INTERTEMPORAL ASSET PRICING THEORY

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Contents

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Abstract

This is a survey of the basic theoretical foundations of intertemporal asset pricing theory. The broader theory is first reviewed in a simple discrete-time setting, emphasizing the key role of state prices. The existence of state prices is equivalent to the absence of arbitrage. State prices, which can be obtained from optimizing investors' marginal rates of substitution, can be used to price contingent claims. In equilibrium, under locally quadratic utility, this leads to Breeden's consumption-based capital asset pricing model. American options call for special handling. After extending the basic modeling approach to continuous-time settings, we turn to such applications as the dynamics of the term structure of interest rates, futures and forwards, option pricing under jumps and stochastic volatility, and the market valuation of corporate securities. The pricing of defaultable corporate debt is treated from a direct analysis of the incentives or ability of the firm to pay, and also by standard reduced-form methods that take as given an intensity process for default. This survey does not consider asymmetric information, and assumes price-taking behavior and the absence of transactions costs and many other market imperfections.

Keywords

1. Introduction

This is a survey of "classical" intertemporal asset pricing theory. A central objective of this theory is to reduce asset-pricing problems to the identification of "state prices", a notion of Arrow (1953) from which any security has an implied value as the weighted sum of its future cash flows, state by state, time by time, with weights given by the associated state prices. Such state prices may be viewed as the marginal rates of substitution among state-time consumption opportunities, for any unconstrained investor, with respect to a numeraire good. Under many types of market imperfections, state prices may not exist, or may be of relatively less use or meaning. While market imperfections constitute an important thrust of recent advances in asset pricing theory, they will play a limited role in this survey, given the limitations of space and the priority that should be accorded to first principles based on perfect markets.

Section 2 of this survey provides the conceptual foundations of the broader theory in a simple discrete-time setting. After extending the basic modeling approach to a continuous-time setting in Section 3, we turn in Section 4 to term-structure modeling, in Section 5 to derivative pricing, and in Section 6 to corporate securities.

The theory of optimal portfolio and consumption choice is closely linked to the theory of asset pricing, for example through the relationship between state prices and marginal rates of substitution at optimality. While this connection is emphasized, for example in Sections 2.3–2.4 and 3.12–3.13, the theory of optimal portfolio and consumption choice, particularly in dynamic incomplete-markets settings, has become so extensive as to defy a proper summary in the context of a reasonably sized survey of asset-pricing theory. The interested reader is especially directed to the treatments of Karatzas and Shreve (1998) and Schroder and Skiadas (1999, 2002).

For ease of reference, as there is at most one theorem per sub-section, we refer to a theorem by its subsection number, and likewise for lemmas and propositions. For example, the unique proposition of Section 2.9 is called "Proposition 2.9".

2. Basic theory

Radner (1967, 1972) originated our standard approach to a dynamic equilibrium of "plans, prices, and expectations," extending the static approach of Arrow (1953) and Debreu (1953).[1] After formulating this standard model, this section provides the equivalence of no arbitrage and state prices, and shows how state prices may be derived from investors' marginal rates of substitution among state-time consumption opportunities. Given state prices, we examine the pricing of derivative securities, such

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[1] The model of Debreu (1953) appears in Chapter 7 of Debreu (1959). For more details in a finance setting, see Dothan (1990). The monograph by Magill and Quinzii (1996) is a comprehensive survey of the theory of general equilibrium in a setting such as this.

as European and American options, whose payoffs can be replicated by trading the underlying primitive securities.

2.1. Setup

We begin for simplicity with a setting in which uncertainty is modeled as some finite set Ω of states, with associated probabilities. We fix a set $\mathcal{F}$ of events, called a *tribe*, also known as a σ-algebra, which is the collection of subsets of Ω that can be assigned a probability. The usual rules of probability apply.[2] We let $P(A)$ denote the probability of an event A.

There are $T + 1$ dates: $0, 1, \ldots, T$. At each of these, a tribe $\mathcal{F}_t \subset \mathcal{F}$ is the set of events corresponding to the information available at time t. Any event in $\mathcal{F}_t$ is known at time t to be true or false. We adopt the usual convention that $\mathcal{F}_t \subset \mathcal{F}_s$ whenever $t \leqslant s$, meaning that events are never "forgotten". For simplicity, we also take it that events in $\mathcal{F}_0$ have probability 0 or 1, meaning roughly that there is no information at time $t = 0$. Taken altogether, the *filtration* $\mathbb{F} = \{\mathcal{F}_0, \ldots, \mathcal{F}_T\}$, sometimes called an information structure, represents how information is revealed through time. For any random variable Y, we let $E_t(Y) = E(Y \mid \mathcal{F}_t)$ denote the conditional expectation of Y given $\mathcal{F}_t$. In order to simplify things, for any two random variables Y and Z, we always write "$Y = Z$" if the probability that $Y \neq Z$ is zero.

An *adapted process* is a sequence $X = \{X_0, \ldots, X_T\}$ such that, for each t, X_t is a random variable with respect to $(\Omega, \mathcal{F}_t)$. Informally, this means that X_t is observable at time t. An adapted process X is a *martingale* if, for any times t and $s > t$, we have $E_t(X_s) = X_t$.

A security is a claim to an adapted *dividend process*, say δ, with δ_t denoting the dividend paid by the security at time t. Each security has an adapted *security-price process* S, so that S_t is the price of the security, *ex dividend*, at time t. That is, at each time t, the security pays its dividend δ_t and is then available for trade at the price S_t. This convention implies that δ_0 plays no role in determining ex-dividend prices. The *cum-dividend* security price at time t is $S_t + \delta_t$.

We suppose that there are N securities defined by an $\mathbb{R}^N$-valued adapted dividend process $\delta = (\delta^{(1)}, \ldots, \delta^{(N)})$. These securities have some adapted price process $S = (S^{(1)}, \ldots, S^{(N)})$. A *trading strategy* is an adapted process θ in $\mathbb{R}^N$. Here, θ_t represents the portfolio held after trading at time t. The dividend process δ^θ *generated* by a trading strategy θ is defined by

$$\delta_t^\theta = \theta_{t-1} \cdot (S_t + \delta_t) - \theta_t \cdot S_t, \tag{1}$$

with "θ_{-1}" taken to be zero by convention.

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[2] The triple $(\Omega, \mathcal{F}, P)$ is a probability space, as defined for example by Jacod and Protter (2000).

2.2. Arbitrage, state prices, and martingales

Given a dividend–price pair (δ, S) for N securities, a trading strategy θ is an *arbitrage* if $\delta^\theta > 0$ (that is, if $\delta^\theta \geqslant 0$ and $\delta^\theta \neq 0$). An arbitrage is thus a trading strategy that costs nothing to form, never generates losses, and, with positive probability, will produce strictly positive gains at some time. One of the precepts of modern asset pricing theory is a notion of efficient markets under which there is no arbitrage. This is a reasonable axiom, for in the presence of an arbitrage, any rational investor who prefers to increase his dividends would undertake such arbitrages without limit, so markets could not be in equilibrium, in a sense that we shall see more formally later in this section. We will first explore the implications of no arbitrage for the representation of security prices in terms of "state prices", the first step toward which is made with the following result.

Proposition. *There is no arbitrage if and only if there is a strictly positive adapted process π such that, for any trading strategy θ,*

$$E\left(\sum_{t=0}^{T} \pi_t \delta_t^\theta\right) = 0.$$

Proof: Let Θ denote the space of trading strategies. For any θ and φ in Θ and scalars a and b, we have $a\delta^\theta + b\delta^\varphi = \delta^{a\theta + b\varphi}$. Thus, the *marketed subspace* $M = \{\delta^\theta \colon \theta \in \Theta\}$ of dividend processes generated by trading strategies is a linear subspace of the space L of adapted processes.

Let $L_+ = \{c \in L \colon c \geqslant 0\}$. There is no arbitrage if and only if the cone L_+ and the marketed subspace M intersect precisely at zero. Suppose there is no arbitrage. The Separating Hyperplane Theorem, in a version for closed convex cones that is sometimes called Stiemke's Lemma (see Appendix B of Duffie (2001)) implies the existence of a nonzero linear functional F such that $F(x) < F(y)$ for each x in M and each nonzero y in L_+. Since M is a linear subspace, this implies that $F(x) = 0$ for each x in M, and thus that $F(y) > 0$ for each nonzero y in L_+. This implies that F is strictly increasing. By the Riesz representation theorem, for any such linear function F there is a unique adapted process π, called the Riesz representation of F, such that

$$F(x) = E\left(\sum_{t=0}^{T} \pi_t x_t\right), \qquad x \in L.$$

As F is strictly increasing, π is strictly positive, that is, $P(\pi_t > 0) = 1$ for all t.

The converse follows from the fact that if $\delta^\theta > 0$ and π is a strictly positive process, then $E(\sum_{t=0}^{T} \pi_t \delta_t^\theta) > 0$. $\square$

For convenience, we call any strictly positive adapted process a *deflator*. A deflator π is a *state-price density* if, for all t,

$$S_t = \frac{1}{\pi_t} E_t \left(\sum_{j=t+1}^{T} \pi_j \delta_j \right). \tag{2}$$

A state-price density is sometimes called a *state-price deflator*, a *pricing kernel*, or a *marginal-rate-of-substitution process*.

For $t = T$, the right-hand side of Equation (2) is zero, so $S_T = 0$ whenever there is a state-price density. It can be shown as an exercise that a deflator π is a state-price density if and only if, for any trading strategy θ,

$$\theta_t \cdot S_t = \frac{1}{\pi_t} E_t \left(\sum_{j=t+1}^{T} \pi_j \delta_j^{\theta} \right), \qquad t < T, \tag{3}$$

meaning roughly that the market value of a trading strategy is, at any time, the state-price discounted expected future dividends generated by the strategy.

The *gain process* G for (δ, S) is defined by $G_t = S_t + \sum_{j=1}^{t} \delta_j$, the price plus accumulated dividend. Given a deflator γ, the *deflated gain process* G^{γ} is defined by $G_t^{\gamma} = \gamma_t S_t + \sum_{j=1}^{t} \gamma_j \delta_j$. We can think of deflation as a change of numeraire.

Theorem. *The dividend–price pair (δ, S) admits no arbitrage if and only if there is a state-price density. A deflator π is a state-price density if and only if $S_T = 0$ and the state-price-deflated gain process G^{π} is a martingale.*

Proof: It can be shown as an easy exercise that a deflator π is a state-price density if and only if $S_T = 0$ and the state-price-deflated gain process G^{π} is a martingale.

Suppose there is no arbitrage. Then $S_T = 0$, for otherwise the strategy θ is an arbitrage when defined by $\theta_t = 0$, $t < T$, $\theta_T = -S_T$. By the previous proposition, there is some deflator π such that $E(\sum_{t=0}^{T} \delta_t^{\theta} \pi_t) = 0$ for any strategy θ.

We must prove Equation (2), or equivalently, that G^{π} is a martingale. Doob's Optional Sampling Theorem states that an adapted process X is a martingale if and only if $E(X_\tau) = X_0$ for any stopping time $\tau \leqslant T$. Consider, for an arbitrary security n and an arbitrary stopping time $\tau \leqslant T$, the trading strategy θ defined by $\theta^{(k)} = 0$ for $k \neq n$ and $\theta_t^{(n)} = 1$, $t < \tau$, with $\theta_t^{(n)} = 0$, $t \geqslant \tau$. Since $E(\sum_{t=0}^{T} \pi_t \delta_t^{\theta}) = 0$, we have

$$E \left(-S_0^{(n)} \pi_0 + \sum_{t=1}^{\tau} \pi_t \delta_t^{(n)} + \pi_\tau S_\tau^{(n)} \right) = 0,$$

implying that the π-deflated gain process $G^{n,\pi}$ of security n satisfies $G_0^{n,\pi} = E(G_\tau^{n,\pi})$. Since τ is arbitrary, $G^{n,\pi}$ is a martingale, and since n is arbitrary, G^{π} is a martingale.

This shows that absence of arbitrage implies the existence of a state-price density. The converse is easy. $\square$

The proof is motivated by those of Harrison and Kreps (1979) and Harrison and Pliska (1981) for a similar result to follow in this section regarding the notion of an "equivalent martingale measure". Ross (1987), Prisman (1985), Kabanov and Stricker (2001), and Schachermayer (2001) show the impact of taxes or transactions costs on the state-pricing model.

2.3. Individual agent optimality

We introduce an agent, defined by a strictly increasing[3] utility function U on the set L_+ of nonnegative adapted "consumption" processes, and by an *endowment process* e in L_+. Given a dividend-price process (δ, S), a trading strategy θ leaves the agent with the total consumption process $e + \delta^\theta$. Thus the agent has the *budget-feasible consumption set*

$$C = \{e + \delta^\theta \in L_+: \theta \in \Theta\},$$

and the problem

$$\sup_{c \in C} U(c). \tag{4}$$

The existence of a solution to Problem (4) implies the absence of arbitrage. Conversely, if U is continuous,[4] then the absence of arbitrage implies that there exists a solution to Problem (4). (This follows from the fact that the feasible consumption set C is compact if and only if there there is no arbitrage.)

Assuming that (4) has a strictly positive solution c^* and that U is continuously differentiable at c^*, we can use the first-order conditions for optimality to characterize security prices in terms of the derivatives of the utility function U at c^*. Specifically, for any c in L, the derivative of U at c^* in the direction c is $g'(0)$, where $g(\alpha) = U(c^* + \alpha c)$ for any scalar α sufficiently small in absolute value. That is, $g'(0)$ is the marginal rate of improvement of utility as one moves in the direction c away from c^*. This directional derivative is denoted $\nabla U(c^*; c)$. Because U is continuously differentiable at c^*, the function that maps c to $\nabla U(c^*; c)$ is linear. Since δ^θ is a budget-feasible direction of change for any trading strategy θ, the first-order conditions for optimality of c^* imply that

$$\nabla U(c^*; \delta^\theta) = 0, \qquad \theta \in \Theta.$$

We now have a characterization of a state-price density.

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[3] A function $f: L \to \mathbb{R}$ is strictly increasing if $f(c) > f(b)$ whenever $c > b$.

[4] For purposes of checking continuity or the closedness of sets in L, we will say that c_n converges to c if $E[\sum_{t=0}^{T} |c_n(t) - c(t)|] \to 0$. Then U is continuous if $U(c_n) \to U(c)$ whenever $c_n \to c$.

Proposition. *Suppose that Problem (4) has a strictly positive solution c^* and that U has a strictly positive continuous derivative at c^*. Then there is no arbitrage and a state-price density is given by the Riesz representation π of $\nabla U(c^*)$, defined by*

$$\nabla U(c^*; x) = E\left(\sum_{t=0}^{T} \pi_t x_t\right), \qquad x \in L.$$

The Riesz representation of the utility gradient is also sometimes called the marginal-rates-of-substitution process. Despite our standing assumption that U is strictly increasing, $\nabla U(c^*; \cdot)$ need not in general be strictly increasing, but is so if U is concave.

As an example, suppose U has the *additive* form

$$U(c) = E\left[\sum_{t=0}^{T} u_t(c_t)\right], \qquad c \in L_+, \tag{5}$$

for some $u_t: \mathbb{R}_+ \to \mathbb{R}$, $t \geq 0$. It is an exercise to show that if $\nabla U(c)$ exists, then

$$\nabla U(c; x) = E\left[\sum_{t=0}^{T} u_t'(c_t) x_t\right]. \tag{6}$$

If, for all t, u_t is concave with an unbounded derivative and e is strictly positive, then any solution c^* to Equation (4) is strictly positive.

Corollary. *Suppose U is defined by Equation (5). Under the conditions of the Proposition, for any time $t < T$,*

$$S_t = \frac{1}{u_t'(c_t^*)} E_t \left[u_{t+1}'(c_{t+1}^*)(S_{t+1} + \delta_{t+1}) \right].$$

This result is often called the *stochastic Euler equation*, made famous in a time-homogeneous Markov setting by Lucas (1978). A precursor is due to LeRoy (1973).

2.4. Habit and recursive utilities

The additive utility model is extremely restrictive, and routinely found to be inconsistent with experimental evidence on choice under uncertainty, as for example in Plott (1986). We will illustrate the state pricing associated with some simple extensions of the additive utility model, such as "habit-formation" utility and "recursive utility".

An example of a *habit-formation* utility is some $U: L_+ \to \mathbb{R}$ with

$$U(c) = E\left[\sum_{t=0}^{T} u(c_t, h_t)\right],$$

where $u: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable and, for any t, the "habit" level of consumption is defined by $h_t = \sum_{j=1}^{t} \alpha_j c_{t-j}$ for some $\alpha \in \mathbb{R}_+^T$. For example, we could take $\alpha_j = \gamma^j$ for $\gamma \in (0, 1)$, which gives geometrically declining weights on past consumption. A natural motivation is that the relative desire to consume may be increased if one has become accustomed to high levels of consumption. By applying the chain rule, we can calculate the Riesz representation π of the gradient of U at a strictly positive consumption process c as

$$\pi_t = u_c(c_t, h_t) + E_t\left(\sum_{s>t} u_h(c_s, h_s)\, \alpha_{s-t}\right),$$

where u_c and u_h denote the partial derivatives of u with respect to its first and second arguments, respectively. The habit-formation utility model was developed by Dunn and Singleton (1986) and in continuous time by Ryder and Heal (1973), and has been applied to asset-pricing problems by Constantinides (1990), Sundaresan (1989) and Chapman (1998).

Recursive utility, inspired by Koopmans (1960), Kreps and Porteus (1978) and Selden (1978), was developed for general discrete-time multi-period asset-pricing applications by Epstein and Zin (1989), who take a utility of the form $U(c) = V_0$, where the "utility process" V is defined recursively, backward in time from T, by

$$V_t = F(c_t, \sim V_{t+1} \mid \mathcal{F}_t),$$

where $\sim V_{t+1} \mid \mathcal{F}_t$ denotes the probability distribution of V_{t+1} given $\mathcal{F}_t$, where F is a measurable real-valued function whose first argument is a non-negative real number and whose second argument is a probability distribution, and finally where we take V_{T+1} to be a fixed exogenously specified random variable. One may view V_t as the utility at time t for present and future consumption, noting the dependence on the future consumption stream through the conditional distribution of the following period's utility. As a special case, for example, consider

$$F(x, m) = f(x, E[h(Y_m)]),\tag{7}$$

where f is a function in two real variables, $h(\cdot)$ is a "felicity" function in one variable, and Y_m is any random variable whose probability distribution is m. This special case of the "Kreps–Porteus utility" aggregates the role of the conditional distribution of future consumption through an "expected utility of next period's utility". If h and J

are concave and increasing functions, then U is concave and increasing. If $h(v) = v$ and if $f(x, y) = u(x) + \beta y$ for some $u: \mathbb{R}_+ \to \mathbb{R}$ and constant $\beta > 0$, then (for $V_{T+1} = 0$) we recover the special case of additive utility given by

$$U(c) = E \left[\sum_t \beta^t u(c_t) \right].$$

"Non-expected-utility" aggregation of future consumption utility can be based, for example, upon the local-expected-utility model of Machina (1982) and the betweenness-certainty-equivalent model of Chew (1983, 1989), Dekel (1989) and Gul and Lantto (1990). With recursive utility, as opposed to additive utility, it need not be the case that the degree of risk aversion is completely determined by the elasticity of intertemporal substitution.

For the special case (Equation 7) of expected-utility aggregation, and with differentiability throughout, we have the utility gradient representation

$$\pi_t = f_1\left(c_t, E_t\left[h\left(V_{t+1}\right)\right]\right) \prod_{s<t} f_2\left(c_s, E_s\left[h\left(V_{s+1}\right)\right]\right) E_s\left[h'\left(V_{s+1}\right)\right],$$

where f_i denotes the partial derivative of f with respect to its ith argument.

Recursive utility allows for preference over early or late resolution of uncertainty (which have no impact on additive utility). This is relevant for asset prices, as for example in the context of remarks by Ross (1989), and as shown by Skiadas (1998) and Duffie, Schroder and Skiadas (1997). Grant, Kajii and Polak (2000) have more to say on preferences for the resolution of information.

The equilibrium state-price density associated with recursive utility is computed in a Markovian setting by Kan (1995).[5] For further justification and properties of recursive utility, see Chew and Epstein (1991) and Skiadas (1997, 1998). For further implications for asset pricing, see Epstein (1988, 1992), Epstein and Zin (1999) and Giovannini and Weil (1989).

2.5. Equilibrium and Pareto optimality

Now, we explore the implications of multi-agent equilibrium for state prices. A key objective is to link state prices with important macro-economic variables that are, hopefully, observable, such as total economy-wide consumption.

Suppose there are m agents. Agent i is defined as above by a strictly increasing utility function $U_i: L_+ \to \mathbb{R}$ and an endowment process $e^{(i)}$ in L_+. Given a dividend

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[5] Kan (1993) further explored the utility gradient representation of recursive utility in this setting.

process δ for N securities, an *equilibrium* is a collection $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$, where S is a security-price process and, for each agent i, $\theta^{(i)}$ is a trading strategy solving

$$\sup_{\theta \in \Theta} U_i\left(e^{(i)} + \delta^{\theta}\right),$$

with $\sum_{i=1}^{m} \theta^{(i)} = 0$.

We define markets to be *complete* if, for each process x in L, there is some trading strategy θ with $\delta_t^{\theta} = x_t$, $t \geqslant 1$. Complete markets thus means that any consumption process x can be obtained by investing some amount at time 0 in a trading strategy that, at each future period t, generates the dividend x_t.

The *First Welfare Theorem* is that complete-markets equilibria provide efficient consumption allocations. Specifically, an allocation $(c^{(1)}, \ldots, c^{(m)})$ of consumption processes to the m agents is feasible if $c^{(1)} + \cdots + c^{(m)} \leqslant e^{(1)} + \cdots + e^{(m)}$, and is *Pareto optimal* if there is no feasible allocation $(b^{(1)}, \ldots, b^{(m)})$ such that $U_i(b^{(i)}) \geqslant U_i(c^{(i)})$ for all i, with strict inequality for some i. Any equilibrium $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$ has an associated feasible consumption allocation $(c^{(1)}, \ldots, c^{(m)})$ defined by letting $c^{(i)} - e^{(i)}$ be the dividend process generated by $\theta^{(i)}$.

First Welfare Theorem. *Suppose* $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$ *is an equilibrium and markets are complete. Then the associated consumption allocation is Pareto optimal.*

An easy proof is due to Arrow (1951). Suppose, with the objective of obtaining a contradiction, that $(c^{(1)}, \ldots, c^{(m)})$ is the consumption allocation of a complete-markets equilibrium and that there is a feasible allocation $(b^{(1)}, \ldots, b^{(m)})$ such that $U_i(b^{(i)}) \geqslant U_i(c^{(i)})$ for all i, with strict inequality for some i. Because of equilibrium, there is no arbitrage, and therefore a state-price density π. For any consumption process x, let $\pi \cdot x = E(\sum_t \pi_t x_t)$. We have $\pi \cdot b^{(i)} \geqslant \pi \cdot c^{(i)}$, for otherwise, given complete markets, the utility of $c^{(i)}$ can be increased strictly by some feasible trading strategy generating $b^{(i)} - e^{(i)}$. Similarly, for at least some agent, we also have $\pi \cdot b^{(i)} > \pi \cdot c^{(i)}$. Thus

$$\pi \cdot \sum_i b^{(i)} > \pi \cdot \sum_i c^{(i)} = \pi \cdot \sum_i e^{(i)},$$

the equality from the market-clearing condition $\sum_i \theta^{(i)} = 0$. This is impossible, however, for feasibility implies that $\sum_i b^{(i)} \leqslant \sum_i e^{(i)}$. This contradiction implies the result.

Duffie and Huang (1985) characterize the number of securities necessary for complete markets. Roughly speaking, extending the spanning insight of Arrow (1953) to allow for dynamic spanning, it is necessary (and generically sufficient) that there are at least as many securities as the maximal number of mutually exclusive events of positive conditional probability that could be revealed between two dates. For example, if the information generated at each date is that of a coin toss, then complete markets requires a minimum of two securities, and almost any two will suffice. Cox, Ross

and Rubinstein (1979) provide the classical example in which one of the original securities has "binomial" returns and the other has riskless returns. That is, $S = (Y, Z)$ is strictly positive, and, for all $t < T$, we have $\delta_t = 0$, Y_{t+1}/Y_t is a Bernoulli trial, and Z_{t+1}/Z_t is a constant. More generally, however, to be assured of complete markets given the minimal number of securities, one must verify that the price process, which is endogenous, is not among the rare set that is associated with a reduced market span, a point emphasized by Hart (1975) and dealt with by Magill and Shafer (1990). In general, the dependence of the marketed subspace on endogenous security price processes makes the demonstration and calculation of an equilibrium problematic. Conditions for the generic existence of equilibrium in incomplete markets are given by Duffie and Shafer (1985, 1986). The literature on this topic is extensive. [6]

Hahn (1994) raises some philosophical issues regarding the possibility of complete markets and efficiency, in a setting in which endogenous uncertainty may be of concern to investors. The Pareto inefficiency of incomplete markets equilibrium consumption allocations, and notions of constrained efficiency, are discussed by Hart (1975), Kreps (1979) (and references therein), Citanna, Kajii and Villanacci (1994), Citanna and Villanacci (1993) and Pan (1993, 1995).

The optimality of individual portfolio and consumption choices in incomplete markets in this setting is given a dual interpretation by He and Pagès (1993). [Girotto and Ortu (1994) offer related remarks.] Methods for computation of equilibrium with incomplete markets are developed by Brown, DeMarzo and Eaves (1996a,b), Cuoco and He (1994), DeMarzo and Eaves (1996) and Dumas and Maenhout (2002). Kraus and Litzenberger (1975) and Stapleton and Subrahmanyam (1978) gave early parametric examples of equilibrium.

2.6. Equilibrium asset pricing

We will review a representative-agent state-pricing model of Constantinides (1982). The idea is to deduce a state-price density from aggregate, rather than individual, consumption behavior. Among other advantages, this allows for a version of the

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[6] Bottazzi (1995) has a somewhat more advanced version of existence in single-period multiple-commodity version. Related existence topics are studied by Bottazzi and Hens (1996), Hens (1991) and Zhou (1997). The literature is reviewed in depth by Geanakoplos (1990). Alternative proofs of existence of equilibrium are given in the 2-period version of the model by Geanakoplos and Shafer (1990), Hirsch, Magill and Mas-Colell (1990) and Husseini, Lasry and Magill (1990); and in a T-period version by Florenzano and Gourdel (1994). If one defines security dividends in nominal terms, rather than in units of consumption, then equilibria always exist under standard technical conditions on preferences and endowments, as shown by Cass (1984), Werner (1985), Duffie (1987) and Gottardi and Hens (1996), although equilibrium may be indeterminate, as shown by Cass (1989) and Geanakoplos and Mas-Colell (1989). On this point, see also Kydland and Prescott (1991), Mas-Colell (1991) and Cass (1991). Surveys of general equilibrium models in incomplete markets settings are given by Cass (1991), Duffie (1992), Geanakoplos (1990), Magill and Quinzii (1996) and Magill and Shafer (1991). Hindy and Huang (1993) show the implications of linear collateral constraints on security valuation.

consumption-based capital asset pricing model of Breeden (1979) in the special case of locally-quadratic utility.

We define, for each vector λ in $\mathbb{R}^m_+$ of "agent weights", the utility function $U_\lambda \colon L_+ \to \mathbb{R}$ by

$$U_\lambda(x) = \sup_{(c^{(1)}, \ldots, c^{(m)})} \sum_{i=1}^m \lambda_i U_i(c^i) \quad \text{subject to} \quad c^{(1)} + \cdots + c^{(m)} \leqslant x. \tag{8}$$

Proposition. *Suppose for all i that U_i is concave and strictly increasing. Suppose that $(\theta^{(1)}, \ldots, \theta^{(m)}, S)$ is an equilibrium and that markets are complete. Then there exists some nonzero $\lambda \in \mathbb{R}^m_+$ such that $(0, S)$ is a (no-trade) equilibrium for the one-agent economy $[(U_\lambda, e), \delta]$, where $e = e^{(1)} + \cdots + e^{(m)}$. With this λ and with $x = e = e^{(1)} + \cdots + e^{(m)}$, problem (8) is solved by the equilibrium consumption allocation.*

A method of proof, as well as the intuition for this proposition, is that with complete markets, a state-price density π represents Lagrange multipliers for consumption in the various periods and states for all of the agents simultaneously, as well as for some *representative agent* (U_λ, e), whose agent-weight vector λ defines a hyperplane separating the set of feasible utility improvements from $\mathbb{R}^m_+$. [See, for example, Duffie (2001) for details. This notion of "representative agent" is weaker than that associated with aggregation in the sense of Gorman (1953).]

Corollary 1. *If, moreover, U_λ is continuously differentiable at e, then λ can be chosen so that a state-price density is given by the Riesz representation of $\nabla U_\lambda(e)$.*

Corollary 2. *Suppose, for each i, that U_i is of the additive form*

$$U_i(c) = E\left[\sum_{t=0}^T u_{it}(c_t)\right].$$

Then U_λ is also additive, with

$$U_\lambda(c) = E\left[\sum_{t=0}^T u_{\lambda t}(c_t)\right],$$

where

$$u_{\lambda t}(y) = \sup_{x \in \mathbb{R}^m_+} \sum_{i=1}^m \lambda_i u_{it}(x_i) \quad \text{subject to} \quad x_1 + \cdots + x_m \leqslant y.$$

In this case, the differentiability of U_λ at e implies that for any times t and $\tau \geqslant t$,

$$S_t = \frac{1}{u'_{\lambda t}(e_t)} E_t\left[u'_{\lambda \tau}(e_\tau) S_\tau + \sum_{j=t+1}^\tau u'_{\lambda j}(e_j)\delta_j\right]. \tag{9}$$

2.7. Breeden's consumption-based CAPM

The consumption-based capital asset-pricing model (CAPM) of Breeden (1979) extends the results of Rubinstein (1976) by showing that if agents have additive utility that is, locally quadratic, then expected asset returns are linear with respect to their covariances with aggregate consumption, as will be stated more carefully shortly. Notably, the result does *not* depend on complete markets. Locally quadratic additive utility is an extremely strong assumption. (It does not violate monotonicity, as utility need not be quadratic at all levels). Breeden actually worked in a continuous-time setting of Brownian information, reviewed shortly, within which smooth additive utility functions are automatically locally quadratic, in a sense that is sufficient to recover a continuous-time analogue of the following consumption-based CAPM.[7] In a one-period setting, the consumption-based CAPM corresponds to the classical CAPM of Sharpe (1964).

First, we need some preliminary definitions. The *return* at time $t + 1$ on a trading strategy θ whose market value $\theta_t \cdot S_t$ is non-zero is

$$R^\theta_{t+1} = \frac{\theta_t \cdot (S_{t+1} + \delta_{t+1})}{\theta_t \cdot S_t}.$$

There is *short-term riskless borrowing* if, for each given time $t < T$, there is a trading strategy θ with $\mathcal{F}_t$-conditionally deterministic return, denoted r_t. We refer to the sequence $\{r_0, r_1, \ldots, r_{T-1}\}$ of such short-term risk-free returns as the associated "short-rate process", even though r_T is not defined. Conditional on $\mathcal{F}_t$, we let $\mathrm{var}_t(\cdot)$ and $\mathrm{cov}_t(\cdot)$ denote variance and covariance, respectively.

Proposition: Consumption-based CAPM. *Suppose, for each agent i, that the utility $U_i(\cdot)$ is of the additive form $U_i(c) = E[\sum_{t=0}^{T} u_{it}(c_t)]$, and moreover that, for equilibrium consumption processes $c^{(1)}, \ldots, c^{(m)}$, we have $u'_{it}(c^{(i)}_t) = a_{it} + b_{it}c^{(i)}_t$, where a_{it} and $b_{it} > 0$ are constants. Let S be the associated equilibrium price process of the securities. Then, for any time t,*

$$S_t = A_t E_t (\delta_{t+1} + S_{t+1}) - B_t E_t [(S_{t+1} + \delta_{t+1})\, e_{t+1}],$$

for adapted strictly positive scalar processes A and B. For a given time t, suppose that there is riskless borrowing at the short rate r_t. Then there is a trading strategy with the property that its return R^*_{t+1} has maximal $\mathcal{F}_t$-conditional correlation with the aggregate consumption e_{t+1} (among all trading strategies). Suppose, moreover, that

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[7] For a theorem and proof, see Duffie and Zame (1989).

there is riskless borrowing at the short rate r_t and that $\mathrm{var}_t(R^*_{t+1})$ is strictly positive. Then, for any trading strategy θ with return R^{θ}_{t+1},

$$E_t\left(R^{\theta}_{t+1} - r_t\right) = \beta^{\theta}_t E_t\left(R^*_{t+1} - r_t\right),$$

where

$$\beta^{\theta}_t = \frac{\mathrm{cov}_t(R^{\theta}_{t+1}, R^*_{t+1})}{\mathrm{var}_t(R^*_{t+1})}.$$

The essence of the result is that the expected return of any security, in excess of risk-free rates, is increasing in the degree to which the security's return depends (in the sense of regression) on aggregate consumption. This is natural; there is an average preference in favor of securities that are hedges against aggregate economic performance. While the consumption-based CAPM does not depend on complete markets, its reliance on locally-quadratic expected utility, and otherwise perfect markets, is limiting, and its empirical performance is mixed, at best. For some evidence, see for example Hansen and Jaganathan (1990).

2.8. Arbitrage and martingale measures

This section shows the equivalence between the absence of arbitrage and the existence of "risk-neutral" probabilities, under which, roughly speaking, the price of a security is the sum of its expected discounted dividends. This idea, stemming from Cox and Ross (1976), was developed into the notion of equivalent martingale measures by Harrison and Kreps (1979).

We suppose throughout this subsection that there is short-term riskless borrowing at some uniquely defined short-rate process r. We can define, for any times t and $\tau \leqslant T$,

$$R_{t,\tau} = (1 + r_t)(1 + r_{t+1}) \ldots (1 + r_{\tau-1}),$$

the payback at time τ of one unit of account borrowed risklessly at time t and "rolled over" in short-term borrowing repeatedly until date τ.

It would be a simple situation, both computationally and conceptually, if any security's price were merely the expected discounted dividends of the security. Of course, this is unlikely to be the case in a market with risk-averse investors. We can nevertheless come close to this sort of characterization of security prices by adjusting the original probability measure P. For this, we define a new probability measure Q to be *equivalent* to P if Q and P assign zero probabilities to the same events. An equivalent probability measure Q is an *equivalent martingale measure* if

$$S_t = E^Q_t\left(\sum_{j=t+1}^{T} \frac{\delta_j}{R_{t,j}}\right), \quad t < T,$$

where E^Q denotes expectation under Q, and $E^Q_t(X) = E^Q(X \mid \mathcal{F}_t)$ for any random variable X.

It is easy to show that Q is an equivalent martingale measure if and only if, for any trading strategy θ,

$$\theta_t \cdot S_t = E_t^Q \left(\sum_{j=t+1}^{T} \frac{\delta_j^\theta}{R_{t,j}} \right), \quad t < T. \tag{10}$$

We will show that the absence of arbitrage is equivalent to the existence of an equivalent martingale measure.

The deflator γ defined by $\gamma_t = R_{0,t}^{-1}$ defines the discounted gain process G^γ, by $G_t^\gamma = \gamma_t S_t + \sum_{j=1}^{t} \gamma_j \delta_j$. The word "martingale" in the term "equivalent martingale measure" comes from the following equivalence.

Lemma. *A probability measure Q equivalent to P is an equivalent martingale measure for (δ, S) if and only if $S_T = 0$ and the discounted gain process G^γ is a martingale with respect to Q.*

If, for example, a security pays no dividends before T, then the property described by the lemma is that the discounted price process is a Q-martingale.

We already know that the absence of arbitrage is equivalent to the existence of a state-price density π. A probability measure Q equivalent to P can be defined in terms of a Radon–Nikodym derivative, a strictly positive random variable $\frac{dQ}{dP}$ with $E(\frac{dQ}{dP}) = 1$, via the definition of expectation with respect to Q given by $E^Q(Z) = E(\frac{dQ}{dP} Z)$, for any random variable Z. We will consider the measure Q defined by $\frac{dQ}{dP} = \xi_T$, where

$$\xi_T = \frac{\pi_T R_{0,T}}{\pi_0}.$$

(Indeed, one can check by applying the definition of a state-price density to the payoff $R_{0,T}$ that ξ_T is strictly positive and of expectation 1.) The *density process* ξ for Q is defined by $\xi_t = E_t(\xi_T)$. Bayes Rule implies that for any times t and $j > t$, and any $\mathcal{F}_j$-measurable random variable Z_j,

$$E_t^Q(Z_j) = \frac{1}{\xi_t} E_t(\xi_j Z_j). \tag{11}$$

Fixing some time $t < T$, consider a trading strategy θ that invests one unit of account at time t and repeatedly rolls the value over in short-term riskless borrowing until time T, with final value $R_{t,T}$. That is, $\theta_t \cdot S_t = 1$ and $\delta_T^\theta = R_{t,T}$. Relation (3) then implies that

$$\pi_t = E_t \left(\pi_T R_{t,T} \right) = \frac{E_t \left(\pi_T R_{0,T} \right)}{R_{0,t}} = \frac{E_t \left(\xi_T \pi_0 \right)}{R_{0,t}} = \frac{\xi_t \pi_0}{R_{0,t}}. \tag{12}$$

From Equations (11), (12), and the definition of a state-price density, Equation (10) is satisfied, so Q is indeed an equivalent martingale measure. We have shown the following result.

Theorem. *There is no arbitrage if and only if there exists an equivalent martingale measure. Moreover, π is a state-price density if and only if an equivalent martingale measure Q has the density process ξ defined by $\xi_t = R_{0,t}\pi_t/\pi_0$.*

This martingale approach simplifies many asset-pricing problems that might otherwise appear to be quite complex, and applies much more generally than indicated here. For example, the assumption of short-term borrowing is merely a convenience, and one can typically obtain an equivalent martingale measure after normalizing prices and dividends by the price of some particular security (or trading strategy). Girotto and Ortu (1996) present general results of this type for this finite-dimensional setting. Dalang, Morton and Willinger (1990) gave a general discrete-time result on the equivalence of no arbitrage and the existence of an equivalent martingale measure, covering even the case with infinitely many states.

2.9. Valuation of redundant securities

Suppose that the dividend–price pair (δ, S) for the N given securities is arbitrage-free, with an associated state-price density π. Now consider the introduction of a new security with dividend process $\hat{\delta}$ and price process $\widehat{S}$. We say that $\hat{\delta}$ is *redundant* given (δ, S) if there exists a trading strategy θ, with respect to only the original security dividend–price process (δ, S), that *replicates* $\hat{\delta}$, in the sense that $\delta_t^\theta = \hat{\delta}_t$, $t \geqslant 1$.

If $\hat{\delta}$ is redundant given (δ, S), then the absence of arbitrage for the "augmented" dividend–price process $[(\delta, \hat{\delta}), (S, \widehat{S})]$ implies that $\widehat{S}_t = Y_t$, where

$$ Y_t = \frac{1}{\pi_t} E_t \left(\sum_{j=t+1}^{T} \pi_j \hat{\delta}_j \right), \quad t < T. $$

If this were not the case, there would be an arbitrage, as follows. For example, suppose that for some stopping time τ, we have $\widehat{S}_\tau > Y_\tau$, and that $\tau \leqslant T$ with strictly positive probability. We can then define the strategy:

(a) Sell the redundant security $\hat{\delta}$ at time τ for $\widehat{S}_\tau$, and hold this position until T.
(b) Invest $\theta_\tau \cdot S_\tau$ at time τ in the replicating strategy θ, and follow this strategy until T.

Since the dividends generated by this combined strategy (a)–(b) after τ are zero, the only dividend is at τ, for the amount $\widehat{S}_\tau - Y_\tau > 0$, which means that this is an arbitrage. Likewise, if $\widehat{S}_\tau < Y_\tau$ for some non-trivial stopping time τ, the opposite strategy is an arbitrage. We have shown the following.

Proposition. *Suppose (δ, S) is arbitrage-free with state-price density π. Let $\hat{\delta}$ be a redundant dividend process with price process $\widehat{S}$. Then the augmented dividend–price pair $[(\delta, \hat{\delta}), (S, \widehat{S})]$ is arbitrage-free if and only if it has π as a state-price density.*

In applications, it is often assumed that (δ, S) generates complete markets, in which case any additional security is redundant, as in the classical "binomial" model of

Cox, Ross and Rubinstein (1979), and its continuous-time analogue, the Black–Scholes option pricing model, coming up in the next section.

Complete markets means that every new security is redundant.

Theorem. *Suppose that $\mathcal{F}_T = \mathcal{F}$ and there is no arbitrage. Then markets are complete if and only if there is a unique equivalent martingale measure.*

Banz and Miller (1978) and Breeden and Litzenberger (1978) explore the ability to deduce state prices from the valuation of derivative securities.

2.10. American exercise policies and valuation

We now extend our pricing framework to include a family of securities, called "American," for which there is discretion regarding the timing of cash flows.

Given an adapted process X, each finite-valued stopping time τ generates a dividend process $\delta^{X,\tau}$ defined by $\delta_t^{X,\tau} = 0$, $t \neq \tau$, and $\delta_\tau^{X,\tau} = X_\tau$. In this context, a finite-valued stopping time is an *exercise policy*, determining the time at which to accept payment. Any exercise policy τ is constrained by $\tau \leqslant \overline{\tau}$, for some *expiration time* $\overline{\tau} \leqslant T$. (In what follows, we might take $\overline{\tau}$ to be a stopping time, which is useful for the case of certain *knockout options*.)

We say that $(X, \overline{\tau})$ defines an *American security*. The exercise policy is selected by the holder of the security. Once exercised, the security has no remaining cash flows. A standard example is an American put option on a security with price process p. The American put gives the holder of the option the right, but not the obligation, to sell the underlying security for a fixed exercise price at any time before a given expiration time $\overline{\tau}$. If the option has an exercise price K and expiration time $\overline{\tau} < T$, then $X_t = (K - p_t)^+$, $t \leqslant \overline{\tau}$, and $X_t = 0$, $t > \overline{\tau}$.

We will suppose that, in addition to an American security $(X, \overline{\tau})$, there are securities with an arbitrage-free dividend-price process (δ, S) that generates complete markets. The assumption of complete markets will dramatically simplify our analysis since it implies, for any exercise policy τ, that the dividend process $\delta^{X,\tau}$ is redundant given (δ, S). For notational convenience, we assume that $0 < \overline{\tau} < T$.

Let π be a state-price density associated with (δ, S). From Proposition 2.9, given any exercise policy τ, the American security's dividend process $\delta^{X,\tau}$ has an associated cum-dividend price process, say V^τ, which, in the absence of arbitrage, satisfies

$$V_t^\tau = \frac{1}{\pi_t} E_t \left(\pi_\tau X_\tau \right), \quad t \leqslant \tau.$$

This value does not depend on which state-price density is chosen because, with complete markets, state-price densities are identical up to a positive scaling.

We consider the optimal stopping problem

$$V_0^* \equiv \max_{\tau \in \mathcal{T}(0)} V_0^\tau, \tag{13}$$

where, for any time $t \leqslant \overline{\tau}$, we let $\mathcal{T}(t)$ denote the set of stopping times bounded below by t and above by $\overline{\tau}$. A solution to Equation (13) is called a *rational exercise policy*

for the American security X, in the sense that it maximizes the initial arbitrage-free value of the resulting claim. Merton (1973) was the first to attack American option valuation systematically using this arbitrage-based viewpoint.

We claim that, in the absence of arbitrage, the actual initial price V_0 for the American security must be V_0^*. In order to see this, suppose first that $V_0^* > V_0$. Then one could buy the American security, adopt for it a rational exercise policy τ, and also undertake a trading strategy replicating $-\delta^{X,\tau}$. Since $V_0^* = E(\pi_\tau X_\tau)/\pi_0$, this replication involves an initial payoff of V_0^*, and the net effect is a total initial dividend of $V_0^* - V_0 > 0$ and zero dividends after time 0, which defines an arbitrage. Thus the absence of arbitrage easily leads to the conclusion that $V_0 \geqslant V_0^*$. It remains to show that the absence of arbitrage also implies the opposite inequality $V_0 \leqslant V_0^*$.

Suppose that $V_0 > V_0^*$. One could sell the American security at time 0 for V_0. We will show that for an initial investment of V_0^*, one can "super-replicate" the payoff at exercise demanded by the holder of the American security, *regardless of the exercise policy used*. Specifically, a *super-replicating trading strategy* for $(X, \overline{\tau}, \delta, S)$ is a trading strategy θ involving only the securities with dividend-price process (δ, S) that has the following properties:

(a) $\delta_t^\theta = 0$ for $0 < t < \overline{\tau}$, and

(b) $V_t^\theta \geqslant X_t$ for all $t \leqslant \overline{\tau}$,

where V_t^θ is the cum-dividend market value of θ at time t. Regardless of the exercise policy τ used by the holder of the security, the payment of X_τ demanded at time τ is dominated by the market value V_t^θ of a super-replicating strategy θ. (In effect, one modifies θ by liquidating the portfolio θ_τ at time τ, so that the actual trading strategy φ associated with the arbitrage is defined by $\varphi_t = \theta_t$ for $t < \tau$ and $\varphi_t = 0$ for $t \geqslant \tau$.) Now, suppose θ is super-replicating, with $V_0^\theta = V_0^*$. If, indeed, $V_0 > V_0^*$ then the strategy of selling the American security and adopting a super-replicating strategy, liquidating at exercise, effectively defines an arbitrage.

This notion of arbitrage for American securities, an extension of the definition of arbitrage used earlier, is reasonable because a super-replicating strategy does not depend on the exercise policy adopted by the holder (or sequence of holders over time) of the American security. It would be unreasonable to call a strategy involving a short position in the American security an "arbitrage" if, in carrying it out, one requires knowledge of the exercise policy for the American security that will be adopted by other agents that hold the security over time, who may after all act "irrationally."

The approach to American security valuation given here is similar to the continuous-time treatments of Bensoussan (1984) and Karatzas (1988), who do not formally connect the valuation of American securities with the absence of arbitrage, but rather deal with the similar notion of "fair price".

Proposition. *Given $(X, \overline{\tau}, \delta, S)$, suppose (δ, S) is arbitrage free and generates complete markets. Then there is a super-replicating trading strategy θ for $(X, \overline{\tau}, \delta, S)$ with the initial value $V_0^\theta = V_0^*$.*

In order to construct a super-replicating strategy with the desired property, we will make a short excursion into the theory of optimal stopping. For any process Y in L, the *Snell envelope* W of Y is defined by

$$W_t = \max_{\tau \in \mathcal{T}(t)} E_t(Y_\tau), \quad 0 \leqslant t \leqslant \bar{\tau}.$$

It can be shown that, naturally, for any $t < \bar{\tau}$, $W_t = \max[Y_t, E_t(W_{t+1})]$, which can be viewed as the Bellman equation for optimal stopping. Thus $W_t \geqslant E_t(W_{t+1})$, implying that W is a supermartingale, implying that we can decompose W in the form $W = Z - A$, for some martingale Z and some increasing adapted[8] process A with $A_0 = 0$.

In order to prove the above proposition, we define Y by $Y_t = X_t \pi_t$, and let W, Z, and A be defined as above. By the definition of complete markets, there is a trading strategy θ with the property that

- $\delta_t^\theta = 0$ for $0 < t < \bar{\tau}$;
- $\delta_{\bar{\tau}}^\theta = Z_{\bar{\tau}}/\pi_{\bar{\tau}}$;
- $\delta_t^\theta = 0$ for $t > \bar{\tau}$.

Property (a) defining a super-replicating strategy is satisfied by this strategy θ. From the fact that Z is a martingale and the definition of a state-price density, the cum-dividend value V^θ satisfies

$$\pi_t V_t^\theta = E_t \left(\pi_{\bar{\tau}} \delta_{\bar{\tau}}^\theta \right) = E_t (Z_{\bar{\tau}}) = Z_t, \quad t \leqslant \bar{\tau}. \tag{14}$$

From Equation (14) and the fact that $A_0 = 0$, we know that $V_0^\theta = V_0^*$ because $Z_0 = W_0 = \pi_0 V_0^*$. Since $Z_t - A_t = W_t \geqslant Y_t$ for all t, from Equation (14) we also know that

$$V_t^\theta = \frac{Z_t}{\pi_t} \geqslant \frac{1}{\pi_t} (Y_t + A_t) = X_t + \frac{A_t}{\pi_t} \geqslant X_t, \quad t \leqslant \bar{\tau},$$

the last inequality following from the fact that $A_t \geqslant 0$ for all t. Thus the dominance property (b) defining a super-replicating strategy is also satisfied, and θ is indeed a super-replicating strategy with $V_0^\theta = V_0^*$. This proves the proposition and implies that, unless there is an arbitrage, the initial price V_0 of the American security is equal to the market value V_0^* associated with a rational exercise policy.

The Snell envelope W is also the key to showing that a rational exercise policy is given by the dynamic-programming solution $\tau^0 = \min\{t: W_t = Y_t\}$. In order to verify this, suppose that τ is a rational exercise policy. Then $W_\tau = Y_\tau$. (This can be seen

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[8] More can be said, in that A_t can be taken to be $\mathcal{F}_{t-1}$-measurable.

from the fact that $W_\tau \geqslant Y_\tau$, and if $W_\tau > Y_\tau$ then τ cannot be rational.) From this fact, any rational exercise policy τ has the property that $\tau \geqslant \tau^0$. For any such τ, we have

$$E_{\tau^0}\left[Y(\tau)\right] \leqslant W(\tau^0) = Y(\tau^0),$$

and the law of iterated expectations implies that

$$E[Y(\tau)] \leqslant E[Y(\tau^0)],$$

so τ^0 is indeed rational. We have shown the following.

Theorem. *Given* $(X, \overline{\tau}, \delta, S)$, *suppose that* (δ, S) *admits no arbitrage and generates complete markets. Let* π *be a state-price deflator. Let* W *be the Snell envelope of* $X\pi$ *up to the expiration time* $\overline{\tau}$. *Then a rational exercise policy for* $(X, \overline{\tau}, \delta, S)$ *is given by* $\tau^0 = \min\{t: W_t = \pi_t X_t\}$. *The unique initial cum-dividend arbitrage-free price of the American security is*

$$V_0^* = \frac{1}{\pi_0} E\left[X(\tau^0)\,\pi(\tau^0)\right].$$

In terms of the equivalent martingale measure Q defined in Section 2.8, we can also write the optimal stopping problem (13) in the form

$$V_0^* = \max_{\tau \in \mathcal{T}(0)} E^Q\left(\frac{X_\tau}{R_{0,\tau}}\right). \tag{15}$$

An optimal exercise time is $\tau^0 = \min\{t: V_t^* = X_t\}$, where $V_t^* = W_t/\pi_t$ is the price of the American option at time t. This representation of the rational-exercise problem is sometimes convenient. For example, let us consider the case of an American call option on a security with price process p. We have $X_t = (p_t - K)^+$ for some exercise price K. Suppose the underlying security has no dividends before or at the expiration time $\overline{\tau}$. We suppose positive interest rates, meaning that $R_{t,s} \geqslant 1$ for all t and $s \geqslant t$. With these assumptions, we will show that it is never optimal to exercise the call option before its expiration date $\overline{\tau}$. This property is sometimes called "no early exercise", or "better alive than dead".

We define the "discounted price process" p^* by $p_t^* = p_t/R_{0,t}$. The fact that the underlying security pays dividends only after the expiration time $\overline{\tau}$ implies, by Lemma 2.8, that p^* is a Q-martingale at least up to the expiration time $\overline{\tau}$. That is, for $t \leqslant s \leqslant \overline{\tau}$, we have $E_t^Q(p_s^*) = p_t^*$.

With positive interest rates, we have, for any stopping time $\tau \leqslant \bar{\tau}$,

$$
\begin{aligned}
E^Q \left[\frac{1}{R_{0,\tau}} (p_\tau - K)^+ \right] &= E^Q \left[\left(p_\tau^* - \frac{K}{R_{0,\tau}} \right)^+ \right] \\
&= E^Q \left[E_\tau^Q \left(\left(p_\tau^* - \frac{K}{R_{0,\tau}} \right)^+ \right) \right] \\
&\leqslant E^Q \left[E_\tau^Q \left(\left(p_{\bar{\tau}}^* - \frac{K}{R_{0,\tau}} \right)^+ \right) \right] \\
&= E^Q \left[\left(p_{\bar{\tau}}^* - \frac{K}{R_{0,\tau}} \right)^+ \right] \\
&\leqslant E^Q \left[\left(p_{\bar{\tau}}^* - \frac{K}{R_{0,\bar{\tau}}} \right)^+ \right] \\
&= E^Q \left[\frac{1}{R_{0,\bar{\tau}}} (p_{\bar{\tau}} - K)^+ \right],
\end{aligned}
$$

the first inequality by Jensen's inequality, the second by the positivity of interest rates. It follows that $\bar{\tau}$ is a rational exercise policy. In typical cases, $\bar{\tau}$ is the unique rational exercise policy.

If the underlying security pays dividends before expiration, then early exercise of the American call is, in certain cases, optimal. From the fact that the put payoff is increasing in the strike price (as opposed to decreasing for the call option), the second inequality above is reversed for the case of a put option, and one can guess that early exercise of the American put is sometimes optimal.

Difficulties can arise with the valuation of American securities in incomplete markets. For example, the exercise policy may play a role in determining the marketed subspace, and therefore a role in pricing securities. If the state-price density depends on the exercise policy, it could even turn out that the notion of a rational exercise policy is not well defined.

3. Continuous-time modeling

Many problems are more tractable, or have solutions appearing in a more natural form, when treated in a continuous-time setting. We first introduce the Brownian model of uncertainty and continuous security trading, and then derive partial differential equations for the arbitrage-free prices of derivative securities. The classic example is the Black–Scholes option-pricing formula. We then examine the connection between equivalent martingale measures and the "market price of risk" that arises from Girsanov's Theorem. Finally, we briefly connect the theory of security valuation with that of optimal portfolio and consumption choice, using the elegant martingale approach of Cox and Huang (1989).

3.1. Trading gains for Brownian prices

We fix a probability space $(\Omega, \mathcal{F}, P)$. A *process* is a measurable[9] function on $\Omega \times [0, \infty)$ into $\mathbb{R}$. The value of a process X at time t is the random variable variously written as X_t, $X(t)$, or $X(\cdot, t)$: $\Omega \to \mathbb{R}$. A *standard Brownian motion* is a process B defined by the following properties:

(a) $B_0 = 0$ almost surely;

(b) Normality: for any times t and $s > t$, $B_s - B_t$ is normally distributed with mean zero and variance $s - t$;

(c) Independent increments: for any times $t_0, \ldots, t_n$ such that $0 \leqslant t_0 < t_1 < \cdots < t_n < \infty$, the random variables $B(t_0)$, $B(t_1) - B(t_0)$, $\ldots$, $B(t_n) - B(t_{n-1})$ are independently distributed; and

(d) Continuity: for each ω in Ω, the *sample path* $t \mapsto B(\omega, t)$ is continuous.

It is a nontrivial fact, whose proof has a colorful history, that $(\Omega, \mathcal{F}, P)$ can be constructed so that there exist standard Brownian motions. In perhaps the first scientific work involving Brownian motion, Bachelier (1900) proposed Brownian motion as a model of stock prices. We will follow his lead for the time being and suppose that a given standard Brownian motion B is the price process of a security. Later we consider more general classes of price processes.

We fix the *standard filtration* $\mathbb{F} = \{\mathcal{F}_t : t \geqslant 0\}$ of B, defined for example in Protter (1990). Roughly speaking,[10] $\mathcal{F}_t$ is the set of events that can be distinguished as true or false by observation of B until time t.

Our first task is to build a model of trading gains based on the possibility of continual adjustment of the position held. A *trading strategy* is an adapted process θ specifying at each state ω and time t the number $\theta_t(\omega)$ of units of the security to hold. If a strategy θ is a constant, say $\overline{\theta}$, between two dates t and $s > t$, then the total gain between those two dates is $\overline{\theta}(B_s - B_t)$, the quantity held multiplied by the price change. So long as the trading strategy θ is piecewise constant, we would have no difficulty in defining the total gain between any two times. For example, suppose, for some stopping times $T_0, \ldots, T_N$ with $0 = T_0 < T_1 < \cdots < T_N = T$, and for any n, we have $\theta(t) = \theta(T_{n-1})$ for all $t \in [T_{n-1}, T_n)$. Then we define the total gain from trade as

$$\int_0^T \theta_t \, dB_t = \sum_{n=1}^N \theta(T_{n-1}) [B(T_n) - B(T_{n-1})]. \tag{16}$$

More generally, in order to make for a good model of trading gains for trading strategies that are not necessarily piecewise constant, a trading strategy θ is required to satisfy the technical condition that $\int_0^T \theta_t^2 \, dt < \infty$ almost surely for each T. We let $\mathcal{L}^2$ denote the space of adapted processes satisfying this integrability restriction.

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[9] See Duffie (2001) for technical definitions not provided here.

[10] The standard filtration is augmented, so that $\mathcal{F}_t$ contains all null sets of $\mathcal{F}$.

For each θ in $\mathcal{L}^2$ there is an adapted process with continuous sample paths, denoted $\int \theta \, dB$, that is called the *stochastic integral* of θ with respect to B. A full definition of $\int \theta \, dB$ is outlined in a standard source such as Karatzas and Shreve (1988).

The value of the stochastic integral $\int \theta \, dB$ at time T is usually denoted $\int_0^T \theta_t \, dB_t$, and represents the total gain generated up to time T by trading the security with price process B according to the trading strategy θ. The stochastic integral $\int \theta \, dB$ has the properties that one would expect from a good model of trading gains. In particular, Equation (16) is satisfied for piece-wise constant θ, and in general the stochastic integral is linear, in that, for any θ and φ in $\mathcal{L}^2$ and any scalars a and b, the process $a\theta + b\varphi$ is also in $\mathcal{L}^2$, and, for any time $T > 0$,

$$\int_0^T (a\theta_t + b\varphi_t) \, dB_t = a \int_0^T \theta_t \, dB_t + b \int_0^T \varphi_t \, dB_t. \tag{17}$$

3.2. Martingale trading gains

The properties of standard Brownian motion imply that B is a martingale. (This follows basically from the property that its increments are independent and of zero expectation.) One must impose technical conditions on θ, however, in order to ensure that $\int \theta \, dB$ is also a martingale. This is natural; it should be impossible to generate an expected profit by trading a security that never experiences an expected price change. The following basic proposition can be found, for example, in Protter (1990).

Proposition. *If* $E\left[\left(\int_0^T \theta_t^2 \, dt\right)^{1/2}\right] < \infty$ *for all* $T > 0$, *then* $\int \theta \, dB$ *is a martingale.*

As a model of security-price processes, standard Brownian motion is too restrictive for most purposes. Consider, more generally, an *Ito process*, meaning a process S of the form

$$S_t = x + \int_0^t \mu_s \, ds + \int_0^t \sigma_s \, dB_s, \tag{18}$$

where x is a real number, σ is in $\mathcal{L}^2$, and μ is in $\mathcal{L}^1$, meaning that μ is an adapted process such that $\int_0^t |\mu_s| \, ds < \infty$ almost surely for all t. It is common to write Equation (18) in the informal "differential" form

$$dS_t = \mu_t \, dt + \sigma_t \, dB_t.$$

One often thinks intuitively of dS_t as the "increment" of S at time t, made up of two parts, the "locally riskless" part $\mu_t \, dt$, and the "locally uncertain" part $\sigma_t \, dB_t$.

In order to further interpret this differential representation of an Ito process, suppose that σ and μ have continuous sample paths and are bounded. It is then literally the case that for any time t,

$$\frac{\mathrm{d}}{\mathrm{d}\tau} E_t(S_\tau)\Big|_{\tau=t} = \mu_t \qquad \text{almost surely,} \tag{19}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \mathrm{var}_t(S_\tau)\Big|_{\tau=t} = \sigma_t^2 \qquad \text{almost surely,} \tag{20}$$

where the derivatives are taken from the right, and where, for any random variable X with finite variance, $\mathrm{var}_t(X) \equiv E_t(X^2) - [E_t(X)]^2$ is the $\mathcal{F}_t$-conditional variance of X. In this sense of Equations (19) and (20), we can interpret μ_t as the rate of change of the expectation of S, conditional on information available at time t, and likewise interpret σ_t^2 as the rate of change of the conditional variance of S at time t. One sometimes reads the associated abuses of notation "$E_t(\mathrm{d}S_t) = \mu_t\,\mathrm{d}t$" and "$\mathrm{var}_t(\mathrm{d}S_t) = \sigma_t^2\,\mathrm{d}t$". Of course, $\mathrm{d}S_t$ is not even a random variable, so this sort of characterization is not rigorously justified and is used purely for its intuitive content. We will refer to μ and σ as the *drift* and *diffusion* processes of S, respectively.

For an Ito process S of the form (18), let $\mathcal{L}(S)$ be the set whose elements are processes θ with $\{\theta_t\mu_t: t \geqslant 0\}$ in $\mathcal{L}^1$ and $\{\theta_t\sigma_t: t \geqslant 0\}$ in $\mathcal{L}^2$. For θ in $\mathcal{L}(S)$, we define the stochastic integral $\int \theta\,\mathrm{d}S$ as the Ito process $\int \theta\,\mathrm{d}S$ given by

$$\int_0^T \theta_t\,\mathrm{d}S_t = \int_0^T \theta_t\mu_t\,\mathrm{d}t + \int_0^T \theta_t\sigma_t\,\mathrm{d}B_t, \qquad T \geqslant 0.$$

Assuming no dividends, we also refer to $\int \theta\,\mathrm{d}S$ as the *gain process generated by the trading strategy* θ, given the price process S.

We will have occasion to refer to adapted processes θ and φ that are equal *almost everywhere*, by which we mean that $E(\int_0^\infty |\theta_t - \varphi_t|\,\mathrm{d}t) = 0$. In fact, we shall write "$\theta = \varphi$" whenever $\theta = \varphi$ almost everywhere. This is a natural convention, for suppose that X and Y are Ito processes with $X_0 = Y_0$ and with $\mathrm{d}X_t = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t$ and $\mathrm{d}Y_t = a_t\,\mathrm{d}t + b_t\,\mathrm{d}B_t$. Since stochastic integrals are defined for our purposes as continuous sample-path processes, it turns out that $X_t = Y_t$ for all t almost surely if and only if $\mu = a$ almost everywhere and $\sigma = b$ almost everywhere. We call this the *unique decomposition property* of Ito processes.

Ito's Formula is the basis for explicit solutions to asset-pricing problems in a continuous-time setting.

Ito's Formula. *Suppose X is an Ito process with $\mathrm{d}X_t = \mu_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}B_t$ and $f: \mathbb{R}^2 \to \mathbb{R}$ is twice continuously differentiable. Then the process Y, defined by $Y_t = f(X_t, t)$, is an Ito process with*

$$\mathrm{d}Y_t = \left[f_x(X_t, t)\,\mu_t + f_t(X_t, t) + \tfrac{1}{2}f_{xx}(X_t, t)\,\sigma_t^2\right]\,\mathrm{d}t + f_x(X_t, t)\,\sigma_t\,\mathrm{d}B_t.$$

A generalization of Ito's Formula appears later in this section.

3.3. The Black–Scholes option-pricing formula

We turn to one of the most important ideas in finance theory, the model of Black and Scholes (1973) for pricing options. Together with the method of proof provided by Robert Merton, this model revolutionized the practice of derivative pricing and risk management, and has changed the entire path of asset-pricing theory.

Consider a security, to be called a *stock*, with price process

$$S_t = x \, e^{\alpha t + \sigma B(t)}, \qquad t \geqslant 0,$$

where $x > 0$, α, and σ are constants. Such a process, called a *geometric Brownian motion*, is often called *log-normal* because, for any t, $\log(S_t) = \log(x) + \alpha t + \sigma B_t$ is normally distributed. Moreover, since $X_t \equiv \alpha t + \sigma B_t = \int_0^t \alpha \, ds + \int_0^t \sigma \, dB_s$ defines an Ito process X with constant drift α and diffusion σ, Ito's Formula implies that S is an Ito process and that

$$dS_t = \mu S_t \, dt + \sigma S_t \, dB_t; \qquad S_0 = x,$$

where $\mu = \alpha + \sigma^2/2$. From Equations (19) and (20), at any time t, the rate of change of the conditional mean of S_t is μS_t, and the rate of change of the conditional variance is $\sigma^2 S_t^2$, so that, per dollar invested in this security at time t, one may think of μ as the "instantaneous" expected rate of return, and σ as the "instantaneous" standard deviation of the rate of return. The coefficient σ is also known as the *volatility* of S. A geometric Brownian motion is a natural two-parameter model of a security-price process because of these simple interpretations of μ and σ.

Consider a second security, to be called a *bond*, with the price process β defined by

$$\beta_t = \beta_0 \, e^{rt}, \qquad t \geqslant 0,$$

for some constants $\beta_0 > 0$ and r. We have the obvious interpretation of r as the *continually compounding short rate*. Since $\{rt: t \geqslant 0\}$ is trivially an Ito process, β is also an Ito process with

$$d\beta_t = r\beta_t \, dt.$$

A pair (a, b) consisting of trading strategies a for the stock and b for the bond is said to be *self-financing* if it generates no dividends before T (either positive or negative), meaning that, for all t,

$$a_t S_t + b_t \beta_t = a_0 S_0 + b_0 \beta_0 + \int_0^t a_u \, dS_u + \int_0^t b_u \, d\beta_u. \qquad (21)$$

This self-financing condition, conveniently defined by Harrison and Kreps (1979), is merely a statement that the current portfolio value (on the left-hand side) is precisely

the initial investment plus any trading gains, and therefore that no dividend "inflow" or "outflow" is generated.

Now consider a third security, an *option*. We begin with the case of a European call option on the stock, giving its owner the right, but not the obligation, to buy the stock at a given exercise price K on a given exercise date T. The option's price process Y is as yet unknown except for the fact that $Y_T = (S_T - K)^+ \equiv \max(S_T - K, 0)$, which follows from the fact that the option is rationally exercised if and only if $S_T > K$.

Suppose that the option is redundant, in that there exists a self-financing trading strategy (a, b) in the stock and bond with $a_T S_T + b_T \beta_T = Y_T$. If $a_0 S_0 + b_0 \beta_0 < Y_0$, then one could sell the option for Y_0, make an initial investment of $a_0 S_0 + b_0 \beta_0$ in the trading strategy (a, b), and at time T liquidate the entire portfolio $(-1, a_T, b_T)$ of option, stock, and bond with payoff $-Y_T + a_T S_T + b_T \beta_T = 0$. The initial profit $Y_0 - a_0 S_0 - b_0 \beta_0 > 0$ is thus riskless, so the trading strategy $(-1, a, b)$ would be an arbitrage. Likewise, if $a_0 S_0 + b_0 \beta_0 > Y_0$, the strategy $(1, -a, -b)$ is an arbitrage. Thus, if there is no arbitrage, $Y_0 = a_0 S_0 + b_0 \beta_0$. The same arguments applied at each date t imply that in the absence of arbitrage, $Y_t = a_t S_t + b_t \beta_t$. A full and careful definition of continuous-time arbitrage will be given later, but for now we can proceed without much ambiguity at this informal level. Our immediate objective is to show the following.

The Black–Scholes Formula. *If there is no arbitrage, then, for all* $t < T$, $Y_t = C(S_t, t)$, *where*

$$C(x, t) = x \Phi(z) - e^{-r(T-t)} K \Phi \left(z - \sigma \sqrt{T - t} \right), \tag{22}$$

with

$$z = \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}},$$

where Φ *is the cumulative standard normal distribution function.*

The Black and Scholes (1973) formula was extended by Merton (1973, 1977), and subsequently given literally hundreds of further extensions and applications. Cox and Rubinstein (1985) is a standard reference on options, while Hull (2000) has further applications and references.

We will see different ways to arrive at the Black–Scholes formula. Although not the shortest argument, the following is perhaps the most obvious and constructive.[11]

We start by assuming that $Y_t = C(S_t, t)$, $t < T$, without knowledge of the function C aside from the assumption that it is twice continuously differentiable on $(0, \infty) \times [0, T)$

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[11] The line of exposition here is based on Gabay (1982) and Duffie (1988). Andreasen, Jensen and Poulsen (1998) provide numerous alternative methods of deriving the Black–Scholes Formula. The basic approach of using continuous-time self-financing strategies as the basis for making arbitrage arguments is due to Merton (1977).

(allowing an application of Ito's Formula). This will lead us to deduce Equation (22), justifying the assumption and proving the result at the same time.

Based on our assumption that $Y_t = C(S_t, t)$ and Ito's Formula,

$$dY_t = \mu_Y(t)\, dt + C_x(S_t, t)\, \sigma S_t\, dB_t, \qquad t < T, \tag{23}$$

where

$$\mu_Y(t) = C_x(S_t, t)\, \mu S_t + C_t(S_t, t) + \tfrac{1}{2} C_{xx}(S_t, t)\, \sigma^2 S_t^2.$$

Now suppose there is a self-financing trading strategy (a, b) with

$$a_t S_t + b_t \beta_t = Y_t, \qquad t \in [0, T]. \tag{24}$$

This assumption will also be justified shortly. Equations (21) and (24), along with the linearity of stochastic integration, imply that

$$dY_t = a_t\, dS_t + b_t\, d\beta_t = (a_t \mu S_t + b_t \beta_t r)\, dt + a_t \sigma S_t\, dB_t. \tag{25}$$

Based on the unique decomposition property of Ito processes, in order that the trading strategy (a, b) satisfies both Equation (23) and Equation (25), we must "match coefficients separately in both dB_t and dt". Specifically, we choose a_t so that $a_t \sigma S_t = C_x(S_t, t)\, \sigma S_t$; for this, we let $a_t = C_x(S_t, t)$. From Equation (24) and $Y_t = C(S_t, t)$, we then have $C_x(S_t, t)\, S_t + b_t \beta_t = C(S_t, t)$, or

$$b_t = \frac{1}{\beta_t}\, [C(S_t, t) - C_x(S_t, t)\, S_t]. \tag{26}$$

Finally, "matching coefficients in dt" from Equations (23) and (25) leaves, for $t < T$,

$$-rC(S_t, t) + C_t(S_t, t) + rS_t C_x(S_t, t) + \tfrac{1}{2} \sigma^2 S_t^2 C_{xx}(S_t, t) = 0. \tag{27}$$

In order for Equation (27) to hold, it is enough that C satisfies the partial differential equation (PDE)

$$-rC(x, t) + C_t(x, t) + rx C_x(x, t) + \tfrac{1}{2}\sigma^2 x^2 C_{xx}(x, t) = 0, \tag{28}$$

for $(x, t) \in (0, \infty) \times [0, T)$. The fact that $Y_T = C(S_T, T) = (S_T - K)^+$ supplies the *boundary condition*:

$$C(x, T) = (x - K)^+, \qquad x \in (0, \infty). \tag{29}$$

By direct calculation of derivatives, one can show as an exercise that Equation (22) is a solution to Equations (28) and (29). All of this seems to confirm that $C(S_0, 0)$, with C defined by the Black–Scholes formula (22), is a good candidate for the initial price of

the option. In order to confirm this pricing, suppose to the contrary that $Y_0 > C(S_0, 0)$, where C is defined by Equation (22). Consider the strategy $(-1, a, b)$ in the option, stock, and bond, with $a_t = C_x(S_t, t)$ and b_t given by Equation (26) for $t < T$. We can choose a_T and b_T arbitrarily so that Equation (24) is satisfied; this does not affect the self-financing condition (21) because the value of the trading strategy at a single point in time has no effect on the stochastic integral. The result is that (a, b) is self-financing by construction and that $a_T S_T + b_T \beta_T = Y_T = (S_T - K)^+$. This strategy therefore nets an initial riskless profit of

$$Y_0 - a_0 S_0 - b_0 \beta_0 = Y_0 - C(S_0, 0) > 0,$$

which defines an arbitrage. Likewise, if $Y_0 < C(S_0, 0)$, the trading strategy $(+1, -a, -b)$ is an arbitrage. Thus, it is indeed a necessary condition for the absence of arbitrage that $Y_0 = C(S_0, 0)$. Sufficiency is a more delicate matter. Under mild technical conditions on trading strategies that will follow, the Black–Scholes formula for the option price is also sufficient for the absence of arbitrage.

Transactions costs play havoc with the sort of reasoning just applied. For example, if brokerage fees are any positive fixed fraction of the market value of stock trades, the stock-trading strategy a constructed above would call for infinite total brokerage fees, since, in effect, the number of shares traded is infinite! Leland (1985) has shown, nevertheless, that the Black–Scholes formula applies approximately, for small proportional transacations costs, once one artificially elevates the volatility parameter to compensate for the transactions costs.

3.4. Ito's Formula

Ito's Formula is extended to the case of multidimensional Brownian motion as follows. A standard Brownian motion in $\mathbb{R}^d$ is defined by $B = (B^1, \ldots, B^d)$ in $\mathbb{R}^d$, where $B^1, \ldots, B^d$ are independent standard Brownian motions. We fix a standard Brownian motion B in $\mathbb{R}^d$, restricted to some time interval $[0, T]$, on a given probability space $(\Omega, \mathcal{F}, P)$. We also fix the standard filtration $\mathbb{F} = \{\mathcal{F}_t : t \in [0, T]\}$ of B. For simplicity, we take $\mathcal{F}$ to be $\mathcal{F}_T$. For an $\mathbb{R}^d$-valued process $\theta = (\theta^{(1)}, \ldots, \theta^{(d)})$ with $\theta^{(i)}$ in $\mathcal{L}^2$ for each i, the stochastic integral $\int \theta \, dB$ is defined by

$$\int_0^t \theta_s \, dB_s = \sum_{i=1}^d \int_0^t \theta_s^{(i)} \, dB_s^i. \tag{30}$$

An Ito process is now defined as one of the form

$$X_t = x + \int_0^t \mu_s \, ds + \int_0^t \theta_s \, dB_s,$$

where μ is a drift (with $\int_0^t |\mu_s| \, ds < \infty$ almost surely) and $\int_0^t \theta_s \, dB_s$ is defined as in Equation (30). In this case, we call θ the diffusion of X.

We say that $X = (X^{(1)}, \ldots, X^{(N)})$ an Ito process in $\mathbb{R}^N$ if, for each i, $X^{(i)}$ is an Ito process. The drift of X is the $\mathbb{R}^N$-valued process μ whose ith coordinate is the drift of $X^{(i)}$. The diffusion of X is the $\mathbb{R}^{N \times d}$-matrix-valued process σ whose ith row is the diffusion of $X^{(i)}$. In this case, we use the notation

$$dX_t = \mu_t \, dt + \sigma_t \, dB_t. \tag{31}$$

Ito's Formula. *Suppose X is the Ito process in $\mathbb{R}^N$ given by Equation (31) and $f: \mathbb{R}^N \times [0, \infty) \times \mathbb{R}$ is $C^{2,1}$; that is, f has at least two continuous derivatives with respect to its first (x) argument, and at least one continuous derivative with respect to its second (t) argument. Then $\{f(X_t, t): t \geqslant 0\}$ is an Ito process and, for any time t,*

$$f(X_t, t) = f(X_0, 0) + \int_0^t \mathcal{D}f(X_s, s) \, ds + \int_0^t f_x(X_s, s) \, \theta_s \, dB_s,$$

where

$$\mathcal{D}f(X_t, t) = f_x(X_t, t) \, \mu_t + f_t(X_t, t) + \tfrac{1}{2} \text{tr} \left[\sigma_t \sigma_t^{\top} f_{xx}(X_t, t) \right].$$

Here, f_x, f_t, and f_{xx} denote the obvious partial derivatives of f, valued in $\mathbb{R}^N$, $\mathbb{R}$, and $\mathbb{R}^{N \times N}$ respectively, and tr(A) denotes the *trace* of a square matrix A (the sum of its diagonal elements).

If X is an Ito process in $\mathbb{R}^N$ with $dX_t = \mu_t \, dt + \sigma_t \, dB_t$ and $\theta = (\theta^1, \ldots, \theta^N)$ is a vector of adapted processes such that $\theta \cdot \mu$ is in $\mathcal{L}^1$ and, for each i, $\theta \cdot \sigma^i$ is in $\mathcal{L}^2$, then we say that θ is in $\mathcal{L}(X)$, which means that the stochastic integral $\int \theta \, dX$ exists as an Ito process when defined by

$$\int_0^T \theta_t \, dX_t \equiv \int_0^T \theta_t \cdot \mu_t \, dt + \int_0^T \sigma_t^{\top} \theta_t \, dB_t, \qquad T \geqslant 0.$$

If X and Y are real-valued Ito processes with $dX_t = \mu_X(t) \, dt + \sigma_X(t) \, dB_t$ and $dY_t = \mu_Y(t) \, dt + \sigma_Y(t) \, dB_t$, then Ito's Formula (for $N = 2$) implies that the product $Z = XY$ is an Ito process, with

$$dZ_t = X_t \, dY_t + Y_t \, dX_t + \sigma_X(t) \cdot \sigma_Y(t) \, dt. \tag{32}$$

If μ_X, μ_Y, σ_X, and σ_Y are bounded and have continuous sample paths (weaker conditions would suffice), then it follows from Equation (32) that

$$\frac{d}{ds} \text{cov}_t(X_s, Y_s) \Big|_{s=t} = \sigma_X(t) \cdot \sigma_Y(t) \qquad \text{almost surely,}$$

where $\text{cov}_t(X_s, Y_s) = E_t(X_s Y_s) - E_t(X_s) E_t(Y_s)$, and where the derivative is taken from the right, extending the intuition developed with Equations (19) and (20).

3.5. Arbitrage modeling

Now, we turn to a more careful definition of arbitrage for purposes of establishing a close link between the absence of arbitrage and the existence of state prices.

Suppose the price processes of N given securities form an Ito process $X = (X^{(1)}, \ldots, X^{(N)})$ in $\mathbb{R}^N$. We suppose, for technical regularity, that each security price process is in the space H^2 containing any Ito process Y with $\mathrm{d}Y_t = a(t)\,\mathrm{d}t + b(t)\,\mathrm{d}B(t)$ for which

$$E\left[\left(\int_0^t a(s)\,\mathrm{d}s\right)^2\right] < \infty \quad \text{and} \quad E\left[\int_0^t b(s) \cdot b(s)\,\mathrm{d}s\right] < \infty.$$

We will suppose that the securities pay no dividends during the time interval $[0, T)$, and that X_T is the vector of cum-dividend security prices at time T.

A trading strategy θ is an $\mathbb{R}^N$-valued process θ in $\mathcal{L}(X)$, meaning simply that the stochastic integral $\int \theta\,\mathrm{d}X$ defining trading gains is well defined. A trading strategy θ is *self-financing* if

$$\theta_t \cdot X_t = \theta_0 \cdot X_0 + \int_0^t \theta_s\,\mathrm{d}X_s, \qquad t \leqslant T. \tag{33}$$

We suppose that there is some *short-rate process*, a process r with the property that $\int_0^T |r_t|\,\mathrm{d}t$ is finite almost surely and, for some security with strictly positive price process β,

$$\beta_t = \beta_0 \exp\left(\int_0^t r_s\,\mathrm{d}s\right), \qquad t \in [0, T]. \tag{34}$$

In this case, $\mathrm{d}\beta_t = r_t\,\beta_t\,\mathrm{d}t$, allowing us to view r_t as the riskless short-term continuously compounding rate of interest, in an instantaneous sense, and to view β_t as the market value of an account that is continually reinvested at the short-term interest rate r.

A self-financing strategy θ is an *arbitrage* if $\theta_0 \cdot X_0 < 0$ and $\theta_T \cdot X_T \geqslant 0$, or if $\theta_0 \cdot X_0 \leqslant 0$ and $\theta_T \cdot X_T > 0$. Our first goal is to characterize the properties of a price process X that admits no arbitrage, at least after placing some reasonable restrictions on trading strategies.

3.6. Numeraire invariance

It is often convenient to renormalize all security prices, sometimes relative to a particular price process. A *deflator* is a strictly positive Ito process. We can deflate the previously given security price process X by a deflator Y to get the new price process X^Y defined by $X_t^Y = X_t Y_t$. Such a renormalization has essentially no economic effects, as suggested by the following result.

Numeraire Invariance Theorem. *Suppose Y is a deflator. Then a trading strategy θ is self-financing with respect to X if and only if θ is self-financing with respect to X^Y.*

The proof is an application of Ito's Formula. We have the following corollary, which is immediate from the Numeraire Invariance Theorem, the strict positivity of Y, and the definition of an arbitrage. On numeraire invariance in more general settings, see Huang (1985a) and Protter (2001). [12]

Corollary. *Suppose Y is a deflator. A trading strategy is an arbitrage with respect to X if and only if it is an arbitrage with respect to the deflated price process X^Y.*

3.7. State prices and doubling strategies

Paralleling the terminology of Section 2.2, a *state-price density* is a deflator π with the property that the deflated price process X^π is a martingale. Other terms used for this concept in the literature are *state-price deflator*, *marginal-rate-of-substitution process*, and *pricing kernel*. In the discrete-state discrete-time setting of Section 2, we found that there is a state-price density if and only if there is no arbitrage. In a general continuous-time setting, this result is "almost" true, up to some technical issues.

A technical nuisance in a continuous-time setting is that, without some frictions limiting trade, arbitrage is to be expected. For example, one may think of a series of bets on fair and independent coin tosses at times 1/2, 3/4, 7/8, and so on. Suppose one's goal is to earn a riskless profit of α by time 1, where α is some arbitrarily large number. One can bet α on heads for the first coin toss at time 1/2. If the first toss comes up heads, one stops. Otherwise, one owes α to one's opponent. A bet of 2α on heads for the second toss at time 3/4 produces the desired profit if heads comes up at that time. In that case, one stops. Otherwise, one is down 3α and bets 4α on the third toss, and so on. Because there is an infinite number of potential tosses, one will eventually stop with a riskless profit of α (almost surely), because the probability of losing on every one of an infinite number of tosses is $(1/2) \cdot (1/2) \cdot (1/2) \cdots = 0$. This is a classic "doubling strategy" that can be ruled out either by a technical limitation, such as limiting the total number of bets, or by a credit restriction limiting the total amount that one is allowed to be in debt.

For the case of continuous-time trading strategies, [13] we will eliminate the possibility of "doubling strategies" with a credit constraint, defining the set $\Theta(X)$ of self-financing trading strategies satisfying the non-negative wealth restriction $\theta_t \cdot X_t \geqslant 0$ for all t. An alternative is to restrict trading strategies with a technical integrability condition, as reviewed in Duffie (2001). The next result is based on Dybvig and Huang (1988).

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[12] For more on the role of numeraire, see Geman, El Karoui and Rochet (1995).

[13] An actual continuous-time "doubling" strategy can be found in Karatzas (1993).

Proposition. *If there is a state-price density, then there is no arbitrage in* $\underline{\Theta}(X)$.

Weaker no-arbitrage conditions based on a lower bound on wealth or on integrability conditions, are summarized in Duffie (2001), who provides a standard proof of this result.

3.8. Equivalent martingale measures

In the finite-state setting of Section 2, it was shown that the existence of a state-price deflator is equivalent to the existence of an equivalent martingale measure (after some deflation). Here, we say that Q is an equivalent martingale measure for the price process X if Q is equivalent to P (they have the same events of zero probability), and if X is a martingale under Q.

Theorem. *If the price process X admits an equivalent martingale measure, then there is no arbitrage in* $\underline{\Theta}(X)$.

In most cases, the theorem is applied along the lines of the following corollary, a consequence of the corollary to the Numeraire Invariance Theorem of Section 3.6.

Corollary. *If there is a deflator Y such that the deflated price process X^Y admits an equivalent martingale measure, then there is no arbitrage in* $\underline{\Theta}(X)$.

As in the finite-state case, the absence of arbitrage and the existence of equivalent martingale measures are, in spirit, identical properties, although there are some technical distinctions in this infinite-dimensional setting. Inspired from early work by Kreps (1981), Delbaen and Schachermayer (1998) showed the equivalence, after deflation by a numeraire deflator, between no free lunch with vanishing risk (a slight strengthening of the notion of no arbitrage) and the existence of a local martingale measure.[14]

3.9. Girsanov and market prices of risk

We now look for convenient conditions on X supporting the existence of an equivalent martingale measure. We will also see how to calculate such a measure, and conditions for the uniqueness of such a measure, which is in spirit equivalent to complete markets. This is precisely the case for the finite-state setting of Theorem 2.9.

The basic approach is from Harrison and Kreps (1979) and Harrison and Pliska (1981), who coined most of the terms and developed most of the techniques and basic results. Huang (1985a,b) generalized the basic theory. The development here

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[14] For related results, see Ansel and Stricker (1992a,b), Back and Pliska (1987), Cassese (1996), Duffie and Huang (1986), El Karoui and Quenez (1995), Frittelli and Lakner (1995), Jacod and Shiryaev (1998), Kabanov (1997), Kabanov and Kramkov (1995), Kusuoka (1993), Lakner (1993), Leventhal and Skorohod (1995), Rogers (1994), Schachermayer (1992, 1994, 2002), Schweizer (1992) and Stricker (1990).

differs in some minor ways. Most of the results extend to an abstract filtration, not necessarily generated by Brownian motion, but the following important property of Brownian filtrations is somewhat special.

Martingale Representation Theorem. *For any martingale ξ, there exists some $\mathbb{R}^d$-valued process θ such that the stochastic integral $\int \theta \, dB$ exists and such that, for all t,*

$$\xi_t = \xi_0 + \int_0^t \theta_s \, dB_s.$$

Now, we consider any given probability measure Q equivalent to P, with density process ξ defined by (11). By the martingale representation theorem, we can express the martingale ξ in terms of a stochastic integral of the form

$$d\xi_t = \gamma_t \, dB_t,$$

for some adapted process $\gamma = (\gamma^{(1)}, \ldots, \gamma^{(d)})$ with $\int_0^T \gamma_t \cdot \gamma_t \, dt < \infty$ almost surely. *Girsanov's Theorem* states that a standard Brownian motion B^Q in $\mathbb{R}^d$ under Q is defined by $B_0^Q = 0$ and $dB_t^Q = dB_t + \eta_t \, dt$, where $\eta_t = -\gamma_t/\xi_t$.

Suppose the price process X of the N given securities (possibly after some change of numeraire) is an Ito process in $\mathbb{R}^N$, with

$$dX_t = \mu_t \, dt + \sigma_t \, dB_t.$$

We can therefore write

$$dX_t = (\mu_t - \sigma_t \eta_t) \, dt + \sigma_t \, dB_t^Q.$$

If X is to be a Q-martingale, then its drift under Q must be zero, which means that, almost everywhere,

$$\sigma(\omega, t) \, \eta(\omega, t) = \mu(\omega, t), \qquad (\omega, t) \in \Omega \times [0, T]. \tag{35}$$

Thus, the existence of a solution η to the system (35) of linear equations (almost everywhere) is necessary for the existence of an equivalent martingale measure for X. Under additional technical conditions, we will find that it is also sufficient.

We can also view a solution η to Equation (35) as providing a proportional relationship between mean rates of change of prices (μ) and the amounts (σ) of "risk" in price changes stemming from the underlying d Brownian motions. For this reason, any such solution η is called a *market-price-of-risk process* for X. The idea is that $\eta_i(t)$ is the "unit price", measured in price drift, of bearing exposure to the increment of $B^{(i)}$ at time t.

A *numeraire deflator* is a deflator that is the reciprocal of the price process of one of the securities. It is usually the case that one first chooses some numeraire deflator Y,

and then calculates the market price of risk for the deflated price process X^Y. This is technically convenient because one of the securities, the "numeraire", has a price that is always 1 after such a deflation. If there is a short-rate process r, a typical numeraire deflator is given by Y, where $Y_t = \exp(-\int_0^t r_s \, ds)$.

If there is no market price of risk, one may guess that something is "wrong", as the following result confirms.

Lemma. *Let Y be a numeraire deflator. If there is no market-price-of-risk process for X^Y, then there are arbitrages in $\Theta(X)$, and there is no equivalent martingale measure for X^Y.*

Proof: Suppose X^Y has drift process μ^Y and diffusion σ^Y, and that there is no solution η to $\sigma^Y \eta = \mu^Y$. Then, as a matter of linear algebra, there exists an adapted process θ taking values that are row vectors in $\mathbb{R}^N$ such that $\theta \sigma^Y \equiv 0$ and $\theta \mu^Y \neq 0$. By replacing $\theta(\omega, t)$ with zero for any (ω, t) such that $\theta(\omega, t) \mu^Y(\omega, t) < 0$, we can arrange to have $\theta \mu^Y > 0$. (This works provided the resulting process θ is not identically zero; in that case the same procedure applied to $-\theta$ works.) Finally, because the numeraire security associated with the deflator has a price that is identically equal to 1 after deflation, we can also choose the trading strategy for the numeraire so that, in addition to the above properties, θ is self-financing. That is, assuming without loss of generality that the numeraire security is the last security, we can let

$$
\theta_t^{(N)} = \left[-\sum_{i=1}^{N-1} \theta_t^{(i)} X_t^{Y,(i)} + \int_0^t \theta_s^{(i)} \, dX_s^{Y,(i)} \right].
$$

It follows that θ is a self-financing trading strategy with $\theta_0 \cdot X_0^Y = 0$, whose wealth process W, defined by $W_t = \theta_t \cdot X_t^Y$, is increasing and not constant. In particular, θ is in $\Theta(X^Y)$. It follows that θ is an arbitrage for X^Y, and therefore (by Numeraire Invariance) for X.

Finally, the reasoning leading to Equation (35) implies that if there is no market-price-of-risk process, then there can be no equivalent martingale measure for X^Y. $\square$

For any $\mathbb{R}^d$-valued adapted process η in $\mathcal{L}(B)$, we let ξ^η be defined by

$$
\xi_t^\eta = \exp\left(-\int_0^t \eta_s \, dB_s - \frac{1}{2} \int_0^t \eta_s \cdot \eta_s \, ds \right). \tag{36}
$$

Ito's Formula implies that $d\xi_t^\eta = -\xi_t^\eta \eta_t \, dB_t$. Novikov's Condition, a sufficient technical condition for ξ to be a martingale, is that

$$
E\left(\exp\left[\frac{1}{2} \int_0^T \eta_s \cdot \eta_s \, ds \right] \right) < \infty.
$$

Theorem. *If X has a market price of risk process η satisfying Novikov's condition, and moreover ξ_T^η has finite variance, then there is an equivalent martingale measure for X, and there is no arbitrage in $\Theta(X)$.*

Proof: By Novikov's Condition, ξ^η is a positive martingale. We have $\xi_0^\eta = e^0 = 1$, so ξ^η is indeed the density process of an equivalent probability measure Q defined by $\frac{dQ}{dP} = \xi_T^\eta$.

By Girsanov's Theorem, a standard Brownian motion B^Q in $\mathbb{R}^d$ under Q is defined by $dB_t^Q = dB_t + \eta_t \, dt$. Thus $dX_t = \sigma_t \, dB_t^Q$. As $\frac{dQ}{dP}$ has finite variance and each security price process $X^{(i)}$ is by assumption in H^2, we know by the Cauchy–Schwartz Inequality that

$$
E^Q \left[\left(\int_0^T \sigma^{(i)}(t) \cdot \sigma^{(i)}(t) \, dt \right)^{1/2} \right] = E^P \left[\left(\int_0^T \sigma^{(i)}(t) \cdot \sigma^{(i)}(t) \, dt \right)^{1/2} \frac{dQ}{dP} \right],
$$

is finite. Thus, $X^{(i)}$ is a Q-martingale by Proposition 3.2, and Q is therefore an equivalent martingale measure. The lack of arbitrage in $\underline{\Theta}(X)$ follows from Theorem 3.8. $\square$

Putting this result together with the previous lemma, we see that the existence of a market-price-of-risk process is necessary and, coupled with a technical integrability condition, sufficient for the absence of "well-behaved" arbitrages and the existence of an equivalent martingale measure. Huang and Pagès (1992) give an extension to the case of an infinite-time horizon.

For uniqueness of equivalent martingale measures, we can use the fact that, for any such measure Q, Girsanov's Theorem implies that we must have $\frac{dQ}{dP} = \xi_T^\eta$, for some market price of risk η. If $\sigma(\omega, t)$ is of maximal rank d, however, there can be at most one solution $\eta(\omega, t)$ to Equation (35). This maximal rank condition is equivalent to the condition that the span of the rows of $\sigma(\omega, t)$ is all of $\mathbb{R}^d$.

Proposition. *If $rank(\sigma) = d$ almost everywhere, then there is at most one market price of risk and at most one equivalent martingale measure. If there is a unique market-price-of-risk process, then $rank(\sigma) = d$ almost everywhere.*

With incomplete markets, significant attention in the literature has been paid to the issue of "which equivalent martingale measure to use" for the purpose of pricing contingent claims that are not redundant. Babbs and Selby (1996), Bühlmann, Delbaen, Embrechts and Shiryaev (1998), and Föllmer and Schweizer (1990) suggest some selection criteria or parameterization for equivalent martingale measures in incomplete markets. In particular, Artzner (1995), Bajeux-Besnainou and Portait (1997), Dijkstra (1996), Johnson (1994) and Long (1990) address the *numeraire portfolio*, also called *growth-optimal portfolio*, as a device for selecting a state-price density. Little of this literature offers an economic theory for the use of a particular measure for pricing new contingent claims that are not already traded (or replicated) by the given primitive securities.

3.10. Black–Scholes again

Suppose the given security-price process is $X = (S^{(1)}, \ldots, S^{(N-1)}, \beta)$, where, for $S = (S^{(1)}, \ldots, S^{(N-1)})$,

$$dS_t = \mu_t \, dt + \sigma_t \, dB_t,$$

and

$$d\beta_t = r_t \beta_t \, dt; \qquad \beta_0 > 0,$$

where μ, σ, and r are adapted processes (valued in $\mathbb{R}^{N-1}$, $\mathbb{R}^{(N-1)\times d}$, and $\mathbb{R}$, respectively). We also suppose for technical convenience that the short-rate process r is bounded. Then $Y = \beta^{-1}$ is a convenient numeraire deflator, and we let $Z = SY$. By Ito's Formula,

$$dZ_t = \left(-r_t Z_t + \frac{\mu_t}{\beta_t} \right) dt + \frac{\sigma_t}{\beta_t} \, dB_t.$$

In order to apply Theorem 3.9 to the deflated price process $\widehat{X} = (Z, 1)$, it would be enough to know that Z has a market price of risk η and that the variance of ξ_T^η is finite. Given this, there would be an equivalent martingale measure Q and no arbitrage in $\Theta(X)$. Suppose, for the moment, that this is the case. By Girsanov's Theorem, there is a standard Brownian motion B^Q in $\mathbb{R}^d$ under Q such that

$$dZ_t = \frac{\sigma_t}{\beta_t} \, dB_t^Q.$$

Because $S = \beta Z$, another application of Ito's Formula yields

$$dS_t = r_t S_t \, dt + \sigma_t \, dB_t^Q. \tag{37}$$

Equation (37) is an important intermediate result for arbitrage-free asset pricing, giving an explicit expression for security prices under a probability measure Q with the property that the "discounted" price process S/β is a martingale. For example, this leads to an easy recovery of the Black–Scholes formula, as follows.

Suppose that, of the securities with price processes $S^{(1)}, \ldots, S^{(N-1)}$, one is a call option on another. For convenience, we denote the price process of the call option by U and the price process of the underlying security by V, so that $U_T = (V_T - K)^+$, for expiration at time T with some given exercise price K. Because UY is by assumption a martingale under Q, we have

$$U_t = \beta_t E_t^Q \left(\frac{U_T}{\beta_T} \right) = E_t^Q \left(\exp\left[-\int_t^T r(s) \, ds \right] (V_T - K)^+ \right). \tag{38}$$

The reader may verify that this is the Black–Scholes formula for the case of $d = 1$, $V_0 > 0$, and with constants $\bar{r}$ and non-zero $\bar{\sigma}$ such that for all

t, $r_t = \bar{r}$ and $dV_t = V_t \mu_V(t)\, dt + V_t \bar{\sigma}\, dB_t$, where μ_V is a bounded adapted process. Indeed, in this case, Z has a market-price-of-risk process η such that ξ_T^η has finite variance, an exercise, so the assumption of an equivalent martingale measure is justified. More precisely, it is sufficient for the absence of arbitrage that the option-price process is given by Equation (38). Necessity of the Black–Scholes formula for the absence of arbitrages in $\Theta(X)$ is addressed in Duffie (2001). We can already see, however, that the expectation in Equation (38) defining the Black–Scholes formula does not depend on which equivalent martingale measure Q one chooses, so one should expect that the Black–Scholes formula (38) is also necessary for the absence of arbitrage. If Equation (38) is not satisfied, for instance, there cannot be an equivalent martingale measure for S/β. Unfortunately, and for purely technical reasons, this is not enough to imply directly the necessity of Equation (38) for the absence of well-behaved arbitrage, because we do not have a precise equivalence between the absence of arbitrage and the existence of equivalent martingale measures.

In the Black–Scholes setting, σ is of maximal rank $d = 1$ almost everywhere. Thus, from Proposition 3.9, there is exactly one equivalent martingale measure.

The detailed calculations of Girsanov's Theorem appear nowhere in the actual solution (37) for the "risk-neutral behavior" of arbitrage-free security prices, which can be given by inspection in terms of σ and r only.

3.11. Complete markets

We say that a random variable W can be *replicated* by a self-financing trading strategy θ if $W = \theta_T \cdot X_T$. Our basic objective in this section is to give a simple spanning condition on the diffusion σ of the price process X under which, up to technical integrability conditions, any random variable can be replicated (without resorting to "doubling strategies").

Proposition. *Suppose Y is a numerator deflator and Q is an equivalent martingale measure for the deflated price process X^Y. Suppose the diffusion σ^Y of X^Y is of maximal rank d almost everywhere. Let W be any random variable with $E^Q(|WY|) < \infty$. Then there is a self-financing trading strategy θ that replicates W and whose deflated market-value process $\{\theta_t \cdot X_t^Y : \leqslant t \leqslant T\}$ is a Q-martingale.*

Proof: Without loss of generality, the numeraire is the last of the N securities, so we write $X^Y = (Z, 1)$. Let B^Q be the standard Brownian motion in $\mathbb{R}^d$ under Q obtained by Girsanov's Theorem. The martingale representation property implies that, for any Q-martingale, there is some φ such that

$$E_t^Q(WY_T) = E^Q(WY_T) + \int_0^t \varphi_s\, dB_s^Q, \quad t \in [0, T]. \tag{39}$$

By the rank assumption on σ^Y and the fact that $\sigma_{Nt}^Y = 0$, there are adapted processes $\theta^{(1)}, \ldots, \theta^{(N-1)}$ solving

$$\sum_{j=1}^{N-1} \theta_t^{(j)} \sigma_{jt}^Y = \varphi_t^\top, \qquad t \in [0, T]. \tag{40}$$

Let $\theta^{(N)}$ be defined by

$$\theta_t^{(N)} = E^Q(WY_T) + \sum_{i=1}^{N-1} \left(\int_0^t \theta_s^{(i)} \, dZ_s^{(i)} - \theta_t^{(i)} Z_t^{(i)} \right). \tag{41}$$

Then $\theta = (\theta^{(1)}, \ldots, \theta^{(N)})$ is self-financing and $\theta_T \cdot X_T^Y = WY_T$. By the Numeraire Invariance Theorem, θ is also self-financing with respect to X and $\theta_T \cdot X_T = W$. As $\int \varphi \, dB^Q$ is by construction a Q-martingale, Equations (39–41) imply that $\{\theta_t \cdot X_t^Y : 0 \leqslant t \leqslant T\}$ is a Q-martingale. $\square$

The property that the deflated market-value process $\{\theta_t \cdot X_t^Y : 0 \leqslant t \leqslant T\}$ is a Q-martingale ensures that there is no use of doubling strategies. For example, if $W \geqslant 0$, then the martingale property implies that $\theta_t \cdot X_t \geqslant 0$ for all t.

Analogues to some of the results in this section for the case of market imperfections such as portfolio constraints or transactions costs are provided by Ahn, Dayal, Grannan and Swindle (1995), Bergman (1995), Constantinides and Zariphopoulou (1999, 2001), Cvitanić and Karatzas (1993), Davis and Clark (1993), Grannan and Swindle (1996), Henrotte (1991), Jouini and Kallal (1993), Karatzas and Kou (1998), Kusuoka (1992, 1995), Soner, Shreve and Cvitanić (1994) and Whalley and Wilmott (1997). Many of these results are asymptotic, for "small" proportional transactions costs, based on the approach of Leland (1985).

3.12. Optimal trading and consumption

We now apply the "martingale" characterization of the cost of replicating an arbitrary payoff, given in the last proposition, to the problem of optimal portfolio and consumption processes.

The setting is *Merton's problem*, as formulated and solved in certain settings, for geometric Brownian prices, by Merton (1971). Merton used the method of dynamic programming, solving the associated Hamilton–Jacobi–Bellman (HJB) equation. [15] A major alternative method is the martingale approach to optimal investment, which reached a key stage of development with Cox and Huang (1989), who treat the agent's candidate consumption choice as though it is a derivative security, and maximize

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[15] The book of Fleming and Soner (1993) treats HJB equations, stochastic control problems, emphasizing the use of viscosity methods.

the agent's utility subject to a wealth constraint on the arbitrage-free price of the consumption. Since that price can be calculated in terms of the given state-price density, the result is a simple static optimization problem.[16] Karatzas and Shreve (1998) provide a comprehensive treatment of optimal portfolio and consumption processes in this setting.

Fixing a probability space $(\Omega, \mathcal{F}, P)$ and the standard filtration $\{\mathcal{F}_t : t \geqslant 0\}$ of a standard Brownian motion B in $\mathbb{R}^d$, we suppose that $X = (X^{(0)}, X^{(1)}, \ldots, X^{(N)})$ is an Ito process in $\mathbb{R}^{N+1}$ for the prices of $N + 1$ securities, with

$$\mathrm{d}X_t^{(i)} = \mu_t^{(i)} X_t^{(i)} \, \mathrm{d}t + X_t^{(i)} \sigma_t^{(i)} \, \mathrm{d}B_t; \qquad X_0^{(i)} > 0, \tag{42}$$

where $\mu = (\mu^{(0)}, \ldots, \mu^{(N)})$ and the $\mathbb{R}^{N \times d}$-valued process σ are bounded adapted processes. Letting $\sigma^{(i)}$ denote the ith row of σ, we suppose that $\sigma^{(0)} = 0$, so that we can treat $\mu^{(0)}$ as the short-rate process r. A special case of this setup is to have geometric Brownian security prices and a constant short rate, which was the setting of Merton's original problem.

We assume for simplicity that $N = d$. The excess expected returns of the "risky" securities are defined by the $\mathbb{R}^N$-valued process λ given by $\lambda_t^{(i)} = \mu_t^{(i)} - r_t$. A deflated price process $\widehat{X}$ is defined by $\widehat{X}_t = X_t \exp(-\int_0^t r_s \, \mathrm{d}s)$. We assume that σ is invertible (almost everywhere) and that the market-price-of-risk process η for $\widehat{X}$, defined by $\eta_t = \sigma_t^{-1} \lambda_t$, is bounded. It follows that markets are complete (in the sense of Proposition 3.11) and that there are no arbitrages meeting the standard credit constraint of non-negative wealth.

In this setting, a state-price density π is defined by

$$\pi_t = \exp\left(-\int_0^t r_s \, \mathrm{d}s\right) \xi_t, \tag{43}$$

where ξ^η is the density process defined by Equation (36) for an equivalent martingale measure Q, after deflation by $\exp[\int_0^t -r(s) \, \mathrm{d}s]$.

Utility is defined over the space D of *consumption* pairs (c, Z), where c is an adapted nonnegative consumption-rate process with $\int_0^T c_t \, \mathrm{d}t < \infty$ almost surely, and Z is an $\mathcal{F}_T$-measurable nonnegative random variable describing terminal lump-sum consumption. Specifically, $U: D \to \mathbb{R}$ is defined by

$$U(c, Z) = E\left[\int_0^T u(c_t, t) \, \mathrm{d}t + F(Z)\right], \tag{44}$$

where
- $F: \mathbb{R}_+ \to \mathbb{R}$ is increasing and concave with $F(0) = 0$;

---

[16] The related literature is immense, and includes Cox (1983), Pliska (1986), Cox and Huang (1991), Back (1986, 1991), Back and Pliska (1987), Duffie and Skiadas (1994), Foldes (1978a,b, 1990, 1991a,b, 1992, 2001), Harrison and Kreps (1979), Huang (1985b), Huang and Pagès (1992), Karatzas, Lehoczky and Shreve (1987), Lakner and Slud (1991), Pagès (1987) and Xu and Shreve (1992).

- u: $\mathbb{R}_+ \times [0, T] \to \mathbb{R}$ is continuous and, for each t in $[0, T]$, $u(\cdot, t)$: $\mathbb{R}_+ \to \mathbb{R}$ is increasing and concave, with $u(0, t) = 0$;
- F is strictly concave or zero, or for each t in $[0, T]$, $u(\cdot, t)$ is strictly concave or zero.
- At least one of u and F is non-zero.

A trading strategy is a process $\theta = (\theta^{(0)}, \ldots, \theta^{(N)})$ in $\mathcal{L}(X)$, meaning merely that the gain-from-trade stochastic integral $\int \theta \, dX$ exists. Given an initial wealth $w > 0$, we say that (c, Z, θ) is *budget-feasible* if (c, Z) is a consumption choice in D and θ is a trading strategy satisfying

$$\theta_t \cdot X_t = w + \int_0^t \theta_s \, dX_s - \int_0^t c_s \, ds \geqslant 0, \qquad t \in [0, T], \tag{45}$$

and

$$\theta_T \cdot X_T \geqslant Z. \tag{46}$$

The first restriction (45) is that the current market value $\theta_t \cdot X_t$ of the trading strategy is non-negative, a credit constraint, and is equal to its initial value w, plus any gains from security trade, less the cumulative consumption to date. The second restriction (46) is that the terminal portfolio value is sufficient to cover the terminal consumption. We now have the problem, for each initial wealth w,

$$\sup_{(c, Z, \theta) \in \Lambda(w)} U(c, Z), \tag{47}$$

where $\Lambda(w)$ is the set of budget-feasible choices at wealth w. First, we state an extension of the numeraire invariance result of Section 3.4, which obtains from an application of Ito's Formula.

Lemma. *Let Y be any deflator. Given an initial wealth $w \geqslant 0$, a strategy (c, Z, θ) is budget-feasible given price process X if and only if it is budget feasible after deflation, that is,*

$$\theta_t \cdot X_t^Y = wY_0 + \int_0^t \theta_s \, dX_s^Y - \int_0^t Y_s c_s \, ds \geqslant 0, \qquad t \in [0, T], \tag{48}$$

and

$$\theta_T \cdot X_T^Y \geqslant ZY_T. \tag{49}$$

With numeraire invariance, we can reduce the dynamic trading and consumption problem to a static optimization problem subject to an initial wealth constraint, as follows.

Proposition. *Given a consumption choice* (c, Z) *in D, there exists a trading strategy* θ *such that* (c, Z, θ) *is budget-feasible at initial wealth w if and only if*

$$E \left(\pi_T Z + \int_0^T \pi_t c_t \, dt \right) \leqslant w. \tag{50}$$

Proof: Suppose (c, Z, θ) is budget-feasible. Applying the previous numeraire-invariance lemma to the state-price deflator π, and using the fact that $\pi_0 = \xi_0 = 1$, we have

$$w + \int_0^T \theta_t \, dX_t^\pi \geqslant \pi_T Z + \int_0^T \pi_t c_t \, dt. \tag{51}$$

Because X^π is a martingale under P, the process M, defined by $M_t = w + \int_0^t \theta_s \, dX_s^\pi$, is a non-negative local martingale, and therefore a supermartingale. For the definitions of local martingale and supermartingale, and for this property, see for example Protter (1990). By the supermartingale property, $M_0 \geqslant E(M_T)$. Taking expectations through Equation (51) thus leaves Equation (50).

Conversely, suppose (c, Z) satisfies Equation (50), and let M be the Q-martingale defined by

$$M_t = E_t^Q \left(e^{-rT} Z + \int_0^T e^{-rt} c_t \, dt \right).$$

By Girsanov's Theorem, a standard Brownian motion B^Q in $\mathbb{R}^d$ under Q is defined by $dB_t^Q = dB_t + \eta_t \, dt$, and B^Q has the martingale representation property. Thus, there is some $\varphi = (\varphi^{(1)}, \ldots, \varphi^{(d)})$ in $\mathcal{L}(B^Q)$ such that

$$M_t = M_0 + \int_0^t \varphi_s \, dB_s^Q, \qquad t \in [0, T],$$

where $M_0 \leqslant w$. For the deflator Y defined by $Y_t = \exp[-\int_0^t r(s) \, ds]$, we also know that $\widehat{X} = X^Y$ is a Q-martingale. From the definitions of the market price of risk η and of B^Q,

$$d\widehat{X}_t^{(i)} = \widehat{X}_t^{(i)} \sigma_t^{(i)} \, dB_t^Q, \qquad 1 \leqslant i \leqslant N.$$

Because σ_t is invertible and $\widehat{X}$ is strictly positive with continuous sample paths, we can choose $\theta^{(i)}$ in $\mathcal{L}(X^{(i)})$ for each $i \leqslant N$ such that

$$\left(\theta_t^{(1)} \widehat{X}_t^{(1)}, \ldots, \theta_t^{(N)} \widehat{X}_t^{(N)} \right) \sigma_t = \varphi_t^\top, \qquad t \in [0, T].$$

This implies that

$$M_t = M_0 + \sum_{i=1}^{N} \int_0^t \theta_s^{(i)} \, d\widehat{X}_s^{(i)}. \tag{52}$$

We can also let

$$\theta_t^{(0)} = w + \sum_{i=1}^{N} \int_0^t \theta_s^{(i)} \, d\widehat{X}_s^{(i)} - \sum_{i=1}^{N} \theta_t^{(i)} \widehat{X}_t^{(i)} - \int_0^t e^{-rs} c_s \, ds. \tag{53}$$

From Equation (50) and the fact that $\xi_t = \pi_t \exp[\int_0^t r(s) \, ds]$ defines the density process for Q,

$$M_0 = E^Q \left(e^{-rT} Z + \int_0^T e^{-rt} c_t \, dt \right) \leqslant w. \tag{54}$$

From Equations (53) and (52), and the fact that $\int \theta^{(0)} \, d\widehat{X}^{(0)} = 0$,

$$\theta_t \cdot \widehat{X}_t = w + \int_0^t \theta_s \, d\widehat{X}_s - \int_0^t e^{-rs} c_s \, ds,$$

$$= w + M_t - M_0 - \int_0^t e^{-rs} c_s \, ds,$$

$$= w - M_0 + E_t^Q \left(\int_t^T e^{-rs} c_s \, ds + e^{-rT} Z \right) \geqslant 0,$$

using Equation (54). With numeraire invariance, Equation (45) follows. We can also use the same inequality for $t = T$, Equation (54), and the fact that $M_T = \exp[-\int_0^T r(s) \, ds] Z + \int_0^T \exp[-\int_0^t r(s) \, ds] c_t \, dt$ to obtain Equation (46). Thus, (c, Z, θ) is budget-feasible. $\square$

Corollary. *Given a consumption choice* (c^*, Z^*) *in D and some initial wealth w, there exists a trading strategy* θ^* *such that* (c^*, Z^*, θ^*) *solves Merton's problem (47) if and only if* (c^*, Z^*) *solves the problem*

$$\sup_{(c,Z) \in D} U(c, Z) \quad \text{subject to} \quad E\left(\int_0^T \pi_t c_t \, dt + \pi_T Z \right) \leqslant w. \tag{55}$$

3.13. Martingale solution to Merton's problem

We are now in a position to obtain a relatively explicit solution to Merton's problem (47) by using the equivalent formulation (55).

By the Saddle Point Theorem and the strict monotonicity of U, (c^*, Z^*) solves (55) if and only if there is a scalar Lagrange multiplier $\gamma^* > 0$ such that, *first*: (c^*, Z^*) solves the unconstrained problem

$$\sup_{(c,Z) \in D} \mathcal{L}(c, Z; \gamma^*), \tag{56}$$

where, for any $\gamma \geqslant 0$,

$$\mathcal{L}(c, Z; \gamma) = U(c, Z) - \gamma E \left(\pi_T Z + \int_0^T \pi_t c_t \, dt - w \right), \tag{57}$$

and *second*, (c^*, Z^*) satisfies the complementary-slackness condition

$$E \left(\pi_T Z^* + \int_0^T \pi_t c_t^* \, dt \right) = w. \tag{58}$$

We can summarize our progress on Merton's problem (47) as follows.

Proposition. *Given some (c^*, Z^*) in D, there is a trading strategy θ^* such that (c^*, Z^*, θ^*) solves Merton's problem (47) if and only if there is a constant $\gamma^* > 0$ such that (c^*, Z^*) solves Equation (56) and $E(\pi_T Z^* + \int_0^T \pi_t c_t^* \, dt) = w$.*

In order to obtain intuition for the solution of (56), we begin with some arbitrary $\gamma > 0$ and treat $U(c, Z) = E[\int_0^T u(c_t, t) \, dt + F(Z)]$ intuitively by thinking of "E" and "$\int$" as finite sums, in which case the first-order conditions for optimality of $(c^*, Z^*) \gg 0$ for the problem $\sup_{(c,Z)} \mathcal{L}(c, Z; \gamma)$, assuming differentiability of u and F, are

$$u_c(c_t^*, t) - \gamma \pi_t = 0, \qquad t \in [0, T], \tag{59}$$

and

$$F'(Z^*) - \gamma \pi_T = 0. \tag{60}$$

Solving, we have

$$c_t^* = I(\gamma \pi_t, t), \qquad t \in [0, T], \tag{61}$$

and

$$Z^* = I_F(\gamma \pi_T), \tag{62}$$

where $I(\cdot, t)$ inverts [17] $u_c(\cdot, t)$ and where I_F inverts F'. We will confirm these conjectured forms (61) and (62) of the solution in the next theorem. Under strict

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[17] If $u = 0$, we take $I = 0$. If $F = 0$, we take $I_F = 0$.

concavity of u or F, the inversions $I(\cdot, t)$ and I_F, respectively, are continuous and strictly decreasing. A decreasing function $\hat{w} \colon (0, \infty) \to \overline{\mathbb{R}}$ is therefore defined by

$$\hat{w}(\gamma) = E \left[\int_0^T \pi_t I\left(\gamma \pi_t, t\right) \, dt + \pi_T I_F\left(\gamma \pi_T\right) \right]. \tag{63}$$

(We have not yet ruled out the possibility that the expectation may be $+\infty$). All of this implies that (c^*, Z^*) of Equations (61) and (62) solves Problem (55) provided the required initial investment $\hat{w}(\gamma)$ is equal to the endowed initial wealth w. This leaves an equation $\hat{w}(\gamma) = w$ to solve for the "correct" Lagrange multiplier γ^*, and with that an explicit solution to the optimal consumption policy for Merton's problem.

We now consider properties of u and F guaranteeing that $\hat{w}(\gamma) = w$ can be solved for a unique $\gamma^* > 0$. A strictly concave increasing function $F \colon \mathbb{R}_+ \to \mathbb{R}$ that is differentiable on $(0, \infty)$ satisfies *Inada conditions* if $\inf_x F'(x) = 0$ and $\sup_x F'(x) = +\infty$. If F satisfies these Inada conditions, then the inverse I_F of F' is well defined as a strictly decreasing continuous function on $(0, \infty)$ whose image is $(0, \infty)$.

Condition A. *Either F is zero or F is differentiable on $(0, \infty)$, strictly concave, and satisfies Inada conditions. Either u is zero or, for all t, $u(\cdot, t)$ is differentiable on $(0, \infty)$, strictly concave, and satisfies Inada conditions. For each $\gamma > 0$, $\hat{w}(\gamma)$ is finite.*

We recall the standing assumption that at least one of u and F is nonzero. The assumption of finiteness of $\hat{w}(\cdot)$ has been shown by Kramkov and Schachermayer (1999) to follow from natural regularity conditions.

Theorem. *Under Condition A and the standing conditions on μ, σ, and r, for any $w > 0$, Merton's problem has the optimal consumption policy given by Equations (61) and (62) for a unique scalar $\gamma > 0$.*

Proof: Under Condition A, the Dominated Convergence Theorem implies that $\hat{w}(\cdot)$ is continuous. Because one or both of $I(\cdot, t)$ and $I_F(\cdot)$ have $(0, \infty)$ as their image and are strictly decreasing, $\hat{w}(\cdot)$ inherits these two properties. From this, given any initial wealth $w > 0$, there is a unique γ^* with $\hat{w}(\gamma^*) = w$. Let (c^*, Z^*) be defined by Equation (61) and (62), taking $\gamma = \gamma^*$. The previous proposition tells us there is a trading strategy θ^* such that (c^*, Z^*, θ^*) is budget-feasible. Let (θ, c, Z) be any budget-feasible choice. The previous proposition also implies that (c, Z) satisfies Equation (50). For each (ω, t), the first-order conditions (59) and (60) are sufficient (by concavity of u and F) for optimality of $c^*(\omega, t)$ and $Z^*(\omega)$ in the problems

$$\sup_{\bar{c} \in [0, \infty)} u\left(\bar{c}, t\right) - \gamma^* \pi(\omega, t) \bar{c},$$

and

$$\sup_{\bar{Z} \in [0, \infty)} F\left(\bar{Z}\right) - \gamma^* \pi(\omega, T) \bar{Z},$$

respectively. Thus,

$$u(c_t^*, t) - \gamma^* \pi_t c_t^* \geqslant u(c_t, t) - \gamma^* \pi_t c_t, \quad 0 \leqslant t \leqslant T, \tag{64}$$

and

$$F(Z^*) - \gamma^* \pi_T Z^* \geqslant F(Z) - \gamma^* \pi_T Z. \tag{65}$$

Integrating Equation (64) from 0 to T, adding Equation (65), taking expectations, and then applying the complementary slackness condition (58) and the budget constraint (50), leaves $U(c^*, Z^*) \geqslant U(c, Z)$. As (c, Z, θ) is arbitrary, this implies the optimality of (c^*, Z^*, θ^*). $\square$

In practice, solving the equation $\hat{w}(\gamma^*) = w$ for γ^* may require a one-dimensional numerical search, which is straightforward because $\hat{w}(\cdot)$ is strictly monotone.

This result, giving a relatively explicit consumption solution to Merton's problem, has been extended in many directions, even generalizing the assumption of additive utility to allow for habit-formation or recursive utility, as shown by Schroder and Skiadas (1999).

For a specific example, we treat terminal consumption only by taking $u \equiv 0$, and we let $F(w) = w^\alpha / \alpha$ for $\alpha \in (0, 1)$. Then $c^* = 0$ and the calculations above imply that $\hat{w}(\gamma) = E[\pi_T(\gamma \pi_T)^{1/(\alpha-1)}]$. Solving $\hat{w}(\gamma^*) = w$ for γ^* leaves

$$\gamma^* = w^{\alpha-1} E\left(\pi_T^{\alpha/(\alpha-1)}\right)^{1-\alpha}.$$

From Equation (62),

$$Z^* = I_F\left(\gamma^* \pi_T\right).$$

Although this approach generates a straightforward solution for the optimal consumption policy, the form of the optimal trading strategy can be difficult to determine. For the special case of geometric Brownian price processes (constant μ and σ) and a constant short rate r, we can calculate that $Z^* = W_T$ where W is the geometric Brownian wealth process obtained from

$$dW_t = W_t(r + \overline{\varphi} \cdot \lambda)\, dt + W_t \overline{\varphi}^\top \sigma\, dB_t; \qquad W_0 = w,$$

where $\overline{\varphi} = (\sigma\sigma^\top)^{-1} \lambda/(1 - \alpha)$ is the vector of fixed optimal portfolio fractions.

More generally, in a Markov setting, one can derive a PDE for the wealth process, as for the pricing approach to Black–Scholes option pricing formula, and from the derivatives of the solution function obtain the associated trading strategy. Merton's original stochastic-control approach, in a Markov setting, gives explicit solutions for the optimal trading strategy in terms of the derivatives of the value function solving the HJB equation. Although there are only a few examples in which these derivatives are

known explicitly, they can be approximated by a numerical solution of the Hamilton–Jacobi–Bellman equation.

This martingale approach to solving Problem (47) has been extended with duality techniques and other methods to cases of investment with constraints, including incomplete markets. See, for example, Cvitanić and Karatzas (1996), Cvitanić, Wang and Schachermayer (2001), Cuoco (1997), and the many sources cited by Karatzas and Shreve (1998).

4. Term-structure models

This section reviews models of the term structure of interest rates. These models are used to analyze the dynamic behavior of bond yields and their relationships with macro-economic covariates, and also for the pricing and hedging of fixed-income securities, those whose future payoffs are contingent on future interest rates. Term-structure modeling is one of the most active and sophisticated areas of application of financial theory to everyday business problems, ranging from managing the risk of a bond portfolio to the design and pricing of collateralized mortgage obligations. In this section, we treat default-free instruments. In Section 6, we turn to defaultable bonds. This section provides only a small skeleton of the extensive literature on term-structure models. More extensive notes to the literature are found in Duffie (2001) and in the surveys by Dai and Singleton (2003) and Piazzesi (2002).

We first treat the standard "single-factor" examples of Merton (1974), Cox, Ingersoll and Ross (1985a), Dothan (1978), Vasicek (1977), Black, Derman and Toy (1990), and some of their variants. These models treat the entire term structure of interest rates at any time as a function of a single state variable, the short rate of interest. We will then turn to multi-factor models, including multi-factor affine models, extending the Cox–Ingersoll–Ross and Vasicek models. Finally, we turn to the term-structure framework of Heath, Jarrow and Morton (1992), which allows, under technical conditions, any initial term structure of forward interest rates and any process for the conditional volatilities and correlations of these forward rates.

Numerical tractability is essential for practical and econometric applications. One must fit model parameters from time-series or cross-sectional data on bond and derivative prices. A fitted model may be used to price or hedge related contingent claims. Typical numerical methods include "binomial trees," Fourier-transform methods, Monte-Carlo simulation, and finite-difference solution of PDEs. Even the "zero curve" of discounts must be fitted to the prices of coupon bonds.[18] In

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[18] See Adams and Van Deventer (1994), Coleman, Fisher and Ibbotson (1992), Diament (1993), Fisher, Nychka and Zervos (1994), Jaschke (1996), Konno and Takase (1995, 1996) and Svensson and Dahlquist (1996). Consistency of the curve-fitting method with an underlying term-structure model is examined by Björk and Christensen (1999), Björk and Gombani (1999) and Filipović (1999).

econometric applications, bond or option prices must be solved repeatedly for a large sample of dates and instruments, for each of many candidate parameter choices.

We fix a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathbb{F} = \{\mathcal{F}_t: 0 \leqslant t \leqslant T\}$ satisfying the usual conditions,[19] as well as a short-rate process r. We have departed from a dependence on Brownian information in order to allow for "surprise jumps", which are important in certain applications.

A *zero-coupon bond* maturing at some future time $s > t$ pays no dividends before time s, and offers a fixed lump-sum payment at time s that we can take without loss of generality to be 1 unit of account. Although it is not always essential to do so, we assume throughout that such a bond exists for each maturity date s. One of our main objectives is to characterize the price $\Lambda_{t,s}$ at time t of the s-maturity bond, and its behavior over time.

We fix some equivalent martingale measure Q, after taking as a numeraire for deflation purposes the market value $\exp[\int_0^t r(s)\,ds]$ of investments rolled over at the short-rate process r. The price at time t of the zero-coupon bond maturing at s is then

$$\Lambda_{t,s} \equiv E_t^Q \left(\exp\left[-\int_t^s r(u)\,du \right] \right). \tag{66}$$

The term structure is often expressed in terms of the *yield curve*. The *continuously compounding yield* $y_{t,\tau}$ on a zero-coupon bond maturing at time $t + \tau$ is defined by

$$y_{t,\tau} = -\frac{\log\left(\Lambda_{t,\,t+\tau}\right)}{\tau}.$$

The term structure can also be represented in terms of forward interest rates, as explained later in this section.

4.1. One-factor models

A *one-factor term-structure model* means a model of r that satisfies a stochastic differential equation (SDE) of the form

$$dr_t = \mu(r_t, t)\,dt + \sigma(r_t, t)\,dB_t^Q, \tag{67}$$

where B^Q is a standard Brownian motion under Q and where $\mu\colon \mathbb{R} \times [0, T] \to \mathbb{R}$ and $\sigma\colon \mathbb{R} \times [0, T] \to \mathbb{R}^d$ satisfy technical conditions guaranteeing the existence of a solution to Equation (67) such that, for all t and $s \geqslant t$, the price $\Lambda_{t,s}$ of the zero-coupon bond maturing at s is finite and well defined by Equation (66).

The one-factor models are so named because the Markov property (under Q) of the solution r to Equation (67) implies, from Equation (66), that the short rate is the only

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[19] For these technical conditions, see for example, Protter (1990).

Table 1
Common single-factor model parameters, Equation (68)

Model	K_0	K_1	K_2	H_0	H_1	ν
Cox, Ingersoll and Ross (1985a)	•	•			•	0.5
Pearson and Sun (1994)	•	•		•	•	0.5
Dothan (1978)					•	1.0
Brennan and Schwartz (1977)	•	•			•	1.0
Merton (1974), Ho and Lee (1986)	•			•		1.0
Vasicek (1977)	•	•		•		1.0
Black and Karasinski (1991)		•	•		•	1.0
Constantinides and Ingersoll (1984)					•	1.5

state variable, or "factor", on which the current yield curve depends. That is, for all t and $s \geqslant t$, we can write $y_{t,s} = F(t,s,r_t)$, for some fixed $F: [0,T] \times [0,T] \times \mathbb{R} \to \mathbb{R}$.

Table 1 shows many of the parametric examples of one-factor models appearing in the literature, with their conventional names. Each of these models is a special case of the SDE

$$dr_t = [K_{0t} + K_{1t}r_t + K_{2t}r_t \log(r_t)] \, dt + [H_{0t} + H_{1t}r_t]^\nu \, dB_t^Q, \tag{68}$$

for deterministic coefficients $K_{0t}, K_{1t}, K_{2t}, H_{0t}$ and H_{1t} depending continuously on t, and for some exponent $\nu \in [0.5, 1.5]$. Coefficient restrictions, and restrictions on the space of possible short rates, are needed for the existence and uniqueness of solutions. For each model, Table 1 shows the associated exponent ν, and uses the symbol "•" to indicate those coefficients that appear in nonzero form. We can view a negative coefficient K_{1t} as a *mean-reversion* parameter, in that a higher short rate generates a lower drift, and vice versa. Empirically speaking, mean reversion is widely believed to be a useful attribute to include in single-factor short-rate models.[20]

Non-parametric single-factor models are estimated by Aït-Sahalia (1996a,b, 2002). The empirical evidence, as examined for example by Dai and Singleton (2000), however, points strongly toward multifactor extensions, to which we will turn shortly.

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[20] In most cases, the original versions of these models had constant coefficients, and were only later extended to allow K_{it} and H_{it} to depend on t, for practical reasons, such as calibration of the model to a given set of bond and option prices. The Gaussian short-rate model of Merton (1974), who originated much of the approach taken here, was extended by Ho and Lee (1986), who developed the idea of calibration of the model to the current yield curve. The calibration idea was further developed by Black, Derman and Toy (1990), Hull and White (1990, 1993) and Black and Karasinski (1991), among others. Option evaluation and other applications of the Gaussian model is provided by Carverhill (1988), Jamshidian (1989a,b,c, 1991a, 1993b) and El Karoui and Rochet (1989). A popular special case of the Black–Karasinski model is the Black–Derman–Toy model.

For essentially any single-factor model, the term structure can be computed (numerically, if not explicitly) by taking advantange of the Feynman–Kac relationship between SDEs and PDEs. Fixing for convenience the maturity date s, the Feynman–Kac approach implies from Equation (66), under technical conditions on μ and σ, for all t, that $\Lambda_{t,s} = f(r_t, t)$, where $f \in C^{2,1}(\mathbb{R} \times [0, T))$ solves the PDE

$$\mathcal{D}f(x, t) - x f(x, t) = 0, \qquad (x, t) \in \mathbb{R} \times [0, s), \tag{69}$$

with boundary condition

$$f(x, s) = 1, \qquad x \in \mathbb{R},$$

where

$$\mathcal{D}f(x, t) = f_t(x, t) + f_x(x, t) \mu(x, t) + \tfrac{1}{2} f_{xx}(x, t) \sigma(x, t)^2.$$

This PDE can be quickly solved using standard finite-difference numerical algorithms.

A subset of the models considered in Table 1, those with $K_2 = H_1 = 0$, are *Gaussian.*[21] Special cases are the models of Merton (1974) (often called "Ho–Lee") and Vasicek (1977). For a Gaussian model, we can show that bond-price processes are log-normal (under Q) by defining a new process y satisfying $dy_t = -r_t\, dt$, and noting that (r, y) is a two-dimensional Gaussian Markov process. Thus, for any t and $s \geqslant t$, the random variable $y_s - y_t = -\int_t^s r_u\, du$ is normally distributed under Q, with a mean $m(s - t)$ and variance $v(s - t)$, conditional on $\mathcal{F}_t$, that are easily computed in terms of r_t, K_0, K_1, and H_0. The conditional variance $v(s - t)$ is deterministic. The conditional mean $m(t, s)$ is of the form $a(s - t) + \beta(s - t) r_t$, for coefficients $a(s - t)$ and $\beta(s - t)$ whose calculation is left to the reader. It follows that

$$\Lambda_{t,s} = E_t^Q \left[\exp\left(-\int_t^s r_u\, du \right) \right],$$

$$= \exp\left(m(t, s) + \frac{v(s - t)}{2} \right),$$

$$= \exp\left[\alpha(s - t) + \beta(s - t) r(t) \right],$$

where $\alpha(s - t) = a(s - t) + v(s - t)/2$. Because r_t is normally distributed under Q, this means that any zero-coupon bond price is log-normally distributed under Q. Using this property, one can compute bond-option prices in this setting using the original Black–Scholes formula. For this, a key simplifying trick of Jamshidian (1989b) is to adopt as a new numeraire the zero-coupon bond maturing at the expiration date of the option. The associated equivalent martingale measure is sometimes called the *forward measure.*

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[21] By a Gaussian process, we mean that the short rates $r(t_1), \ldots, r(t_k)$ at any finite set $\{t_1, \ldots, t_k\}$ of times have a joint normal distribution under Q.

Under the new numeraire and the forward measure, the price of the bond underlying the option is log-normally distributed with a variance that is easily calculated, and the Black–Scholes formula can be applied. Aside from the simplicity of the Gaussian model, this explicit computation is one of its main advantages in applications.

An undesirable feature of the Gaussian model, however, is that it implies that the short rate and yields on bonds of any maturity are negative with positive probability at any future date. While negative interest rates are sometimes plausible when expressed in "real" (consumption numeraire) terms, it is common in practice to express term structures in nominal terms, relative to the price of money. In nominal terms, negative bond yields imply a kind of arbitrage. In order to describe this arbitrage, we can formally view *money* as a security with no dividends whose price process is identically equal to 1. (This definition in itself is an arbitrage!) If a particular zero-coupon bond were to offer a negative yield, consider a short position in the bond (that is, borrowing) and a long position of an equal number of units of money, both held to the maturity of the bond. With a negative bond yield, the initial bond price is larger than 1, implying that this position is an arbitrage. To address properly the role of money in supporting nonnegative interest rates would, however, require a rather wide detour into monetary theory and the institutional features of money markets. Let us merely leave this issue with the sense that allowing negative interest rates is not necessarily "wrong," but is somewhat undesirable. Gaussian short-rate models are nevertheless frequently used because they are relatively tractable and in light of the low likelihood that they would assign to negative interest rates within a reasonably short time, with reasonable choices for the coefficient functions.

One of the best-known single-factor term-structure models is that of Cox, Ingersoll and Ross (1985b), the "CIR model," which exploits the stochastic properties of the diffusion model of population sizes of Feller (1951). For constant coefficient functions K_0, K_1, and H_1, the CIR drift and diffusion functions, μ and σ, may be written in the form

$$\mu(x, t) = \kappa(\bar{x} - x); \qquad \sigma(x, t) = C\sqrt{x}, \qquad x \geqslant 0, \tag{70}$$

for constants κ, $\bar{x}$, and C. Provided κ and $\bar{x}$ are non-negative, there is a nonnegative solution to the associated SDE (67). (Karatzas and Shreve (1988) offer a standard proof.) Given r_0, provided $\kappa\bar{x} > C^2$, we know that r_t has a non-central χ^2 distribution under Q, with parameters that are known explicitly. The drift $\kappa(\bar{x} - r_t)$ indicates reversion of r_t toward a stationary risk-neutral mean $\bar{x}$ at a rate κ, in the sense that

$$E^Q(r_t) = \bar{x} + e^{-\kappa t}(r_0 - \bar{x}),$$

which tends to $\bar{x}$ as t goes to $+\infty$. Cox, Ingersoll and Ross (1985b) show how the coefficients κ, $\bar{x}$, and C can be calculated in a general equilibrium setting in terms of the utility function and endowment of a representative agent. For the CIR model,

it can be verified by direct computation of the derivatives that the solution for the term-structure PDE (69) is

$$f(x, t) = \exp\left[\alpha(s - t) + \beta(s - t)x\right],$$ (71)

where

$$\alpha(u) = \frac{2\kappa\bar{x}}{C^2}\left[\log\left(2\gamma\exp\left[(\gamma + \kappa)u/2\right]\right) - \log\left((\gamma + \kappa)(e^{\gamma u} - 1) + 2\gamma\right)\right],$$

$$\beta(u) = \frac{2(1 - e^{\gamma u})}{(\gamma + \kappa)(e^{\gamma u} - 1) + 2\gamma},$$

for $\gamma = (\kappa^2 + 2C^2)^{1/2}$.

The Gaussian and Cox–Ingersoll–Ross models are special cases of single-factor models with the property that the solution f of the term-structure PDE (69) is given by the exponential-affine form (71) for some coefficients $\alpha(\cdot)$ and $\beta(\cdot)$ that are continuously differentiable. For all t, the yield $-\log[f(x, t)]/(s - t)$ obtained from Equation (71) is affine in x. We therefore call any such model an *affine term-structure model*. (A function $g: \mathbb{R}^k \to \mathbb{R}$, for some k, is *affine* if there are constants a and b in $\mathbb{R}^k$ such that for all x, $g(x) = a + b \cdot x$.)

It turns out that, technicalities aside, μ and σ^2 are affine in x if and only if the term structure is itself affine in x. The idea that an affine term-structure model is typically associated with affine drift μ and squared diffusion σ^2 is foreshadowed in Cox, Ingersoll and Ross (1985b) and Hull and White (1990), and is explicit in Brown and Schaefer (1994). Filipović (2001a) provides a definitive result for affine term structure models in a one-dimensional state space. We will get to multi-factor models shortly. The special cases associated with the Gaussian model and the CIR model have explicit solutions for α and β.

Cherif, El Karoui, Myneni and Viswanathan (1995), Constantinides (1992), El Karoui, Myneni and Viswanathan (1992), Jamshidian (1996) and Rogers (1995) characterize a model in which the short rate is a linear-quadratic form in a multivariate Markov Gaussian process. This "LQG" class of models overlaps with the general affine models, as for example in Piazzesi (1999), although it remains to be seen how we would maximally nest the affine and quadratic Gaussian models in a simple and tractable framework.

4.2. Term-structure derivatives

An important application of term-structure models is the arbitrage-free valuation of derivatives. Some of the most common derivatives are listed below, abstracting from many institutional details that can be found in a standard reference such as Sundaresan (1997).

(a) A European option expiring at time s on a zero-coupon bond maturing at some later time u, with strike price p, is a claim to $(\Lambda_{s,u} - p)^+$ at s.

(b) A *forward-rate agreement* (FRA) calls for a net payment by the fixed-rate payer of $c^* - c(s)$ at time s, where c^* is a fixed payment and $c(s)$ is a floating-rate payment for a time-to-maturity δ, in *arrears*, meaning that $c(s) = \Lambda_{s-\delta,s}^{-1} - 1$ is the simple interest rate applying at time $s - \delta$ for loans maturing at time s. In practice, we usually have a time to maturity, δ, of one quarter or one half year. When originally sold, the fixed-rate payment c^* is usually set so that the FRA is *at market*, meaning of zero market value. Cox, Ingersoll and Ross (1981), Duffie and Stanton (1988) and Grinblatt and Jegadeesh (1996) consider the relative pricing of futures and forwards.

(c) An *interest-rate swap* is a portfolio of FRAs maturing at a given increasing sequence $t(1), t(2), \ldots, t(n)$ of coupon dates. The inter-coupon interval $t(i) - t(i-1)$ is usually 3 months or 6 months. The associated FRA for date $t(i)$ calls for a net payment by the fixed-rate payer of $c^* - c(t(i))$, where the floating-rate payment received is $c(t(i)) = \Lambda_{t(i-1),t(i)}^{-1} - 1$, and the fixed-rate payment c^* is the same for all coupon dates. At initiation, the swap is usually *at market*, meaning that the fixed rate c^* is chosen so that the swap is of zero market value. Ignoring default risk and market imperfections, this would imply that the fixed-rate coupon c^* is the par coupon rate. That is, the at-market swap rate c^* is set at the origination date t of the swap so that

$$1 = c^* \left(\Lambda_{t,t(1)} + \cdots + \Lambda_{t,t(n)} \right) + \Lambda_{t,t(n)},$$

meaning that c^* is the coupon rate on a *par* bond, one whose face value and initial market value are the same. Swap markets are analyzed by Brace and Musiela (1994), Carr and Chen (1996), Collin-Dufresne and Solnik (2001), Duffie and Huang (1996), Duffie and Singleton (1997), El Karoui and Geman (1994) and Sundaresan (1997). For institutional and general economic features of swap markets, see Lang, Litzenberger and Liu (1998) and Litzenberger (1992).

(d) A *cap* can be viewed as portfolio of "caplet" payments of the form $(c(t(i)) - c^*)^+$, for a sequence of payment dates $t(1), t(2), \ldots, t(n)$ and floating rates $c(t(i))$ that are defined as for a swap. The fixed rate c^* is set with the terms of the cap contract. For the valuation of caps, see, for example, Chen and Scott (1995), Clewlow, Pang and Strickland (1997), Miltersen, Sandmann and Sondermann (1997), and Scott (1997). The basic idea is to view a caplet as a put option on a zero-coupon bond.

(e) A *floor* is defined symmetrically with a cap, replacing $(c(t(i)) - c^*)^+$ with $(c^* - c(t(i)))^+$.

(f) A *swaption* is an option to enter into a swap at a given strike rate c^* at some exercise time. If the future time is fixed, the swaption is European. Pricing of European swaptions is developed in Gaussian settings by Jamshidian (1989a,b,c, 1991a), and more generally in affine settings by Berndt (2002), Collin-Dufresne and Goldstein (2002) and Singleton and Umantsev (2003). An important variant, the *Bermudan swaption*, allows exercise at any of a given set of successive coupon

dates. For valuation methods, see Andersen and Andreasen (2000b) and Longstaff and Schwartz (2001).

Jamshidian (2001) and Rutkowski (1996, 1998) offer general treatments of LIBOR (London Interbank Offering Rate) derivative modeling.[22] Path-dependent derivative securities, such as mortgage-backed securities, sometimes call for additional state variables.[23]

In a one-factor setting, suppose a derivative has a payoff at some given time s defined by $g(r_s)$. By the definition of an equivalent martingale measure, the price at time t for such a security is

$$F(r_t, t) \equiv E_t^Q \left[\exp\left(- \int_t^s r_u \, du \right) g(r_s) \right].$$

Under technical conditions on μ, σ and g, we know that F solves the PDE, for $(x,t) \in \mathbb{R} \times [0,s)$,

$$F_t(x,t) + F_x(x,t)\,\mu(x,t) + \tfrac{1}{2}F_{xx}(x,t)\,\sigma(x,t)^2 - xF(x,t) = 0, \tag{72}$$

with boundary condition

$$F(x,s) = g(x), \qquad x \in \mathbb{R}.$$

For example, the valuation of a zero-coupon bond option is given, in a one-factor setting, by the solution F to Equation (72), with boundary value $g(x) = [f(x,s) - p]^+$, where $f(x,s)$ is the price at time s of a zero-coupon bond maturing at u.

4.3. Fundamental solution

Under technical conditions, we can also express the solution F of the PDE (72) for the value of a derivative term-structure security in the form

$$F(x,t) = \int_{-\infty}^{+\infty} G(x,t,y,s)\,g(y)\,dy, \tag{73}$$

where G is the *fundamental solution* of the PDE (72). One may think of $G(x,t,y,s)\,dy$ as the price at time t, state x, of an "infinitesimal security" paying one unit of account in

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[22] On the valuation of other specific forms of term-structure derivatives, see Artzner and Roger (1993), Bajeux-Besnainou and Portait (1998), Brace and Musiela (1994), Chacko and Das (2002), Chen and Scott (1992, 1993), Cherubini and Esposito (1995), Chesney, Elliott and Gibson (1993), Cohen (1995), Daher, Romano and Zacklad (1992), Décamps and Rochet (1997), El Karoui, Lepage, Myneni, Roseau and Viswanathan (1991a,b), and Turnbull (1993), Fleming and Whaley (1994) (wildcard options), Ingersoll (1977) (convertible bonds), Jamshidian (1993a, 1994) (diff swaps and quantos), Jarrow and Turnbull (1994), Longstaff (1990) (yield options), and Turnbull (1995).

[23] The pricing of mortgage-backed securities based on term-structure models is pursued by Boudoukh, Richardson, Stanton and Whitelaw (1997), Cheyette (1996), Jakobsen (1992), Stanton (1995) and Stanton and Wallace (1995, 1998), who also review some of the related literature.

the event that the state is at level y at time s, and nothing otherwise. One can compute the fundamental solution G by solving a PDE that is "dual" to Equation (72), in the following sense. Under technical conditions, for each (x, t) in $\mathbb{R} \times [0, T)$, a function $\psi \in C^{2,1}(\mathbb{R} \times (0, T])$ is defined by $\psi(y, s) = G(x, t, y, s)$, and solves the *forward Kolmogorov equation* (also known as the *Fokker–Planck equation*):

$$\mathcal{D}^* \psi(y, s) - y \psi(y, s) = 0, \tag{74}$$

where

$$\mathcal{D}^* \psi(y, s) = -\psi_s(y, s) - \frac{\partial}{\partial y} \left[\psi(y, s) \mu(y, s) \right] + \tfrac{1}{2} \frac{\partial^2}{\partial y^2} \left[\psi(y, s) \sigma(y, s)^2 \right].$$

The "intuitive" boundary condition for Equation (74) is obtained from the role of G in pricing securities. Imagine that the current short rate at time t is x, and consider an instrument that pays one unit of account immediately, if and only if the current short rate is some number y. Presumably this contingent claim is valued at 1 unit of account if $x = y$, and otherwise has no value. From continuity in s, one can thus think of $\psi(\cdot, s)$ as the density at time s of a measure on $\mathbb{R}$ that converges as $s \downarrow t$ to a probability measure ν with $\nu(\{x\}) = 1$, sometimes called the *Dirac measure* at x. This initial boundary condition on ψ can be made more precise. See, for example, Karatzas and Shreve (1988) for details.

Applications to term-structure modeling of the fundamental solution, sometimes erroneously called the "Green's function," are illustrated by Dash (1989), Beaglehole (1990), Beaglehole and Tenney (1991), Büttler and Waldvogel (1996), Dai (1994) and Jamshidian (1991b). For example, Beaglehole and Tenney (1991) show that the fundamental solution G of the Cox–Ingersoll–Ross model (70) is given explicitly in terms of the parameters κ, $\bar{x}$ and C by

$$G(x, 0, y, t) = \frac{\varphi(t) I_q \left(\varphi(t) \sqrt{xy\, \mathrm{e}^{-\gamma t}} \right)}{\exp \left[\varphi(t)(y + x\, \mathrm{e}^{-\gamma t}) - \eta(x + \kappa \bar{x} t - y) \right]} \left(\frac{\mathrm{e}^{\gamma t} y}{x} \right)^{q/2},$$

where $\gamma = (\kappa^2 + 2C^2)^{1/2}$, $\eta = (\kappa - \gamma)/C^2$,

$$\varphi(t) = \frac{2\gamma}{C^2(1 - \mathrm{e}^{-\gamma t})}, \qquad q = \frac{2\kappa \bar{x}}{C^2} - 1,$$

and $I_q(\cdot)$ is the modified Bessel function of the first kind of order q. For time-independent μ and σ, as with the CIR model, we have, for all t and $s > t$, $G(x, t, y, s) = G(x, 0, y, s - t)$.

The fundamental solution for the Dothan (log-normal) short-rate model can be deduced from the form of the solution by Hogan and Weintraub (1993) of what he calls the "conditional discounting function". Chen (1996) provides the fundamental

solution for his 3-factor affine model. Van Steenkiste and Foresi (1999) provide a general treatment of fundamental solutions of the PDE for affine models. For more technical details and references see, for example, Karatzas and Shreve (1988).

Given the fundamental solution G, the derivative asset-price function F is more easily computed by numerically integrating Equation (73) than from a direct numerical attack on the PDE (72). Thus, given a sufficient number of derivative securities whose prices must be computed, it may be worth the effort to compute G.

4.4. Multifactor term-structure models

The one-factor model (67) for the short rate is limiting. Even a casual review of the empirical properties of the term structure, for example as reviewed in the surveys of Dai and Singleton (2003) and Piazzesi (2002), shows the significant potential improvements in fit offered by a *multifactor term-structure model*. While terminology varies from place to place, by a "multifactor" model we mean a model in which the short rate is of the form $r_t = R(X_t, t)$, $t \geqslant 0$, where X is a Markov process with a state space D that is some subset of $\mathbb{R}^k$, for $k > 1$. For example, in much of the literature, X is an Ito process solving a stochastic differential equation of the form

$$\mathrm{d}X_t = \mu(X_t, t)\, \mathrm{d}t + \sigma(X_t, t)\, \mathrm{d}B_t^Q, \tag{75}$$

where B^Q is a standard Brownian motion in $\mathbb{R}^d$ under Q and the given functions R, μ and σ on $D \times [0, \infty)$ into $\mathbb{R}$, $\mathbb{R}^k$ and $\mathbb{R}^{k \times d}$, respectively, satisfy enough technical regularity to guarantee that Equation (75) has a unique solution and that the term structure (66) is well defined. In empirical applications, one often supposes that the state process X also satisfies a stochastic differential equation under the probability measure P, in order to exploit the time-series behavior of observed prices and price-determining variables in estimating the model.

There are various approaches for identifying the state vector X_t. In certain models, some or all elements of the state vector X_t are *latent*, that is, unobservable to the modeler, except insofar as they can be inferred from prices that depend on the levels of X. For example, k state variables might be identified from bond yields at k distinct maturities. Alternatively, one might use both bond and bond option prices, as in Singleton and Umantsev (2003) or Collin-Dufresne and Goldstein (2001b, 2002). This is typically possible once one knows the parameters, as explained below, but the parameters must of course be estimated at the same time as the latent states are estimated. This latent-variable approach has nevertheless been popular in much of the empirical literature. Notable examples include Dai and Singleton (2000), and references cited by them.

Another approach is to take some or all of the state variables to be directly observable variables, such as macro-economic determinants of the business cycle

and inflation, that are thought to play a role in determining the term structure. This approach has also been explored by Piazzesi (1999), among others.[24]

A derivative security, in this setting, can often be represented in terms of some real-valued terminal payment function g on $\mathbb{R}^k$, for some maturity date $s \leqslant T$. By the definition of an equivalent martingale measure, the associated derivative security price is

$$F(X_t, t) = E_t^Q \left[\exp \left(-\int_t^s R(X_u, u) \, \mathrm{d}u \right) g(X_s) \right].$$

For the case of a diffusion state process X satisfying Equation (75), extending Equation (72), under technical conditions we have the PDE characterization

$$\mathcal{D}F(x, t) - R(x, t) F(x, t) = 0, \qquad (x, t) \in D \times [0, s), \tag{76}$$

with boundary condition

$$F(x, s) = g(x), \qquad x \in D, \tag{77}$$

where

$$\mathcal{D}F(x, t) = F_t(x, t) + F_x(x, t) \, \mu(x, t) + \tfrac{1}{2}\mathrm{tr} \left[\sigma(x, t) \, \sigma(x, t)^\top F_{xx}(x, t) \right].$$

The case of a zero-coupon bond is $g(x) \equiv 1$. Under technical conditions, we can also express the solution F, as in Equation (73), in terms of the fundamental solution G of the PDE (76).

4.5. Affine models

Many financial applications including term-structure modeling are based on a state process that is Markov, under some reference probability measure that, depending on the application, may or may not be an equivalent martingale measure. We will fix the probability measure P for the current discussion.

A useful assumption is that the Markov state process is "affine". While several equivalent definitions of the class of affine processes can be usefully applied, perhaps the simplest definition of the affine property for a Markov process X in a state space $D \subset \mathbb{R}^d$ is that its conditional characteristic function is of the form, for any $u \in \mathbb{R}^d$,

$$E \left(\exp \left[iu \cdot X(t) \right] \mid X(s) \right) = \exp \left[\varphi(t - s, u) + \psi(t - s, u) \cdot X(s) \right]. \tag{78}$$

for some deterministic coefficients $\varphi(t - s, u)$ and $\psi(t - s, u)$. Duffie, Filipović and Schachermayer (2003) show that, for a time-homogeneous[25] affine process X with a

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[24] See also Babbs and Webber (1994), Balduzzi, Bertola, Foresi and Klapper (1998) and Piazzesi (1997). On modeling the term-structure of real interest rates, see Brown and Schaefer (1996) and Pennacchi (1991).

[25] Filipović (2001b) extends to the time inhomogeneous case.

state space of the form $\mathbb{R}^n_+ \times \mathbb{R}^{d-n}$, provided the coefficients $\varphi(\cdot)$ and $\psi(\cdot)$ of the characteristic function are differentiable and their derivatives are continuous at 0, the affine process X must be a jump-diffusion process, in that

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t + dJ_t, \tag{79}$$

for a standard Brownian motion B in $\mathbb{R}^d$ and a pure-jump process J, and moreoever the drift $\mu(X_t)$, the "instantaneous" covariance matrix $\sigma(X_t)\,\sigma(X_t)'$, and the jump measure associated with J must all have affine dependence on the state X_t. This result also provides necessary and sufficient conditions on the coefficients of the drift, diffusion, and jump measure for the process to be a well defined affine process, and provides that the coefficients $\varphi(\cdot, u)$ and $\psi(\cdot, u)$ of the characteristic function satisfy a certain (generalized Riccati) ordinary differential equation (ODE), the key to tractability for this class of processes.[26] Conversely, any jump-diffusion whose coefficients are of this affine class is an affine process in the sense of Equation (78). A complete statement of this result is found in Duffie, Filipović and Schachermayer (2003).

Simple examples of affine processes used in financial modeling are the Gaussian Ornstein–Uhlenbeck model, applied to interest rates by Vasicek (1977), and the Feller (1951) diffusion, applied to interest-rate modeling by Cox, Ingersoll and Ross (1985b), as already mentioned in the context of one-factor models. A general multivariate class of affine term-structure jump-diffusion models was introduced by Duffie and Kan (1996) for term-structure modeling. Dai and Singleton (2000) classified 3-dimensional affine diffusion models, and found evidence in U.S. swap rate data that both time-varying conditional variances and negatively correlated state variables are essential ingredients to explaining the historical behavior of term structures.

For option pricing, there is a substantial literature building on the particular affine stochastic-volatility model for currency and equity prices proposed by Heston (1993). Bates (1997), Bakshi, Cao and Chen (1997), Bakshi and Madan (2000) and Duffie, Pan and Singleton (2000) brought more general affine models to bear in order to allow for stochastic volatility and jumps, while maintaining and exploiting the simple property (78).

A key property related to Equation (78) is that, for any affine function $R: D \to \mathbb{R}$ and any $w \in \mathbb{R}^d$, subject only to technical conditions reviewed in Duffie, Filipović and Schachermayer (2003),

$$E_t \left(\exp\left[\int_t^s -R(X(u))\,du + w \cdot X(s) \right] \right) = \exp\left[\alpha(s - t) + \beta(s - t) \cdot X(t)\right], \tag{80}$$

for coefficients $\alpha(\cdot)$ and $\beta(\cdot)$ that satisfy generalized Riccati ODEs (with real boundary conditions) of the same type solved by φ and ψ of Equation (78), respectively.

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[26] Recent work, yet to be distributed, by Martino Graselli of CREST, Paris, and Claudio Tebaldi, provides explicit solutions for the Riccati equations of multi-factor affine diffusion processes.

In order to get a quick sense of how the Riccati equations for $\alpha(\cdot)$ and $\beta(\cdot)$ arise, we consider the special case of an affine diffusion process X solving the stochastic differential equation (79), with state space $D = \mathbb{R}_+$, and with $\mu(x) = a + bx$ and $\sigma^2(x) = cx$, for constant coefficients a, b and c. (This is the continuous branching process of Feller (1951).) We let $R(x) = \rho_0 + \rho_1 x$, for constants ρ_0 and ρ_1, and apply the Feynman–Kac partial differential equation (PDE) (69) to the candidate solution $\exp[\alpha(s - t) + \beta(s - t) \cdot x]$ of Equation (80). After calculating all terms of the PDE and then dividing each term of the PDE by the common factor $\exp[\alpha(s - t) + \beta(s - t) \cdot x]$, we arrive at

$$-\alpha'(z) - \beta'(z)x + \beta(z)(a + bx) + \tfrac{1}{2}\beta(z)^2 c^2 x - \rho_0 - \rho_1 x = 0, \qquad (81)$$

for all $z \geqslant 0$. Collecting terms in x, we have

$$u(z)x + v(z) = 0, \qquad (82)$$

where

$$u(z) = -\beta'(z) + \beta(z)b + \tfrac{1}{2}\beta(z)^2 c^2 - \rho_1, \qquad (83)$$

$$v(z) = -\alpha'(z) + \beta(z)a - \rho_0. \qquad (84)$$

Because Equation (82) must hold for all x, it must be the case that $u(z) = v(z) = 0$. This leaves the Riccati equations:

$$\beta'(z) = \beta(z)b + \tfrac{1}{2}\beta(z)^2 c^2 - \rho_1, \qquad (85)$$

$$\alpha'(z) = \beta(z)a - \rho_0, \qquad (86)$$

with the boundary conditions $\alpha(0) = 0$ and $\beta(0) = w$, from Equation (80) for $s = t$. The explicit solutions for $\alpha(z)$ and $\beta(z)$ were stated earlier for the CIR model (the case $w = 0$), and are given explicitly in a more general case with jumps, called a "basic affine process", in Duffie and Gârleanu (2001).

Beyond the Gaussian case, any Ornstein–Uhlenbeck process, whether driven by a Brownian motion (as for the Vasicek model) or by a more general Lévy process with jumps, as in Sato (1999), is affine. Moreover, any continuous-branching process with immigration (CBI process), including multi-type extensions of the Feller process, is affine. [See Kawazu and Watanabe (1971).] Conversely, an affine process in $\mathbb{R}_+^d$ is a CBI process.

For term-structure modeling,[27] the state process X is typically assumed to be affine under a given equivalent martingale measure Q. For econometric modeling of

[27] Special cases of affine term-structure models include those of Balduzzi, Das and Foresi (1998), Balduzzi, Das, Foresi and Sundaram (1996), Baz and Das (1996), Berardi and Esposito (1999), Chen (1996), Cox, Ingersoll and Ross (1985b), Das (1993, 1995, 1997, 1998), Das and Foresi (1996), Duffie and Kan (1996), Duffie, Pedersen and Singleton (2003), Heston (1988), Langetieg (1980), Longstaff and Schwartz (1992, 1993), Pang and Hodges (1995) and Selby and Strickland (1993).

bond yields, the affine assumption is sometimes also made under the data-generating measure P, although Duffee (1999b) suggests that this is overly restrictive from an empirical viewpoint, at least for 3-factor models of interest rates in the USA that do not have jumps. For general reviews of this issue, and summaries of the empirical evidence on affine term structure models, see Dai and Singleton (2003) and Piazzesi (2002). The affine class allows for the analytic calculation of bond option prices on zero-coupon bonds and other derivative securities, as reviewed in Section 5, and extends to the case of defaultable models, as we show in Section 6. For related computational results, see Liu, Pan and Pedersen (1999) and Van Steenkiste and Foresi (1999). Singleton (2001) exploits the explicit form of the characteristic function of affine models to provide a class of moment conditions for econometric estimation.

4.6. The HJM model of forward rates

We turn to the term structure model of Heath, Jarrow and Morton (1992). Until this point, we have taken as the primitive a model of the short-rate process of the form $r_t = R(X_t, t)$, where (under some equivalent martingale measure) X is a finite-dimensional Markov process. This approach has analytical advantages, especially for derivative pricing and statistical modeling. A more general approach that is especially popular in business applications is to directly model the risk-neutral stochastic behavior of the entire term structure of interest rates. This is the essence of the Heath–Jarrow–Morton (HJM) model. The remainder of this section is a summary of the basic elements of the HJM model.

If the discount $\Lambda_{t,s}$ is differentiable with respect to the maturity date s, a mild regularity, we can write

$$\Lambda_{t,s} = \exp\left(-\int_t^s f(t,u)\,du\right),$$

where

$$f(t,u) = -\frac{1}{\Lambda_{t,u}}\frac{\partial \Lambda_{t,u}}{\partial u}.$$

The term structure can thus be represented in terms of the *instantaneous forward rates*, $\{f(t,u): u \geqslant t\}$.

The HJM approach is to take as primitive a particular stochastic model of these forward rates. First, for each fixed maturity date s, one models the one-dimensional forward-rate process $f(\cdot, s) = \{f(t,s): 0 \leqslant t \leqslant s\}$ as an Ito process, in that

$$f(t,s) = f(0,s) + \int_0^t \mu(u,s)\,du + \int_0^t \sigma(u,s)\,dB_u^Q, \qquad 0 \leqslant t \leqslant s, \tag{87}$$

where $\mu(\cdot, s) = \{\mu(t, s) : 0 \leqslant t \leqslant s\}$ and $\sigma(\cdot, s) = \{\sigma(t, s) : 0 \leqslant t \leqslant s\}$ are adapted processes valued in $\mathbb{R}$ and $\mathbb{R}^d$, respectively, such that Equation (87) is well defined.[28] Under purely technical conditions, it must be the case that

$$\mu(t, s) = \sigma(t, s) \cdot \int_t^s \sigma(t, u) \, du. \tag{88}$$

In order to confirm this key risk-neutral drift restriction (88), consider the Q-martingale M defined by

$$
\begin{aligned}
M_t &= E_t^Q \left[\exp\left(-\int_0^s r_u \, du \right) \right] \\
&= \exp\left(-\int_0^t r_u \, du \right) \Lambda_{t,s} \\
&= \exp\left(X_t + Y_t \right),
\end{aligned} \tag{89}
$$

where

$$X_t = -\int_0^t r_u \, du; \qquad Y_t = -\int_t^s f(t, u) \, du.$$

We can view Y as an infinite sum of the Ito processes for forward rates over all maturities ranging from t to s. Under technical conditions[29] for Fubini's Theorem for stochastic integrals, we thus have

$$dY_t = \mu_Y(t) \, dt + \sigma_Y(t) \, dB_t^Q,$$

where

$$\mu_Y(t) = f(t, t) - \int_t^s \mu(t, u) \, du,$$

and

$$\sigma_Y(t) = -\int_t^s \sigma(t, u) \, du.$$

We can then apply Ito's Formula in the usual way to $M_t = e^{X(t) + Y(t)}$ and obtain the drift under Q of M as

$$\mu_M(t) = M_t \left(\mu_Y(t) + \tfrac{1}{2} \sigma_Y(t) \cdot \sigma_Y(t) - r_t \right).$$

Because M is a Q-martingale, we must have $\mu_M = 0$, so, substituting $\mu_Y(t)$ into this equation, we obtain

$$\int_t^s \mu(t, u) \, du = \tfrac{1}{2} \left(\int_t^s \sigma(t, u) \, du \right) \cdot \left(\int_t^s \sigma(t, u) \, du \right).$$

Taking the derivative of each side with respect to s then leaves the risk-neutral drift restriction (88) which in turn provides, naturally, the property that $r(t) = f(t, t)$.

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[28] The necessary and sufficient condition is that, almost surely, $\int_0^s |\mu(t, s)| \, dt < \infty$ and $\int_0^s \sigma(t, s) \cdot \sigma(t, s) \, dt < \infty$.

[29] In addition to measurability, it suffices that $\mu(t, u, \omega)$ and $\sigma(t, u, \omega)$ are uniformly bounded and, for each ω, continuous in (t, u). For weaker conditions, see Protter (1990).

Thus, the initial forward rates $\{f(0, s): 0 \leqslant s \leqslant T\}$ and the forward-rate "volatility" process σ can be specified with nothing more than technical restrictions, and these are enough to determine all bond and interest-rate derivative price processes. Aside from the Gaussian special case associated with deterministic volatility $\sigma(t, s)$, however, most valuation work in the HJM setting is typically done by Monte Carlo simulation. Special cases aside,[30] there is no finite-dimensional state variable for the HJM model, so PDE-based computational methods cannot be used.

The HJM model has been extensively treated in the case of Gaussian instantaneous forward rates by Jamshidian (1989b), who developed the forward-measure approach, and Jamshidian (1989a,c, 1991a) and El Karoui and Rochet (1989), and extended by El Karoui, Lepage, Myneni, Roseau and Viswanathan (1991a,b), El Karoui and Lacoste (1992), Frachot (1995), Frachot, Janci and Lacoste (1993), Frachot and Lesne (1993) and Miltersen (1994). A related model of log-normal discrete-period interest rates, the "market model," was developed by Miltersen, Sandmann and Sondermann (1997).[31]

Musiela (1994b) suggested treating the entire forward-rate curve

$$g(t, u) = \{f(t, t + u): 0 \leqslant u \leqslant \infty\},$$

itself as a Markov process. Here, u indexes time to maturity, *not* date of maturity. That is, we treat the term structure $g(t) = g(t, \cdot)$ as an element of some convenient state space S of real-valued continuously differentiable functions on $[0, \infty)$. Now, letting $v(t, u) = \sigma(t, t + u)$, the risk-neutral drift restriction (88) on f, and enough regularity, imply the *stochastic partial differential equation* (SPDE) for g given by

$$dg(t, u) = \frac{\partial g(t, u)}{\partial u} dt + V(t, u) dt + v(t, u) dB_t^Q,$$

where

$$V(t, u) = v(t, u) \cdot \int_0^u v(t, z) dz.$$

This formulation is an example of a rather delicate class of SPDEs that are called "hyperbolic". Existence is usually not shown, or shown only in a "weak sense", as by Kusuoka (2000). The idea is nevertheless elegant and potentially important in getting a parsimonious treatment of the yield curve as a Markov process. One may even allow

[30] See Au and Thurston (1993), Bhar and Chiarella (1995), Cheyette (1995), Jeffrey (1995), Musiela (1994b), Ritchken and Sankarasubramaniam (1992) and Ritchken and Trevor (1993).

[31] See also Andersen and Andreasen (2000a), Brace and Musiela (1995), Dothan (1978), Goldberg (1998), Goldys, Musiela and Sondermann (1994), Hansen and Jorgensen (2000), Hogan and Weintraub (1993), Jamshidian (1997a,b, 2001), Sandmann and Sondermann (1997), Miltersen, Sandmann and Sondermann (1997), Musiela (1994a) and Vargiolu (1999). A related log-normal futures-price term structure model is due to Heath (1998).

the Brownian motion B^Q to be "infinite-dimensional". For related work in this setting, sometimes called a string, random field, or SPDE model of the term structure, see Cont (1998), Jong and Santa-Clara (1999), Goldstein (1997, 2000), Goldys and Musiela (1996), Hamza and Klebaner (1995), Kennedy (1994), Kusuoka (2000), Musiela and Sondermann (1994), Pang (1996), Santa-Clara and Sornette (2001) and Sornette (1998).

5. Derivative pricing

We turn to a review of the pricing of derivative securities, taking first futures and forwards, and then turning to options. The literature is immense, and we shall again merely provide a brief summary of results. Again, we fix a probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\mathbb{F} = \{\mathcal{F}_t : 0 \leqslant t \leqslant T\}$ satisfying the usual conditions, as well as a short-rate process r.

5.1. Forward and futures prices

We briefly address the pricing of forward and futures contracts, an important class of derivatives.

The forward contract is the simpler of these two closely related securities. Let W be an $\mathcal{F}_T$-measurable finite-variance random variable underlying the claim payable to a holder of the forward contract at its delivery date T. For example, with a forward contract for delivery of a foreign currency at time T, the random variable W is the market value at time T of the foreign currency. The forward-price process F is defined by the fact that one forward contract at time t is a commitment to pay the net amount $F_t - W$ at time T, with no other cash flows at any time. In particular, the true price of a forward contract, at the contract date, is zero.

We fix an equivalent martingale measure Q for the available securities, after deflation by $\exp[-\int_0^t r(u)\,du]$, where r is a short-rate process that, for convenience, is assumed to be bounded. The dividend process H defined by the forward contract made at time t is given by $H_s = 0$, $s < T$, and $H_T = W - F_t$. Because the true price of the forward contract at t is zero,

$$0 = E_t^Q \left[\exp\left(-\int_t^T r_s \, ds\right)(W - F_t) \right].$$

Solving for the forward price,

$$F_t = \frac{E_t^Q \left[\exp\left(-\int_t^T r_s \, ds\right) W \right]}{E_t^Q \left[\exp\left(-\int_t^T r_s \, ds\right) \right]}.$$

If we assume that there exists at time t a zero-coupon riskless bond maturing at time T, with price $\Lambda_{t,T}$, then

$$F_t = \frac{1}{\Lambda_{t,T}} E_t^Q \left[\exp\left(-\int_t^T r_s \, ds\right) W \right].$$

If r and W are statistically independent with respect to Q, we have the simplified expression $F_t = E_t^Q(W)$, implying that the forward price is a Q-martingale. This would be true, for instance, if the short-rate process r is deterministic.

As an example, suppose that the forward contract is for delivery at time T of one unit of a particular security with price process S and cumulative dividend process D. In particular, $W = S_T$. We can obtain a more concrete representation of the forward price, as follows. We have

$$F_t = \frac{1}{\Lambda_{t,T}} \left(S_t - E_t^Q \left[\int_t^T \exp\left(-\int_t^s r_u \, du\right) dD_s \right] \right).$$

If the short-rate process r is deterministic, we can simplify further to

$$F_t = \frac{S_t}{\Lambda_{t,T}} - E_t^Q \left[\int_t^T \exp\left(\int_s^T r_u \, du\right) dD_s \right], \tag{90}$$

which is known as the *cost-of-carry formula* for forward prices for the case in which interest rates and dividends are deterministic.

As with a forward contract, a futures contract with delivery date T is keyed to some delivery value W, which we take to be an $\mathcal{F}_T$-measurable random variable with finite variance. The contract is completely defined by a *futures-price process* Φ with the property that $\Phi_T = W$. As we shall see, the contract is literally a security whose price process is zero and whose cumulative dividend process is Φ. In other words, changes in the futures price are credited to the holder of the contract as they occur.

This definition is an abstraction of the traditional notion of a futures contract, which calls for the holder of one contract at the delivery time T to accept delivery of some asset (whose spot market value at T is represented here by W) in return for simultaneous payment of the current futures price Φ_T. Likewise, the holder of -1 contract, also known as a *short position* of 1 contract, is traditionally obliged to make delivery of the same underlying assset in exchange for the current futures price Φ_T. This informally justifies the property $\Phi_T = W$ of the futures-price process Φ given in the definition above. Roughly speaking, if Φ_T is not equal to W (and if we continue to neglect transactions costs and other details), there is a *delivery arbitrage*. We won't explicitly define a delivery arbitrage since it only complicates the analysis of futures prices that follows. Informally, however, in the event that $W > \Phi_T$, one could buy at time T the deliverable asset for W, simultaneously sell one futures contract, and make immediate delivery for a profit of $W - \Phi_T$. Thus, the potential of delivery

arbitrage will naturally equate Φ_T with the delivery value W. This is sometimes known as the principle of *convergence*.

Many modern futures contracts have streamlined procedures that avoid the delivery process. For these, the only link that exists with the notion of delivery is that the terminal futures price Φ_T is contractually equated to some such variable W, which could be the price of some commodity or security, or even some abstract variable of general economic interest such as a price deflator. This procedure, finessing the actual delivery of some asset, is known as *cash settlement*. In any case, whether based on cash settlement or the absence of delivery arbitrage, we shall always take it by definition that the delivery futures price Φ_T is equal to the given delivery value W.

The institutional feature of futures markets that is central to our analysis of futures prices is *resettlement*, the process that generates daily or even more frequent payments to and from the holders of futures contracts based on changes in the futures price. As with the expression "forward price", the term "futures price" can be misleading in that the futures price Φ_t at time t is not at all the price of the contract. Instead, at each resettlement time t, an investor who has held $\bar{\theta}$ futures contracts since the last resettlement time, say s, receives the resettlement payment $\bar{\theta}(\Phi_t - \Phi_s)$, following the simplest resettlement recipe. More complicated resettlement arrangements often apply in practice. The continuous-time abstraction is to take the futures-price process Φ to be an Ito process and a *futures position process* to be some θ in $\mathcal{L}(\Phi)$ generating the resettlement gain $\int \theta \, d\Phi$ as a cumulative-dividend process. In particular, as we have already stated in its definition, the futures-price process Φ is itself, formally speaking, the cumulative dividend process associated with the contract. The true price process is zero, since (again ignoring some of the detailed institutional procedures), there is no payment against the contract due at the time a contract is bought or sold.

The futures-price process Φ can now be characterized as follows. We suppose that the short-rate process r is bounded. For all t, let $Y_t = \exp[-\int_0^t r(s) \, ds]$. Because Φ is strictly speaking the cumulative-dividend process associated with the futures contract, and since the true-price process of the contract is zero, from the fact that the risk-neutral discounted gain is a martingale,

$$0 = E_t^Q \left(\int_t^T Y_s \, d\Phi_s \right), \qquad t \leqslant T,$$

from which it follows that the stochastic integral $\int Y \, d\Phi$ is a Q-martingale. Because r is bounded, there are constants $k_1 > 0$ and k_2 such that $k_1 \leqslant Y_t \leqslant k_2$ for all t. The process $\int Y \, d\Phi$ is therefore a Q-martingale if and only if Φ is also a Q-martingale. Since $\Phi_T = W$, we have deduced a convenient representation for the futures-price process:

$$\Phi_t = E_t^Q(W), \qquad t \in [0, T]. \tag{91}$$

If r and W are statistically independent under Q, the futures-price process Φ and the forward-price process F are thus identical. Otherwise, as pointed out by Cox, Ingersoll

and Ross (1981), there is a distinction based on correlation between changes in futures prices and interest rates.

5.2. Options and stochastic volatility

The Black–Scholes formula, which treats option prices under constant volatility, can be extended to cases with stochastic volatility, which is crucial in many markets from an empirical viewpoint. We will briefly examine several basic approaches, and then turn to the computation of option prices using the Fourier-transform method introduced by Stein and Stein (1991), and then first exploited in an affine setting by Heston (1993).

We recall that the Black–Scholes option-pricing formula is of the form $C(x, p, \bar{r}, t, \overline{\sigma})$, for $C \colon \mathbb{R}_+^5 \to \mathbb{R}_+$, where x is the current underlying asset price, p is the exercise price, $\bar{r}$ is the constant short rate, t is the time to expiration, and $\overline{\sigma}$ is the volatility coefficient for the underlying asset. For each fixed $(x, p, \bar{r}, t)$ with non-zero x and t, the map from $\overline{\sigma}$ to $C(x, p, \bar{r}, t, \overline{\sigma})$ is strictly increasing, and its range is unbounded. We may therefore invert and obtain the volatility from the option price. That is, we can define an *implied volatility* function $I \colon \mathbb{R}_+^5 \to \mathbb{R}_+$ by

$$c = C\left(x, p, \bar{r}, t, I\left(x, p, \bar{r}, t, c\right)\right), \tag{92}$$

for all sufficiently large $c \in \mathbb{R}_+$.

If c_1 is the Black–Scholes price of an option on a given asset at strike p_1 and expiration t_1, and c_2 is the Black–Scholes price of an option on the same asset at strike p_2 and expiration t_2, then the associated implied volatilities $I(x, p_1, \bar{r}, t_1, c_1)$ and $I(x, p_2, \bar{r}, t_2, c_2)$ must be identical, if indeed the assumptions underlying the Black–Scholes formula apply literally, and in particular if the underlying asset-price process has the constant volatility of a geometric Brownian motion. It has been widely noted, however, that actual market prices for European options on the same underlying asset have associated Black–Scholes implied volatilities that vary with both exercise price and expiration date. For example, in certain markets at certain times, the implied volatilities of options with a given exercise date depend on strike prices in a manner that is often termed a *smile curve*. Figure 1 illustrates the dependence of Black–Scholes implied volatilities on moneyness (the ratio of strike price to futures price), for various S&P 500 index options on November 2, 1993. Other forms of systematic deviation from constant implied volatilities have been noted, both over time and across various derivatives at a point in time.

Three major lines of modeling address these systematic deviations from the assumptions underlying the Black–Scholes model. In all of these, a key step is to generalize the underlying log-normal price process by replacing the constant volatility parameter $\overline{\sigma}$ of the Black–Scholes model with $\sqrt{V_t}$, an adapted non-negative process V with $\int_0^T V_t \, dt < \infty$ such that the underlying asset price process S satisfies

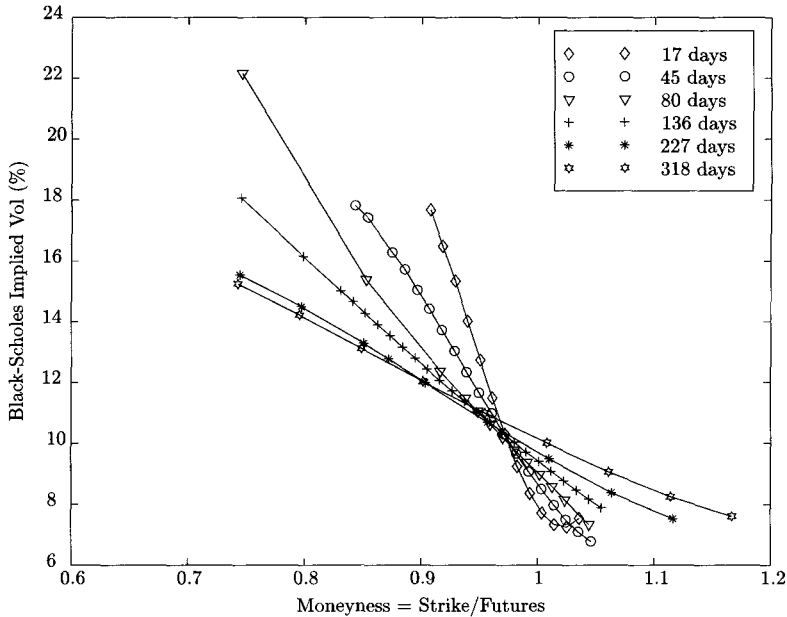$$dS_t = r_t S_t \, dt + S_t \sqrt{V_t} \, d\epsilon_t^S, \tag{93}$$

Fig. 1. "Smile curves" implied by SP500 Index options of 6 different times to expiration, from market data for November 2, 1993.

where B^Q is a standard Brownian motion in $\mathbb{R}^d$ under the given equivalent martingale measure Q, and $\epsilon^S = c_S \cdot B^Q$ is a standard Brownian motion under Q obtained from any c_S in $\mathbb{R}^d$ with unit norm.

In the first class of models, $V_t = v(S_t, t)$, for some function v: $\mathbb{R} \times [0, T] \to \mathbb{R}$ satisfying technical regularity conditions. In practical applications, the function v, or its discrete-time discrete-state analogue, is often "calibrated" to the available option prices. This approach, sometimes referred to as the *implied-tree* model, was developed by Dupire (1994), Rubinstein (1995) and Jackwerth and Rubinstein (1996).

For a second class of models, called *autoregressive conditional heteroscedastic*, or ARCH, the volatility depends on the path of squared returns, as formulated by Engle (1982). The GARCH (generalized ARCH) variant has the squared volatility V_t at time t of the discrete-period return $R_{t+1} = \log S_{t+1} - \log S_t$ adjusting according to the recursive formula

$$V_t = a + bV_{t-1} + cR_t^2, \tag{94}$$

for fixed coefficients a, b and c satisfying regularity conditions. By taking a time period of length h, normalizing in a natural way, and taking limits, a natural continous-time limiting behavior for volatility is simply a deterministic mean-reverting process V satisfying the ordinary differential equation

$$\frac{dV(t)}{dt} = \kappa\left(\bar{v} - V(t)\right). \tag{95}$$

Corradi (2000) explains that this deterministic continuous-time limit is more natural than the stochastic limit of Nelson (1990). For both the implied-tree approach and the GARCH approach, the volatility process V depends only on the underlying asset prices; volatility is not a separate source of risk.

In a third approach, however, the increments of the squared-volatility process V depend on Brownian motions that are *not* perfectly correlated with ϵ^S. For example, in a simple "one-factor" setting,

$$dV_t = \mu_V(V_t)\, dt + \sigma_V(V_t)\, d\epsilon_t^V, \tag{96}$$

where $\epsilon^V = c_V \cdot B^Q$ is a standard Brownian motion under Q, for some constant vector c_V of unit norm. As we shall see, the correlation parameter $c_{SV} = c_S \cdot c_V$ has an important influence on option prices.

The price of a European option at exercise price p and expiration at time t is

$$f(S_s, V_s, s) = E_s^Q\left(\exp\left[-\bar{r}(t-s)\right](S_t - p)^+\right),$$

which can be solved, for example, by reducing to a PDE and applying, if necessary, a finite-difference approach.

In many settings, a pronounced *skew* to the smile, as in Figure 1, indicates an important potential role for correlation between the increments of the return-driving and volatility-driving Brownian motions, ϵ^S and ϵ^V. This role is borne out directly by the correlation apparent from time-series data on implied volatilities and returns for certain important asset classes, as indicated for example by Pan (2002).

A tractable model that allows for the skew effects of correlation is the *Heston model*, the special case of Equation (96) for which

$$dV_t = \kappa(\bar{v} - V_t)\, dt + \sigma_v \sqrt{V_t}\, d\epsilon_t^V, \tag{97}$$

for positive coefficients κ, $\bar{v}$ and σ_v that play the same respective roles for V as for a Cox–Ingersoll–Ross interest-rate model. Indeed, this Feller diffusion model of volatility (97) is sometimes called a "CIR volatility model." In the original Heston model, the short rate is a constant, say $\bar{r}$, and option prices can be computed analytically, using transform methods explained later in this section, in terms of the parameters $(\bar{r}, c_{SV}, \kappa, \bar{v}, \sigma_v)$ of the Heston model, as well as the initial volatility V_0, the initial underlying price S_0, the strike price, and the expiration time.

Figure 2 shows the "smile curves," for the same options illustrated in Figure 1, that are implied by the Heston model for parameters, including V_0, chosen to minimize the sum of squared differences between actual and theoretical option prices, a calibration approach popularized for this application by Bates (1997). Notably, the distinctly downward slopes, often called skews, are captured with a negative correlation coefficient c_{SV}. Adopting a short rate $\bar{r} = 0.0319$ that roughly captures the effects of contemporary short-term interest rates, the remaining coefficients of the Heston model are calibrated to $c_{SV} = -0.66$, $\kappa = 19.66$, $\bar{v} = 0.017$, $\sigma_v = 1.516$, and $\sqrt{V_0} = 0.094$.
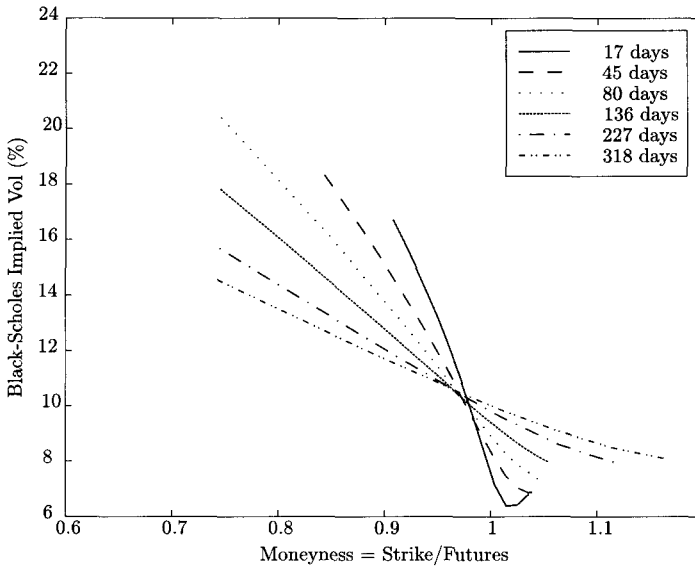
Fig. 2. "Smile curves" calculated for SP500 Index options of 6 different exercise dates, November 2, 1993, using the Heston Model.

Going beyond the calibration approach, time-series data on both options and underlying prices have been used simultaneously to fit the parameters of various stochastic-volatility models, for example by Aït-Sahalia, Wang and Yared (2001), Benzoni (2002), Chernov and Ghysels (2000), Guo (1998), Pan (2002), Poteshman (1998) and Renault and Touzi (1992). The empirical evidence for S&P 500 index returns and option prices suggests that the Heston model is overly restrictive for these data. For example, Pan (2002) rejects the Heston model in favor of a generalization with jumps in returns, proposed by Bates (1997), that is a special case of the affine model for option pricing to which we now turn.

5.3. Option valuation by transform analysis

We now address the calculation of option prices with stochastic volatility and jumps in an affine setting of the type already introduced for term-structure modeling, a special case being the model of Heston (1993). We use an approach based on transform analysis that was initiated by Stein and Stein (1991) and Heston (1993), allowing for relatively rich and tractable specifications of stochastic interest rates and volatility, and for jumps. This approach and the underlying stochastic models were subsequently generalized by Bakshi, Cao and Chen (1997), Bakshi and Madan (2000), Bates (1997) and Duffie, Pan and Singleton (2000).

We assume that there is a state process X that is affine under Q in a state space $D \subset \mathbb{R}^k$, and that the short-rate process r is of the affine form $r_t = \rho_0 + \rho_1 \cdot X_t$,

for coefficients ρ_0 in $\mathbb{R}$ and ρ_1 in $\mathbb{R}^k$. The price process S underlying the options in question is assumed to be of the exponential-affine form $S_t = \exp[a(t) + b(t) \cdot X(t)]$, for potentially time-dependent coefficients $a(t)$ in $\mathbb{R}$ and $b(t)$ in $\mathbb{R}^k$. An example would be the price of an equity, a foreign currency, or, as shown earlier in the context of affine term-structure models, the price of a zero-coupon bond.

The Heston model (97) is a special case, for an affine process $X = (X^{(1)}, X^{(2)})$, with $X_t^{(1)} = Y_t \equiv \log(S_t)$, and $X_t^{(2)} = V_t$, and with a constant short rate $\bar{r} = \rho_0$. From Ito's Formula,

$$\mathrm{d}Y_t = \left(\bar{r} - \tfrac{1}{2}V_t\right)\,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}\epsilon_t^S, \tag{98}$$

which indeed makes the state vector $X_t = (Y_t, V_t)$ an affine process, whose state space is $D = \mathbb{R} \times [0, \infty)$, as we can see from the fact that the drift and instantaneous covariance matrix of X_t are affine with respect to X_t. The underlying asset price is indeed of the desired exponential-affine form because $S_t = e^{Y(t)}$. We will return to the Heston model shortly with some explicit results on option valuation.

One of the affine models generalizing Heston's that was tested by Pan (2002) took

$$\mathrm{d}Y_t = \left(\bar{r} - \tfrac{1}{2}V_t\right)\,\mathrm{d}t + \sqrt{V_t}\,\mathrm{d}\epsilon_t^S + \mathrm{d}Z_t, \tag{99}$$

where, under the equivalent martingale measure Q, Z is a pure-jump process whose jump times have an arrival intensity (as defined in Section 6) that is affine with respect to the volatility process V, and whose jump sizes are independent normals.

For the general affine case, suppose we are interested in valuing a European call option on the underlying security, with strike price p and exercise date t. We have the initial option price

$$U_0 = E^Q\left[\exp\left(-\int_0^t r_u\,\mathrm{d}u\right)(S_u - p)^+\right].$$

Letting A denote the exercise event $\{\omega: S(\omega, t) \geqslant p\}$, we have the option price

$$U_0 = E^Q\left[\exp\left(-\int_0^t r_s\,\mathrm{d}s\right)(S_t 1_A - p 1_A)\right].$$

Because $S(t) = \exp[a(t) + b(t) \cdot X(t)]$,

$$\begin{aligned} U_0 &= e^{a(t)} G\left(-\log p + a(t);\ t, b(t), -b(t)\right) \\ &\quad - p G\left(-\log p + a(t);\ t, 0, -b(t)\right), \end{aligned} \tag{100}$$

where, for any $y \in \mathbb{R}$ and for any coefficient vectors d and δ in $\mathbb{R}^k$,

$$G(y;\ t, d, \delta) = E^Q\left[\exp\left(-\int_0^t r_s\,\mathrm{d}s\right)\exp[d \cdot X(t)]\,1_{\delta \cdot X(t) \leqslant y}\right]. \tag{101}$$

So, if we can compute the function G, we can obtain the prices of options of any strike and exercise date. Likewise, the prices of European puts, interest-rate caps,

chooser options, and many other derivatives can be derived in terms of G. For example, following this approach of Heston (1993), the valuation of discount bond options and caps in an affine setting was undertaken by Chen and Scott (1995), Duffie, Pan and Singleton (2000), Nunes, Clewlow and Hodges (1999) and Scaillet (1996).

We note, for fixed (t, d, δ), assuming $E^Q(\exp[-\int_0^t r(u)\, du]\, \exp[d \cdot X(t)]) < \infty$, that $G(\cdot; t, d, \delta)$ is a bounded increasing function. For any such function $g \colon \mathbb{R} \to [0, \infty)$, an associated *transform* $\hat{g} \colon \mathbb{R} \to \mathbb{C}$, where $\mathbb{C}$ is the set of complex numbers, is defined by

$$\hat{g}(z) = \int_{-\infty}^{+\infty} e^{izy}\, dg(y), \tag{102}$$

where i is the usual imaginary number, often denoted $\sqrt{-1}$. Depending on one's conventions, one may refer to $\hat{g}$ as the *Fourier transform* of g. Under the technical condition that $\int_{-\infty}^{+\infty} |\hat{g}(z)|\, dz < \infty$, we have the Lévy Inversion Formula

$$g(y) = \frac{\hat{g}(0)}{2} - \frac{1}{\pi} \int_0^\infty \frac{1}{z} \mathrm{Im}\left[e^{-izy} \hat{g}(z) \right]\, dz, \tag{103}$$

where $\mathrm{Im}(c)$ denotes the imaginary part of a complex number c.

For the case $g(\cdot) = G(\cdot; t, d, \delta)$, with the associated transform $\widehat{G}(\cdot; t, d, \delta)$, we can compute $G(y; t, d, \delta)$ from Equation (103), typically by computing the integral in Equation (103) numerically, and thereby obtain option prices from Equation (100). Our final objective is therefore to compute the transform $\widehat{G}$. Fixing z, and applying Fubini's Theorem to Equation (102), we have $\widehat{G}(z; t, d, \delta) = f(X_0, 0)$, where $f \colon D \times [0, t] \to \mathbb{C}$ is defined by

$$f(X_s, s) = E^Q \left(\exp\left[-\int_s^t r(u)\, du \right] \exp\left[d \cdot X(t) \right] \exp\left[iz\delta \cdot X(t) \right] \Big| X_s \right). \tag{104}$$

From Equation (104), the same separation-of-variables arguments used to treat the affine term-structure models imply, under technical regularity conditions, that

$$f(x, s) = \exp\left[\alpha(t - s) + \beta(t - s) \cdot x \right], \tag{105}$$

where (α, β) solves the generalized Riccati ordinary differential equation (ODE) associated with the affine model and the coefficients ρ_0 and ρ_1 of the short rate. The solutions for $\alpha(\cdot)$ and $\beta(\cdot)$ are complex numbers, in light of the complex boundary condition $\beta(0) = d + iz\delta$. For technical details, see Duffie, Filipović and Schachermayer (2003).

Thus, under technical conditions, we have our transform $\widehat{G}(z; t, d, \delta)$, evaluated at a particular z. We then have the option-pricing formula (100), where $G(y; t, d, \delta)$ is obtained from the inversion formula (103) applied to the transforms $\widehat{G}(\cdot; t, b(t), -b(t))$ and $\widehat{G}(\cdot; t, 0, -b(t))$.

For option pricing with the Heston model, we require only the transform $\psi(u) = e^{-\bar{r}t}E^Q(\exp[uY(t)])$, for some particular choices of $u \in \mathbb{C}$. Heston (1993) solved the Riccati equation for this case, arriving at

$$\psi(u) = \exp\left[\bar{\alpha}(t,u) + uY(0) + \bar{\beta}(t,u)V(0)\right],$$

where, letting $b = u\sigma_v c_{SV} - \kappa$, $a = u(1-u)$, and $\gamma = \sqrt{b^2 + a\sigma_v^2}$,

$$\bar{\beta}(t,u) = -\frac{a\left(1 - e^{-\gamma t}\right)}{2\gamma - (\gamma + b)(1 - e^{-\gamma t})},$$

$$\bar{\alpha}(t,u) = \bar{r}t(u-1) - \kappa\bar{v}\left(\frac{\gamma+b}{\sigma_v^2}t + \frac{2}{\sigma_v^2}\log\left[1 - \frac{\gamma+b}{2\gamma}\left(1 - e^{-\gamma t}\right)\right]\right).$$

Other special cases for which one can compute explicit solutions are cited in Duffie, Pan and Singleton (2000).

6. Corporate securities

This section offers a basic review of the valuation of equities and corporate liabilities, beginning with some standard issues regarding the capital structure of a firm. Then, we turn to models of the valuation of defaultable debt that are based on an assumed stochastic arrival intensity of the stopping time defining default. The use of intensity-based defaultable bond pricing models was instigated by Artzner and Delbaen (1990, 1992, 1995), Lando (1994, 1998) and Jarrow and Turnbull (1995), and has become commonplace in business applications among banks and investment banks.

We begin with an extremely simple model of the stochastic behavior of the market values of assets, equity, and debt. We may think of equity and debt, at this first pass, as derivatives with respect to the total market value of the firm, as proposed by Black and Scholes (1973) and Merton (1974). In the simplest case, equity is merely a call option on the assets of the firm, struck at the level of liabilities, with possible exercise at the maturity date of the debt.[32]

At first, we are in a setting of perfect capital markets, where the results of Modigliani and Miller (1958) imply the irrelevance of capital structure for the total market value of the firm. Later, we introduce market imperfections and increase the degree of control that may be exercised by holders of equity and debt. With this, the theory becomes more complex and less like a derivative valuation model. There are many more interesting variations than could be addressed well in the space available here.

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[32] Geske (1977) used compound option modeling so as to extend to the Black–Scholes–Merton model to cases of debt at various maturities.

Our objective is merely to convey some sense of the types of issues and standard modeling approaches.

We let B be a standard Brownian motion in $\mathbb{R}^d$ on a complete probability space $(\Omega, \mathcal{F}, P)$, and fix the standard filtration $\{\mathcal{F}_t: t \geqslant 0\}$ of B. Later, we allow for information revealed by "Poisson-like arrivals", in order to tractably model "sudden-surprise" defaults that cannot be easily treated in a setting of Brownian information.

6.1. Endogenous default timing

We assume a constant short rate r and take as given a martingale measure Q, in the infinite-horizon sense of Huang and Pagès (1992), after deflation by e^{-rt}.

The resources of a given firm are assumed to consist of cash flows at a rate δ_t for each time t, where δ is an adapted process with $\int_0^t |\delta_s| \, ds < \infty$ almost surely for all t. The market value of the assets of the firm at time t is defined as the market value A_t of the future cash flows. That is,

$$A_t = E_t^Q \left(\int_t^\infty \exp\left[-r(s-t)\right] \delta_s \, ds \right). \tag{106}$$

We assume that A_t is well defined and finite for all t. The martingale representation theorem implies that

$$dA_t = (rA_t - \delta_t) \, dt + \sigma_t \, dB_t^Q, \tag{107}$$

for some adapted $\mathbb{R}^d$-valued process σ such that $\int_0^T \sigma_t \cdot \sigma_t \, dt < \infty$ for all $T \in [0, \infty)$, and where B^Q is the standard Brownian motion in $\mathbb{R}^d$ under Q obtained from B and Girsanov's Theorem.[33]

We suppose that the original owners of the firm chose its capital structure to consist of a single bond as its debt, and pure equity, defined in detail below. The bond and equity investors have already paid the original owners for these securities. Before we consider the effects of market imperfections, the total of the market values of equity and debt must be the market value A of the assets, which is a given process, so the design of the capital structure is irrelevant from the viewpoint of maximizing the total value received by the original owners of the firm.

For simplicity, we suppose that the bond promises to pay coupons at a constant total rate c, continually in time, until default. This sort of bond is sometimes called a *consol*. Equityholders receive the residual cash flow in the form of dividends at the rate $\delta_t - c$ at time t, until default. At default, the firm's future cash flows are assigned to debtholders.

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[33] For an explanation of how Girsanov's Theorem applies in an infinite-horizon setting, see for example the last section of Chapter 6 of Duffie (2001), based on Huang and Pagès (1992).

The equityholders' dividend rate, $\delta_t - c$, may have negative outcomes. It is commonly stipulated, however, that equity claimants have *limited liability*, meaning that they should not experience negative cash flows. One can arrange for limited liability by dilution of equity. [34]

Equityholders are assumed to have the contractual right to declare default at any stopping time T, at which time equityholders give up to debtholders the rights to all future cash flows, a contractual arrangement termed *strict priority,* or sometimes *absolute priority*. We assume that equityholders are not permitted to delay liquidation after the value A of the firm reaches 0, so we ignore the possibility that $A_T < 0$. We could also consider the option of equityholders to change the firm's production technology, or to call in the debt for some price.

The bond contract may convey to debtholders, under a *protective covenant,* the right to force liquidation at any stopping time τ at which the asset value A_τ is as low or lower than some stipulated level. We ignore this feature for brevity.

6.2. Example: Brownian dividend growth

We turn to a specific model proposed by Fisher, Heinkel and Zechner (1989), and explicitly solved by Leland (1994), for optimal default timing and for the valuation of equity and debt. Once we allow for taxes and bankruptcy distress costs, [35] capital structure matters, and, within the following simple parametric framework, Leland (1994) calculated the initial capital structure that maximizes the total initial market value of the firm.

Suppose the cash-flow rate process δ is a geometric Brownian motion under Q, in that

$$d\delta_t = \mu\delta_t \, dt + \sigma\delta_t \, dB_t^Q,$$

for constants μ and σ, where B^Q is a standard Brownian motion under Q. We assume throughout that $\mu < r$, so that, from Equation (106), A is finite and

$$dA_t = \mu A_t \, dt + \sigma A_t \, dB_t^Q.$$

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[34] That is, so long as the market value of equity remains strictly positive, newly issued equity can be sold into the market so as to continually finance the negative portion $(c - \delta_t)^+$ of the residual cash flow. While dilution increases the quantity of shares outstanding, it does not alter the total market value of all shares, and so is a relatively simple modeling device. Moreover, dilution is irrelevant to individual shareholders, who would in any case be in a position to avoid negative cash flows by selling their own shares as necessary to finance the negative portion of their dividends, with the same effect as if the firm had diluted their shares for this purpose. We are ignoring here any frictional costs of equity issuance or trading.

[35] The model was further elaborated to treat coupon debt of finite maturity in Leland and Toft (1996), endogenous calling of debt and re-capitalization in Leland (1998) and Uhrig-Homburg (1998), incomplete observation by bond investors, with default intensity, in Duffie and Lando (2001), and alternative approaches to default recovery by Anderson and Sundaresan (1996), Anderson, Pan and Sundaresan (2001), Décamps and Faure-Grimaud (2000, 2002), Fan and Sundaresan (2000), Mella-Barral (1999) and Mella-Barral and Perraudin (1997).

We calculate that $\delta_t = (r - \mu) A_t$.

For any given constant $K \in (0, A_0)$, the market value of a security that claims one unit of account at the hitting time $\tau(K) = \inf\{t: A_t \leqslant K\}$ is, at any time $t < \tau(K)$,

$$E_t^Q \left(\exp\left[-r(\tau(K) - t)\right] \right) = \left(\frac{A_t}{K} \right)^{-\gamma}, \tag{108}$$

where

$$\gamma = \frac{m + \sqrt{m^2 + 2r\sigma^2}}{\sigma^2},$$

and where $m = \mu - \sigma^2/2$. This can be shown by applying Ito's Formula to see that $e^{-rt}(A_t/K)^{-\gamma}$ is a Q-martingale.

Let us consider for simplicity the case in which bondholders have no protective covenant. Then, equityholders declare default at a stopping time that attains the maximum equity valuation

$$w(A_0) \equiv \sup_{T \in \mathcal{T}} \ E^Q \left[\int_0^T e^{-rt} (\delta_t - c) \, dt \right], \tag{109}$$

where $\mathcal{T}$ is the set of stopping times.

We naturally conjecture that the maximization problem (109) is solved by a hitting time of the form $\tau(A_B) = \inf\{t: A_t \leqslant A_B\}$, for some default-triggering level A_B of assets to be determined. Black and Cox (1976) developed the idea of default at the first passage of assets to a sufficiently low level, but used an exogenous default boundary. Longstaff and Schwartz (1995) extended this approach to allow for stochastic default-free interest rates. Their work was then refined by Collin-Dufresne and Goldstein (2001a).

Given this conjectured form $\tau(A_B)$ for the optimal default time, we further conjecture from Ito's Formula that the equity value function $w: (0, \infty) \to [0, \infty)$ defined by Equation (109) solves the ODE

$$\mathcal{D}w(x) - rw(x) + (r - \mu)x - c = 0, \quad x > A_B, \tag{110}$$

where

$$\mathcal{D}w(x) = w'(x)\,\mu x + \tfrac{1}{2} w''(x)\,\sigma^2 x^2, \tag{111}$$

with the absolute-priority boundary condition

$$w(x) = 0, \quad x \leqslant A_B. \tag{112}$$

Finally, we conjecture the *smooth-pasting condition*

$$w'(A_B) = 0, \tag{113}$$

based on Equation (112) and continuity of the first derivative $w'(\cdot)$ at A_B. Although not an obvious requirement for optimality, the smooth-pasting condition, sometimes

called the *high-order-contact* condition, has proven to be a fruitful method by which to conjecture solutions, as follows.

If we are correct in conjecturing that the optimal default time is of the form $\tau(A_B) = \inf\{t: A_t \leqslant A_B\}$, then, given an initial asset level $A_0 = x > A_B$, the value of equity must be

$$w(x) = x - A_B \left(\frac{x}{A_B}\right)^{-\gamma} - \frac{c}{r}\left[1 - \left(\frac{x}{A_B}\right)^{-\gamma}\right]. \tag{114}$$

This conjectured value of equity is merely the market value x of the total future cash flows of the firm, less a deduction equal to the market value of the debtholders' claim to A_B at the default time $\tau(A_B)$ using Equation (108), less another deduction equal to the market value of coupon payments to bondholders before default. The market value of those coupon payments is easily computed as the present value c/r of coupons paid at the rate c from time 0 to time $+\infty$, less the present value of coupons paid at the rate c from the default time $\tau(A_B)$ until $+\infty$, again using Equation (108). In order to complete our conjecture, we apply the smooth-pasting condition $w'(A_B) = 0$ to this functional form (114), and by calculation obtain the conjectured default triggering asset level as

$$A_B = \beta c, \tag{115}$$

where

$$\beta = \frac{\gamma}{r(1+\gamma)}. \tag{116}$$

We are ready to state and verify this result of Leland (1994).

Proposition. *The default timing problem (109) is solved by* $\inf\{t: A_t \leqslant \beta c\}$. *The associated initial market value* $w(A_0)$ *of equity is* $W(A_0, c)$, *where*

$$W(x, c) = 0, \quad x \leqslant \beta c, \tag{117}$$

and

$$W(x, c) = x - \beta c \left(\frac{x}{\beta c}\right)^{-\gamma} - \frac{c}{r}\left[1 - \left(\frac{x}{\beta c}\right)^{-\gamma}\right], \quad x \geqslant \beta c. \tag{118}$$

The initial value of debt is $A_0 - W(A_0, c)$.

Proof: First, it may be checked by calcuation that $W(\cdot, c)$ satisfies the differential equation (110) and the smooth-pasting condition (113). Ito's Formula applies to C^2 (twice continuously differentiable) functions. In our case, although $W(\cdot, c)$ need not

be C^2, it is convex, is C^1, and is C^2 except at βc, where $W_x(\beta c, c) = 0$. Under these conditions, we obtain the result of applying Ito's Formula as

$$W(A_s, c) = W(A_0, c) + \int_0^s \mathcal{D}W(A_t, c) \, dt + \int_0^s W_x(A_t, c) \, \sigma A_t \, dB_t^Q,$$

where $\mathcal{D}W(x, c)$ is defined as usual by

$$\mathcal{D}W(x, c) = W_x(x, c) \mu x + \tfrac{1}{2} W_{xx}(x, c) \sigma^2 x^2,$$

except at $x = \beta c$, where we may replace "$W_{xx}(\beta c, c)$" with zero. [This slight extension of Ito's Formula is found, for example, in Karatzas and Shreve (1988), p. 219.]

For each time t, let

$$q_t = e^{-rt} W(A_t, c) + \int_0^t e^{-rs} ((r - \mu) A_s - c) \, ds.$$

From Ito's Formula,

$$dq_t = e^{-rt} f(A_t) \, dt + e^{-rt} W_x(A_t, c) \, \sigma A_t \, dB_t^Q, \qquad (119)$$

where

$$f(x) = \mathcal{D}W(x, c) - rW(x, c) + (r - \mu) x - c.$$

Because W_x is bounded, the last term of Equation (119) defines a Q-martingale. For $x \leqslant \beta c$, we have both $W(x, c) = 0$ and $(r - \mu) x - c \leqslant 0$, so $f(x) \leqslant 0$. For $x > \beta c$, we have Equation (110), and therefore $f(x) = 0$. The drift of q is therefore never positive, and for any stopping time T we have $q_0 \geqslant E^Q(q_T)$, or equivalently,

$$W(A_0, c) \geqslant E^Q \left[\int_0^T e^{-rs} (\delta_s - c) \, ds + e^{-rT} W(A_T, c) \right].$$

For the particular stopping time $\tau(\beta c)$, we have

$$W(A_0, c) = E^Q \left[\int_0^{\tau(\beta c)} e^{-rs} (\delta_s - c) \, ds \right],$$

using the boundary condition (117) and the fact that $f(x) = 0$ for $x > \beta c$. So, for any stopping time T,

$$W(A_0, c) = E^Q \left[\int_0^{\tau(\beta c)} e^{-rs} (\delta_s - c) \, ds \right]$$

$$\geqslant E^Q \left[\int_0^T e^{-rs} (\delta_s - c) \, ds + e^{-rT} W(A_T, c) \right]$$

$$\geqslant E^Q \left[\int_0^T e^{-rs} (\delta_s - c) \, ds \right],$$

using the non-negativity of W for the last inequality. This implies the optimality of the stopping time $\tau(\beta c)$ and verification of the proposed solution $W(A_0, c)$ of Equation (109). $\square$

Boyarchenko and Levendorskiĭ (2002), Hilberink and Rogers (2002) and Zhou (2000) extend this first passage model of optimal default timing to the case of jump-diffusion asset processes.

6.3. Taxes, bankruptcy costs, capital structure

In order to see how the original owners of the firm may have a strict but limited incentive to issue debt, we introduce two market imperfections:
- A tax deduction, at a tax rate of θ, on interest expense, so that the after-tax effective coupon rate paid by the firm is $(1 - \theta)c$.
- Bankruptcy costs, so that, with default at time t, the assets of the firm are disposed of at a salvage value of $\hat{A}_t \leqslant A_t$, where $\hat{A}$ is a given continuous adapted process.

We also consider more carefully the formulation of an equilibrium, in which equityholders and bondholders each exercise their own rights so as to maximize the market values of their own securities, given correct conjectures regarding the equilibrium policy of the other claimant. Because the total of the market values of equity and debt is not the fixed process A, new considerations arise, including inefficiencies. That is, in an equilibrium, the total of the market values of equity and bond may be strictly less than maximal, for example because of default that is premature from the viewpoint of maximizing the total value of the firm. An unrestricted central planner could in such a case split the firm's cash flows between equityholders and bondholders so as to achieve strictly larger market values for each than the equilibrium values of their respective securities.

Absent the tax shield on debt, the original owner of the firm, who selects a capital structure at time 0 so as to maximize the total initial market value of all corporate securities, would have avoided a capital structure that involves an inefficiency of this type. For example, an all-equity firm would avoid bankruptcy costs.

In order to illustrate the endogenous choice of capital structure based on the tradeoff between the values of tax shields and of bankruptcy losses, we extend the example of Section 6.2 by assuming a tax rate of $\theta \in (0, 1)$ and bankruptcy recovery $\hat{A} = \epsilon A$, for a constant fractional recovery rate $\epsilon \in [0, 1]$. For simplicity, we assume no protective covenant.

The equity valuation and optimal default timing problem is identical to Equation (109), except that equityholders treat the effective coupon rate as the after-tax rate $c(1 - \theta)$. Thus, the optimal equity market value is $W(A_0, c(1 - \theta))$, where $W(x, y)$ is given by Equations (117) and (118). The optimal default time is

$$T^* = \inf\{t : A_t \leqslant \beta(1 - \theta)c\}.$$

For a given coupon rate c, the bankruptcy recovery rate ϵ has no effect on the equity value. The market value $U(A_0, c)$ of debt, at asset level A_0 and coupon rate c, is indeed affected by distress costs, in that

$$U(x, c) = \epsilon x, \quad x \leqslant \beta(1 - \theta)c, \tag{120}$$

and, for $x \geqslant \beta(1 - \theta)c$,

$$U(x,c) = \epsilon\beta c(1-\theta)\left(\frac{x}{\beta c(1-\theta)}\right)^{-\gamma} + \frac{c}{r}\left[1 - \left(\frac{x}{\beta c(1-\theta)}\right)^{-\gamma}\right]. \tag{121}$$

The first term of Equation (121) is the market value of the payment of the recovery value $\epsilon A(T^*) = \epsilon\beta c(1 - \theta)$ at default, using Equation (108). The second term is the market value of receiving the coupon rate c until T^*.

The capital structure that maximizes the market value received by the initial owners for sale of equity and debt can now be determined from the coupon rate c^* solving

$$\sup_c \left\{ U\left(A_0, c\right) + W\left(A_0, (1 - \theta)c\right) \right\}. \tag{122}$$

Leland (1994) provides an explicit solution for c^*, which then allows one to easily examine the resolution of the tradeoff between the market value

$$H\left(A_0, c\right) = \frac{\theta c}{r}\left[1 - \left(\frac{A_0}{\beta c(1-\theta)}\right)^{-\gamma}\right],$$

of tax shields and the market value

$$h\left(A_0, c\right) = \epsilon\beta c(1-\theta)\left(\frac{A_0}{\beta c(1-\theta)}\right)^{-\gamma},$$

of financial distress costs associated with bankruptcy. The coupon rate that solves Equation (122) is that which maximizes $H(A_0, c) - h(A_0, c)$, the benefit–cost difference. Although the tax shield is valuable to the firm, it is merely a transfer from somewhere else in the economy. The bankruptcy distress cost, however, involves a net social cost, illustrating one of the inefficiencies caused by taxes.

Leland and Toft (1996) extend the model so as to treat bonds of finite maturity with discrete coupons. One can also allow for multiple classes of debtholders, each with its own contractual cash flows and rights. For example, bonds are conventionally classified by priority, so that, at liquidation, senior bondholders are contractually entitled to cash flows resulting from liquidation up to the total face value of senior debt (in proportion to the face values of the respective senior bonds, and normally without regard to maturity dates). If the most senior class of debtholders can be paid off in full, the next most senior class is assigned liquidation cash flows, and so on, to the lowest subordination class. Some bonds may be secured by certain identified assets, or *collateralized*, in effect giving them seniority over the liquidation value resulting from those cash flows, before any unsecured bonds may be paid according to the seniority of unsecured claims. In practice, the overall priority structure may be rather complicated.

Corporate bonds are often callable, within certain time restrictions. Not infrequently, corporate bonds may be converted to equity at pre-arranged conversion ratios (number

of shares for a given face value) at the timing option of bondholders. Such *convertible bonds* present a challenging set of valuation issues, some examined by Brennan and Schwartz (1980) and Nyborg (1996). Occasionally, corporate bonds are *puttable,* that is, may be sold back to the issuer at a pre-arranged price at the option of bondholders.

One can also allow for adjustments in capital structure, normally instigated by equityholders, that result in the issuing and retiring of securities, subject to legal restrictions, some of which may be embedded in debt contracts.

6.4. Intensity-based modeling of default

This section introduces a model for a default time as a stopping time τ with a given intensity process λ, as defined below. From the joint behavior of λ, the short-rate process r, the promised payment of the security, and the model of recovery at default, as well as risk premia, one can characterize the stochastic behavior of the term structure of yields on defaultable bonds.

In applications, default intensities may be modeled as functions of observable variables that are linked with the likelihood of default, such as debt-to-equity ratios, asset volatility measures, other accounting measures of indebtedness, market equity prices, bond yield spreads, industry performance measures, and macroeconomic variables related to the business cycle. This dependence could, but in practice does not usually, arise endogenously from a model of the ability or incentives of the firm to make payments on its debt. Because the approach presented here does not depend on the specific setting of a firm, it has also been applied to the valuation of defaultable sovereign debt, as in Duffie, Pedersen and Singleton (2003) and Pagès (2000).

We fix a complete probability space $(\Omega, \mathcal{F}, P)$ and a filtration $\{\mathcal{G}_t: t \geqslant 0\}$ satisfying the usual conditions. At some points, it will be important to make a distinction between an adapted process and a *predictable process*. A predictable process is, intuitively speaking, one whose value at any time t depends only on the information in the underlying filtration that is available up to, but not including, time t. Protter (1990) provides a full definition.

A non-explosive counting process K (for example, a Poisson process) has an intensity λ if λ is a predictable non-negative process satisfying $\int_0^t \lambda_s \, ds < \infty$ almost surely for all t, with the property that a local martingale M, the *compensated counting process*, is given by

$$M_t = K_t - \int_0^t \lambda_s \, ds. \tag{123}$$

The compensated counting process M is a martingale if, for all t, we have $E(\int_0^t \lambda_s \, ds) < \infty$. A standard reference on counting processes is Brémaud (1981).

For simplicity, we will say that a stopping time τ has an *intensity* λ if τ is the first jump time of a non-explosive counting process whose intensity process is λ. The accompanying intuition is that, at any time t and state ω with $t < \tau(\omega)$, the

$\mathcal{G}_t$-conditional probability of an arrival before $t + \Delta$ is approximately $\lambda(\omega, t)\Delta$, for small Δ. This intuition is justified in the sense of derivatives if λ is bounded and continuous, and under weaker conditions.

A stopping time τ is *non-trivial* if $P(\tau \in (0, \infty)) > 0$. If a stopping time τ is non-trivial and if the filtration $\{\mathcal{G}_t\colon t \geqslant 0\}$ is the standard filtration of some Brownian motion B in $\mathbb{R}^d$, then τ could not have an intensity. We know this from the fact that, if $\{\mathcal{G}_t\colon t \geqslant 0\}$ is the standard filtration of B, then the associated compensated counting process M of Equation (123) (indeed, any local martingale) could be represented as a stochastic integral with respect to B, and therefore cannot jump, but M must jump at τ. In order to have an intensity, a stopping time τ must be *totally inaccessible*, roughly meaning that it cannot be "foretold" by an increasing sequence of stopping times that converges to τ. An inaccessible stopping time is a "sudden surprise", but there are no such surprises on a Brownian filtration!

As an illustration, we could imagine that the firm's equityholders or managers are equipped with some Brownian filtration for purposes of determining their optimal default time τ, but that bondholders have imperfect monitoring, and may view τ as having an intensity with respect to the bondholders' own filtration $\{\mathcal{G}_t\colon t \geqslant 0\}$, which contains less information than the Brownian filtration. Such a situation arises in Duffie and Lando (2001).

We say that τ is *doubly stochastic* with intensity λ if the underlying counting process whose first jump time is τ is doubly stochastic with intensity λ. This means roughly that, conditional on the intensity process, the counting process is a Poisson process with that same (conditionally deterministic) intensity. The doubly-stochastic property thus implies that, for $t < \tau$, using the law of iterated expectations,

$$\begin{aligned}
P(\tau > s \mid \mathcal{G}_t) &= E\left[P(\tau > s \mid \mathcal{G}_t, \{\lambda_u\colon t \leqslant u \leqslant s\}) \mid \mathcal{G}_t\right] \\
&= E\left(\exp\left[-\int_t^s \lambda(u)\,\mathrm{d}u\right] \,\bigg|\, \mathcal{G}_t\right),
\end{aligned} \tag{124}$$

using the fact that the probability of no jump between t and s of a Poisson process with time-varying (deterministic) intensity λ is $\exp[-\int_t^s \lambda(u)\,\mathrm{d}u]$. This property (124) is convenient for calculations, because evaluating $E(\exp[-\int_t^s \lambda(u)\,\mathrm{d}u] \mid \mathcal{G}_t)$ is computationally equivalent to the pricing of a default-free zero-coupon bond, treating λ as a short rate. Indeed, this analogy is also quite helpful for intuition and suggests tractable models for intensities based on models of the short rate that are tractable for default-free term structure modeling.

As we shall see, it would be sufficient for Equation (124) that $\lambda_t = \Lambda(X_t, t)$ for some measurable $\Lambda\colon \mathbb{R}^n \times [0, \infty) \to [0, \infty)$, where X in $\mathbb{R}^d$ solves a stochastic differential equation of the form

$$\mathrm{d}X_t = \mu(X_t, t)\,\mathrm{d}t + \sigma(X_t, t)\,\mathrm{d}B_t, \tag{125}$$

for some $(\mathcal{G}_t)$-standard Brownian motion B in $\mathbb{R}^d$. More generally, Equation (124) follows from assuming that the doubly-stochastic counting process K whose first jump

time is τ is *driven by some filtration* $\{\mathcal{F}_t: t \geqslant 0\}$. This means roughly that, for any t, conditional on $\mathcal{F}_t$, the distribution of K during $[0, t]$ is that of a Poisson process with time-varying conditionally deterministic intensity λ. A complete definition is provided in Duffie (2001).[36]

For purposes of the market valuation of bonds and other securities whose cash flows are sensitive to default timing, we would want to have a *risk-neutral intensity process*, that is, an intensity process λ^Q for the default time τ that is associated with $(\Omega, \mathcal{F}, Q)$ and the given filtration $\{\mathcal{G}_t: t \geqslant 0\}$, where Q is an equivalent martingale measure. In this case, we call λ^Q the *Q-intensity* of τ. (As usual, there may be more than one equivalent martingale measure.) Such an intensity always exists, as shown by Artzner and Delbaen (1995), but the doubly-stochastic property may be lost with a change of measure [Kusuoka (1999)]. The ratio λ^Q/λ (for λ strictly positive) is in some sense a multiplicative risk premium for the uncertainty associated with the timing of default. This issue is pursued by Jarrow, Lando and Yu (2003), who provide sufficient conditions for no default-timing risk premium (but allowing nevertheless a default risk premium).

6.5. Zero-recovery bond pricing

We fix a short-rate process r and an equivalent martingale measure Q after deflation by $\exp[-\int_0^t r(u)\,du]$. We consider the valuation of a security that pays $F1_{\{\tau > s\}}$ at a given time $s > 0$, where F is a $\mathcal{G}_T$-measurable bounded random variable. Because $1_{\{\tau > s\}}$ is the random variable that is 1 in the event of no default by s and zero otherwise, we may view F as the contractually promised payment of the security at time s, with default by s leading to no payment. The case of a defaultable zero-coupon bond is treated by letting $F = 1$. In the next sub-section, we will consider recovery at default.

From the definition of Q as an equivalent martingale measure, the price S_t of this security at any time $t < s$ is

$$S_t = E_t^Q \left(\exp\left[-\int_t^s r(u)\,du\right] 1_{\{\tau > s\}} F \right), \tag{126}$$

where E_t^Q denotes $\mathcal{G}_t$-conditional expectation under Q. From Equation (126) and the fact that τ is a stopping time, S_t must be zero for all $t \geqslant \tau$.

Under Q, the default time τ is assumed to have a Q-intensity process λ^Q.

Theorem. *Suppose that F, r and λ^Q are bounded and that τ is doubly stochastic under Q driven by a filtration $\{\mathcal{F}_t: t \geqslant 0\}$ such that r is $(\mathcal{F}_t)$-adapted and F is $\mathcal{F}_s$-measurable. Fix any $t < s$. Then, for $t \geqslant \tau$, we have $S_t = 0$, and for $t < \tau$,*

$$S_t = E_t^Q \left(\exp\left[-\int_t^s \left(r(u) + \lambda^Q(u)\right)\,du\right] F \right). \tag{127}$$

---

[36] Included in the definition is the condition that λ is $(\mathcal{F}_t)$-predictable, that $\mathcal{F}_t \subset \mathcal{G}_t$, and that $\{\mathcal{F}_t: t \geq 0\}$ satisfies the usual conditions.

This theorem is based on Lando (1998).[37] The idea of this representation (127) of the pre-default price is that discounting for default that occurs at an intensity is analogous to discounting at the short rate r.

Proof: From Equation (126), the law of iterated expectations, and the assumption that r is $(\mathcal{F}_t)$-adapted and F is $\mathcal{F}_s$-measurable,

$$S_t = E^Q \left(E^Q \left\{ \exp\left[-\int_t^s r(u)\,du \right] 1_{\{\tau > s\}} F \mid \mathcal{F}_s \vee \mathcal{G}_t \right\} \mid \mathcal{G}_t \right)$$
$$= E^Q \left(\exp\left[-\int_t^s r(u)\,du \right] F E^Q \left[1_{\{\tau > s\}} \mid \mathcal{F}_s \vee \mathcal{G}_t \right] \mid \mathcal{G}_t \right).$$

The result then follows from the implication of double stochasticity that $Q(\tau > s \mid \mathcal{F}_s \vee \mathcal{G}_t) = \exp[\int_t^s \lambda^Q(u)\,du]$. $\square$

As a special case, suppose the filtration $\{\mathcal{F}_t : t \geqslant 0\}$ is that generated by a process X that is affine under Q and valued in $D \subset \mathbb{R}^d$. It is natural to allow dependence of λ^Q, r and F on the state process X in the sense that

$$\lambda_t^Q = \Lambda(X_t), \quad r_t = \rho(X_t), \quad F = \exp[f(X(T))], \tag{128}$$

where Λ, ρ and f are affine on D.

Under the technical regularity in Duffie, Filipović and Schachermayer (2003), relation (127) then implies that, for $t < \tau$, we have

$$S_t = \exp[\alpha(T - t) + \beta(T - t) \cdot X(t)], \tag{129}$$

for coefficients $\alpha(\cdot)$ and $\beta(\cdot)$ that are computed from the associated Generalized Riccati equations.

6.6. Pricing with recovery at default

The next step is to consider the recovery of some random payoff W at the default time τ, if default occurs before the maturity date s of the security. We adopt the assumptions of Theorem 6.5, and add the assumption that $W = w_\tau$, where w is a bounded predictable process that is also adapted to the driving filtration $\{\mathcal{F}_t : t \geqslant 0\}$.

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[37] Additional work in this vein is by Bielecki and Rutkowski (1999a,b, 2001), Cooper and Mello (1991, 1992), Das and Sundaram (2000), Das and Tufano (1995), Davydov, Linetsky and Lotz (1999), Duffie (1998), Duffie and Huang (1996), Duffie, Schroder and Skiadas (1996), Duffie and Singleton (1999), Elliott, Jeanblanc and Yor (2000), Hull and White (1992, 1995), Jarrow and Yu (2001), Jeanblanc and Rutkowski (2000), Madan and Unal (1998) and Nielsen and Ronn (1995).

The market value at any time $t < \min(s, \tau)$ of any default recovery is, by definition of the equivalent martingale measure Q, given by

$$J_t = E_t^Q \left(\exp\left[\int_t^\tau -r(u)\, du \right] 1_{\{\tau \leqslant s\}} w_\tau \right). \tag{130}$$

The doubly-stochastic assumption implies that τ has a probability density under Q, at any time u in $[t, s]$, conditional on $\mathcal{G}_t \vee \mathcal{F}_s$, and on the event that $\tau > t$, of

$$q(t, u) = \exp\left[\int_t^u -\lambda^Q(z)\, dz \right] \lambda^Q(u).$$

Thus, using the same iterated-expectations argument of the proof of Theorem 6.5, we have, on the event that $\tau > t$,

$$\begin{aligned} J_t &= E^Q \left(E^Q \left[\exp\left(\int_t^\tau -r(z)\, dz \right) 1_{\{\tau \leqslant s\}} w_\tau \,\Big|\, \mathcal{F}_s \vee \mathcal{G}_t \right] \,\Big|\, \mathcal{G}_t \right) \\ &= E^Q \left(\int_t^s \exp\left[\int_t^u -r(z)\, dz \right] q(t, u)\, w_u\, du \,\Big|\, \mathcal{G}_t \right) \\ &= \int_t^s \Phi(t, u)\, du, \end{aligned}$$

using Fubini's Theorem, where

$$\Phi(t, u) = E_t^Q \left(\exp\left[-\int_t^u [\lambda^Q(z) + r(z)]\, dz \right] \lambda^Q(u)\, w(u) \right). \tag{131}$$

We summarize the main defaultable valuation result as follows.

Theorem. *Consider a security that pays F at s if $\tau > s$, and otherwise pays w_τ at τ. Suppose that w, F, λ^Q and r are bounded. Suppose that τ is doubly stochastic under Q, driven by a filtration $\{\mathcal{F}_t: t \geqslant 0\}$ with the property that r and w are $(\mathcal{F}_t)$-adapted and F is $\mathcal{F}_s$-measurable. Then, for $t \geqslant \tau$, we have $S_t = 0$, and for $t < \tau$,*

$$S_t = E_t^Q \left\{ \exp\left[-\int_t^s (r(u) + \lambda^Q(u))\, du \right] F \right\} + \int_t^s \Phi(t, u)\, du. \tag{132}$$

These results are based on Duffie, Schroder and Skiadas (1996) and Lando (1994, 1998). Schönbucher (1998) extends to treat the case of recovery W which is not of the form w_τ for some predictable process w, but rather allows the recovery to be revealed just at the default time τ. For details on this construction, see Duffie (2002).

In the affine state-space setting described at the end of the previous section, $\Phi(t, u)$ can be computed by our usual "affine" methods, provided that w is of form $w_t = e^{a + b \cdot X(t)}$ for constant coefficients a and b. In this case, under technical regularity,

$$\Phi(t, u) = \exp\left[\alpha(u - t) + \beta(u - t) \cdot X(t)\right]\left[c(u - t) + C(u - t) \cdot X(t)\right], \tag{133}$$

for readily computed deterministic coefficients α, β, c and C, as in Duffie, Pan and Singleton (2000). This still leaves the task of numerical computation of the integral $\int_t^s \Phi(t, u)\, du$.

For the price of a typical defaultable bond promising periodic coupons followed by its principal at maturity, one may sum the prices of the coupons and of the principal, treating each of these payments as though it were a separate zero-coupon bond. An often-used assumption, although one that need not apply in practice, is that there is no default recovery for coupons remaining to be paid as of the time of default, and that bonds of different maturities have the same recovery of principal. In any case, convenient parametric assumptions, based for example on an affine driving process X, lead to straightforward computation of a term structure of defaultable bond yields that may be applied in practical situations, such as the valuation of *credit derivatives*, a class of derivative securities designed to transfer credit risk that is treated in Duffie and Singleton (2003).

For the case of defaultable bonds with embedded American options, the most typical cases being callable or convertible bonds, the usual resort is valuation by some numerical implementation of the associated dynamic programming problems. See Berndt (2002).

6.7. Default-adjusted short rate

In the setting of Theorem 6.6, a particularly simple pricing representation can be based on the definition of a predictable process ℓ for the fractional loss in market value at default, according to

$$(1 - \ell_\tau)(S_{\tau-}) = w_\tau. \tag{134}$$

Manipulation left to the reader shows that, under the conditions of Theorem 6.6, for $t < \tau$,

$$S_t = E_t^Q\left(\exp\left[\int_t^s -\left(r(u) + \ell(u)\,\lambda^Q(u)\right)\, du\right] F\right). \tag{135}$$

This valuation model (135) is from Duffie and Singleton (1999), and based on a precursor of Pye (1974). This representation (135) is particularly convenient if we take ℓ as an exogenously given fractional loss process, as it allows for the application of standard valuation methods, treating the payoff F as default-free, but accounting for the

intensity and severity of default losses through the "default-adjusted" short-rate process $r + \ell\lambda^Q$. The adjustment $\ell\lambda^Q$ is in fact the risk-neutral mean rate of proportional loss in market value due to default.

Notably, the dependence of the bond price on the intensity λ^Q and fractional loss ℓ at default is only through the product $\ell\lambda^Q$. For example, doubling λ^Q and halving ℓ has no effect on the bond price process.

Suppose, for example, that τ is doubly stochastic driven by the filtration of a state process X that is affine under Q, and we take $r_t + \ell_t\lambda_t^Q = R(X_t)$ and $F = \exp[f(X(T))]$, for affine $R(\cdot)$ and $f(\cdot)$. Then, under regularity conditions, we obtain at each time t before default a bond price of the simple form (129), again for coefficients solving the associated Generalized Riccati equation.

Using this affine approach to default-adjusted short rates, Duffee (1999a) provides an empirical model of risk-neutral default intensities for corporate bonds.[38]

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[38] For related empirical work on sovereign debt, see Duffie, Pedersen and Singleton (2003) and Pagès (2000).

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