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OPTIMAL HEDGING AND EQUILIBRIUM IN A DYNAMIC FUTURES MARKET*

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This paper considers an agent maximizing the expected utility of the sum of the terminal value of a fixed portfolio of spot market assets and the terminal value of a margin account on a futures trading position. Closed-form solutions for the optimal hedging strategy are provided in several special cases.

1. Introduction

This paper solves the optimal futures hedging problem in several simple continuous-time settings, and examines the resultant equilibrium in one case. Spot and futures prices are described by vector diffusion processes. A hedge is a vector stochastic process specifying a futures position in each futures market. Hedging profits and losses are marked to market in an interest-bearing (or interest-paying) margin account. A hedge is optimal if it maximizes the expected utility of terminal wealth, which is the market value of a committed portfolio of spot market assets plus the terminal value of the margin account.

The special cases solved in this paper are quite restrictive. In particular, in all of the cases, futures prices are either martingales or have independent normally distributed price increments. In some cases, it has been difficult to empirically reject the martingale hypothesis for many contracts. [See, for example, Cornell (1977), Dusak (1973), Hansen and Hodrick (1980), and Jackson (1985).] Nevertheless, the martingale assumption is extremely restrictive from a theoretical point of view. The Gaussian price process assumption,

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which unfortunately allows negative prices with nonzero probability, leads to a myopic hedging problem at each time: an agent is hedging only local changes in wealth. In these cases, the optimal hedges are therefore the same as the corresponding static hedges, as in Anderson and Danthine (1981). This is not the case with the log-normal price process, which we also examine in special cases.

Our work was completed independently of the paper by Karp (1986) which has one of our results in a discrete-time approximation sense. Svensson (1988) has complimentary results on a related problem and surveys the literature on this topic. Our paper is restricted to explicit solutions to dynamic hedging problems for which no published proofs were available. Most other papers on dynamic hedging, such as Ho (1984), Breeden (1984), and Adler and DeTemple (1988), instead characterize optimal hedging policies in terms of the derivatives of the value function assumed to solve the Bellman Equation for optimal control.

The paper is organized as follows. The next section outlines the general model. Section 3 presents five special cases and their solutions. Section 4 exploits the solution in the Gaussian-exponential case to demonstrate equilibria in dynamic futures markets in closed form. Section 5 presents proofs.

2. The basic model

We consider a single agent choosing a futures trading strategy to maximize expected utility of wealth at a future time T, in the following setup.

(A) Let $B=(B^1,\ldots,B^N)$ denote a Standard Brownian Motion in $\mathbb{R}^N$ which is a martingale with respect to the agent's filtered probability space.[1] Throughout, probabilistic statements are in the context of this filtered probability space. For technical convenience, let L denote the space of predictable square-integrable processes. That is,

$$L \equiv \left\{\text{predictable } v:[0,T]\times\Omega\to\mathbb{R} \mid \mathrm{E}\left[\int_0^t v_s^2\,\mathrm{d}s\right]<\infty,\ t\in[0,T]\right\},$$

where $T\in\mathbb{R}_+$ is a time, Ω is the state space, and predictable means measurable with respect to the σ-algebra generated by left-continuous processes adapted to the agent's filtration (or, roughly speaking, that v_t depends only on information available up to time t).

[1] The reader is referred to Krylov (1988) for definitions that we do not provide.

(B) There exist M assets to be hedged. The value of these assets is described by an M-dimensional Ito process S, with the stochastic differential representation

$$\mathrm{d}S_t = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t, \tag{1}$$

where μ is M-dimensional, σ is $(M \times N)$-dimensional, and $\mu^m \in L$ and $\sigma^{mn} \in L$ for all m and n (which assures that the Ito process S is well defined).

(C) There are K futures contracts available for trade. The futures prices are given by a K-dimensional Ito process F with the stochastic differential representation

$$\mathrm{d}F_t = m_t\,\mathrm{d}t + v_t\,\mathrm{d}B_t, \tag{2}$$

where $m^k \in L$ and $v^{kn} \in L$ for all k and n. A futures position is taken by marking to market a margin account according to a K-dimensional process $\theta = (\theta^1, \dots, \theta^K)$, with the property that $\theta^\top m$ as well as each element of $\theta^\top v$ is in L (where $^\top$ indicates transpose). The space Θ of all such futures position strategies is then described by

$$\Theta = \left\{ \theta \,|\, \theta^\top m \in L \text{ and } \theta^\top v^n \in L, \forall n \right\}.$$

At time t, the position θ_t in the K contracts is credited with any gains or losses incurred by futures price changes, the credits (or debits) are added to the agent's margin account, and the margin account's current value, denoted X_t^θ, is credited with interest at the constant continuously compounding rate $r \geq 0$. We assume that losses bringing the account to a negative level are covered by borrowing at the same interest rate, and ignore transactions costs and other institutional features. In a continuous-time model, the margin account then has the form

$$X_t^\theta = \int_0^t \mathrm{e}^{r(t-s)}\,\theta_s\,\mathrm{d}F_s, \tag{3}$$

indicating that the 'increment' $\theta_s\,\mathrm{d}F_s$ to the margin account at time s is re-invested at the rate r, implying a corresponding increment of $\mathrm{e}^{r(t-s)}\theta_s\,\mathrm{d}F_s$ to the margin account by time t. It is useful for dynamic programming purposes to apply Ito's Lemma in order to obtain the equivalent stochastic differential representation

$$\mathrm{d}X_t^\theta = \left(rX_t^\theta + \theta^\top m_t\right)\mathrm{d}t + \theta^\top v_t\,\mathrm{d}B_t.$$

(D) The agent in question is committed to receiving the value at time T of a position in these assets represented by a fixed portfolio $\pi \in \mathbb{R}^M$, leaving the terminal value $\pi^\top S_T$. Given a futures position strategy θ, the total wealth of the agent at time T is then W_T^θ, where W^θ is the Ito process having the stochastic differential representation

$$\mathrm{d}W_t^\theta = \pi^\top \,\mathrm{d}S_t + \mathrm{d}X_t^\theta. \tag{4}$$

(E) Preferences of the agent over wealth at time T are given by a strictly concave function $U: \mathbb{R} \to \mathbb{R}$, for a von Neumann–Morgenstern utility representation $\mathrm{E}[U(\cdot)]$. This leaves the problem

$$\max_{\theta \in \Theta} \mathrm{E}\left[U\left(W_T^\theta\right)\right]. \tag{5}$$

A futures position strategy θ is defined to be optimal if it solves (5).

3. Cases and solutions

We will delineate special cases of the problem defined in the previous section, along with their solutions. Proofs appear in the final section. The proofs use several different methods, verification of the necessity and sufficiency of the Bellman Equation, direct calculation, and the theory of second-order stochastic dominance.

Case 1: Gaussian prices – Martingales futures – Smooth utility

Our assumption here is that the processes S and F are Gaussian. That is, μ, σ, m, and v are deterministic[2] processes which, for technical convenience, are bounded, with $v_t v_t^\top$ nonsingular for all t. (The family of solutions is also easily derived when $v_t v_t^\top$ is singular.) For this case, we also assume that $m = 0$, the martingale futures price hypothesis discussed in the introduction. Finally, we assume that U is 'smooth', meaning monotonic, twice continuously differentiable, satisfying a Lipschitz condition, with U' and U'' each satisfying a (linear) growth condition. [The function U satisfies a Lipschitz condition if there exists a constant $k \in \mathbb{R}$ such that $|U(w) - U(w')| \leq k|w - w'|$ for all real numbers w and w'. A function $f: \mathbb{R} \to \mathbb{R}$ satisfies a (linear) growth condition if there exists a constant $k \in \mathbb{R}$ such that $|f(x)| \leq k(1 + |x|)$ for all $x \in \mathbb{R}$.] We have our first result.

[2] That is, for example, there exists a measurable function $f: [0, T] \to \mathbb{R}^M$ such that $\mu_t = f(t)$.

Solution 1. Under the assumptions of case 1, the optimal futures position strategy is θ^*, where

$$\theta_t^* = -\mathrm{e}^{-r(T-t)}\left(v_t v_t^\top\right)^{-1} v_t \sigma_t^\top \pi. \tag{6}$$

In this case, the solution does not depend on μ or U. Since $m = 0$, the demand for futures is based only on the hedge they provide. That is, futures are only used to control 'noise' in the portfolio process. The optimal manner of doing so depends solely on the structure of the 'noise' in the price processes, not on the structure of utility (given that it is concave), nor on the drift of the assets' price processes.

Case 2: Gaussian prices – Exponential utility

Our assumption in case 2 is that prices are Gaussian, in the sense of case 1, and that $U(w) = -\mathrm{e}^{-\gamma w}$, where $\gamma > 0$ is a constant measure of risk aversion.

Solution 2. Under the assumptions of case 2, the optimal futures position strategy is θ^*, where

$$\theta_t^* = -\mathrm{e}^{-r(T-t)}\left(v_t v_t^\top\right)^{-1}\left(v_t \sigma_t^\top \pi - m_t/\gamma\right). \tag{7}$$

This is completely analogous with the static hedge, as shown for example, in Bray (1981). One can also write out an obvious analogue in the discrete-time version of case 2.

Case 3: Martingale prices – Mean–variance preferences

This case is of limited theoretical interest, since we make two unrealistic assumptions. First, prices are martingales ($\mu = m = 0$). Second, the agent's utility is mean–variance, meaning U is quadratic. Under these two assumptions, problem (5) is equivalent to the problem

$$\min_{\theta \in \Theta} \operatorname{var}\left(W_T^\theta\right), \tag{8}$$

where $\operatorname{var}(\cdot)$ denotes variance. Again for technical convenience, we assume, for all t, that $v_t v_t^\top$ is nonsingular.

Solution 3. Under the assumptions of case 3, the optimal futures position strategy θ^* is given by

$$\theta_t^* = -\mathrm{e}^{-r(T-t)}\left(v_t v_t^\top\right)^{-1} v_t \sigma_t^\top \pi.$$

Case 4: Log-normal asset prices – Martingale futures prices – Mean–variance utility

In the previous special cases, the solutions are directly comparable to analogous solutions in the static and discrete-time cases, as in the results of Anderson and Danthine (1981, 1983a, b). Now, however, we assume that asset price increments are lognormal. Taking the mth asset for example, this means that there exist deterministic processes[3] g^m and h^m such that

$$\mathrm{d}S_t^m = S_t^m g_t^m \,\mathrm{d}t + S_t^m h_t^m \,\mathrm{d}B_t. \tag{9}$$

[A special case is geometric Brownian Motion.] Furthermore, for this case, futures prices are martingales ($m = 0$), and utility is mean–variance, in the sense of case 3. In this case, the continuous-time solution is not obtained by a simple analogy from the discrete-time case, nor is this discrete-time solution as convenient to represent as the continuous-time solution.

Solution 4. Under the assumptions of case 4, the optimal futures position strategy is θ^*, where

$$\theta_t^* = -\mathrm{e}^{-r(T-t)}\left(v_t v_t^{\top}\right)^{-1} v_t H_t^{\top} \pi, \tag{10}$$

where H_t is the $M \times N$ matrix whose mth row is $h_t^m S_t^m \exp[\int_t^T g_s^m - \frac{1}{2} h_s^{m\top} h_s^m \,\mathrm{d}s]$.

For the case of $M = K = 1$, with $\mathrm{d}S_t = S_t \bar{g}\,\mathrm{d}t + S_t \bar{h}\,\mathrm{d}B_t$ and $\mathrm{d}F_t = F_t \bar{f}\,\mathrm{d}B_t$, for constant $\bar{f}$, $\bar{g}$, and $\bar{h}$, we then have the simple hedging calculation:

$$\theta_t^* = -\exp\left(-\left[r - \bar{g} + \tfrac{1}{2}\bar{h}^{\top}\bar{h}\right](T-t)\right)\frac{S_t}{F_t}\frac{\bar{f}\cdot\bar{h}}{\bar{f}\cdot\bar{f}}\pi. \tag{11}$$

In subsequent work, Duffie and Richardson (1989) have extended the solution for this case to that for nonmartingale futures prices, using a different solution technique.

Case 5: Log-normal asset prices – Delivery basis risk only

We take the case of log-normal asset price increments, as given by (9), and assume that the futures contracts are for delivery of the same respective assets at some time $\tau \geq T$. Of course, at delivery, $F_\tau = S_\tau$. For case 5, we assume the so-called expectations hypothesis: at any time t, we have $F_t = \mathrm{E}_t[S_\tau]$, where E_t

[3] As earlier, $g^m : [0, T] \to \mathbb{R}$ and $h^m : [0, T] \to \mathbb{R}^N$ are bounded measurable functions.

denotes conditional expectation[4] at time t. No (additional) utility assumptions are invoked.

Solution 5. Under the assumptions of case 5, the optimal futures position strategy is θ^*, where

$$\theta_t^* = -e^{-r(T-t)} G_t \pi, \tag{12}$$

where G_t is the $M \times M$ diagonal matrix with mth *diagonal element* $\exp[\int_t^T g_s^m \, ds]$.

4. Equilibrium in the Gaussian model

Consider an economy with a finite number, I, of agents. Let p_{it} be agent i's spot commitment at time t (where $p_i : [0, T] \to \mathbb{R}^M$ is a bounded measurable function) and let $U_i(w) = -e^{-\gamma_i w}$ represent the agent's Von Neumann–Morgenstern utility for terminal wealth. Then, by an easy extension of case 2, the agent's optimal hedging strategy at time t, assuming $r = 0$, is[5]

$$\theta_{it} = -\left(v_t v_t^\top\right)^{-1}\left[v_t \sigma_t^\top p_{it} - m_t/\gamma_i\right].$$

Market clearing, $\sum_{i=1}^I \theta_{it} = 0$, implies that

$$m_t = \frac{v_t \sigma_t^\top \sum_{i=1}^{I} p_{it}}{\sum_{i=1}^{I} (1/\gamma_i)}. \tag{13}$$

One can view a futures contract as specified exogenously by the process v_t, and treat m_t as chosen so that markets clear. We have the following properties in equilibrium. First, m_t is proportional to the net spot position $\sum_{i=1}^I p_{it}$. Second, m_t is proportional to the covariance between the futures contracts and spot prices. A high covariance term indicates that the futures contract provides a good hedge, increasing the demand for hedging. The higher expected return to the futures contract (which is of a sign that attracts investors to positions opposite the excess demand for hedges) offsets the excess demand. Finally, m_t is proportional to the risk aversion of investors. Higher

[4] Formally, $F_t = E[S_T | \mathcal{F}_t]$, where $\mathcal{F}_t$ denotes the σ-algebra generated by $\{B_s : 0 \le s \le t\}$, representing information available at time t.

[5] The case of $r \neq 0$ can also be solved. The solution is the obvious extension of what we obtain if T is the same for all agents. If T differs across agents the solution is still straightforward to compute.

levels of risk aversion (lower γ_i's) correspond to a higher (in absolute value) m_t.

Conditions under which the expectations hypothesis holds follow quite naturally from (13). Any of the following is sufficient: (i) $\sum_{i=1}^{I} p_{it} = 0$, (ii) $v_t \sigma_t^{\top} = 0$, or (iii) $\gamma_i = 0$ for some i. In case (i) there is no excess demand for hedging. This means that agents can costlessly insure themselves since there exists someone who wishes to take an opposite position[6] and so there is no need to attract 'speculators' into the market. [For a similar result see Anderson and Danthine (1983a).] In case (ii) the futures provide no hedge, while in case (iii) there is a risk-neutral agent who drives out any expected returns.

We remark that, substituting (13) into the expression for θ_{it}, equilibrium implies that

$$\theta_{it} = -\left(v_t v_t^{\top}\right)^{-1} v_t \sigma_t^{\top} \left[p_{it} - \frac{\sum_{i=1}^{I} p_{it}}{\gamma_i \sum_{i=1}^{I} (1/\gamma_i)} \right], \tag{14}$$

which leaves a formula for open interest as a function of exogenous parameters.

5. Proofs

Case 1 is proved using Bellman's Principle. Cases 2, 3, and 4 are solved by direct calculation. Case 5 is solved by observing that the theory of second-order stochastic dominance applies in a simple way.

Case 1

Let $W^{\theta t}$ define the wealth process that would obtain starting at time t and with futures strategy θ, translating time parameters back t time units to time 0, or

$$\mathrm{d}W_s^{\theta t} = a_{t+s}\,\mathrm{d}s + b_{t+s}\,\mathrm{d}B_s,$$

where $a_{t+s} = rX_s^{\theta t} + \pi^{\top}\mu_{t+s} + \theta_{t+s}^{\top} m_{t+s}$ and $b_{t+s} = \pi^{\top}\sigma_{t+s} + \theta_{t+s}^{\top} v_{t+s}$ and $X^{\theta t}$

[6] Remark that in solution 2 the agent's risk aversion enters only through the term linked to the expected return m. The agent's 'hedging demand' depends only on the spot position and covariance structure of the prices and is independent of risk aversion. The intuition for this is the same as that given following solution 1.

is the t-translate of the process X defined by

$$\mathrm{d}X_s^{\theta t} = \alpha_{t+s}\mathrm{d}s + \beta_{t+s}\mathrm{d}B_s,$$

where $\alpha_{t+s} = rX_s^{\theta t} + \theta_{t+s}^{\top}m_{t+s}$ and $\beta_{t+s} = \theta_{t+s}^{\top}v_{t+s}$. We define the value function $V:\mathbb{R}^2 \times [0,T] \to \mathbb{R}$ by

$$V(w,x,t) = \sup_{\theta \in \Theta} \mathrm{E}\left[U\left(W_{T-t}^{\theta t}\right)\right],$$

where $W_0^{\theta t} = w$ and $X_0^{\theta t} = x$. We then have the Bellman equation:

$$\sup_{\theta_t \in \mathbb{R}^K} \left[\frac{\partial V}{\partial t} + \frac{\partial V}{\partial w}a_t + \frac{\partial V}{\partial x}\alpha_t \right.$$

$$\left. + \tfrac{1}{2}\operatorname{tr}\left(\frac{\partial^2 V}{\partial w^2}b_t^{\top}b_t + 2\frac{\partial^2 V}{\partial w \partial x}b_t^{\top}\beta_t + \frac{\partial^2 V}{\partial x^2}\beta_t^{\top}\beta_t \right)\right] = 0, \tag{15}$$

where $\operatorname{tr}(\cdot)$ indicates trace, as a necessary and sufficient condition for optimality under our assumptions [provided that V satisfies $V(T,w) = U(w)$ and a growth condition]. The reader is referred to Krylov (1980, especially theorem 5.3.14) for a rigorous derivation of (15).

We now demonstrate that the futures position θ^* defined in solution 1 induces a function which satisfies (15) and is the value function. We have assumed that the utility function $U:\mathbb{R} \to \mathbb{R}$ is twice continuously differentiable and satisfies a Lipschitz condition. We have also assumed that U' and U'' satisfy a growth condition. Let θ^* be defined as in solution 1, and let $\hat{V}:\mathbb{R}^2 \times [0,T] \to \mathbb{R}$ be defined by

$$\hat{V}(w,x,t) = \mathrm{E}\left[U\left(W_{T-t}^{\theta^* t}\right)\right],$$

where $W_0^{\theta^* t} = w$ and $X_0^{\theta^* t} = x$. The drift function a_t and the diffusion function b_t are then bounded measurable functions of time. It follows from standard partial differential equations theory [for example, Krylov (1980, theorem 2.9.10)] that $\hat{V}$ is the solution of the partial differential equation:

$$\frac{\partial \hat{V}}{\partial t} + \frac{\partial \hat{V}}{\partial w}a_t + \frac{\partial \hat{V}}{\partial x}\alpha_t$$

$$+ \tfrac{1}{2}\operatorname{tr}\left(\frac{\partial^2 \hat{V}}{\partial w^2}b_t^{\top}b_t + 2\frac{\partial^2 \hat{V}}{\partial w \partial x}b_t^{\top}\beta_t + \frac{\partial^2 \hat{V}}{\partial x^2}\beta_t^{\top}\beta_t \right) = 0, \tag{16}$$

where a_t and b_t are evaluated for strategy θ^* with boundary condition $\hat{V}(w, x, T) = U(w)$, $(w, x) \in \mathbb{R}^2$, and furthermore, that $\hat{V}$ satisfies growth and Lipschitz conditions.

We differentiate inside the expectations operator to observe that

$$\frac{\partial^2 \hat{V}}{\partial w^2}[\exp(r[T-t]) - 1]^2 = \frac{\partial^2 \hat{V}}{\partial w \partial x}[\exp(r[T-t]) - 1] = \frac{\partial^2 \hat{V}}{\partial x^2}.$$

An argument justifying differentiation inside the expectations operator appears in the appendix. Since $\partial^2 \hat{V}/\partial w^2 = \mathrm{E}[\partial^2 U/\partial w^2] < 0$ (again, we differentiate inside the expectations sign by an argument presented in the appendix) and $m_t = 0$, it follows that $\theta_t \in \mathbb{R}^K$ maximizes

$$\frac{\partial \hat{V}}{\partial t} + \frac{\partial \hat{V}}{\partial w} a_t + \frac{\partial \hat{V}}{\partial x}\alpha_t + \tfrac{1}{2}\operatorname{tr}\left(\frac{\partial^2 \hat{V}}{\partial w^2} b_t^\top b_t + 2\frac{\partial^2 \hat{V}}{\partial w \partial x} b_t^\top \beta_t + \frac{\partial^2 \hat{V}}{\partial x^2}\beta_t^\top \beta_t \right),$$

if and only if it minimizes

$$\operatorname{tr}\left(b_t^\top b_t + 2 b_t^\top \beta_t [\exp(r[T-t]) - 1] + \beta_t^\top \beta_t [\exp(r[T-t]) - 1]^2 \right). \tag{17}$$

Therefore, since θ_t^* minimizes (17), it follows from (16) that $\hat{V}$ satisfies (15) and we can apply stochastic control verification theorem [for example, Krylov (1980, theorem 5.3.14)]. Thus $\hat{V} = V$ and θ^* is the optimal control.

Case 2

Let $Y_s^{\theta t} = \exp(-\gamma W_s^{\theta t})$. We apply Ito's Lemma to find that

$$Y_{T-t}^{\theta t} = Y_0^{\theta t} + \int_0^{T-t} Y_s^{\theta t}\left[\frac{\gamma^2}{2}\operatorname{tr}\left(b_{t+s}^\top b_{t+s}\right) - \gamma a_{t+s}\right] \mathrm{d}s$$

$$+ \int_0^{T-t} Y_s^{\theta t}(-\gamma b_{t+s})\,\mathrm{d}B_s. \tag{18}$$

Thus,

$$\mathrm{E}\left[Y_{T-t}^{\theta t}\right] = Y_0^{\theta t} + \mathrm{E}\left[\int_0^{T-t} Y_s^{\theta t}\left[\frac{\gamma^2}{2}\operatorname{tr}\left(b_{t+s}^\top b_{t+s}\right) - \gamma a_{t+s}\right]\mathrm{d}s\right].$$

Since θ^* minimizes $(\gamma^2/2)\operatorname{tr}(b_{t+s}^\top b_{t+s}) - \gamma a_{t+s}$ at each s, it follows from the structure of $Y^{\theta t}$ given in (18) that θ^* minimizes $\mathrm{E}[\int_0^{T-t} Y_s^{\theta t}[(\gamma^2/2)\operatorname{tr}(b_{t+s}^\top b_{t+s}) - \gamma a_{t+s}]\,\mathrm{d}s]$. Hence, θ^* maximizes $\mathrm{E}[-\exp(-\gamma W_{T-t}^{\theta t})] = -\mathrm{E}[Y_{T-t}^{\theta t}]$.

Case 3

As described before, in case 3 the agent's expected utility maximization problem is equivalent to (8). For a given futures strategy θ, since $\mu = m = 0$, the definition of variance implies that

$$\operatorname{var}(W_T^\theta) = \mathrm{E}\left[\left(\int_0^T (\theta_t^\top v_t + \pi_t^\top \sigma_t)\,\mathrm{d}B_t\right)^2\right],$$

which can also be written [referring, for example, to Oksendal (1980, ch. 3)] in the form

$$\mathrm{E}\left[\int_0^T (\theta_t^\top v_t + \pi_t^\top \sigma_t)(v_t^\top \theta_t + \sigma_t^\top \pi_t)\,\mathrm{d}t\right].$$

This expression is minimized pointwise by

$$\theta_t = -(v_t v_t^\top)^{-1} v_t \sigma_t^\top \pi_t.$$

In order to account for interest earned on margin, from eq. (3) we replace v_t with $\mathrm{e}^{r(T-t)} v_t$, which provides expression (10).

Case 4

We merely observe that, by Ito's Lemma,

$$\log(S_t^m) = \log(S_0^m) + \int_0^t [g_s^m - \tfrac{1}{2} h_s^{m\top} h_s^m]\,\mathrm{d}s + \int_0^t h_s^m\,\mathrm{d}B_s.$$

We can therefore write

$$S_t^m = \exp\left(\int_0^t [g_s^m - \tfrac{1}{2} h_s^{m\top} h_s^m]\,\mathrm{d}s\right)\hat{S}_t^m,$$

where $\hat{S}^m$ is the martingale defined by

$$\mathrm{d}\hat{S}_t^m = \hat{S}_t^m h_t^m\,\mathrm{d}B_t \quad \text{and} \quad \hat{S}_0^m = S_0^m.$$

We can then apply case 3, after converting to the asset commitment $\hat{\pi}$, defined by

$$\hat{\pi}_m = \pi_m \exp\left[\int_0^T [g_s^m - \tfrac{1}{2}h_s^{m\top}h_s^m]\,\mathrm{d}s,\right.$$

and the diffusion term $\hat{\sigma}$, defined by

$$\hat{\sigma}_t^m = \exp\left(-\int_0^t [g_s^m - \tfrac{1}{2}h_s^{m\top}h_s^m]\,\mathrm{d}s\right)S_t^m h_t^m.$$

Case 5

This case is particularly easy. Under the expectations hypothesis and since S is log-normal, the futures price process is described by

$$F_t^m = \mathrm{E}_t[S_T^m] = \exp\left(\int_t^T g_s^m\,\mathrm{d}s\right)S_t^m.$$

Thus,

$$\mathrm{d}F_t^m = \exp\left(\int_t^T g_s^m\,\mathrm{d}s\right)\mathrm{d}S_t^m - g_t^m \exp\left(\int_t^T g_s^m\,\mathrm{d}s\right)S_t^m\,\mathrm{d}t$$

or

$$\mathrm{d}F_t^m = \exp\left(\int_t^T g_s^m\,\mathrm{d}s\right)S_t^m h_t^m\,\mathrm{d}B_t.$$

Given the expression for $\mathrm{d}F_t$ we note that $\mathrm{var}(W_T^{\theta *})$, the variance of the terminal wealth induced by the trading strategy proposed in (12), is zero; all risk is eliminated. Hence, F is a martingale and it follows that any other futures position strategy induces a terminal wealth which is a mean-preserving spread of $W_T^{\theta *}$. By the theory of second-order stochastic dominance [Rothschild and Stiglitz (1970)] and the concavity of U, the solution is verified.

Appendix

We now discuss sufficient conditions for differentiation inside the expectations operator, which we applied in the proof of case 1. Define an open interval $I = (w_1, w_2)$, $I \subset \mathbb{R}$. The first two conditions required by Lang (1969, p. 375) are: $\mathrm{E}[|U(W_T^{\theta *})|] < \infty$ and $\mathrm{E}[|U'(W_T^{\theta *})|] < \infty$ for all $W_0 \in I$. Since $\mathrm{E}[|W_T^{\theta *}|] < \infty$, these conditions follow from the growth conditions satisfied by

U and U'. The third condition is that there exists a real-valued function h with $\mathrm{E}[|h(W_T^{\theta*})|] < \infty$ such that $|U'(W_T^{\theta*})| \leq |h(W_T^{\theta*})|$ for all $W_0 \in I$. This condition is met when we define h by $h(x) = U'(w_2 + x)$. Lemma 2 of Lang (1969, p. 375) then allows us to differentiate inside the expectations operator for any $W_0 \in I$. Since we can apply this argument for any open interval $I \subset \mathbb{R}$, it holds for all $W_0 \in \mathbb{R}$.

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