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PDE solutions of stochastic differential utility*

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This paper presents conditions for the existence and properties of stochastic differential utility as a solution of a partial differential equation. Stochastic differential utility is an extention of the classical additively-separable utility model that is designed as a platform for new financial asset pricing results. The extension is important, for example, when investors display preference for early or late resolution of uncertainty. The existence conditions admit Kreps–Porteus stochastic differential utility.

1. Introduction

This paper presents conditions for the existence and properties of stochastic differential unity as a solution of a partial differential equation. As such, utility can be represented as a smooth function of Markov state variables. Stochastic differential utility is an extension of the classical additively-separable utility model that was developed by Duffie and Epstein (1992) as a platform for new financial asset pricing results. The additively-separable model of utility, in its classical form, assigns utility

$$U(c)=E\left[\int_0^\infty e^{-\beta t}u(c_t)\,dt\right]$$

to a consumption process c. (Technical details are provided in the following section.) The extension to stochastic differential utility is important, for

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instance, if investors display preference for early or late resolution of uncertainty. In particular, this paper develops conditions for existence and uniqueness of the stochastic differential version of Kreps and Porteus (1978) utility with a constant-elasticity-of-substitution form that cannot be treated by the techniques in Duffie and Epstein (1992). Neither do Duffie and Epstein show whether, in a Markov setting, it is possible to represent utility as a function on the Markov state space.

Section 2 presents the problem, while section 3 contains the main existence and uniqueness conditions. Additional characterization is provided in section 4. Section 5 contains remarks on related results, such as stochastic control and security prices.

2. Stochastic differential utility and its PDE

This section defines stochastic differential utility for diffusion consumption processes and presents the PDE whose solution is the utility function.

2.1. Stochastic differential utility

We fix a probability space $(\Omega, \mathscr{F}, P)$ and a family $\{\mathscr{F}_t: t \in [0,\infty)\}$ of σ-algebras that is the augmented filtration of a standard Brownian motion B in $\mathbb{R}^n$. For each initial point $x \in \mathbb{R}^n$, a process X satisfies the stochastic differential equation

$$\mathrm{d}X_t = b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t, \quad X_0 = x, \tag{1}$$

where $b: \mathbb{R}^n \to \mathbb{R}^n$ and $\sigma: \mathbb{R}^n \to \mathbb{R}^{n\times n}$ satisfy a Lipschitz condition. [A unique (strong) solution exists under weaker conditions.] For simplicity, for now we may think of X_t as the rate of consumption by some economic agent of n different goods and services at time t.

The utility for the consumption process X is determined by the following measurable functions:

(a) An 'instantaneous aggregator' $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$,
(b) A 'variance multiplier' $A: \mathbb{R} \to \mathbb{R}$,
(c) A 'terminal reward' $g: \mathbb{R}^n \to \mathbb{R}$.

First we pick a finite time interval $[0, T]$. An integrable semimartingale V is defined by Duffie and Epstein (1992) to be the T-horizon utility process for X if it uniquely satisfies the family of integral equations

$$V_t = E\left[\int_t^T \left(f(X_s, V_s) + \frac{1}{2} A(V_s) \frac{\mathrm{d}}{\mathrm{d}s}[V]_s \right) \mathrm{d}s + g(X_T) \,\middle|\, \mathscr{F}_t \right] \text{ a.s.,} \quad t \in [0, T], \tag{2}$$

where $[V]$ is the quadratic variation process for V.

In our setting, it can easily be shown that this implies that V is an Ito process with

$$\mathrm{d}V_t = \mu_t \, \mathrm{d}t + \sigma_V(t) \, \mathrm{d}B_t,$$

where μ is a real-valued progressively measurable process and σ_V is an $\mathbb{R}^n$-valued progressively measurable process. From this, we have $[V]_t = \int_0^t \sigma_V(s) \cdot \sigma_V(s) \, \mathrm{d}s$ and

$$\mu_t = -f(X_t, V_t) - \tfrac{1}{2} A(V_t) \sigma_V(t) \cdot \sigma_V(t), \quad t \in [0, T].$$

Technical definitions can be found in Chung and Williams (1983) or Protter (1990). Motivation and some existence results can be found in Duffie and Epstein (1992).

For example, the classical additively-separable utility model has $A = 0$ and $f(x, v) = u(x) - \beta v$, for some suitable function u and scalar $\beta \geqq 0$, in which case (2) is equivalent to

$$V_t = E\left[\int_t^T \mathrm{e}^{-\beta(s-t)} u(X_s) \, \mathrm{d}s + g(X_T) \middle| \mathscr{F}_t \right] \text{ a.s.}, \quad t \in [0, T]. \tag{3}$$

In asset pricing applications, it is often the case that X is a state process for the economy, while the consumption process itself is $\{C(X_t): t \geqq 0\}$ for some suitable function $C: \mathbb{R}^n \to \mathbb{R}$. In that case, our conditions are actually on the instantaneous aggregator $\tilde{f}$ defined by $\tilde{f}(x, v) = f[C(x), v]$.

An integrable semimartingale V is defined to be the infinite-horizon utility process for X if it uniquely satisfies the two conditions:

(a) For all t, $V_t = \lim_T V_t^T$ almost surely, where V^T is the T-horizon utility process.
(b) For all t and $T \geqq t$,

$$V_t = E\left[\int_t^T \left(f(X_s, V_s) + \frac{1}{2} A(V_s) \frac{\mathrm{d}}{\mathrm{d}s} [V]_s \right) \mathrm{d}s + V_T \middle| \mathscr{F}_t \right]. \tag{4}$$

One is interested in conditions on (b, σ, f, A), in both the T-horizon and infinite horizon cases, under which the utility process exists.

2.2. The PDE

Naturally, we expect that when the utility process V for the infinite horizon case is well-defined, there is some measurable function $J: \mathbb{R}^n \to \mathbb{R}$ such that

$$V_t = J(X_t) \text{ a.s.,} \quad t \geqq 0. \tag{5}$$

If J satisfies (5) and is C^2, Itô's Lemma and (4) imply that J satisfies the following PDE in $\mathbb{R}^n$:

$$\mathscr{D}J(x) + f[x, J(x)] + A[J(x)]\left(\sum_{i,j} a_{ij}(x)\partial_i J(x)\partial_j J(x)\right) = 0, \tag{6}$$

where $a(x) = \frac{1}{2}\sigma(x)\sigma(x)^{\mathrm{T}}$ and $\mathscr{D}J = \sum_{i,j} a_{ij}\partial_{ij}J + \sum_i b_i \partial_i J$. Conversely, if J is the unique solution of (6) that satisfies a natural growth condition and $\{J(X_t): t \geqq 0\}$ is integrable, then J defines by (5) the utility process V. An analogous PDE characterizes the T-horizon utility process.

2.3. A convenient change of unknown

We now present a change of unknown that eliminates the variance multiplier A from the PDE (6), converting the equation from quasi-linear to semi-linear. Suppose J satisfies (6) and, for some C^2 strictly increasing function $\phi: \mathbb{R} \to \mathbb{R}$, let $H = \phi \circ J$. Then direct calculation shows that H satisfies

$$\mathscr{D}H(x) + \bar{f}[x, H(x)] + \bar{A}[H(x)]\left(\sum_{i,j} a_{ij}(x)\partial_i H(x)\partial_j H(x)\right) = 0, \tag{7}$$

where

$$f(x, v) = \frac{\bar{f}[x, \phi(v)]}{\phi'(v)} \tag{8}$$

and

$$A(v) = \phi'(v)\bar{A}[\phi(v)] + \frac{\phi''(v)}{\phi'(v)}. \tag{9}$$

Since $\phi' > 0$, it is necessary and sufficient for $\bar{A} = 0$ that $\phi'' = A\phi'$, so a suitable change of unknown is given by

$$\phi(v) = C_2 + C_1 \int_1^v \exp\left[\int_1^s A(t)\,\mathrm{d}t\right]\mathrm{d}s, \tag{10}$$

where $C_1 > 0$ and C_2 are constants. This leaves the semi-linear elliptic PDE

$$\mathscr{D}H(x)+\bar{f}[x,H(x)]=0, \quad x\in\mathbb{R}^n. \tag{11}$$

Precisely the same procedure in the T-horizon case leaves the semi-linear parabolic equation in $\mathbb{R}^n\times[0,T]$:

$$\mathscr{D}H(x,t)+\frac{\partial H(x,t)}{\partial t}+\bar{f}[x,H(x,t)]=0, \tag{12}$$

with boundary condition $H(\cdot,T)=g$. From a PDE viewpoint, several difficulties arise: the nonlinearities are not always monotone, and in addition growths at infinity are allowed and even interesting. These two features make the problem nonstandard.

2.4. Kreps–Porteus stochastic differential utility

Without applying a Markov assumption, Duffie and Epstein (1992) as well as Duffie et al. (1990) provide conditions under which there exist utility processes. Their general existence results rely on the above change of unknown, originating with this paper. The general existence results in the above cited papers, however, require a uniform Lipschitz condition on $\bar{f}(x,\cdot)$ that rules out some of the most promising examples for asset pricing results, including the stochastic differential version of the Kreps–Porteus (1978) utility process. For example, Duffie and Epstein (1992) assume the existence of the Kreps–Porteus utility model in order to provide a two-factor capital asset pricing model and to extend the Cox et al. (1985) model of the term structure of interest rates.

Modulo a change of unknown, the Kreps–Porteus stochastic differential utility model presented in Duffie and Epstein (1992) has $f(x,v)=u(x)-\beta v$ and $A(v)=\mu/v$ for some function $u:\mathbb{R}^n\to\mathbb{R}_+$ and some constant $\mu\leqq 0$. The results in the following section include this as a special case.

Specifically, in the setting of a Markov state process X, the Kreps–Porteus utility model shown in Duffie and Epstein (1992) has $u(x)=kC(x)^\rho$ for some non-zero constant $\rho<1$, some constant $k>0$, and some function $C:\mathbb{R}^n\to\mathbb{R}_+$ defining the rate of consumption of a single commodity as it depends on the exogenous state x of the economy. By rescaling, we can always assume without loss of generality that $\beta=1$. Based on the discussion in Duffie and Epstein (1992) and fixing the coefficient ρ as a measure of intertemporal substitution, we may think of $\alpha=(1-\rho)(1-\mu)/\mu$ as a measure of risk tolerance. In that case, $\mu=-1+\alpha/\rho$. It is seen below that $\mu<-1$, or equivalently $\alpha/\rho<0$, presents a rather delicate special case insofar as existence and uniqueness.

3. Existence and uniqueness results

This section presents existence and uniqueness results for the PDE's (11) and (12) with suitable boundary conditions, in various cases. Throughout, we assume that σ and b are bounded and Lipschitz continuous in $\mathbb{R}^n$ and that a is uniformly elliptic, in that there exists $v>0$ such that for any ζ and x, $\sum_{i,j} a_{ij}(x)\zeta_i\zeta_j \geqq v|\zeta|^2$. The uniform ellipticity may be relaxed in most of the results below and we can also study more general situations than σ and b to be bounded but we will not consider such extensions here.

We also assume throughout that $A \leqq 0$ and that A is bounded for $t \geqq 1$ and that $f(x,v) = u(x) - \beta v$ for some positive scalar β and continuous, positive-valued function $u: \mathbb{R}^n \to \mathbb{R}$. By re-scaling, however, we can always take $\beta = 1$ without loss of generality.

Let us also mention that our results below are organized as follows: in each case, we begin our analysis with the 'simpler' case when $a_{ij} \equiv \frac{1}{2}\delta_{ij}$ and $b_i \equiv 0$ so that $\mathscr{D}$ reduces to $\frac{1}{2}\Delta$. Then, we explain how to extend our analysis to the general case.

3.1. The finite-horizon/parabolic case

Under our assumption on f, the relevant equation for T-horizon utility is

$$\mathscr{D}J(x,t) + \frac{\partial J}{\partial t} - \beta J(x,t) + u(x,t)$$

$$+ A[J(x,t)] \sum_{i,j} a_{ij}(x)\partial_i J(x,t)\partial_j J(x,t) = 0, \tag{13}$$

with boundary condition $J(\cdot, T) = g$. We will instead study the following equation:

$$\mathscr{D}\tilde{J}(x,t) + \frac{\partial \tilde{J}}{\partial t} + u(x,t) + A[\tilde{J}(x,t)] \sum_{i,j} a_{ij}(x)\partial_i \tilde{J}(x,t)\partial_j \tilde{J}(x,t) = 0, \tag{14}$$

with boundary condition $\tilde{J}(\cdot, T) = g$, for continuous, positive-valued $u: \mathbb{R}^n \times [0,T] \to \mathbb{R}$ and $g: \mathbb{R}^n \to \mathbb{R}$. We look only for positive solutions. Of course, (14) contains (13) if $\beta = 0$, in which we have only replaced $u(x)$ by $u(x,t)$. If $\beta > 0$ and if we consider the Kreps–Porteus model, that is $A(v) = \mu/v$ for $\mu < 0$, (14) follows from (13) upon setting $\tilde{J}(x,t) = e^{\beta(T-t)} J(x,t)$ and $u(x,t) = e^{\beta(T-t)} u(x)$.

Next, we make a change of unknown. This change of unknown is in some sense an extension of the classical log transform which reduces a diffusive Burgers equation into a linear heat equation (the so-called Cole–Hopf transform). We let $\phi_A(v)=\int_1^v \exp(\int_1^t A(s)\,ds)\,dt$, and we obtain the equation

$$\mathscr{D}\phi_A(\tilde{J})+\frac{\partial}{\partial t}\phi_A(\tilde{J})+\phi_A'(\tilde{J})u=0, \qquad \phi_A(\tilde{J})|_{t=T}=\phi_A(g), \quad t=T, \tag{15}$$

and $\phi_A'(\tilde{J})=\psi(\phi_A(\tilde{J}))$ with $\phi'(v)=A(\phi_A^{-1}(v))\leqq 0$.

In the Kreps–Porteus model, we find $A(s)=\mu/s$ ($\mu<0$) and $\phi_A(v)=(v^{1+\mu}-1)/(1+\mu)$ if $\mu\neq -1$, and $\phi_A(v)=\log v$ if $\mu=-1$. Notice also that if $\mu<-1$, $\phi_A(v)$ remains bounded as v goes to $+\infty$, or in other words that we have

$$\int_1^\infty \exp\left[\int_1^t A(s)\,ds\right]dt<\infty. \tag{16}$$

As we mentioned in the Introduction of section 3, we begin with the case of $\mathscr{D}=\frac{1}{2}\Delta$, and we thus consider the equation

$$\frac{\partial w}{\partial t}+\frac{1}{2}\Delta w+\psi(w)u=0, \qquad w|_{t=T}=\bar{w}, \tag{17}$$

where $\psi=\phi_A'(\phi_A^{-1}(w))$, $\bar{w}=\phi_A(g)$.

Of course, ψ is not in general defined on the whole real line and we set $\alpha=\lim_{v\downarrow 0}\downarrow\phi_A(v)$ ($\geqq-\infty$), $\beta=\lim_{v\uparrow+\infty}\uparrow\phi_A(v)$ ($\leqq+\infty$). Then, $\alpha<\bar{w}<\beta$ on $\mathbb{R}^n$ and ψ is locally Lipschitz, continuous, and nonincreasing on (α,β).

We will be using the following conditions:

Condition A. If $\alpha=-\infty$ (resp. $\beta=+\infty$), there exist constants $C_0\in(0,\infty)$, $\varepsilon_0\in(0,1/2T)$ such that

$$g\geqq\phi_A^{-1}(-C_0(1+e^{\varepsilon_0|x|^2})) \quad (\text{resp.}\, g\leqq\phi_A^{-1}(C_0(1+e^{\varepsilon_0|x|^2}))) \text{ on } \mathbb{R}^n.$$

Of course, these inequalities translate into

$$\bar{w}\geqq-C_0(1+e^{\varepsilon_0|x|^2}) \quad (\text{resp.}\, \bar{w}\leqq C_0(1+e^{\varepsilon_0|x|^2})) \text{ on } \mathbb{R}^n. \tag{18}$$

In the Kreps–Porteus model, these conditions become:

$$\text{if} \quad \mu<-1, \qquad g\geqq(1-(1+\mu)C_0(1+e^{\varepsilon_0|x|^2}))^{1/(1+\mu)} \text{ on } \mathbb{R}^n;$$

if $\mu = -1$, $\quad \exp(-C_0(1+e^{\varepsilon_0|x|^2})) \leqq g \leqq \exp(C_0(1+e^{\varepsilon_0|x|^2}))$ on $\mathbb{R}^n$;

if $0 > \mu > -1$, $\quad g \leqq (1+(1+\mu)C_0(1+e^{\varepsilon|x|^2}))^{1/(1+\mu)}$ on $\mathbb{R}^n$.

We will also be using:

Condition B. If $\beta = +\infty$, there exist constants $C_0 \in (0,\infty)$, $\varepsilon_0 \in (0, 1/2T)$ such that

$$0 \leqq u(x,t)\psi(C_0(1+e^{\varepsilon_0|x|^2})) \leqq C_0(1+e^{\varepsilon|x|^2}) \text{ on } \mathbb{R}^n \times [0,T].$$

Here and everywhere below we let $B_R = \{x \in \mathbb{R}^n : |x| \leqq R\}$. We will refer to the following spaces of functions:

- $C^{2,1,\alpha}(B_R \times [0,T])$, the space of functions on $B_R \times [0,T]$ into $\mathbb{R}$ with two derivatives with respect to the 'space' variable $x \in B_R$, and one derivative with respect to the 'time' variable $t \in [0,T]$ such that these derivatives are uniformly Hölder continuous with exponent α. For a definition of Hölder continuity and properties of this space, see for example Gilbarg and Trudinger (1983, p. 52).
- $W^{2,1,p}(B_R \times [0,T])$, the Sobolev space of functions on $B_R \times [0,T]$ into $\mathbb{R}$ that have two weak derivatives with respect to $x \in B_R$, one weak derivative with respect to $t \in [0,T]$, such that these derivatives are all in $L^p(B_R \times [0,T])$. [See, for example, Gilbarg and Trudinger (1983, p. 153)].
- $C^{0,\alpha}(B_R \times [0,T])$, the uniformly Hölder continuous (with exponent α) functions on $B_R \times [0,T]$ into $\mathbb{R}$.

Variations on this notation will have the obvious meaning. For example, the subscript 'loc', as in $W^{2,2,p}_{\text{loc}}(B_R \times [0,T])$, means the space of functions on $B_R \times [0,T]$ into $\mathbb{R}$ whose restriction to any compact sub-domain of $B_R \times [0,T]$ has the indicated properties.

In the following, we will refer in several ways to a collection of results known as the 'maximum principle', which states basically that a subharmonic function u on a domain (open connected proper subset of $\mathbb{R}^n$) can only achieve its maximum in the interior of the domain if u is constant. The associated 'comparison principle', in a relatively general setting, runs as follows. Let Ω be a domain in $\mathbb{R}^n$, let $\Gamma = \{(x,z,p,r) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} : r \text{ is symmetric}\}$, and let $F: \Gamma \to \mathbb{R}$ be elliptic, in the sense that the matrix

$$\left[\frac{\partial F(x,z,p,r)}{\partial r_{ij}}\right], \quad i,j = 1,\dots,n,$$

is positive definite for all $(x,z,p,r) \in \Gamma$. An elliptic second-order equation is defined by F, for $u \in C^2(\Omega)$, by $F[u]=0$, where $F[u](x) = F(x, u(x), u_x(x), u_{xx}(x))$.

Suppose that u and v are in $C^2(\Omega)$ and have continuous extensions to the closure of Ω. Suppose, moreover, that F is continuously differentiable with respect to the (z, p, r) variables in Γ and non-increasing with respect to the z variable. Then $F[u] \geqq F[v]$ in Ω, and $u \leqq v$ on the boundary of Ω, together imply that $u \leqq v$ in Ω. This conclusion holds even if F is merely elliptic on the range of $x \mapsto (x, f(x), f_x(x), f_{xx}(x))$ for any f of the form $\theta u + (1-\theta)v$, $0 \leqq \theta \leqq 1$.

This particular consequence of the maximum principle can be found, for example, in Gilbarg and Trudinger (1983, Theorem 17.1). Protter and Weinberger (1967) offer an easily accessible introduction to maximum principles.

Theorem 1. Under conditions A and B, there is a unique solution $\tilde{J}$ of (14) with the following properties:

$$\tilde{J} \in W^{2,1,p}(B_R \times [0,T]) \quad (\forall R < \infty,\ \forall p < \infty),$$

and if $\alpha = -\infty$ (resp. $\beta = +\infty$),

$$\tilde{J} \geqq \phi_A^{-1}(-C_1(1+e^{\varepsilon_1 |x|^2})) \quad (resp.\ \tilde{J} \leqq \phi_A^{-1}(C_1(1+e^{\varepsilon_1 |x|^2}))) \ on\ \mathbb{R}^n \times [0,T],$$

for some constants $C_1 \in (0,\infty)$, $\varepsilon_1 \in (0, 1/2T)$. Furthermore, if $g \in C^{0,\alpha}(\mathbb{R}^n)$ and $u \in C^{0,\alpha}(B_R \times [0,T])$ $(\forall R < \infty)$ for some $\alpha \in (0,s)$,

$$\tilde{J} \in C^{2,1,\alpha}(B_R \times [0,T]) \quad (\forall R < \infty).]$$

Proof. We begin with the uniqueness part which, in fact, does not use Condition B. We first observe that if $\tilde{J}_1, \tilde{J}_2$, are two solutions of (15) with the above properties, then $w_1 = \phi_A(\tilde{J}_1)$, $w_2 = \phi_A(\tilde{J}_2)$ are well defined, belong to $W^{2,1,p}(B_R \times [0,T])$ $(\forall R<\infty, \forall p<\infty)$ and solve (17). Furthermore, $\alpha < w_1, w_2 < \beta$ on $\mathbb{R}^n \times [0,T]$ and thus $w = w_1 - w_2$ solves

$$\frac{\partial w}{\partial t} = \frac{1}{2}\Delta w + a(x,t)w = 0, \qquad w|_{t=T} = 0 \tag{19}$$

for some $a \in L^\infty(B_R \times (0,T))$ $(\forall R<\infty)$, $a \leqq 0$ a.e. In addition, $|w| \leqq |\beta - \alpha|$ if both α and β are finite, or $|w| \leqq |\beta| + C_0(1+e^{\varepsilon_0 |x|^2})$ if β is finite and $\alpha = -\infty$, or $|w| \leqq 2C_0(1+e^{\varepsilon_0|x|^2})$ if $\alpha = -\infty$, $\beta = +\infty$. Here, we used the bounds assumed in Theorem 1. These bounds and (18) yield $w \equiv 0$ in view of standard results on linear parabolic equations.

Next, the regularity of $\tilde{J}$ follows easily from parabolic regularity results and we only have to explain why a locally bounded solution exists. By tedious approximations arguments involving truncation arguments [replace g by $(g+1/n) \wedge n, u$ by $u \wedge n$], we easily obtain a bounded (from below and from above) approximated solution $(w_n, \tilde{J}_n)$ on which we have to show the bounds stated in Theorem 1. Of course, $\alpha < w_n < \beta$ in $\mathbb{R}^n \times [0, T]$.

First, since $\psi_n, u_n \geqq 0$, $w_n \geqq z_n$, where z_n solves

$$\frac{\partial z_n}{\partial t} + \frac{1}{2} W z_n = 0, \quad z_n|_{t=T} = k_A(g_n)$$

and because of Condition A, we obtain the lower bound on $\tilde{J}_n$ stated in Theorem 1 in the event that $\alpha = -\infty$. If $\alpha > -\infty$, we also deduce $\tilde{J}_n \geqq \phi_A^{-1}(z_n)$ which is locally positive, uniformly in n, since $\phi_A(g_n)$ and thus z_n are locally strictly above α, uniformly in n.

Next, if $\beta = +\infty$, we choose $T' > T$ in such a way that the constants ε_0 appearing in Conditions A and B are strictly less than $1/2T'$ and we then consider $\bar{w} = \exp\{\frac{1}{2}(|x|^2 + K)(t + T' - T)^{-1}\}$, where $K > 0$ is to be determined. We compute

$$\frac{\partial \bar{w}}{\partial t} + \frac{1}{2} \Delta \bar{w} = -\frac{\bar{w}}{2}(K - N(t + T' - T)^{-1})$$

and we observe that $\bar{w}|_{t=T} = \exp\{(|x|^2 + K/2T')\}$. We then require K to be larger than $\log 2C_0$ where C_0 is the constant appearing in Conditions A and B so that

$$\phi_A(g_n) \leqq \bar{w}|_{t=T}, \qquad u_n \psi(\bar{w}) \leqq \bar{w} \text{ on } \mathbb{R}^n \times [0, T].$$

We thus deduce

$$\frac{\partial \bar{w}}{\partial t} + \frac{1}{2} \Delta \bar{w} + \psi(\bar{w}) u_n \leqq 0 \text{ in } \mathbb{R}^n \times [0, T], \qquad \bar{w}|_{t=T} \geqq w_n|_{t=T},$$

provided we also require that $K > N/T' + 2$. Then, the uniqueness proof above adapts easily to this situation and yields via the maximum principle: $w_n \leqq \bar{w}$ in $\mathbb{R}^n \times [0, T]$. In particular, $\tilde{J}_n \leqq \phi_A^{-1}(\bar{w})$ and the upper bound stated in Theorem 1 is proven when $\beta = +\infty$.

Finally, if $\beta < +\infty$, we have to obtain local upper bounds on $\tilde{J}_n$ or equivalently, we have to show that $\beta - w_n$ is locally bounded from below, uniformly in n. To this end, we first observe that the above analysis shows that there exists a solution $\bar{w}_n$ of

$$\frac{\partial \bar{w}_n}{\partial t}+\frac{1}{2}\Delta \bar{w}_n+\psi(\bar{w}_n)u_n=0, \qquad \bar{w}_n\big|_{t=T}=\psi_A(g_n \vee 1),$$

which satisfies $0 \leqq \bar{w}_n < \beta$ in $\mathbb{R}^n \times [0,T]$, $w_n \leqq \bar{w}_n$ in $\mathbb{R}^n \times [0,T]$. Then, we remark that $\psi(\beta)=0$, $\psi'(t)=A(\psi_A^{-1}(t))$ is bounded on $[0,\beta)$. Hence we have

$$\frac{\partial}{\partial t}(\beta-\bar{w}_n)+\frac{1}{2}\Delta(\beta-\bar{w}_n)+a_n(\beta-\bar{w}_n)=0, \qquad \beta-\bar{w}_n \geqq 0,$$

where $a_n \in L^\infty(B_R \times (0,T))$ $(\forall R<\infty)$. And we conclude using the strong maximum principle since $\beta-\bar{w}_n\big|_{t=T}=\beta-\phi_A(g_n \vee 1)$ is locally strictly positive, uniformly in n. □

We now go back to the original eq. (13) where we replace as before $u(x)$ by $u(x,t)$ but we insist on keeping the term $-\beta J$. Changing unknowns as before yields the equation

$$\frac{\partial w}{\partial t}+\frac{1}{2}\Delta w+\psi(w)u+\chi(w)=0, \qquad w\big|_{t=T}=\phi_A(g), \tag{20}$$

where $\chi(w)=-\beta(w)\phi_A^{-1}(w)$. We then observe that

$$\chi'(w)=-\beta A(\phi_A^{-1}(w))\phi_A^{-1}(w)-\beta.$$

Thus $\chi'(w)$ is bounded from above as soon as we have

$$A(t) \geqq \frac{\mu}{t} \text{ on } (0,\infty) \tag{21}$$

for some $\mu \in (-\infty,0)$. This property of χ allows us to reproduce the proof of Theorem 1 and yields

Theorem 1′. Under Conditions A and B, Theorem 1 also holds if we replace eq. (14) by eq. (13).

We now turn to the general case of a diffusion operator $\mathscr{D}=a_{ij}\partial_{ij}+b_i\partial_i$. By rescaling, we may always assume

$$\sum_{i,j} a_{ij}(x)x_i x_j \leqq \frac{1}{2}|x|^2 \text{ on } \mathbb{R}^n. \tag{22}$$

Theorem 2. Under Conditions A and B, Theorems 1 and 1′ hold in the case of a general operator $\mathscr{D}$, where the diffusion matrix (q_{ij}) is normalized to satisfy (22).

Proof (Sketch). The proof is essentially the same once we remark that for all $T'>T$, $\delta\in(0,1/T')$, there exists a constant $\bar{K}=\bar{K}(\delta,T'-T)$ such that for $K\geqq\bar{K}$, $\bar{w}=\exp\{\frac{1}{2}(|x|^2+K)\,(T'-T+t)^{-1}-\delta\}$ satisfies

$$\frac{\partial\bar{w}}{\partial t}+\mathscr{D}\bar{w}\leqq-\frac{K}{4(T')^2}\,\bar{w},\qquad \bar{w}|_{t=T}=\exp\left\{\frac{1}{2}\left(\frac{1}{T'}-\delta\right)(|x|^2+K)\right\}.$$

Indeed, we have

$$\begin{aligned}\frac{\partial\bar{w}}{\partial t}+\mathscr{D}\bar{w}=&-\tfrac{1}{2}(|x|^2+K)(T'-T+t)^{-2}\bar{w}+\{(T'-T+t)^{-1}-\delta\}\,\mathrm{Tr}\,(a)\bar{w}\\&+\{(T'-T+t)^{-1}-\delta\}^2\left(\sum_{i,j}a_{ij}x_ix_j\right)\bar{w}+\{(T'-T+t)^{-1}-\delta\}\sum_i b_ix_i\bar{w}\\&\leqq-\tfrac{1}{2}\gamma|x|^2\bar{w}-(T'-T+t)^{-1}\{\tfrac{1}{2}K(T'-T+t)^{-1}-C(1+|x|)\}\bar{w},\end{aligned}$$

in view of (22) and the boundedness of a_{ij}, b_i where $\gamma=\delta(T'-T-t)^{-1}>0$ and $C>0$ is independent of x, b, T', δ and K. Therefore, we deduce

$$\frac{\partial\bar{w}}{\partial t}+\mathscr{D}\bar{w}\leqq-\frac{K}{2}(T'-T+t)^{-2}\bar{w}+\frac{C}{2\delta}(T'-T+t)^{-1}\bar{w}\leqq-\frac{K}{4(T')^2}\,\bar{w},$$

provided we choose $K\geqq 2CT'\delta^{-1}$. □

3.2. *The infinite-horizon Kreps–Porteus utility equation* ($\mu=-1$)

We now study the PDE (6) for the case of $f(x,v)=u(x)v$ and $A(v)=-1/v$, that is

$$\mathscr{D}J-J-\frac{1}{J}\sum_{i,j}a_{ij}\partial_iJ\partial_jJ+u=0\ \text{ in } \mathbb{R}^n. \tag{23}$$

The appropriate change of unknown is $w=\phi_A(J)=\log J$ which yields the equation

$$\mathscr{D}w-1+\mathrm{e}^{-w}u=0\ \text{ in } \mathbb{R}^n. \tag{24}$$

Again, we will start with the special case when $\mathscr{D}=\frac{1}{2}\Delta$, $a_{ij}\equiv\frac{1}{2}\delta_{ij}$ so that (23), (24) reduce respectively to

$$\frac{1}{2}\Delta J-J-\frac{1}{2J}|\Delta J|^2+u=0 \text{ in } \mathbb{R}^n, \tag{25}$$

and

$$\tfrac{1}{2}\Delta w-1+e^{-w}u=0 \text{ in } \mathbb{R}^n. \tag{26}$$

We will use the following:

Condition C. A function $h: \mathbb{R}^n \to \mathbb{R}$ satisfies Condition C if there exist some constants $C>1$, $\delta \in (0,1/N)$ such that $1/C \leqq h(x) \leqq C\,e^{\delta|x|^2}$ on $\mathbb{R}^n$.

Theorem 3. Suppose u satisfies Condition C. Let J_1, J_2 be positive functions on $\mathbb{R}^n$ in $W^{2,p}_{\text{loc}}(\mathbb{R}^n)$ ($\forall p<\infty$) satisfying, respectively,

$$\frac{1}{2}\Delta J_1-J_1-\frac{1}{2J_1}|\nabla J_1|^2+u\geqq 0 \text{ in } \mathbb{R}^n, \tag{27}$$

$$\frac{1}{2}\Delta J_2-J_2-\frac{1}{2J_2}|\nabla J_2|^2+u\leqq 0 \text{ in } \mathbb{R}^n, \tag{28}$$

Then, if J_1 satisfies Condition C and J_2 is bounded away from 0, $J_1 \leqq J_2$ on $\mathbb{R}^n$. In addition there is a unique solution $\bar{J}$ in $W^{2,p}_{\text{loc}}(\mathbb{R}^n)$ ($\forall p<\infty$) of (25) satisfying Condition C.

Proof. We begin with the uniqueness part. Let $w_i=\log J_i$ ($i=1,2$), w_i solve

$$\tfrac{1}{2}\Delta w_1-1+e^{-w_1}u\geqq 0 \text{ in } \mathbb{R}^n, \qquad w_1\in W^{2,p}_{\text{loc}}(\mathbb{R}^n) \ (\forall p<\infty), \tag{29}$$

$$\tfrac{1}{2}\Delta w_2-1+e^{-w_2}u\leqq 0 \text{ in } \mathbb{R}^n, \qquad w_2\in W^{2,p}_{\text{loc}}(\mathbb{R}^n) \ (\forall p<\infty), \tag{30}$$

and

$$w_1(x)\leqq \delta|x|^2+K, \qquad w_2(x)\leqq -K \quad \text{on } \mathbb{R}^n, \tag{31}$$

for some positive constant K.

We then introduce $\bar{w}=\varepsilon_1|x|^2+M$, where $\varepsilon_1\in(\delta,1/N)$ and M is a positive constant to be determined. We observe that

$$\tfrac{1}{2}\Delta\bar{w}-1-e^{-\bar{w}}u=n\varepsilon_1-1-e^{-\varepsilon_1|x|^2-M}u(x)\leqq 0,$$

choosing M large enough so that $u(x)\,\mathrm{e}^{-\varepsilon_1|x|^2-M} \leqq (1-N\varepsilon_1)$ (because of Condition C).

Then, letting $\theta \in (0,1)$, we are going to show that $w_1 \leqq \theta w_2 + (1-\theta)\bar{w}$ on $\mathbb{R}^n$ and we will conclude by letting θ go to 1. To this end, we remark first that (31) and the choice of $\bar{w}$ imply that $w_1 \leqq \theta w_2 + (1-\theta)\bar{w}$ for $|x|$ large, say $|x| \geqq R_0$. Then, we use the convexity of $(w \mapsto \mathrm{e}^{-w})$ to deduce that

$$\tfrac{1}{2}\Delta(\theta w_2 + (1-\theta)\bar{w}) - 1 + \mathrm{e}^{-(\theta\omega_2 + (1-\theta)\bar{w})u}$$

$$\leqq \theta\{\tfrac{1}{2}\Delta w_2 - 1 + \mathrm{e}^{-w_2}u\} + (1-\theta)\{\tfrac{1}{2}\Delta \bar{w} - 1 + \mathrm{e}^{-\bar{w}}u\} \leqq 0.$$

Therefore, denoting by $z = \theta w_2 + (1-\theta)\bar{w}$, we obtain on B_{R_0}

$$\tfrac{1}{2}\Delta z + az \leqq 0, \qquad a \in L^\infty(B_{R_0}), \qquad z\big|_{\partial B_{R_0}} \geqq 0.$$

Since $(w \mapsto \mathrm{e}^{-w})$ is decreasing, we see that $a \leqq 0$. We then conclude by a simple application of the maximum principle.

Next, we prove the existence part. Because of Condition C there exists $\alpha \in (0,1]$ such that $u \geqq \alpha$. Since $(w \mapsto \mathrm{e}^{-w})$ is decreasing, standard results on monotone operators yield the existence and uniqueness of a solution w_R of

$$\tfrac{1}{2}\Delta w_R - 1 + \mathrm{e}^{-w_R}u = 0 \text{ in } B_R, \quad w_R \in W^{2,p}(B_R) \ (\forall p < \infty),$$

$$w_R\big|_{\partial B_R} = \log \alpha, \tag{32}$$

for all $R > 0$. The above argument then shows that $w_R \geqq \log \alpha$ on B_R and $w_R \leqq \bar{w}$ on B_R. These bounds and elliptic estimates allow us to pass to the limit as R goes to $+\infty$, and we recover a solution of

$$\tfrac{1}{2}\Delta w - 1 + \mathrm{e}^{-w}u = 0 \text{ in } \mathbb{R}^n, \quad w \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n) \ (\forall p < 0),$$

which satisfies $\log \alpha \leqq w \leqq \bar{w}$ on $\mathbb{R}^n$. This concludes the proof of Theorem 3, setting $J = \mathrm{e}^w$. □

Remark. Condition C is optimal both for the comparison part and the existence part. Indeed, $J_2 = \exp(|x|^2/N)$ satisfies (28) for all positive u while $J_1 = 2$ satisfies (27) if $u \geqq 2$ on $\mathbb{R}^n$; however, $J_1(0) > J_2(0)$.

Next, if $u \geqq v\exp(|x|^2/n)$ for some $v > 0$, one can show that there does not exist a solution of (25) that is bounded away from 0. Indeed, arguing by contradiction, we would obtain a solution w of

$$\tfrac{1}{2}\Delta w - 1 + \mathrm{e}^{-w}u = 0 \text{ in } \mathbb{R}^n, \ w \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n) \ (\forall p < 0),$$

which is bounded from below, say by $-M$ for some positive constant M. In particular, we have

$$\tfrac{1}{2}\Delta w - 1 + e^{-w} v\, e^{\delta|x|^2} \leqq 0 \ \text{ in } \mathbb{R}^n, \ \forall \delta \in (0, 1/n),$$

while $w_\delta = \delta|x|^2 - \log(1-\delta n) + \log v$ solves

$$\tfrac{1}{2}\Delta w_\delta - 1 + e^{-w_\delta} v\, e^{\delta|x|^2} = 0 \ \text{ in } \mathbb{R}^n.$$

By Theorem 3, we deduce: $w \geqq w_\delta$ on $\mathbb{R}^n$. And we reach a contradiction by letting δ go to $1/n$ since $w_\delta(w) \uparrow +\infty$ as $\delta \uparrow 1/n$.

We next consider general operator $\mathscr{D}$. In that case we have to replace the crucial quantity $|x|^2/n$ by a function w_0 satisfying

$$\mathscr{D} w_0 \leqq 1, \text{ in } \mathbb{R}^n, \qquad w_0 \in W^{2,p}_{\text{loc}}(\mathbb{R}^n) \ (\forall p < 0),$$

$$w_0 > 0 \text{ in } \mathbb{R}^n, \qquad w_0 \to +\infty \text{ as } |x| \to \infty. \tag{33}$$

Let us give a few examples of such a function w_0. First of all, if there exists a point $x_0 \in \mathbb{R}^n$ such that

$$\operatorname{Tr}(a) + b(x)\cdot(x - x_0) \leqq \mu \ \text{ on } \mathbb{R}^n \quad \text{for some} \quad \mu \geqq 0,$$

then we may take $w_0(x) = \frac{1}{2}\lambda|x - x_0|^2$ with $0 < \lambda \leqq 1/\mu$. In general, we may always take $w_0(x) = \lambda(\mu + |x|^2)^{1/2}$, where

$$0 < \lambda < 1/\|b\|_{L^\infty}, \qquad \mu \geqq \lambda^2 \|\operatorname{Tr}(a)\|^2_{L^\infty}(1 - \lambda^2\|b\|^2_{L^\infty})^{-1}.$$

Then, we replace Condition C with

Condition C′. A function $h: \mathbb{R}^n \to \mathbb{R}$ satisfies Condition C′ if there exist some constants $C > 1$, $\delta \in (0,1)$ such that $1/C \leqq h(x) \leqq C\, e^{\delta w_0(x)}$ on $\mathbb{R}^n$.

Theorem 3′. The analogue of Theorem 3 holds for eq. (23) provided we replace Condition C by Condition C′.

We conclude this section by mentioning how the preceding arguments adapt to more general functions A. We assume that A is differentiable on $(0, \infty)$ and that it satisfies

$$A'(t) \geqq 0, \quad (tA(t))' \leqq 0 \qquad \text{on } (0, \infty), \tag{34}$$

and

$$A(t) \geqq -\frac{1}{t} \text{ on } (0, \infty). \tag{35}$$

Then, the PDE (6) with $f(x,0) = u(x) - v$ becomes

$$\mathscr{D}J - J + A(J) \sum_{i,j} a_{ij} \partial_i J \partial_j J + u = 0 \text{ in } \mathbb{R}^n, \tag{36}$$

and after changing variables $w = \phi_A(J)$ reduces to

$$\mathscr{D}w + \psi(w)u - \phi_A^{-1}(w)\psi(w) = 0 \text{ in } \mathbb{R}^n. \tag{37}$$

Then, we see that the nonlinear function $g(x,t) = \psi(t)u(x) - \phi_A^{-1}(t)\psi(t)$ is non-increasing because of (35) and is convex because of (34). These two properties allow us to reproduce the preceding proofs and to obtain the same results as in Theorems 3 and 3′ provided we replace the condition on u by $\inf_{\mathbb{R}^n} u > 0$ and that for some $\delta \in (0,1)$, we can find $K > 0$ such that

$$u(x)\psi(\delta w_0(x) + K) \leqq 1 - \delta \text{ on } \mathbb{R}^n. \tag{38}$$

Similarly, the condition on J in Theorem 3 has to be replaced by

$$J(x) \leqq \phi_A^{-1}(\delta w_0(x) + K) \text{ on } \mathbb{R}^n, \tag{39}$$

for some $\delta \in (0,1)$, $K > 0$.

3.3. The infinite-horizon Kreps–Porteus utility equation $[\mu \in (-1,0)]$

We now consider (6) for the case $f(x,v) = u(x) - v$ and $A(v) = \mu/v$ with $\mu \in (-1,0)$, that is,

$$\mathscr{D}J - J + \frac{\mu}{J} \sum_{i,j} a_{ij} \partial_i J \partial_j J + u = 0 \text{ in } \mathbb{R}^n. \tag{40}$$

The change of unknowns $w = \phi_A(J) = (J^{\mu+1} - 1)/(1+\mu)$ yields the equation

$$\mathscr{D}w + (1 + (1+\mu)w)^{\mu/(1+\mu)}u - 1 - (1+\mu)w = 0 \text{ in } \mathbb{R}^n. \tag{41}$$

Again, the special case of $\mathscr{D} = \frac{1}{2}\Delta$, $a_{ij} = \frac{1}{2}\delta_{ij}$ produces

$$\frac{1}{2}\Delta J - J + \frac{\mu}{2J} |\nabla J|^2 + u = 0 \text{ in } \mathbb{R}^n \tag{42}$$

and

$$\tfrac{1}{2}\Delta w+(1+(1+\mu)w)^{\mu/(1+\mu)}u-1-(1+\mu)w=0 \text{ in } \mathbb{R}^n. \tag{43}$$

Observe that the nonlinear function

$$g(x,t)=(1+(1+\mu)t)^{\mu/(1+\mu)}u-1-(1+\mu)t$$

[defined for $t>-1/(1+\mu)$] is again non-increasing and convex. This allows us to adapt in a straightforward manner the arguments of the preceding section to this case. And we introduce the conditions:

Condition D_1. A function $h:\mathbb{R}^n\to\mathbb{R}$ satisfies Condition D_1 if there exists some constant $C>1$ such that $1/C\leqq h(x)\leqq Cw_0(x)^{-\mu/(1+\mu)}$ on $\mathbb{R}^n$.

Condition D_2. A function $h:\mathbb{R}^n\to\mathbb{R}$ satisfies Condition D_2 if there exists some constant $C>0$ such that $h(x)\leqq Cw_0(x)^{1/(1+\mu)}$ on $\mathbb{R}^n$.

In these two conditions, w_0 is a function satisfying

$$\mathscr{D}w_0-(1+\mu)w_0\leqq 0 \text{ in } \mathbb{R}^n, \qquad w_0\in W^{2,p}_{\text{loc}}(\mathbb{R}^n)\ (\forall p<\infty),$$

$$\text{with}\quad w_0>0 \text{ in } \mathbb{R}^n, \qquad w_0\to+\infty \text{ as } |x|\to\infty. \tag{44}$$

Notice that we might as well require that $\mathscr{D}w_0-(1+\mu)w_0$ is bounded from above (replace w_0 by w_0+K with $K>0$ large...). An example of such a function w_0 is given by $w_0(x)\exp(\lambda\sqrt{1+|x-x_0|^2})+K$, where x_0 is any point in $\mathbb{R}^n$, where K is large enough, and where $\lambda>0$ satisfies

$$\sup_{x\neq x_0}\lambda^2\sum_{i,j}a_{ij}\frac{(x-x_0)_i(x-x_0)_j}{|x-x_0|^2}+\lambda\sum_i b_i\frac{(x-x_0)_i}{|x-x_0|}<(1+\mu).$$

In particular, if $a_{ij}\equiv\delta_{ij}$, $b_i\equiv 0$, that is, $\mathscr{D}=\Delta$, we may take $w_0(x)=\exp(\lambda\sqrt{|x|^2+1})+K$, where $\lambda<(1+\mu)^{1/2}$.

And we obtain by the arguments of the preceding section

Theorem 4. Suppose u satisfies Condition D_1. Let J_1,J_2 be positive functions on $\mathbb{R}^n$ in $W^{2,p}_{\text{loc}}(\mathbb{R}^n)$ $(\forall p<\infty)$ satisfying, respectively,

$$\mathscr{D}J_1-J_1+\frac{\mu}{J_1}\sum_{i,j}a_{ij}\partial_iJ_1\partial_jJ_1+u\geqq 0 \text{ in } \mathbb{R}^n, \tag{45}$$

$$\mathscr{D}J_2 - J_2 + \frac{\mu}{J_2} \sum_{i,j} a_{ij} \partial_i J_2 \partial_j J_2 + u \leqq 0 \text{ in } \mathbb{R}^n. \tag{46}$$

Then, if J_1 satisfies Condition D_2 and J_2 is bounded away from 0, $J_1 \leqq J_2$ on $\mathbb{R}^n$. In addition, there is a unique solution J in $W^{2,p}_{\text{loc}}(\mathbb{R}^n)$ ($\forall p < \infty$) of (40) satisfying Condition D_2 that is bounded away from 0.

In fact, this result can be adapted to more general functions A that are differentiable on $(0, \infty)$ satisfy (34), and

$$A(t) \geqq \frac{\mu}{t} \text{ on } (0, \infty), \tag{47}$$

where $\mu \in (-1, 0)$. Then, the equations are again given by (36) and (37) and the nonlinear function $g(x,t) = \psi(t)u(x) - \phi_A^{-1}(t)\psi(t)$ is still convex because of (34), but it also satisfies

$$\frac{\partial g}{\partial t}(x,t) = \psi'(t)u(x) - 1 - A(\phi_A^{-1}(t))\phi_A^{-1}(t) \leqq -(1+\mu) + \psi'(t)u(x),$$

in view of (47). Notice also that $g(x,0) = u(x) - 1$.

Then, Theorem 4 adapts to this general situation replacing the bounds on u and J implied by Conditions D_1 and D_2, respectively, by the following ones:

$$\psi'(Cw_0(x))u(x) \leqq C \text{ on } \mathbb{R}^n, \tag{48}$$

$$J(x) \leqq \phi_A^{-1}(Cw_0(x)) \text{ on } \mathbb{R}^n, \tag{49}$$

for some $C > 0$.

3.4. *The infinite-horizon Kreps–Porteus utility equation ($\mu < -1$)*

We finally consider the PDE (6) for the case $f(x,v) = u(x) - v$ and $A(v) = \mu/v$ where $\mu < -1$, that is, (40) or (42). The change of unknowns

$$w = \phi_A(J) = \frac{1}{|1+\mu|}(1 - J^{-|\mu+1|}) = \frac{1}{\gamma}(1 - J^{-\gamma}),$$

where $\gamma = |\mu + 1| > 0$, yields the equations

$$\mathscr{D}w + (1 - \gamma w)^{1+1/\gamma} u - 1 + \gamma w = 0 \text{ in } \mathbb{R}^n. \tag{50}$$

Notice that $w<1/\gamma$ if $J>0$, so that the function is well-defined. The function $g(x,t)$ is now given by

$$g(x,t)=(1-\gamma t)^{1+1/\gamma}u-1+\gamma t,$$

and even if it is still convex, it is no more non-increasing. This fact explains why the problem is much more delicate in this case. To fix these ideas, let us also write down for the special case $\mathscr{D}=\frac{1}{2}\Delta$

$$\tfrac{1}{2}\Delta w+(1-\gamma w)^{1+1/\gamma}u-1+\gamma w=0 \text{ in } \mathbb{R}^n. \tag{51}$$

A related difficulty to the loss of monotonicity of the function is the fact that (50) or (51) possess a trivial solution $w\equiv 1/\gamma$ that we have tried to avoid since the change of unknown requires w to be strictly less than $1/\gamma$. Finally, we will in fact work with $z=1-\gamma w=J^{-\gamma}$ rewriting (50), (51) as

$$\mathscr{D}z+\gamma z=\gamma z^{1+1/\gamma}u \text{ in } \mathbb{R}^n, \quad z>0 \text{ in } \mathbb{R}^n, \tag{52}$$

or

$$\tfrac{1}{2}\Delta z+\gamma z=\gamma z^{1+1/\gamma}u \text{ in } \mathbb{R}^n, \quad z>0 \text{ in } \mathbb{R}^n. \tag{53}$$

As we will see below, the existence of solutions will be insured in general; however, the uniqueness and comparison results will depend on the growth and behavior of u and on the operator $\mathscr{D}$. We thus begin with existence questions. To this end, we need to introduce a few notions. We will say that $\bar{J}$ (resp. $\underline{J}$) is a maximum (resp. minimum) solution of (40) if it is a solution and is above all subsolutions of (40) [resp. below all supersolutions of (40)]. Finally, J' is a supersolution of (40) [resp. a subsolution of (40)] if J' satisfies (46) [resp. (45)] (and $\tilde{J}>0$ in $\mathbb{R}^n$). One defines in a similar manner supersolutions, subsolutions, maximum and minimum solutions of (52). We will also need to introduce $\lambda_1(\mathscr{D})$, which we define to be

$$\lim_{R\uparrow+\infty}\ \sup_{x\in\mathbb{R}^n}\lambda_1(\mathscr{D},B_R(x)),$$

where $\lambda_1(\mathscr{D},B_R(x))$ is the first eigenvalue of $-\mathscr{D}$ on the ball $B_R(x)$ with homogeneous Dirichlet conditions. [Remark that $\lambda_1(\mathscr{D})\geqq 0$.]

Theorem 5. Assume that u is strictly positive on $\mathbb{R}^n$. Then there exists a minimum solution $\underline{J}$ in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ $(\forall p<\infty)$ of (40), or equivalently a maximum solution $\bar{z}$ in $W^{2,p}_{\mathrm{loc}}(\mathbb{R}^n)$ $(\forall p<\infty)$ of (52). In addition, if u is bounded away from 0, $\bar{z}$ is bounded, and $\underline{J}$ is also bounded away from 0. Furthermore, if $\lambda_1(\mathscr{D})<\gamma$, *there exists a maximum solution $\bar{J}$ in $W^{2,p}_{\text{loc}}(\mathbb{R}^n)$ ($\forall p<\infty$) of (40), or equivalently a minimum solution $\underline{z}$ in $W^{2,p}_{\text{loc}}(\mathbb{R}^n)$ ($\forall p<\infty$) of (52).*

Proof. It is of course enough to prove the existence of $\bar{z}, \underline{z}$. The existence relies upon a local estimate described in the following.

Lemma 6. Let $z \geqq 0 \in W^{2,p}_{\text{loc}}(\mathbb{R}^n)$ ($\forall p<\infty$) satisfy

$$\mathscr{D}z - \delta z^{1+1/\gamma} \geqq -\frac{1}{\delta} \text{ in } B_M(x_0), \tag{54}$$

for some $M>0$, $\delta \in (0,1)$, $x_0 \in \mathbb{R}^n$. Then, there exists a positive constant dependng only on $\mathscr{D}$, δ, γ and M such that $z(x_0) \leqq C$.

This lemma is more or less standard [see for instance Brézis (1984) for an application of these bounds] and is proven by considering an explosive supersolution $\hat{z}_M = C(M^2 - |x-x_0|^2)^{-\alpha}$, where $\alpha = 2\gamma$ and C is large enough. Clearly, by the maximum principle, $z(x_0) \leqq \hat{z}_M(x_0) = CM^{-2\alpha}$.

We now use this lemma to prove the existence of $\underline{z}, \bar{z}$. To this end, we consider for R large enough the positive solution $\underline{z}_R$ of

$$\mathscr{D}\underline{z}_R + \gamma \underline{z}_R = \gamma \underline{z}_R^{1+1/\gamma} u \text{ in } B_R, \qquad \underline{z}_R|_{\partial B_R} = 0. \tag{55}$$

Such a solution exists [see for example the review article by Lions (1982)] as soon as the first eigenvalue of $\mathscr{D}$ on B_R with homogeneous Dirichlet condition is strictly less than γ and thus in particular for R large enough. Next, any supersolution of (52) is a supersolution of (55) and thus above z_R by Lemma 7 below. We easily obtain the existence of $\underline{z}$ by letting R go to $= +\infty$, using the local bounds provided by Lemma 6 and local elliptic estimates.

Lemma 7. Let $z_1, z_2 \geqq 0 \in W^{2,p}_{\text{loc}}(B_R)$ ($\forall p<\infty$) satisfy $\mathscr{D}z_1 + \gamma z_1 \leqq \gamma z_1^{1+1/\gamma} u$ in B_R, $\mathscr{D}z_2 + \gamma z_2 \geqq \gamma z_2^{1+1/\gamma} u$ in B_R and $z_2 \leqq z_1$ on ∂B_R, $z_1 \not\equiv 0$. Then, $z_2 \leqq z_1$ on B_R.

Again, this is a standard result on semilinear elliptic equations that we will not reprove here [see Lions (1982) for results of this sort].

The existence of $\bar{z}$ follows from a similar approximation procedure. First of all, Lemma 6 implies that all subsolutions of (52) are bounded on B_R by a constant $C_R > 0$. Then we consider the positive solution $\bar{z}_R$ of

$$\mathscr{D}\bar{z}_R + \gamma \bar{z}_R = \gamma \bar{z}_R^{1+1/\gamma} u \text{ in } B_R, \qquad \bar{z}_R|_{\partial B_R} = C_R.$$

By Lemma 7, all subsolutions of (52) are below $\bar{z}_R$ on B_R and we again let R go to $+\infty$ to recover the maximum solution $\bar{z}$.

Finally, we have to prove that $\bar{z}$ is bounded if u is bounded away from 0. This again follows from Lemma 6 since $\bar{z}$ satisfies (54) for all $M>0$, $x_0 \in \mathbb{R}^n$, and a fixed $\delta>0$. Then,

$$\left(\frac{1}{\delta^2}\right)^{1+1/\gamma}+\frac{CM^{3\gamma}}{(M^2-|x-x_0|^2)^{2\gamma}}$$

for some fixed constant $C>0$ is a supersolution of (54), and therefore

$$\bar{z}(x_0) \leqq \left(\frac{1}{\delta^2}\right)^{1+1/\gamma}+C\,\frac{M^{3\gamma}}{M^{4\gamma}}.$$

And we conclude by letting M go to $+\infty$. □

This proof and Lemma 7 shows that the main difficulty with uniqueness or comparison lies with the behavior of solutions at infinity. And the following example shows that it is a real difficulty.

Example. Let $\mathscr{D}=\Delta+b(\partial/\partial x_1)$ for some $b>0$ that we will determine later on. Then, $e^{\lambda x_1}$ is a solution of (40) with $u \equiv 0$ if we have $(1+\mu)\lambda^2+b\lambda-1=0$. This is the case if we fix $\lambda>0$ and choose $b=(1+\gamma\lambda^2)\lambda^{-1}$. In particular, $e^{\lambda x_1}$ is a subsolution of (40) for all positive u. It is not difficult to check that

$$\lambda_1(\mathscr{D})=\frac{b^2}{4}=\frac{(1+\gamma\lambda^2)^2}{4\lambda^2} \geqq \gamma.$$

Notice also that $K e^{\lambda x_1}$ is for all $K>0$ still a subsolution of (40) for all positive u, a fact that shows that $\bar{J}$ does not exist and that Theorem 5 is in some sense optimal. □

This example also illustrates some of the specific difficulties encountered with exponential growths and a general operator $\mathscr{D}$, and might be an explanation for the type of uniqueness results we show below that are of three different types: We first treat the case of 'log-bounded variation' functions u, then we consider functions u that grow in a 'superexponential' fashion, and finally we consider operators that are self-adjoint like Δ and obtain uniqueness results for rather general classes of functions u.

Condition E. A function $h: \mathbb{R}^n \to \mathbb{R}$ satisfies Condition E if there exists a positive constant δ such that

$$u \leqq \delta > 0 \text{ on } \mathbb{R}^n, \quad \frac{u(x)}{u(y)} \geqq \delta \text{ for all } x, y \in \mathbb{R}^n, \quad |x-y| \leqq 1.$$

Condition E simple means that $\log u$ has bounded oscillations on bounded sets. It is satisfied in particular if u is differentiable on $\mathbb{R}^n$ and $|\nabla u(x)| \leqq Cu(x)$ on $\mathbb{R}^n$ for some constant $C>0$.

Theorem 8. Suppose u satisfies Condition E and $\lambda_1(\mathscr{D}) < \gamma$. Then $\bar{J} = \underline{J}$ on $\mathbb{R}^n$, or equivalently $\bar{z} \equiv \underline{z}$ on $\mathbb{R}^n$.

Proof. We are going to show that $\bar{z} \equiv \underline{z}$ on $\mathbb{R}^n$. By Theorem 5, we know that $\underline{z}$ exists, that $\bar{z}$ is bounded on $\mathbb{R}^n$, and of course that $\bar{z} \geqq \underline{z}$ on $\mathbb{R}^n$.

The first step consists in showing that $a(x) = \underline{z}^{1/\gamma} u$ is bounded away from 0 on $\mathbb{R}^n$. In view of the definition of $\lambda_1(\mathscr{D})$, we can find $R_0 > 0$ large enough so that $\sup_{x \in \mathbb{R}^n} \lambda_1(\mathscr{D}, B_{R_0}(x)) < \gamma$. Then, for each $x_0 \in \mathbb{R}^n$, we consider the solution of

$$\mathscr{D} z_{x_0} + \gamma z_{x_0} = \gamma z_{x_0}^{1+1/\gamma} \text{ in } B_{R_0}(x_0), \quad z_{x_0} > 0 \in B_{R_0}(x_0), \quad z_{x_0 | \partial B_{R_0}} = 0. \tag{56}$$

A simple compactness argument shows that there exists some $v > 0$ such that $z_{x_0}(x_0) \geqq v$ for all $x_0 \in \mathbb{R}^n$. And we remark that for all $\lambda > 0$, λz_{x_0} solves

$$\mathscr{D}(\lambda z_{x_0}) + \gamma(\lambda z_{x_0}) = \gamma \lambda^{-1/\gamma} (\lambda z_{x_0})^{1+1/\gamma} \text{ in } B_{R_0}(x_0),$$

with

$$(\lambda z_{x_0}) > 0 \text{ in } B_{R_0}(x_0), \quad \lambda z_{x_0}|_{\partial B_{R_0}}(x_0) = 0.$$

Therefore, Lemma 7 implies that $\underline{z} \geqq \lambda z_{x_0}$ on $B_{R_0}(x_0)$, where $\lambda = [\sup_{B_{R_0}(x_0)} u]^{-\gamma}$. In particular we deduce for all $x_0 \in \mathbb{R}^n$:

$$(\underline{z}(x_0))^{1/\gamma} \sup_{B_{R_0}(x_0)} u \geqq v.$$

In conclusion, Condition E yields

$$a = \underline{z}^{1/\gamma} u \geqq \mu > 0 \text{ on } \mathbb{R}^n. \tag{57}$$

The next step is to show that $\bar{z}^{1/\gamma} u$ is bounded. This is quite similar to the above proof. Recall that we already know that $\bar{z}$ is bounded, say by C_0 for some $C_0 > 0$. We then introduce $\hat{z}_{x_0} > 0$ as a solution of

$$\mathscr{D}\hat{z}_{x_0}+\gamma\hat{z}_{x_0}=\gamma\hat{z}_{x_0}^{1+1/\gamma}\in B_{R_0}(x_0), \qquad \hat{z}_{x_0}\big|_{\partial B_{R_0}(x_0)}=C_0\left(\inf_{B_{R_0}(x_0)} u\right)^{\gamma}.$$

And we observe that $\bar{z}\leqq\lambda\hat{z}_{x_0}$ on $B_{R_0}(x_0)$, where $\lambda=(\inf_{B_{R_0}(x_0)}u)^{-\gamma}$. Since we have

$$\mathscr{D}(\lambda\hat{z}_{x_0})+\gamma(\lambda\hat{z}_{x_0})=\gamma\lambda^{-1/\gamma}(\lambda\hat{z}_{x_0})^{1+1/\gamma}\leqq\gamma u(\lambda\hat{z}_{x_0})^{1+1/\gamma}\in B_{R_0}(x_0),$$

and $\lambda\hat{z}_{x_0}=C_0$ on $\partial B_{R_0}(x_0)$. The comparison between $\bar{z}$ and $\lambda\hat{z}_{x_0}$ then follows from Lemma 6. But, by Lemma 7, we deduce that

$$\bar{z}(x_0)\leqq\lambda\hat{z}_{x_0}(x_0)\leqq C'\lambda$$

for some $C>0$ independent of x_0. And Condition E finally yields

$$b=\bar{z}^{1/\gamma}u<C \text{ on } \mathbb{R}^n, \tag{58}$$

for some $C>0$.

In particular, we deduce from (57) and (58) that

$$\bar{z}\leqq M\underline{z} \text{ on } \mathbb{R}^n, \tag{59}$$

for some $M>0$. We may now complete the uniqueness proof by observing that

$$\mathscr{D}(\bar{z}-\underline{z})+\gamma(\bar{z}-\underline{z})\geqq(\gamma+1)a(\bar{z}-\underline{z})\geqq\gamma a(\bar{z}-\underline{z})+\mu(\bar{z}-\underline{z}) \text{ on } \mathbb{R}^n, \tag{60}$$

while of course,

$$\mathscr{D}\underline{z}+\gamma\underline{z}=a\underline{z} \text{ in } \mathbb{R}^n. \tag{61}$$

We then consider the unique bounded solution $\tilde{z}$ of

$$-\frac{\partial\tilde{z}}{\partial t}+\mathscr{D}\tilde{z}+\gamma\tilde{z}=\gamma a\tilde{z}\in\mathbb{R}^n, \qquad \tilde{z}\big|_{t=0}=(\bar{z}-\underline{z}) \text{ on } \mathbb{R}^n.$$

Such a solution exists since $a\in L^{\infty}_{\mathrm{loc}}$, $a\geqq 0$ [in fact, (58), (59) imply that $a\in L^{\infty}(\mathbb{R}^n)$]. Then, we have on one hand, $\tilde{z}(x,t)\leqq(M-1)\underline{z}(x)$ on $\mathbb{R}^n\times(0,\infty)$, because of (59), (61) and the maximum principle.

On the other hand, $\hat{z}(x,t)=e^{\mu t}(\bar{z}-\underline{z})$ satisfies in view of (60)

$$-\frac{\partial \hat{z}}{\partial t}+\mathscr{D}\hat{z}+\gamma\hat{z}\geqq\gamma a\hat{z} \text{ in } \mathbb{R}^n\times(0,\infty), \qquad \hat{z}\big|_{t=0}=(\bar{z}-\underline{z}) \text{ on } \mathbb{R}^n.$$

Therefore, applying again the maximum principle, we find that $\hat{z}\leqq\tilde{z}$ on $\mathbb{R}^n\times(0,\infty)$. These comparisons yield

$$e^{\mu t}(\bar{z}-\underline{z})\leqq(M-1)\underline{z} \text{ on } \mathbb{R}^n\times(0,\infty),$$

and we conclude that $\bar{z}\equiv\underline{z}$ upon letting t go to $+\infty$. □

We now turn to the second situation that we are able to analyze. In fact, we will only give one example of the type of results that can be achieved by similar arguments. We consider an auxiliary function $\psi\in C^2([0,\infty))$ satisfying:

$$\psi(0)>0, \quad \psi'(0)=0, \quad \psi''>0 \text{ on } [0,\infty), \quad \frac{\psi'}{\psi}(t)\to+\infty \text{ as } t\to+\infty,$$

$$\tfrac{1}{2}(\psi')^2\leqq\psi\psi''\leqq(1+\delta)(\psi')^2 \text{ for } t \text{ large and some } \delta\in(0,\gamma).$$

Typical examples are provided by $\psi(t)=\exp(\lambda t^\alpha)$ with $\lambda>0$, $\alpha>1$. Now we introduce

Condition F. A function $u:\mathbb{R}^n\to\mathbb{R}$ satisfies Condition F if there exist $x_0\in\mathbb{R}^n$, a function ψ satisfying the above properties, and a constant $M\in(1,\infty)$ such that we have $u(x)\psi(|x-x_0|)^{-1}\to+\infty$ as $|x|\to\infty$ and

$$\frac{1}{M}\leqq u\leqq M(1+(\psi'(|x-x_0|))^2\psi(|x-x_0|)^{-1}) \text{ on } \mathbb{R}^n.$$

Theorem 9. Suppose u satisfies Condition F. Then, $\underline{J}$ is the unique positive solution in $W^{2,p}_{\text{loc}}(\mathbb{R}^n)\,(\forall p<\infty)$ of (40), or equivalently, $\bar{z}$ is the unique solution in $W^{2,p}_{\text{loc}}(\mathbb{R}^n)\,(\forall p<\infty)$ of (52).

Remark. Let us emphasize that, unless $\lambda_1(\mathscr{D})<\gamma$, $\underline{J}$ is not in general above all subsolutions of (40) as the example given after the proof of Theorem 5 shows. On the other hand, the proof below shows that $\underline{J}$ is above all subsolutions of (40) that are bounded away from 0.

Proof of Theorem 9. We consider an arbitrary positive solution J of (40), or equivalently, a solution z of (52), and we want to prove that $J=\underline{J}$, or equivalently, that $z=\bar{z}$. Of course, we have $J \geqq \underline{J}$ on $\mathbb{R}^n$, $z \leqq \bar{z}$ on $\mathbb{R}^n$. To simplify notation, we assume that $x_0 = 0$.

We first show that $J\psi^{-1}$ is bounded from above on $\mathbb{R}^n$. This is done by considering $\psi_R = \psi(r(1+1/(R-r)))$, where $R \geqq 1$, $r=|x|$. A horrendous computation shows that there exists some $R_0 > 0$, $\varepsilon_1 > 0$ independent of R such that

$$\mathscr{D}\psi_R - \psi_R + \frac{\mu}{\psi} \sum_{i,j} a_{ij} \partial_i \psi_R \partial_j \psi_R \leqq -\varepsilon_1 \psi_R^{-1} (\psi_R')^2 \quad \text{for} \quad R_0 \leqq |x| < R. \tag{62}$$

On the other hand, by Condition F, we find some $M_0 > 0$ such that

$$u(x) \leqq M_0 (\psi_R')^2 \psi_R^{-1} \quad \text{for} \quad R_0 \leqq |x| < R. \tag{63}$$

Finally, there exists some $K_0 < 0$ such that

$$J(x) \leqq K_0 \psi(x) \leqq K_0 \psi_R(x) \quad \text{for} \quad |x| = R_0. \tag{64}$$

Combining (62), (63), (64) and setting $K = \max(K_0, M_0/\varepsilon_1)$, we find that

$$\mathscr{D}(K\psi_R) + \frac{\mu}{(K\psi_R)} \sum_{i,j} a_{ij} \partial_i (K\psi_R) \partial_j (K\psi_R) + u \leqq 0 \quad \text{for} \quad R_0 \leqq |x| < R,$$

while of course $(K\psi_R)(x) \to +\infty$ as $|x| \to R$, and $J \leqq (K\psi_R)$ for $|x| = R_0$. This yields: $J \leqq K\psi_R$ for $R_0 \leqq |x| < R$. This comparison may be deduced from Lemma 7 after transforming the equations in the 'z-unknowns'. We then conclude that $J \leqq K_\gamma$ upon letting R go to $+\infty$.

In particular, we see that

$$a = z^{1/\gamma} u = \frac{u}{J} \geqq \frac{1}{K} \frac{u}{\psi}$$

satisfies

$$0 < a(x) \text{ on } \mathbb{R}^n, \qquad a(x) \to +\infty \text{ as } |x| \to +\infty. \tag{65}$$

On the other hand, exactly as in the proof of Theorem 8, we have

$$\mathscr{D}(\bar{z}-z) + \gamma(\bar{z}-z) - \gamma a(\bar{z}-z) \geqq a(\bar{z}-z) \text{ on } \mathbb{R}^n, \quad \bar{z}-z \geqq 0 \text{ on } \mathbb{R}^n$$

and

$$\mathscr{D}z + \gamma z - az = 0 \text{ on } \mathbb{R}^n, \quad z > 0 \text{ on } \mathbb{R}^n.$$

But then the asymptotic behavior of a at ∞ allows us to conclude the proof easily in view of well-known results on the first eigenvalue of elliptic operators. □

We finally conclude this section by the third case, which corresponds to self-adjoint operators $\mathscr{D}=(\partial/\partial x_i)a_{ij}(x)(\partial/\partial x_j)$, so that in other words $b_i(x)=\sum_j(\partial/\partial x_j)a_{ij}(x)$ (for all i). This assumption that we make throughout the end of this section allows us to use some special results. Notice already that this implies obviously that $\lambda_1(\mathscr{D})=0$. We introduce

Condition G. There exists a constant $\nu>0$ such that

$$u(x)\geqq \nu(1+|x|)^{\alpha} \text{ on } \mathbb{R}^n,$$

where $\alpha\geqq 0$ satisfies: if $\mu\leqq -2$ and $n>6$, then $|1+\mu|\alpha\geqq\frac{1}{2}(n-6)$, and if $-2\leqq\mu<-1$ and $n>3$, then $|1+\mu|(2+\alpha)\geqq\frac{1}{2}(n-2)$.

Theorem 10. Suppose u satisfies Condition G, then $\bar{J}\equiv\underline{J}$ on $\mathbb{R}^n$, or equivalently, $\bar{z}\equiv\underline{z}$ on $\mathbb{R}^n$.

Proof. We only have to show that $\bar{z}-\underline{z}$, which is nonnegative on $\mathbb{R}^n$, in fact vanishes. We remark that if $\gamma=|\mu+1|\leqq 1$, then

$$\mathscr{D}(\bar{z}-\underline{z})+\gamma(\bar{z}-\underline{z})-\gamma\underline{z}^{1/\gamma}u(\bar{z}-\underline{z})\geqq\gamma u(\bar{z}-\underline{z})^{1+1/\gamma} \text{ on } \mathbb{R}^n, \tag{66}$$

while if $\gamma>1$,

$$\mathscr{D}(\bar{z}-\underline{z})+\gamma(\bar{z}-\underline{z})-\gamma\underline{z}^{1/\gamma}u(\bar{z}-\underline{z})\geqq\frac{(\gamma+1)}{2\gamma}\,u(\bar{z}-\underline{z})^{2\bar{z}(-1+1/\gamma)} \text{ on } \mathbb{R}^n. \tag{67}$$

In addition, since $\underline{z}>0$ satisfies

$$\mathscr{D}\underline{z}+\gamma\underline{z}-\gamma z^{1/\gamma}u\underline{z}=0 \text{ in } \mathbb{R}^n,$$

we deduce that $\lambda_1(\mathscr{D}+\gamma-\gamma\underline{z}^{1/\gamma}u)\geqq 0$. Therefore

$$\int_{\mathbb{R}^n}\sum_{i,j}a_{ij}\partial_i\psi\partial_j\phi+(\gamma\underline{z}^{1/\gamma}u-\gamma)\phi^2\,\mathrm{d}x\geqq 0, \tag{68}$$

for all $\phi\in W^{1,2}(\mathbb{R}^n)$.

Finally, the proof of Theorem 8 shows that, because of Condition G,

$$\bar{z}(x)\leqq C(1+|x|)^{-\alpha\gamma} \text{ on } \mathbb{R}^n,$$

for some $C>0$. Combining this bound with (67), we deduce

$$\mathscr{D}(\bar{z}-\underline{z})+\gamma(\bar{z}-\underline{z})-\gamma\underline{z}^{1/\gamma}u(\bar{z}-\underline{z})\geqq\delta g(\bar{z}-\underline{z})^{\theta} \text{ on } \mathbb{R}^n, \tag{69}$$

where $\delta>0$, $g=(1+|x|)^{\beta}$, $\theta=1+1/\gamma$ and $\beta=\alpha$ if $\gamma\leqq 1$, while $\theta=2$ and $\beta=\alpha\gamma$ if $\gamma>1$. Then, let $\phi\in C_0^{\infty}(\mathbb{R}^n)$, $0\leqq\phi\leqq 1$, $\sup\phi\subset B_2$, $\phi\equiv 1$ on B_1, and for $R>0$, we set $\phi_R(x)=\phi(x/R)$. We then multiply (69) by $\phi_R^2(\bar{z}-\underline{z})$, and integrate by parts in order to find

$$\delta\int_{\mathbb{R}^n} g(\bar{z}-\underline{z})^{1+\theta}\phi_R^2\,\mathrm{d}x$$

$$+\int_{\mathbb{R}^n}\left[\sum_{i,j}a_{ij}\partial_i(\phi_R(\bar{z}-\underline{z}))\partial_j(\phi_R(\bar{z}-\underline{z}))+(\gamma\underline{z}^{1/\gamma}u-\gamma)(\phi_R(\bar{z}-\underline{z}))^2\right]\mathrm{d}x$$

$$\leqq\int_{\mathbb{R}^n}(\bar{z}-\underline{z})^2\,\mathrm{d}x.$$

In view of (68) and the properties of ϕ_R, we deduce from this inequality

$$\int_{|x|\leqq R} g(\bar{z}-\underline{z})^{1+\theta}\,\mathrm{d}x$$

$$\leqq\frac{C}{R^2}\int_{R\leqq|x|\leqq 2R}(\bar{z}-\underline{z})^2\,\mathrm{d}x$$

$$\leqq\frac{C}{R^2}\left(\int_{R\leqq|x|\leqq 2R} g(\bar{z}-\underline{z})^{1+\theta}\,\mathrm{d}x\right)^{2/(1+\theta)}$$

$$\times\left(\int_{R\leqq|x|\leqq 2R} g^{-2/(\theta-1)}\,\mathrm{d}x\right)^{(\theta-1)/(\theta+1)}$$

by Hölder's inequalities, where C denotes in all that follows various constants independent of R. We finally deduce from this inequality and the explicit form of g that

$$\int_{|x|\leqq R} h\,\mathrm{d}x\leqq CR^{n(\theta-1)/(\theta+1)}R^{-2}R^{-2\beta/(\theta+1)}$$

$$\times\left(\int_{R\leqq|x|\leqq 2R} h\,\mathrm{d}x\right)^{2/(1+\theta)}, \tag{70}$$

where $h=g(\bar{z}-\underline{z})^{1+\theta}$.

Notice that the exponent of R, namely

$$n\frac{\theta-1}{\theta+1}-2-\frac{2\beta}{\theta+1}$$

is nonpositive in view of Condition G. Therefore, (70) yields for all $R \geqq 1$,

$$\int_{|x| \leqq R} h \, \mathrm{d}x \leqq C \left(\int_{R \leqq |x| \leqq 2R} h \, \mathrm{d}x \right)^{2/(1+\theta)}. \tag{71}$$

In particular, if we choose successively $R = 2^k$, we find for all $k \geqq 0$,

$$x_{k+1} \geqq \left(\frac{x_k}{C} \right)^{(\theta+1)/2},$$

where $C > 0$, $\theta > 1$ are independent of K. Therefore, $y_k = C^{-(\theta+1)/(\theta-1)} x_k$ satisfies

$$y_{k+1} \geqq (y_k)^{(\theta+1)/2} \quad \text{or} \quad \log y_{k+1} \geqq \tfrac{1}{2}(\theta+1) \log y_k.$$

Hence, either $y_k \leqq 1$ for all k, that is, $x_k \leqq C^{(\theta+1)/(\theta-1)}$ for all k, or $y_{k_0} > 1$ for some k_0 and then

$$y_k \geqq (y_{k_0})^{((\theta+1)/2)^{k-k_0}} \quad \text{for} \quad k \geqq k_0.$$

If the second case were to happen, this would mean that for $k \geqq k_0$

$$\int_{|x| \leqq 2^k} g(\bar{z} - \underline{z})^{1+\theta} \, \mathrm{d}x \geqq C^{(\theta+1)/(\theta-1)} (y_{k_0})^{((\theta+1)/2)^{k-k_0}} \quad \text{for some } y_{k_0} > 1,$$

and we reach a contradiction since $\frac{1}{2}(\theta+1) > 1$, and by (68) and the fact that $\bar{z} - \underline{z}$ is bounded we have for all $k \geqq 0$

$$\int_{|x| \leqq 2^k} g(\bar{z} - \underline{z})^{1+\theta} \, \mathrm{d}x \leqq C(2^{n-2})^k.$$

Therefore, $x_k \leqq C^{(\theta+1)/(\theta-1)}$ for all k and thus $h \in L^1(\mathbb{R}^n)$. But then $\int_{R \leqq |x| \leqq 2R} h \, \mathrm{d}x \to 0$ as $R \to +\infty$. And (70) then implies that $h \equiv 0$, that is, $\bar{z} \equiv \underline{z}$ on $\mathbb{R}^n$. □

4. Remarks and related results

We begin with some applications of the preceding results and their proofs. These applications concern the monotonicity of J with respect to various data such as the terminal reward g or the running reward u, or comparative risk aversion, which is in fact a monotonicity property with respect to A [see Duffie and Epstein (1992)]. All of these properties were shown using probabilistic methods in Duffie and Epstein (1992) by an extension of Gronwall's inequalities but again required Lipschitz conditions that rule out interesting special cases such as Kreps–Porteus utility. They can be summarized by the following inequality:

$$J(g_1, u_1, A_1) \geqq J(g_2, u_2, A_2) \quad \text{if} \quad g_1 \geqq g_2,\ u_1 \geqq u_2,\ A_1 \geqq A_2, \tag{72}$$

in the finite horizon case or

$$J(u_1, A_1) \geqq J(u_2, A_2) \quad \text{if} \quad u_1 \geqq u_2,\ A_1 \geqq A_2, \tag{73}$$

where we recall the dependence of the utility function J upon the relevant data by writing J as function of these data.

These comparison results are true whenever our existence, uniqueness and comparison results proved in the preceding sections apply since $J(g_1, u_1, A_1)$ [resp. $J(u_1, A_1)$] is a supersolution of the problem (g_2, u_2, A_2) [resp. (u_2, A_2)] and conversely, $J(g_2, u_2, A_2)$ [resp. $J(u_2, A_2)$] is a subsolution of the problem (g_1, u_1, A_1) [resp. (u_1, A_1)]. And all our results apply to the comparison of super and subsolutions even if we did not explicitly mention this fact in section 3 for the finite-horizon case. For instance, (72) follows from Theorems 1, 1′, or 2 whenever (g_2, u_1), (g_2, u_2) satisfy Conditions A and B.

Another possible extension concerns the possibility of having degenerate diffusion processes or operator $\mathscr{D}$, or the extension to Hamilton–Jacobi–Bellman equations, extending to this setting the usual HJB equations for stochastic control with additively-separable objectives. Indeed, all of our methods of proofs rely on comparison results involving the maximum principle except for Theorem 10, and this can be combined with the theory of viscosity solutions. [See Crandall and Lions (1983), Lions (1983), Ishii and Lions (1990), Crandall et al. (1990) and the references therein.]

Our final remark concerns the link between our results and the original stochastic problems (2) or (4). By Itô's Lemma we see that the verification of (2) or (4) is straightforward provided we observe that, by easy truncation arguments which reduce the problem to cases with bounded data and passing to the limit while removing the truncations, one just has to check the integrability of $u(X_t)$, $g(X_T)$, $J(X_t)$, and $A(J(X_t))|\nabla J(X_t)|^2$. This can be easily achieved by growth conditions on u and g, such as $u(x)$, $g(x) \leqq C\mathrm{e}^{M|x|}$ for C, $M \geqq 0$, which imply similar bounds on $J(x)$ by the arguments of section 3.1. Then, elliptic and parabolic estimates on the transformed equations yield the necessary bounds on $\nabla J(X_t)$.

Having solved for the utility process $V_t = J(X_t)$, one can solve for the prices of securities in a single-agent setting as in Duffie and Epstein (1992) and, in greater generality, Duffie and Epstein (1992) and, in greater generality, Duffie and Skiadas (1990). As shown in these papers, a security promising consumption dividends (in a numeraire commodity) at a rate given by a process $\{\delta_t\}$ has, assuming integrability, a price process $\{S_t\}$ defined by

$$S_t = \frac{1}{\pi_t} E\left[\int_t^\infty \pi_s \delta_s \, ds \,\middle|\, \mathscr{F}_t\right], \tag{74}$$

where

$$\pi_t = \exp\left(\int_0^t \hat{f}_v[C(X_s), J(X_s)] \, ds\right) \hat{f}_c[C(X_t), J(X_t)], \tag{75}$$

and where $\hat{f}(C(x), v) = \bar{f}(x, v)$ for $\bar{f}$ as in (11) or (12).

For a dividend process of the form $\delta_t = h(X_t)$, one can derive yet another PDE from (74) for the security price S_t as a function of the current state, in the usual manner of the Feynman–Kac formula

An analogous solution for security prices applies in the finite horizon case.

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