{N} \Gamma_{0,n} [f(n) - C(N)] = 0. \tag{4}$$

Also let $\pi(n)$ denote the state-price density[3] for time n contingent claims so that, for example, $\Gamma_{0,n} = E[\pi(n)]$, where E denotes expectation.

For an issuer of given credit quality, Pye (1974) and in a setting of uncertain interest rates and credit quality, Duffie and Singleton (1999) showed simple conditions under which one may price a defaultable claim by treating the claim as default free after an additional discount,

$$D_{0,n} = \prod_{i=0}^{n-1} [1 + s(n)]^{-1}, \tag{5}$$

for contingent cash flows at time n, where $s(n) \geq 0$ is the (state-dependent) short default spread conditional on information at time n.

For example, letting $\Lambda_{m,n}$ denote the price at time m of a zero-coupon bond maturing at time n of the given issuer quality produces

$$\Lambda_{0,n} = E[D_{0,n}\,\pi(n)]. \tag{6}$$

We adopt this defaultable valuation model here. For simplicity, we assume that the short default spread, $s(n)$, does not vary among the claims of the given issuer being considered.[4]

The spread at time 0 on defaultable floating-rate debt of maturity N, $k(N)$, is defined by matching to 1 the price of a defaultable note that obliges the issuer, so long as solvent, to pay $c(n) + k(N)$ at each time $n < N$ and to pay $1 + c(N) + k(N)$ at time N. Thus,

$$k(N) = \frac{1 - \Lambda_{0,N} - \sum_{n=1}^{N} E[D_{0,n}\pi(n)c(n)]}{\sum_{n=1}^{N} \Lambda_{0,n}}. \tag{7}$$

The fixed-rate spread on defaultable debt of maturity N, $K(N)$, is similarly determined by

$$K(N) = \frac{1 - \Lambda_{0,N} - C(N)\sum_{n=1}^{N} \Lambda_{0,n}}{\sum_{n=1}^{N} \Lambda_{0,n}}. \tag{8}$$

The difference between the floating and fixed spreads, $\Delta(N)$, is then

$$\begin{aligned} \Delta(N) &\equiv k(N) - K(N) \\ &= \frac{\sum_{n=1}^{N} \{\Lambda_{0,n} C(N) - E[\pi(n) D_{0,n} c(n)]\}}{\sum_{n=1}^{N} \Lambda_{0,n}}. \end{aligned} \tag{9}$$

Proposition. Suppose for all n that the state-price density, $\pi(n)$, and the default discount, $D_{0,n}$, are uncorrelated. Suppose, moreover, that some n_0 exists such that $f(n) \leq C(N)$ for $n \leq n_0$ and $f(n) \geq C(N)$ for $n < n_0 \leq N$. [It is enough for this that the forward rate, $f(n)$, is increasing in n up to time N.] Then, the floating–fixed spread, $\Delta(N)$, is nonnegative. If, in addition, the short default spread, $s(n)$, is greater than zero with positive probability for each time n before default and if $f(n)$ is not constant in n, then $\Delta(N) > 0$.

We give a continuous-time version of the proposition with a proof in Appendix A. The proof of the discrete version of the proposition is similar. The intuition for the result is given in the introduction. A similar result applies to obtain a negative floating–fixed spread $\Delta(N)$ for an "inverted" forward rate curve.

Floating-Rate Debt in an Affine Setting[5]

To work with an econometrically estimated model of the term structure and to provide for sufficient analytical tractability, we now move to a traditional continuous-time setting that involves a short-rate process, r, and a "risk-neutral" probability measure, Q, defined by the property that any contingent claim paying X at some time T has a price at any time $t < T$ given by

$$E_t^Q[\exp(-\int_t^T r_s ds) X],$$

where E_t^Q is expectation under Q conditional on information available to investors at time t.[6] For example, the default-free zero-coupon bond price in this setting is given by

$$\Gamma_{0,n} = E_0^Q[\exp(-\int_0^n r_t dt)]. \quad (10)$$

Duffie and Singleton (1999) provided conditions under which, for the issuer's given credit quality, there exists a default-risk-adjusted short-rate process $R \geq r$ such that the price at time t of a defaultable claim to X at time T is given by

$$E_t^Q[\exp(-\int_t^T R_s ds)X].$$

That is, one can apply the standard formula for pricing default-free claims to defaultable claims provided the default-free short rate, r, is replaced by the risk-adjusted short rate, R. For example, the defaultable zero-coupon bond price is given by

$$\Lambda_{0,n} = E^Q[\exp(-\int_0^n R_t dt)]. \quad (11)$$

The continuous-time analog of the zero-correlation assumption given in the proposition is that the short spread process, $S = R - r$, is independent of r under Q. We extend that proposition in Appendix A and here explore cases in which this assumption does not necessarily hold.

To tractably value floating-rate debt in a flexible parametric setting, we work with some "state" process X that is a k-dimensional affine jump diffusion, in the sense of Duffie and Kan (1996). That is, X is valued in some appropriate domain D that is a subset of a k-dimensional Euclidian space $\mathbf{R}^k$, with

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t, \quad (12)$$

where

$\mu(x)$ = the risk-neutral drift at x

$\sigma(x)\sigma(x)^T$ = the instantaneous covariance at x

W = a standard Brownian motion in $\mathbf{R}^k$ under Q

J = a pure jump process with risk-neutral jump-arrival intensity of $\{\kappa(X_t): t \geq 0\}$ and with a risk-neutral jump-size probability distribution of υ on $\mathbf{R}^k$

and where $\kappa(x)$, $\mu(x)$, and $C(x) \equiv \sigma(x)\sigma(x)^T$ have affine (constant plus linear) dependence on x.[7] We delete time dependencies in the coefficients for notational simplicity only; the following approach extends to the case of time-dependent coefficients in a straightforward manner.

A classical special case is the "multifactor CIR state process" X valued in $D = \mathbf{R}^k_+$, for which $X^{(1)}$, $X^{(2)}$, . . ., $X^{(k)}$ are risk-neutrally independent processes of the "square-root" type introduced into term-structure modeling by Cox, Ingersoll, and Ross.[8]

We can take advantage of the affine setting for pricing defaultable floating-rate and fixed-rate debt by supposing that the default-adjusted short-rate process R of a given issuer is of the affine form, in that

$$R(t) = A + B[X(t)], \quad (13)$$

where

$$B[X(t)] = B_1 X_t^{(1)} + \ldots + B_k X_t^{(k)}, \quad (14)$$

A is a real number, and $B \in \mathbf{R}^k$. For analytical approaches based on the affine structure just described, one can repeatedly use the following calculation, regularity conditions for which are provided by Proposition 1 of Duffie, Pan, and Singleton (2000). For given times t and $s > t$ and given coefficients $a \in \mathbf{R}$ and $b \in \mathbf{R}^k$, let

$$g(X_t, t) = E^Q\left\langle \exp\left[\int_t^s -R(u)du\right]\exp\{a + b[X(t)]\} \mid X_t \right\rangle. \quad (15)$$

Under technical conditions, there are ordinary differential equations (ODEs) for $\alpha:(0,s) \to \mathbf{R}$ and $\beta:(0,s) \to \mathbf{R}^k$ such that

$$g(x,t) = \exp[\alpha(t) + \beta(t)x], \quad (16)$$

with boundary conditions $\alpha(s) = a$ and $\beta(s) = b$. Provided the Laplace transform of the distribution of the jump size of X is given explicitly, the ODEs for α and β are easily and routinely solved by numerical methods such as that of Runge–Kutta (see Press, Teukolsky, Veterling, and Flannery 1992). Details, with illustrative numerical examples and empirical applications, can be found in Duffie, Pan, and Singleton. For the special multifactor CIR case, explicit closed-form solutions for α and β can be deduced from Cox, Ingersoll, and Ross.

Now, suppose there is a reference discrete-tenor floating rate, such as LIBOR, on which an individual issuer's floating-rate payments are based. For an intercoupon time interval of length δ, such as one-half year, the reference rate $L(t)$ paid at time t on floating-rate loans is the simple rate of interest set at time $t - \delta$ for loans maturing at t, defined by the fact that the price $p_L(t-\delta, t)$ of a zero-coupon reference-quality bond sold at time $t - \delta$ for maturity at time t satisfies

$$1 + L(t) = \frac{1}{p_L(t-\delta, t)}. \quad (17)$$

[We emphasize that $L(t)$ is set at time $t - \delta$ and paid at time t.] If the default risk of an issuer of the reference (say, LIBOR) quality is captured by a default-adjusted short-rate process of the form $R_L = A_L + B_L X$, where $A_L \in \mathbf{R}$ and $B_L \in \mathbf{R}^k$ are fixed (for simplicity), then

$$p_L(t,s) = E^Q\left\{\exp[\int_t^s -R_L(u)du] \mid X_t\right\}. \quad (18)$$

Under the technical regularity conditions of Proposition 1 in Duffie, Pan, and Singleton, from Equation 16, we have

$$p_L(t-\delta, t) = e^{\alpha_L + \beta_L[X(t-\delta)]}, \tag{19}$$

for fixed coefficients α_L and β_L that are easily calculated. Then, from Equation 17,

$$L(t) = e^{-\alpha_L - \beta_L[X(t-\delta)]} - 1. \tag{20}$$

Now, consider a nonreference issuer with default-adjusted short-rate process $R = A + BX$. Let $V(t,\delta,K,n)$ denote the price at time t of a floating-rate note of the same intercoupon period δ as that of reference rate L with spread K to the reference floating rate and with a time to maturity of $n\delta$ for some integer number $n \geq 1$ of coupon periods. This floating-rate note is a defaultable claim to a total coupon payment of $L(t + \delta j) + K$ at coupon date $t + \delta j$ for each $j \leq n$ and a claim to the principal of 1 at the nth (maturity) coupon date. Therefore,

$$V(t, \delta, K, n) = p(t, t+n\delta) + \sum_{j=1}^{n} q(t, t+j\delta), \tag{21}$$

where for any s,

$$p(t, s) = E^Q\left\{\exp\left[\int_t^s -R(u)du\right] \mid X_t\right\} \tag{22}$$

is the market value of a zero-coupon bond of this quality to maturity date s and

$$q(t, t+j\delta) = E^Q\left\{\exp\left[\int_t^{t+j\delta} -R(u)du\right][L(t+j\delta)+K] \mid X_t\right\} \tag{23}$$

is the market value at time t of the jth floating-rate coupon.

We now show how to calculate $p(t,s)$ and $q(t,s)$ for any s, thereby providing a calculation of the value of the floating-rate note $V(t,\delta,K,n)$. From Equations 15 and 16,

$$p(t,s) = e^{c(s-t) + C(s-t)[X(t)]} \tag{24}$$

for some coefficients $c(s-t) \in \mathbf{R}$ and $C(s-t) \in \mathbf{R}^d$ that depend only on $s - t$.

Substituting Equation 20 into Equation 23 produces

$$q(t,s) = (K-1)p(t,s) + u(t,s), \tag{25}$$

where

$$u(t, s) = E^Q\left\{\exp\left[\int_t^s -R(u)du\right] e^{-\alpha_L - \beta_L[X(s-\delta)]} \mid X_t\right\}. \tag{26}$$

Now, by the law of iterated expectations,

$$u(t, s) = E^Q\left\{\left[e^{\int_t^{s-\delta} -R(u)du}\right] \times E^Q\left[e^{\int_{s-\delta}^{s} -R(u)du} e^{-a_L - \beta_L[X(s-\delta)]} \mid X_{s-\delta}\right] \mid X_t\right\}. \tag{27}$$

Because

$$E^Q\{e^{\int_{s-\delta}^{s} -R(u)du} e^{-\alpha_L - \beta_L[X(s-\delta)]} \mid X_{x-\delta}\} \tag{28}$$

$$= e^{-\alpha_L - \beta_L[X(s-\delta)]} p(s-\delta, s)$$

$$= e^{c(\delta) - \alpha_L + [C(\delta) - \beta_L][x(s-\delta)]},$$

another application of Equations 15 and 16 (again under the technical conditions of Duffie, Pan, and Singleton, Proposition 1) implies that new coefficients α and $\tilde{\beta}$ can be calculated so that

$$u(t, s) = e^{\tilde{\alpha} + \tilde{\beta}[X(t)]}. \tag{29}$$

Thus,

$$q(t, s) = (K-1)e^{c(s-t) + C(s-t)[X(t)]} + e^{\tilde{\alpha} + \tilde{\beta}[X(t)]}. \tag{30}$$

Finally, both $p(t,s)$ and $q(t,s)$ are explicit (and easily calculated) and we have $V(t,\delta,K,n)$ from Equation 21. The par floating-rate spread at time t for a time to maturity of $n\delta$ is that spread K with the property that $V(t,\delta,K,n) = 1$. That spread is normally expressed at the annualized rate of K/δ.

Computational Examples

A concrete example will help clarify the procedure. The state process $X = (X_1, X_2, X_3)'$ is made up of three independent CIR processes; that is, for each i,

$$dX_{it} = [\kappa_i\theta_i - (\kappa_i + \lambda_i)X_{it}]\,dt + \sigma_i\sqrt{X_{it}}\,dW_{it}, \tag{31}$$

for given coefficients κ_i, λ_i (a risk premium coefficient), θ_i (a long-run mean), and σ_i, where $W = (W_1, W_2, W_3)$ is a standard Brownian motion in $\mathbf{R}^3$ under Q.[9] For this example, we assume that

$$r = X_1 + X_2 - \bar{y} \tag{32}$$

for a constant $\bar{y}$. **Table 1** shows estimates of the coefficients κ_i, σ_i, θ_i, and λ_i for $i \in \{1, 2\}$ that were estimated from LIBOR swap data at several maturities by Duffie and Singleton (1997). (The coefficient $\bar{y}$ was estimated to be 0.58.)

As for the short-spread process, we assume that

$$S = \gamma_1 X_1 + \gamma_2 X_2 + \gamma_3 X_3, \tag{33}$$

Table 1. Model Parameters for Risk-Free Term Structure

κ_1	0.544
κ_2	0.003
σ_1	0.023
σ_2	0.019
θ_1	0.374
θ_2	0.258
λ_1	–0.036
λ_2	–0.004

where γ_1, γ_2, and γ_3 are coefficients that we adjusted, together with the coefficients and initial condition of X_3, to obtain various alternative credit-spread behaviors.

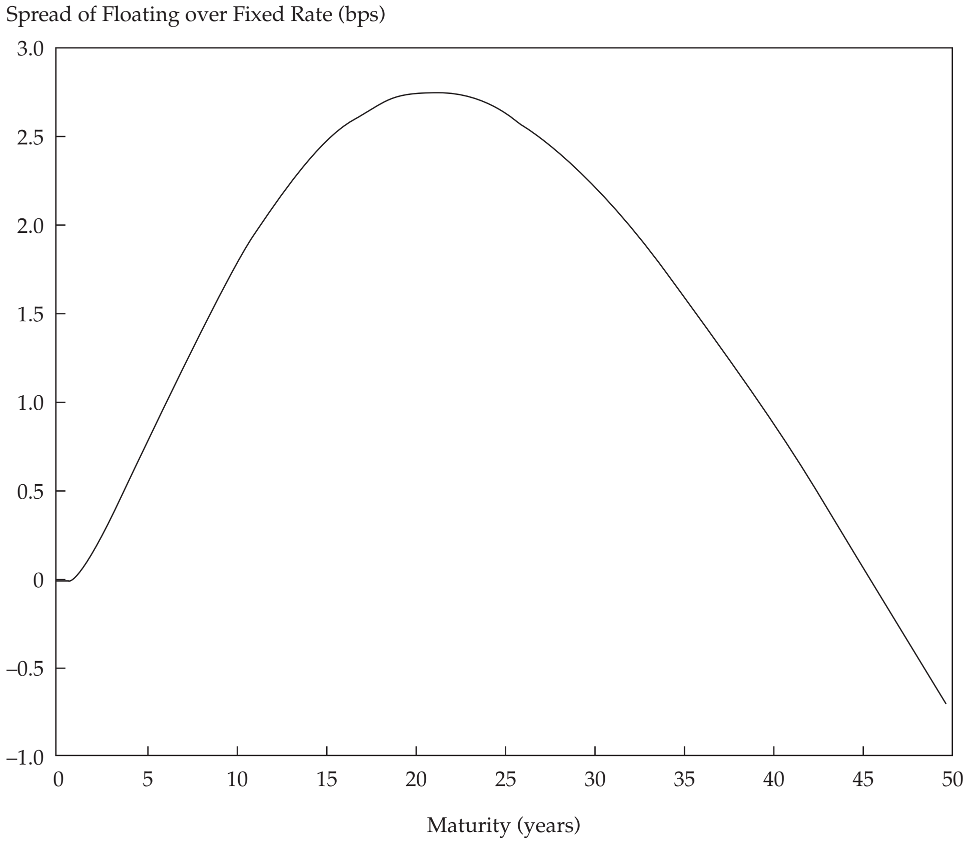
Because all zero-coupon default yields and yield spreads are in closed form for this model, we could easily set up the model for the following given values:[10]

- 3-month zero-coupon yield (10 percent in the base case),
- 10-year yield minus 3-month yield, or slope (1.5 percent in the base case),
- 5-year credit spread, the difference between the 5-year zero-coupon defaultable yield and the 5-year zero-coupon default-free yield (100 bps in the base case), and
- conditional volatility of the 5-year credit spread (48 percent in the base case).

For the CIR model, one can compute par defaultable fixed- and floating-rate spreads explicitly, as shown in Appendix A. For the calculations that follow, we took fixed- and floating-coupon payments to be made continuously in time to simplify the calculations, as shown in Appendix A. The numerical results are roughly the same as for discrete coupon payments except for maturities close to zero. For these results, we kept to the base-case parameters with the exception of the parameter whose level was to be varied in each case.

Figure 1 shows the relationship between maturity and the differential (floating–fixed) spread. As maturity goes to zero, the defaultable fixed and

Figure 1. Differential Spread as a Function of Maturity



Note: For each maturity, the spread indicated is the difference between the par spread on floating-rate debt less the par spread on fixed-rate debt at the base-case parameters. At each maturity, the initial short spread, $S(0)$, was adjusted [through adjustment of $X_3(0)$] to guarantee a zero-coupon yield spread at each maturity of 100 bps.

floating spreads both approach the difference between default-adjusted short rate R_t and risk-free short rate r_t, so the floating–fixed spread approaches zero. The long-maturity behavior in Figure 1 is determined essentially by the shape of the default-free yield curve.

Figure 2 shows the dependence of the floating–fixed spread on the slope of the yield curve. For changes in the level of the yield curve up to 15 percent, with slope held constant, the impact on the floating–fixed spread is, at most, 0.05 bps. **Figure 3** shows the dependence, which is close to linear, on the five-year zero-coupon yield spread of the issuer.

CIR models—in fact, even general affine term-structure models of the sort introduced by Duffie and Kan—have limited flexibility with regard to the correlation between yield spread and default-free yield. For example, one apparently cannot have this correlation be negative, within this class, while guaranteeing that yields and yield spreads remain positive. Therefore, to explore the implications of negative correlation for floating–fixed spreads (and only for that purpose), we used a quadratic-Gaussian term-structure model suggested by El Karoui, Myneni, and Viswanathan (1992). We took

$$r_t = Y_{1t}^2 + Y_{2t}^2 - \bar{y}, \qquad (34)$$

and

$$S_t = Y_{3t}^2, \qquad (35)$$

where the state process $Y = (Y_1, Y_2, Y_3)'$ is of the Ornstein–Uhlenbeck form:

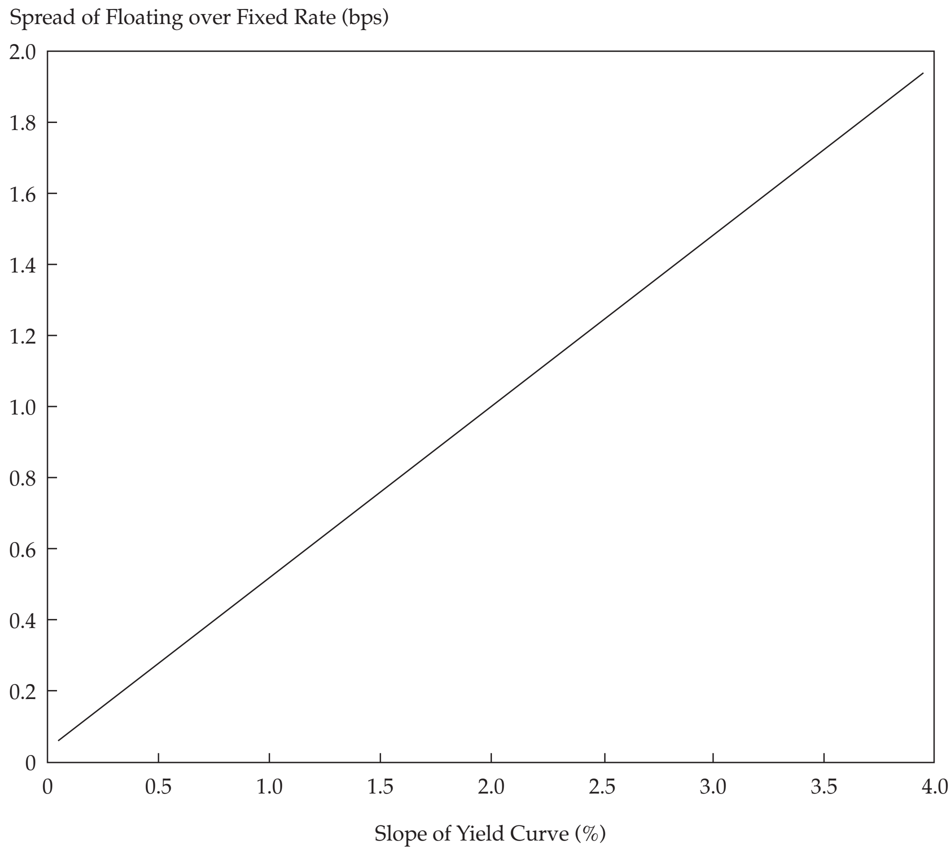
$$dY_t = (\beta - BY_t)dt + \Sigma dW_t, \qquad (36)$$

where B is a diagonal 3×3 matrix, β is a vector in $\mathbf{R}^3$, and

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ \rho_1\sigma_3 & \rho_2\sigma_3 & \sqrt{1 - \rho_1^2 - \rho_2^2\sigma_3} \end{pmatrix}, \qquad (37)$$

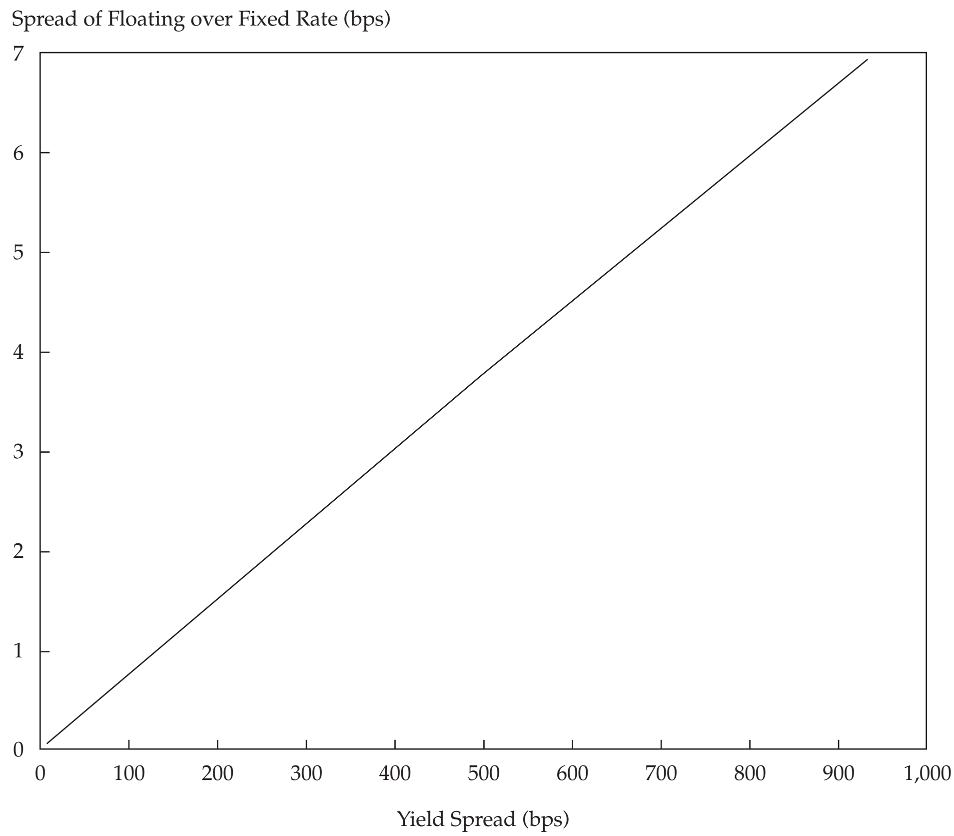
for given σ_i and ρ_i. For this model, zero-coupon yields and yield spreads of maturity t are of the form $\Sigma_i[\phi_0(t) + \phi_{1i}(t)Y_i(0) + \phi_{2i}(t)Y_i(0)^2]$ for $\phi_{ji}(t)$, where $\phi_{ji}(t)$ (for j = 0, 1, 2 and i = 1, 2, 3) solve

Figure 2. Differential Spread as a Function of Yield-Curve Slope



Note: For each slope (10-year minus 3-month yield spread), the spread indicated is the difference between the par spread on floating-rate debt less the par spread on fixed-rate debt at the base-case parameters.

Figure 3. Differential Spread as a Function of Default Spread



Note: For each five-year zero-coupon yield spread, the spread indicated is the difference between the par spread on floating-rate debt less the par spread on fixed-rate debt at the base-case parameters.

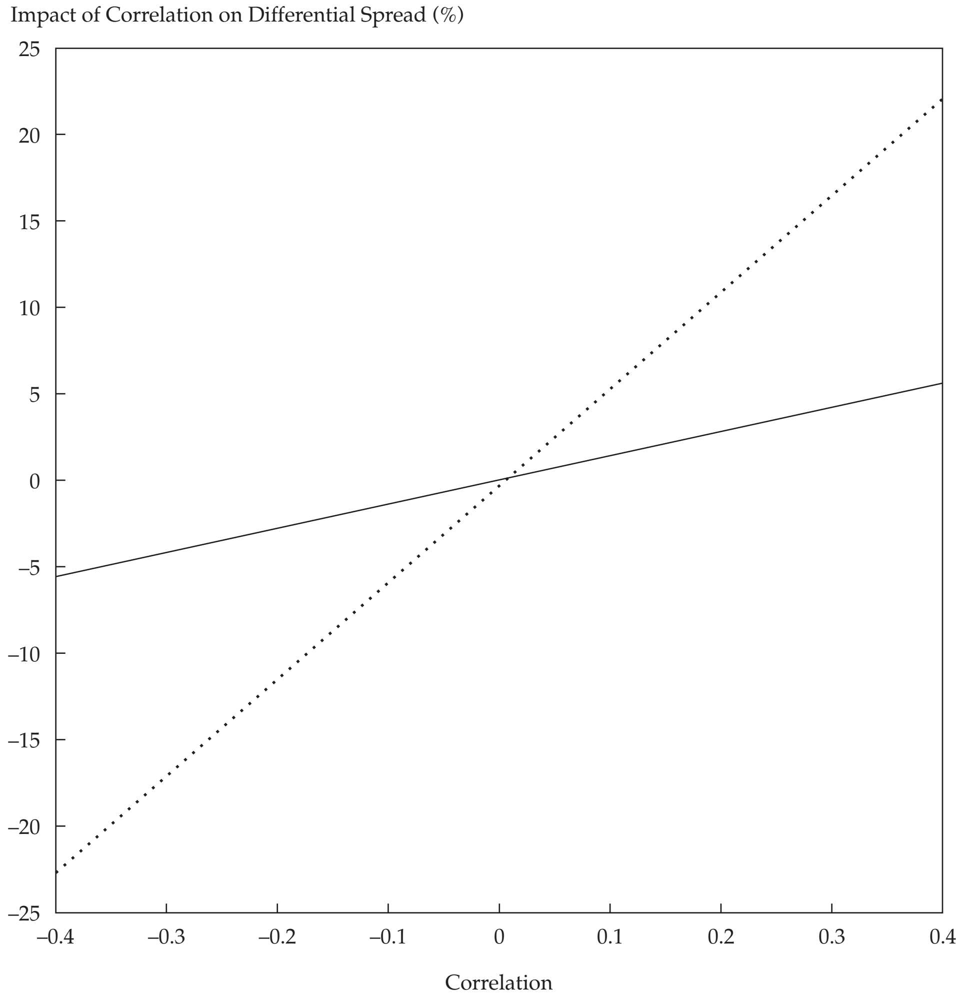
ordinary differential Riccati equations in t that are shown in Appendix A.[11]

Figure 4 shows the relative impact on differential spread of the correlation between credit spread and changes in the default-free yields. Increasing the correlation between changes in the risk-free term structure and default risk implies that, conditional on the event that the payment on floating debt is high, the probability of default is high. The floating spread is, therefore, increasing relative to fixed, intuitively, in this correlation. This effect is indicated in Figure 4, which also shows that the magnitude of the effect, unsurprisingly, grows with the volatility of the default spreads and default-free term structure. For example, increasing the correlation from 0 to 0.4 increases the floating–fixed spread by 6 percent of its base-case level or, if the volatilities are also doubled, by 22 percent of its base-case level. This result is consistent with that of Longstaff and Schwartz (1995), who found that the correlation between default risk and default-free interest rates has a significant effect on the properties of both floating- and fixed-rate spreads.

Conclusion

We examined the term structure of yield spreads between floating-rate and fixed-rate notes of the same credit quality and maturity. Floating–fixed spreads were theoretically characterized in some practical cases and quantified in a simple model in terms of maturity, credit quality, yield volatility, yield-spread volatility, correlation between changes in yield spreads and default-free yields, and other determining variables. We showed that if the issuer's default risk is risk-neutrally independent of interest rates, the sign of floating–fixed spreads is determined by the term structure of the risk-free forward rate. For example, if the term structure of default-free rates is increasing up to some maturity, then spreads on floating-rate

Figure 4. Differential Spread as a Function of Correlation between the Yield and Credit Spreads



Note: The horizontal axis shows the correlation between "instantaneous" increments of the three-month yield and the five-year default spread. The vertical axis shows the percentage difference between the differential spread at the indicated correlation and the differential spread at zero correlation. For example, if the spread is 105 bps at the given correlation and 100 bps at zero correlation, the percentage difference is 5 percent. The solid line is at the base-case parameters; the dotted line shows the same effect with the diffusion coefficients σ_1, σ_2, and σ_3 all doubled.

debt are larger than spreads on fixed-rate debt. Conversely, under the same independence assumption, if the default-free term structure is inverted, floating-rate spreads are smaller than fixed-rate spreads.

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Appendix A. Proof and Model Explanation

Here we provide the proof of the proposition stated in the text and explain the quadratic-Gaussian term-structure model of credit spreads.

Proof of the Proposition. The version of the proposition in this appendix is a continuous-time version. We assume that both $\Gamma(t)$ and $f(t)$ exist, with instantaneous forward rate $f(t)$ defined by

$$f(t) = -\left(\frac{1}{\Gamma(t)}\right)\left(\frac{d\Gamma(t)}{dt}\right). \tag{A1}$$

The coupon rate of a par default-free fixed-rate bond of maturity T, $C(T)$, is defined by

$$C(T) = \frac{E^Q[\exp(-\int_0^T r_s ds)]}{E^Q[\int_0^T \exp(-\int_0^t r_s ds)dt]}. \tag{A2}$$

We now have

$$\int_0^T \Gamma(t)[f(t) - C(T)]dt = 0. \tag{A3}$$

The floating-rate spread for maturity T, $k(T)$, is given by

$$k(T) = \frac{1 - \Lambda(T) - E^Q[\int_0^T \exp(-\int_0^t R_s ds) r_t dt]}{\int_0^T \Lambda(t)dt}. \tag{A4}$$

The fixed-rate spread, $K(T)$, is given by

$$K(T) = \frac{1 - \Lambda(T) - E^Q[\int_0^T \exp(-\int_0^t R_s ds) C(T) dt]}{\int_0^T \Lambda(t)dt}. \tag{A5}$$

The floating–fixed spread is then

$$\Delta(T) \equiv k(T) - K(T) = \frac{E^Q\left\{\int_0^T \exp(-\int_0^t R_s ds)[C(T) - r_t]dt\right\}}{\int_0^T \Lambda(t)dt}. \tag{A6}$$

■ *Proposition*. Suppose t_0 exists such that $f(t) \le C(T)$ for $t \le t_0$ and $f(t) \ge C(T)$ for $t > t_0$. [This is true if f is increasing on $(0,T)$.] If S is independent of r under the risk-neutral probability measure Q, then $\Delta(t) \ge 0$ for all $t \le T$. If, in addition, f is continuous and not constant on $(0,T)$ and S is strictly positive, then $\Delta(t) > 0$.

■ *Proof.* Because r and S are Q-independent,

$$E^Q\left\{\exp[-\int_0^t (r_s + S_s)ds][r_t - C(T)]\right\} = E^Q\left\{\exp[-\int_0^t r_s ds][r_t - C(T)]\right\}E^Q[\exp(-\int_0^t S_s ds)]. \tag{A7}$$

We have

$$E^Q\left\{\exp[-\int_0^t r_s ds][r_t - C(T)]\right\} = \Lambda(t)[f(t) - C(T)], \tag{A8}$$

and

$$E^Q\left\{\int_0^T \exp[-\int_0^t r_s ds][r_t - C(T)]dt\right\} = \int_0^T \Lambda(t)[f(t) - C(T)dt] \\ = 0. \tag{A9}$$

Because $S \geq 0$,

$$g(t) \equiv E^Q[\exp(-\int_0^t S_s ds)] \tag{A10}$$

is decreasing in t. It follows that

$$\begin{aligned} E^Q\left\{\int_0^T \exp[-\int_0^t (r_s + S_s)\, ds]\,[r_t - C(T)]\, dt\right\} &= \int_0^T \Lambda(t)[f(t) - C(t)]\, g(t) dt \\ &= \left(\int_0^{t_0} + \int_{t_0}^T\right)\Lambda(t)[f(t) - C(T)]\, g(t)\, dt. \end{aligned} \tag{A11}$$

We have assumed that $f(t) \leq C(T)$ for $t \leq t_0$, and because $S \geq 0$, we know that $g(t) \geq g(t_0)$ for $t \leq t_0$, so

$$\int_0^{t_0} \Lambda(t)[f(t) - C(T)]\, g(t) dt \leq \int_0^{t_0} \Lambda(t)\,[f(t) - C(T)]\, g(t_0) dt. \tag{A12}$$

Because for $t \geq t_0$, $f(t) \geq C(T)$ and $g(t) \geq g(t_0)$,

$$\int_{t_0}^T \Lambda(t)[f(t) - C(T)]\, g(t) dt \leq \int_{t_0}^T \Lambda(t)\,[f(t) - C(T)]\, g(t_0) dt. \tag{A13}$$

We thus have

$$E^Q\left\{\int_0^T \exp[-\int_0^t (r_s + S_s) ds][r_t - C(T)] dt\right\} \leq \left(\int_0^{t_0} + \int_{t_0}^T\right)\Lambda(t)[f(t) - C(T)]g(t_0) dt = 0. \tag{A14}$$

If S is strictly positive, then $g(t)$ is strictly decreasing. If, in addition, $f(t)$ is continuous and not constant, then at least one of the preceding inequalities is strict and we obtain $\Delta(T) > 0$.

The Quadratic-Gaussian Term-Structure Credit-Spread Model. We can write $r = Y^{\mathrm{T}} \xi Y$ and $R = Y^{\mathrm{T}} \Xi Y$ for diagonal ξ and Ξ.[12] We will use the fact that defaultable forward rate $F(t)$ satisfies

$$F(t) = E^Q[\exp(-\int_0^t R_s ds) Y_t^{\mathrm{T}}\, \delta Y_t] \tag{A15}$$

for diagonal δ.

One can show that

$$\Lambda_{0,t} = \exp[Y_t^{\mathrm{T}} U(t) Y_t + b(t)^{\mathrm{T}} Y_t + a(t)], \tag{A16}$$

and that

$$F(t) = [Y_t^{\mathrm{T}} V(t) Y_t + d(t)^{\mathrm{T}} Y_t + c(t)] \exp[a(t) + b(t)^{\mathrm{T}} Y_t + Y_t^{\mathrm{T}} U(t) Y_t], \tag{A17}$$

for time-dependent coefficients U, V, a, b, c, and d.

Substituting these expressions into the partial differential equation satisfied by Λ and F gives the following ordinary differential equations:

$$\begin{aligned} a' &= \beta^{\mathrm{T}} b + \frac{1}{2} b^{\mathrm{T}} \Sigma\Sigma^{\mathrm{T}} b + \mathrm{tr}\,(\Sigma\Sigma^{\mathrm{T}} U) \\ b' &= -B^{\mathrm{T}} b + 2U^{\mathrm{T}} \beta + 2U\Sigma\Sigma^{\mathrm{T}} b \\ U' &= -(B^{\mathrm{T}} U + UB) + 2U\Sigma\Sigma^{\mathrm{T}} U - \Xi \end{aligned} \tag{A18}$$

and

$$\begin{aligned} c' &= \beta^{\mathrm{T}} d + b^{\mathrm{T}} \Sigma\Sigma^{\mathrm{T}} d + \mathrm{tr}\,(\Sigma\Sigma^{\mathrm{T}} V) \\ d' &= -B^{\mathrm{T}} d + 2V^{\mathrm{T}} \beta + 2\,(V\Sigma\Sigma^{\mathrm{T}} b + U\Sigma\Sigma^{\mathrm{T}} d) \\ V' &= -(B^{\mathrm{T}} V + VB) + 2\,(U\Sigma\Sigma^{\mathrm{T}} V + V\Sigma\Sigma^{\mathrm{T}} U), \end{aligned} \tag{A19}$$

with the initial conditions $a(0) = b(0) = U(0) = c(0) = d(0) = 0$, and $V(0) = \delta$. (These differential equations are solved by Runge–Kutta methods for our examples.)

To compute the correlations between the yields, we use the fact that

$$\begin{aligned} E(Y_{it}) &= \bar{y}_{it} \\ &\equiv \exp(-B_i t) Y_{i0} + \frac{1}{B_i}[1 - \exp(-B_i t)]\beta_i, \end{aligned} \tag{A20}$$

where B_i denotes the ith diagonal element of the diagonal matrix B and

$$\operatorname{cov}(Y_t) = \int_0^t \exp[-B(t-s)]\Sigma\Sigma^T \exp[-B(t-s)]\, ds. \tag{A21}$$

For our special example of Σ, this covariance matrix, $\operatorname{cov}(Y_t) \equiv \Sigma_Y(t)$, is computed as

$$\begin{pmatrix} \dfrac{\sigma_1^2 e^{-2B_1 t}}{2B_1} & 0 & \dfrac{\rho_1\sigma_1\sigma_3\left[1 - e^{-(B_1+B_3)t}\right]}{B_1 + B_3} \\ 0 & \dfrac{\sigma_2^2 e^{-2B_2 t}}{2B_2} & \dfrac{\rho_2\sigma_1\sigma_3\left[1 - e^{-(B_2+B_3)t}\right]}{B_2 + B_3} \\ \dfrac{\rho_1\sigma_1\sigma_3\left[1 - e^{-(B_1+B_3)t}\right]}{B_1 + B_3} & \dfrac{\rho_2\sigma_1\sigma_3\left[1 - e^{-(B_2+B_3)t}\right]}{B_2 + B_3} & \dfrac{\sigma_3^2\left(1 - e^{-2B_3 t}\right)}{2B_3} \end{pmatrix}. \tag{A22}$$

Let

$$w_1(t) = Y_t^T U(t) Y_t + b(t)^T Y_t + a(t) \tag{A23a}$$

and

$$w_2(t) = Y_t^T V(t) Y_t + d(t)^T Y_t + c(t). \tag{A23b}$$

Then, suppressing t from the notation, we have

$$\operatorname{cov}(w_1, w_2) = 2\operatorname{tr}(U\Sigma_Y V\Sigma_Y) + (2U\bar{y} + b)^T \Sigma_y (2V\bar{y} + d), \tag{A24}$$

where $\operatorname{tr}(\mathbf{A})$ denotes the trace of a matrix $\mathbf{A}$. This allows the computation of yield correlations and thus enables us to "calibrate" coefficients to given yield correlations.

Notes

1. Previous work on this topic by Cooper and Mello (1988) pointed to differences between fixed- and floating-rate spreads that are at least an order of magnitude larger than those we found. One possible explanation is the artificial definition of a floating-rate note that Cooper and Mello used for illustration. Longstaff and Schwartz (1995) provided some model results for floating- and fixed-rate pricing but not in a format that would allow a direct calculation of the spread between floating-rate and fixed-rate debt of the same maturity and issuer.
2. The parameters of the CIR model are as estimated by Duffie and Singleton (1997).
3. We fix a probability space. The existence of a state-price density, a positive random variable sometimes called a "state-price deflator," or "state-price kernel," is implied by the absence of arbitrage and mild integrability conditions.
4. This assumption is found in most reduced-form defaultable valuation models, such as that of Duffie and Singleton (1999) or Jarrow and Turnbull (1995).
5. An affine setting is one in which zero-coupon yields are linear with respect to underlying state variables. For a more precise definition, see Duffie and Kan (1996).
6. Underlying the model is a probability space (Ω,F,P) and a filtration $(F_t{:}t \geq 0)$ satisfying the usual conditions, as stated for example in Protter (1990). The probability measure Q is equivalent to P, and integrability is assumed as required for the analysis shown. Expectation under Q given F_t is denoted E_t^Q. The short-rate process r is assumed to be progressively measurable.
7. This is made precise by defining the generator A of X by

$$Af(x) = f_x(x)\mu(x) + \frac{1}{2}\sum_{ij} C_{ij}(x) f_{x_i x_j}(x) + \kappa(x)\int [f(x+z) - f(x)]\, d\nu(z),$$

for any C^2 function f with compact support. One may add time dependencies to these coefficients. Conditions must be imposed for existence and uniqueness of solutions, as indicated by Duffie and Kan. Generalizations are discussed in Duffie, Pan, and Singleton.
8. That is $dX_t^{(i)} = \kappa_i\left[\bar{x}_i - X_t^{(i)}\right]dt + \sigma_i\sqrt{Y_t^i}\, dW_t^{(i)}$ for some given constants $\kappa_i > 0$, $\bar{x}_i > 0$, and σ_i.
9. The risk-premium coefficients λ_1, λ_2, and λ_3 can be used to determine the behavior under the original probability measure P, as in the conventional model of Cox, Ingersoll, and Ross, but we have no need for that here.
10. The base-case parameters for S were $\gamma_1 = 0.005$, $\gamma_2 = 0.01$, $\gamma_3 = 1$, $\lambda_3 = 0$, $\kappa_3 = 0.01$, $\theta_3 = 0.005$, and $\sigma_3 = 0.0015$. We adjusted $X_3(0)$ for the desired five-year zero-coupon yield spread.
11. The base-case parameters are $\beta_1 = 0.165$, $B_1 = 0.504$, $\sigma_1 = 0.07$, $\beta_2 = 0.0001$, $B_2 = 0.001$, $\sigma_2 = 0.001$, $\beta_3 = 0.01$, $B_3 = 0.5$, $\sigma_3 = 0.05$, and $\rho_1 = \rho_2 = 0$. These parameters were chosen so as to match by "calibration" to our base-case CIR model. Because Y_i^2 behaves approximately like X_i, we chose σ_i for Y_i to be half of the corresponding coefficient for X_i, $B_i = \kappa_i + \lambda_i$, $\beta_i / B_i = [\kappa_i\theta_i/(\kappa_i + \lambda_i)]^2$, and $Y_i(0) = \sqrt{X_i(0)}$.
12. One can introduce terms linear in y and a constant term without any difficulty.

References

Cooper, Ian, and Antonio S. Mello. 1988. "Default Spreads in the Fixed and in the Floating Interest Rate Markets: A Contingent Claims Approach." *Advances in Futures and Options Research*, vol. 3, no. 1:269–289.

Cox, J, J. Ingersoll, and S. Ross. 1985. "A Theory of the Term Structure of Interest Rates." *Econometrica*, vol. 53, no. 2 (March):385–408.

Duffie, Darrell, and Rui Kan. 1996. "A Yield-Factor Model of Interest Rates." *Mathematical Finance*, vol. 6, no. 4 (October):379–406.

Duffie, Darrell, and Kenneth J. Singleton. 1997. "An Econometric Model of the Term Structure of Interest Rate Swap Yields." *Journal of Finance*, vol. 52, no. 4 (September):1287–1321.

———. 1999. "Modeling Term Structure of Defaultable Bonds." *Review of Financial Studies*, vol. 12, no. 4 (Special):687–720.

Duffie, Darrell, Jun Pan, and Kenneth Singleton. 2000. "Transform Analysis and Asset Pricing for Affine Jump-Diffusions." *Econometrica*, vol. 68, no. 6 (November):1343–76.

El Karoui, Nicole, Ravi Myneni, and Ravi Viswanathan. 1992. "Arbitrage Pricing and Hedging of Interest Rate Claims with State Variables I: Theory." Working paper, Laboratoire de Probabilités, Université de Paris.

Jarrow, Robert A., and Stuart M. Turnbull. 1995. "Pricing Derivatives on Financial Securities Subject to Credit Risk." *Journal of Finance*, vol. 50, no. 1 (February):53–85.

Longstaff, Francis A., and Eduardo S. Schwartz. 1995. "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt." *Journal of Finance*, vol. 50, no. 3 (June):789–819.

Press, William H., Saul A. Teukolsky, William T. Veterling, and Brian P. Flannery. 1992. *Numerical Recipes in FORTRAN*. Cambridge, UK: Cambridge University Press.

Protter, P. 1990. *Stochastic Integration and Differential Equations*. New York: Springer-Verlag.

Pye, Gordon. 1974. "Gauging the Default Premium." *Financial Analysts Journal*, vol. 30, no. 1 (January/February):49–50.