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ABSTRACT

This article presents a model for valuing claims subject to default by both contracting parties, such as swaps and forwards. With counterparties of different default risk, the promised cash flows of a swap are discounted by a switching discount rate that, at any given state and time, is equal to the discount rate of the counterparty for whom the swap is currently out of the money (that is, a liability). The impact of credit-risk asymmetry and of netting is presented through both theory and numerical examples, which include interest rate and currency swaps.

THIS ARTICLE PRESENTS A model for valuing claims subject to default by both contracting parties, such as swaps and forwards. This extends the valuation model for defaultable claims proposed by Duffie and Singleton (1994) to cases in which the two counterparties have asymmetric default risk. The extension permits a reexamination of the impact of credit risk on swap rates. While the valuation model applies to all forms of contingent claims in which both contracting parties are at risk to default, such as forward contracts, we focus on swaps for purposes of illustration.

For example, consider a 5-year interest rate swap between a given party paying a floating rate such as the London Inter Bank Offered Rate (LIBOR) and another counterparty paying a fixed rate. Replacing the given fixed-rate counterparty with a "lower-quality" counterparty, whose bond yields are 100 basis points higher, increases the swap rate by roughly 1 basis point, using our model and typical parameters for LIBOR rate processes. This credit impact on swap rates is approximately linear within the range of normally encountered credit quality. For a 5-year currency swap, given a foreign exchange rate with 15 percent volatility, our model shows the impact of credit risk asymmetry on the market swap rate to be roughly 10-fold greater than that for interest rate swaps; that is, approximately 10 basis points in swap rate per 100 basis points in bond yield credit spread. The main goal of this article is to provide a simple and theoretically consistent model allowing such computations.

I. The Basic Model

The basic idea behind our model is that the impact of credit risk on swap rates depends on the probability distribution of the path taken by the value of the swap

itself. When the swap value is positive for a given counterparty, it is the default characteristics (default hazard rate and fractional loss given default) of the *other counterparty* that are relevant for the backward recursive computation of the current swap value given its value at the next point in time. The basic idea for this recursion was developed by Rendleman (1992). Rendleman's model, however, is based on the impact of the swap value on the balance sheets of the counterparties, and considers the direct implications for structural insolvency (liabilities exceed assets). In order to address the problem of determining market swap rates, for which one cannot normally analyze the financial statements of the counterparties on a case-by-case basis with ease or accuracy, we develop a reduced-form model in which the default characteristics of the counterparties are directly estimated in terms of credit spreads. For other approaches, see Abken (1993), Cooper and Mello (1991), Hull and White (1995), Li (1995), Solnik (1990), Sorensen and Bollier (1994), and Sundaresan (1991).

The discrete-time intuition behind our model is as follows. At any given time t, the current market value of the swap, assuming that it has not yet defaulted, is denoted V_t. We suppose that V_t is the value to counterparty A, and therefore that $-V_t$ is the value to the other counterparty, B. If $V_t > 0$, then counterparty A is at risk to the default of counterparty B between t and $t + 1$. Thus, under risk-neutral probabilities, V_t is the probability that B defaults between t and $t + 1$ multiplied by the market value given default by B, plus the probability that B does not default between t and $t + 1$, multiplied by the market value given no default by B. The market value given no default is the risk-neutral expected present value of receiving V_{t+1} at $t + 1$, plus any net dividends paid to A by B between t and $t + 1$, under the terms of the swap. The market value given default is some fraction, associated with the credit quality of B, of the market value given no default. If, on the other hand, $V_t < 0$, then this recursive method for computing V_t from V_{t+1} is the same, except for the fact that B is at risk to default by A in this case, so the probability of default and fractional recovery on default used in the recursion are those of A. Our model has been extended in a background research report, Duffie and Huang (1994), to variations of the default settlement provisions of the swap contract standardized by the International Swap and Derivatives Association (ISDA).

The effective credit quality of a counterparty in our model is the spread s in the short rate of interest that applies for swap liabilities of that counterparty, over the usual (default-free) short rate r. In continuous-time, this spread was demonstrated by Duffie and Singleton (1994) to be $s_t = (1 - \varphi_t)h_t$, where φ is the stochastic process for fractional recovery rates given default and h is the risk-neutral hazard rate process. In effect, $h_t \Delta t$ is approximately equal to the conditional (risk-neutral) probability at time t of default over the next interval of "small" length Δt. The default-adjusted effective short rate is $R = r + s$. As shown by Duffie and Singleton (1994), term-structure models for default-free bonds are valid as well for defaultable bonds when substituting the default-adjusted short rate R (appropriate for bonds of the given credit quality) for the usual short rate r.

We consider special cases in which counterparty A is always of higher credit quality than counterparty B, in the sense that $s^A \leq s^B$, where s^A is the short credit spread for A and s^B is the short credit spread for B. In these cases, we show that netting across swap portfolios always increases the market value of the portfolio for the higher-quality counterparty A (and therefore reduces the market value for the lower quality counterparty B). Of course, the diversification of credit risk associated with netting is usually beneficial to both counterparties. We also show that the party of higher credit quality prefers to delay the release of information that may have an impact on swap values. We provide a relatively explicit formula for the marginal impact of an increase in the credit-risk asymmetry $s^B - s^A$ on the market value of a swap. The distinguishing feature of this formula is the appearance of an expectation of an integral over time of $(s_t^A - s_t^B)V_t^+$, the credit spread multiplied by the positive part of the market value of the swap contract, showing the importance to both counterparties of the volatility of the market value of the swap. Indeed, in an example involving a currency swap, we are able to exploit the Black-Scholes formula to compute explicitly this marginal impact of credit-risk asymmetry, and thereby deduce the marginal impact on the swap rate.

In Section II, we present and characterize a two-counterparty defaultable claim valuation model, extending results from Duffie and Singleton (1994) and Duffie, Schroder, and Skiadas (1995). In Section III, we apply our model to the case of interest-rate swaps and calculate the impact on swap rates of asymmetric credit quality. In Section IV, we apply the model to foreign currency swaps. All proofs are in the appendices.

II. Valuation of Defaultable Swaps

One-party defaultable claim valuation problems, in our "reduced-form" setting, have been studied by Artzner and Delbaen (1992); Duffie, Schroder, and Skiadas (1995); Duffie and Singleton (1994); Jarrow, Lando, and Turnbull (1993); Jarrow and Turnbull (1995); Lando (1993, 1994); and Ramaswamy and Sundaresan (1986). Our focus is the valuation of two-party contingent claims that can be either an asset or a liability to each party during the life of the contract. Without loss of generality, this problem is equivalent to the valuation of defaultable swaps.

A. Basic Setup

We begin with a probability space $(\Omega, \mathscr{F}, P)$ and a family $\mathbf{F} = \{\mathscr{F}_t : t \geq 0\}$ of sub-σ-algebras of $\mathscr{F}$ satisfying the *usual* conditions. (See, for example, Protter (1990) for technical details.) The filtration $\mathbf{F}$ represents the arrival of information over time. We also assume the existence of a short rate process r (progressively measurable and integrable), so that an investor can place one unit of account in riskless deposits at any time t and roll over the proceeds until time $s \geq t$ for a (time-s) market value of $\exp(\int_t^s r_u \, du)$.

We study the valuation directly under an equivalent martingale measure, denoted Q, relative to the short rate process r. By assumption, this means that, for a security defined by a cumulative dividend process $\{X_t : 0 \leq t \leq T\}$ (adapted, RCLL, with finite variation), the market value of the security at time t is

$$S_t = \mathrm{E}_Q\left[\int_t^T \exp\left(-\int_t^s r_u \, du \right) dX_s \,\middle|\, \mathscr{F}_t \right], \tag{2.1}$$

where E_Q denotes expectation under Q. We do not deal directly with the existence of an equivalent martingale measure, a property essentially equivalent to the absence of arbitrage, as shown by Harrison and Kreps (1979), nor with the identification of some particular equivalent martingale measure from market prices. For example, consider a swap whose cumulative net cash flows to a given counterparty, including the effects of default by either counterparty, are given by X. Then an outside investor who is unconstrained from borrowing at the risk-free rate would be willing, at time t, to pay S_t, at the margin, to receive the same net cash flows X of the swap. Our task is to compute S_t in terms of the promised cash flows, the short rate process r, and the default risks, defined below, of the two counterparties.

B. Swaps and Default Characteristics

Consider two counterparties denoted, respectively, as party A and B. The stochastic default time of party i, for $i \in \{A, B\}$, is an **F**-stopping time τ^i valued in $[0, \infty]$.[1] The default time for the swap is defined as $\tau = \tau^A \wedge \tau^B$, the minimum of τ^A and τ^B. The event $\{\tau > T\}$ is then the event of no default. In this setup, a swap with maturity T initiated at time zero between these two counterparties is formally defined by a predefault payment by party B to party A (until default) of a cumulative dividend process $\{D_t : 0 \leq t \leq T\}$, where D is a semimartingale of finite variation[2] such that[3]

$$\mathrm{E}_Q\left[\int_0^T \exp\left(-\int_0^t r_u \, du \right) |dD_t| \right] < \infty. \tag{2.2}$$

For example, if δ_t is the fixed-rate coupon payment by counterparty B minus the floating-rate coupon payment by counterparty A on an interest rate swap

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[1] Throughout, we use superscripts A and B to denote counterparties. The event $\tau^i = \infty$ means no default.

[2] See, for example, Protter (1990) for a technical definition of a semimartingale. We always assume without loss of generality that a semimartingale is RCLL (right-continuous with left limits).

[3] This integrability condition is satisfied if the market value of the promised gross payment of each counterparty, if assumed to be default-free, is finite.

at time t, with payments at times 1, 2, ..., n, then $D_t = \Sigma_{i \le t}\delta_i$. Our definition of a swap includes a forward contract as a special case.

Some conditions are placed on the default times τ^A and τ^B. We introduce, for each i, the *default indicator functions*, $H_t^i = 1_{\{t \ge \tau^i\}}$, a stochastic process that is equal to one if default by party i has occurred, and zero otherwise. The Doob-Meyer decomposition implies that H^i can be uniquely decomposed as $H^i = A^i + M^i$, where A^i is a predictable and right-continuous increasing process with $A_0^i = 0$, and M^i is a Q-martingale. We assume that

$$A_t^i = \int_0^t h_s^i 1_{\{s < \tau^i\}} ds, \quad t \ge 0,$$

for a bounded predictable hazard rate process h^i. The hazard rate model is traditional for inaccessible stopping times, such as the interarrival times of a Poisson process, which are defined by having a constant hazard rate. The hazard rate model for default times is also exploited by Artzner and Delbaen (1992), Duffie and Singleton (1994), Duffie, Schroder, and Skiadas (1995), Jarrow, Lando, and Turnbull (1993), Jarrow and Turnbull (1995), Lando (1993, 1994), and Madan and Unal (1993). Artzner and Delbaen (1995) give technical conditions under which a hazard rate exists under one probability measure if and only if a hazard rate exists under an equivalent probability measure.

We let S denote the price process for the swap (to counterparty A), a semimartingale that is to be determined by equation (2.1), once we complete the specification of the dividend process X for the swap to account for payments on default. We let S_{t-} denote the left limit of the swap value at time t. Under the simple "no-fault" model of swap settlement that we treat here, there are two possibilities to consider for recovery on default, depending on whether the value of the swap is positive or negative for the defaulting counterparty.

(i) For the case of a default at a time t when the value of the swap for the defaulting counterparty i is positive, we assume termination settlement by payment in full of the market value of the swap just prior to default, S_{t-}.

(ii) For the case of a default at a time t when the value of the swap for the defaulting counterparty i is negative, we assume that the nondefaulting counterparty recovers a fraction, denoted φ_t^i, of the market value of the swap immediately prior to default.

Alternative default recovery models are considered in Duffie and Huang (1994). The fractional recovery processes, φ^A and φ^B, are assumed to be bounded nonnegative and predictable stochastic processes. Duffie and Huang (1994) consider an extension of this model that allows the recovery fraction φ_t^i to depend on the market value S_{t-} of the swap immediately prior to default. Duffie and Huang (1994) also consider cases in which the hazard rate h^i may depend on the swap price itself. For simplicity, however, we will consider here

only cases in which both h^i and φ^i are exogenously given processes. In some sense, this means that the swap is "small" relative to the total portfolios of the counterparties. The default characteristics (h^i, φ^i) are nevertheless permitted to be correlated with variables determining default risk, such as market interest rates.

The cumulative dividend process X of the swap for counterparty A, including the effects of default, can now be defined by

$$X_t = \int_0^t (1 - H_u) \, \mathrm{d}D_u + \int_0^t S_{u-}(\gamma_u^A + \gamma_u^B) \, \mathrm{d}H_u, \qquad (2.3)$$

where $H_t = 1_{\{t \geq \tau\}}$ and

$$\gamma_t^A = 1_{\{t = \tau^A\}}(1_{\{S_{t-} < 0\}}\varphi_t^A + 1_{\{S_{t-} \geq 0\}})$$

$$\gamma_t^B = 1_{\{t = \tau^B\}}(1_{\{S_{t-} \geq 0\}}\varphi_t^B + 1_{\{S_{t-} < 0\}}).$$

The first term is the prearranged swap payment before default. The second term is the settlement payoff in the two default scenarios: party A defaults or party B defaults. We assume for simplicity that the two counterparties default simultaneously with zero probability.

One will note that the definition of the swap dividend process X, including the effects of default, involves the price process S of the swap itself, which in turn is determined in terms of X by equation (2.1). This is analogous to the valuation of an American option, for which the value at a given time depends on what the value would be at the next point in time if not exercised, which in turn is determined by the following values, and so on. In order to characterize the swap valuation in this recursive setting, we will work with the "predefault" value process V, defined as the solution of the recursive integral equation:

$$V_t = \mathrm{E}_Q\left[\int_t^T - R(V_s, s)V_s \, \mathrm{d}s + \mathrm{d}D_s \,\middle|\, \mathscr{F}_t\right], \quad t \leq T; \qquad (2.5)$$

where

$$R(v, \omega, t) = r_t(\omega) + s_t^A(\omega) 1_{\{v < 0\}} + s_t^B(\omega) 1_{\{v \geq 0\}}, \qquad (2.6)$$

with

$$s_t^i = (1 - \varphi_t^i)h_{t-}^i. \qquad (2.7)$$

PROPOSITION 1. *Suppose r, s^A, and s^B are bounded. The swap price process S exists and is uniquely defined by equations (2.1) and (2.3). There is a unique solution V to (2.5). If $\Delta V_\tau = 0$ almost surely, then $S_t = V_t$ for all $t < \tau$.*

$$V_t = \mathbf{E}_Q \left[\int_t^T e^{-\int_t^s R_u du} dD_s \,\middle|\, \mathcal{F}_t \right], \quad t \leq T. \tag{2.5'}$$

The process s^i defined by equation (2.7) is the risk-neutral expected fractional rate of loss associated with default by counterparty i. Extending from Duffie and Singleton (1994) and Duffie, Schroder, and Skiadas (1995), we may therefore think of $R(V_t, t)$ as the short rate after adjustment for the effect of default risk (see also equation (2.5')). In practice, default may occur on coupon dates, when, according to equation (2.5), V will jump. This would violate the assumption in Proposition 1 that $\Delta V_\tau = 0$. This assumption can be avoided by taking the slightly more cumbersome approach of defining V to be the cum dividend value and by adding the weak condition that the dividend lump ΔD_t is measurable with respect to $\mathcal{F}_{t-}$, the information available "just before" t. Further remedies are considered in Duffie and Huang (1994). Aside from lump-sum payment dates, there is no obvious reason to expect V to jump at default. For example, we can rule out any jumps for V at default times in diffusion-style models such as those treated in Section III. For further discussion of this issue, see Duffie and Huang (1994).

If the short rate r is unbounded, Proposition 1 remains valid under additional technical integrability conditions. We maintain the assumptions of Proposition 1 from this point, unless otherwise indicated.

The default-adjusted short rate R has a switching-type dependence on the swap value V. The default-free short rate r is adjusted by the default spread s^i of that counterparty i with negative swap value. When the two parties have asymmetric default risk (that is, different default spreads), the value of the swap therefore does not depend linearly on the *promised* cash exchange D. We explore this nonlinearity in more detail in the following proposition, which shows that, if party A always has a lower default spread than party B, then the value to party A of a swap portfolio with a netting provision is always weakly higher than that of the same swap portfolio without a netting provision. Furthermore, this comparison is strict if, given the information available at any t, the values of swaps in the portfolio (calculated separately without a netting provision) can, with positive probability, offset each other in the future.

PROPOSITION 2. *Let S^a, S^b, and S^{ab} be, respectively, the value processes (to party A) of swaps with cumulative dividend processes D^a, D^b, and $D^a + D^b$. Suppose that $s^A \leq s^B$. Then $S^{ab} \geq S^a + S^b$. Furthermore, for given $t < \tau$, $S_t^{ab} > S_t^a + S_t^b$*

on the event Γ *defined by*

$$\Lambda = \{(\omega, u) : u \geq t, \qquad s_u^A(\omega) < s_u^B(\omega), \qquad S_u^a(\omega)S_u^b(\omega) < 0\}$$

$$\Gamma = \left\{ \omega : \mathbf{E}_Q\left[\int_t^T 1_\Lambda(\cdot, u) \, du \,\bigg|\, \mathscr{F}_t \right] > 0 \right\}.$$

Proposition 2 captures the market value of the favorable industry practice of attempting, if possible, to arrange swaps of offsetting default risk in a portfolio with netting provisions, particularly with the same counterparty of a lower credit rating. (See, for example, Ruml (1992).) Of course, swap diversification is also a useful means of credit risk management, independently of its impact on market values. In a similar spirit, Cooper and Mello (1992) show that a bank that already owns a claim on a counterparty can afford to offer a more competitive rate to the same counterparty on a forward contract for hedging purpose. A numerical example is given in Section III.

C. Early Resolution of Information and Default Spread Asymmetry

Nabar, Stapleton, and Subramanyam (1988) and Duffie, Schroder, and Skiadas (1995) have pointed out that the future timing of resolution of information may influence the current market price of a defaultable claim whose default hazard rate or payoff upon default may depend on the price of the claim. Our defaultable swaps valuation model provides an example of such an effect. The intuition is that the party with a lower default spread prefers later resolution of uncertainty because it causes the swap value to deviate from its initial (zero) value more slowly and therefore leaves the two parties exposed to default risk for a shorter period of time.

To illustrate this effect, we compare defaultable swap prices in two markets. Market F, with filtration $\mathbf{F}$, is the one that we have been studying. Market G, with filtration $\mathbf{G} = \{\mathscr{G}_t : t \in [0, T]\}$, is identical to market F except that it has earlier resolution of uncertainty. That is, $\mathscr{F}_t \subseteq \mathscr{G}_t$ for all t while $\mathscr{F}_0 = \mathscr{G}_0$. The equivalent martingale measure Q is assumed to apply to both markets, as we are interested in the pure effect of information. (One can imagine, for example, a setting with risk-neutral investors.) Consider a swap of a cumulative pre-default dividend process D, a semimartingale of integrable variation with respect to both $\mathbf{F}$ and $\mathbf{G}$. All assumptions for Proposition 1 are assumed to apply in both markets F and G. The following proposition shows that, if party A always has a lower default spread, the time-zero value of the swap (to party A) in market F is higher than it is in market G. For example, counterparty A would be averse to the early public release of an audit of counterparty B's credit worthiness, even if such an audit could reveal favorable news regarding the credit quality of counterparty B. The proposition can be deduced from Duffie, Schroder, and Skiadas (1995), whose technical convexity assumption is naturally satisfied with our model of asymmetric default risk. The converse result, for higher default risk by counterparty A, is easy to deduce.

PROPOSITION 3. *Suppose that $s^A \leq s^B$. Let S^F and S^G denote, respectively, the market value processes (for counterparty A) for the given swap in markets F and G. Then $S_0^F \geq S_0^G$.*

As with Proposition 2, a strict version of the inequality can be developed. The proposition is interesting in light of the model of Ross (1989), which deals with the irrelevance of early resolution of uncertainty.

D. The Marginal Impact of Credit Quality

Next, we study the marginal price impact of default-spread asymmetry. As one might expect, each party's market value for the swap is monotonically decreasing with respect to the other party's default spread.

To formally describe this monotonicity, we will compute the Gateaux derivative of the swap value with respect to $\eta \equiv s^B - s^A$, the default-spread asymmetry of the two parties, at a given asymmetry $\bar{\eta}$. This derivative is defined (when it exists) as a process $\nabla V(\bar{\eta}; \eta)$ such that

$$\lim_{\varepsilon \downarrow 0} \sup_t \left| \nabla V_t(\bar{\eta}; \eta) - \frac{V_t(\bar{\eta} + \varepsilon\eta) - V_t(\bar{\eta})}{\varepsilon} \right| = 0,$$

for any bounded predictable process η, where $V(\eta)$ denotes the predefault value process, that process solving equation (2.5) for spread asymmetry η. For this calculation, s^A is held fixed, and s^B is varied to determine η.

PROPOSITION 4. *For any bounded predictable process η, the Gateaux derivative $\nabla V(\bar{\eta}; \eta)$ exists and is given by*

$$0 \geq \nabla V_t(\bar{\eta}; \eta) = -E_Q\left[\int_t^T \exp\left[-\int_t^s R(V_u, u) \, du \right] \max(V_s, 0)\eta_s \, ds \, \middle| \, \mathscr{F}_t \right],$$

$$(2.8)$$

where $V = V(\bar{\eta})$.

This calculation follows from the gradient calculation for stochastic differential utility given by Duffie and Skiadas (1994). Relation (2.8) shows that increasing the default spread of counterparty B relative to that of counterparty A reduces the swap value to counterparty A. As expected, the impact of default spread is more dramatic in more volatile markets, given the appearance of $\max(V_s, 0)$ in relation (2.8).

In Section IV, we derive an explicit expression for $\nabla V_t(0; \eta)$ in an example involving fixed-coupon currency swaps. This formula is then used to estimate default spreads for currency swaps. Hull and White (1995) treat the case in which counterparty A is default-free and arrive at an impact of default risk on swap values similar in spirit to relation (2.8), with $s^A = 0$. The two models, however, are quite different.

The result here is also reminiscent of the study by Sorensen and Bollier (1994) in which they view counterparty A's exposure to the credit risk of counterparty B as a series of European-style options (written by A to B) to terminate the swap, exercisable by B only when it defaults. Conceptually, this approach at best represents a first-order approximation since the exact values of these options depend on the value of the swap and the valuation problem is thus recursive in nature. Furthermore, the approach is reasonable only if one treats the credit risk of each party as that of a continuous series of options (see relation (2.8)). There is also significant numerical difference between our results. Sorensen and Bollier (1994) state that the swap credit spread between AAA and BBB parties should be about 10 basis points. Our model, applied to a standard term structure setup with typical parameters, and assuming a bond yield spread of 100 basis points between the two counterparties, implies a swap credit spread of less than 1 basis point. (See Section III.) Sorensen and Bollier (1994) also claimed that, for the same two counterparties, the swap credit spread can vary 10 to 15 basis points as the yield curve goes from steep to inverted. Our result in Section III.F shows that the impact of the yield curve on swap credit spread is rather small (less than 2 basis points as the difference between the 5-year LIBOR rate and the short LIBOR rate varies by 650 basis points).

III. Term Swap Rate Credit Spreads

In this section, we apply the general framework of valuation of defaultable swaps developed in the last section to illustrate the pricing of interest rate swaps involving an exchange of floating and fixed interest payments between two counterparties with asymmetric default risk. In particular, we are interested in the quantitative relationship between the term swap rate spread that counterparties with different credit ratings face in the fixed-for-floating swap market (against the same counterparty), and the yield spreads they face in the corporate bond market.

A. Coupon Swap and Model Specification

We select as our object of study a "plain vanilla" coupon swap with semiannual exchanges of fixed-rate payments for floating-rate payments on a constant notional amount. We assume that counterparty A pays the six-month LIBOR rate and counterparty B pays the fixed term swap rate. For our purpose, we fix counterparty A and vary the credit quality of counterparty B. We then study how the fixed coupon rate of an at-market swap is related to the credit spreads of the two parties in the bond market. The impact of changing the relative credit quality of the counterparties on the fixed coupon rate is called the swap credit spread.

For our base case, we assume that the credit quality of counterparty A is such that its default-adjusted short rate, $r_t + s_t^A$, is always equal to the short

term LIBOR rate, which we represent by ρ_t. (We call such a counterparty a LIBOR party. In fact, the LIBOR rate is a "replenished" rate, that is, the rate on current borrowing by a (roughly) AA-quality firm.) Third, we assume that the default spread η between counterparty B and the LIBOR party is a function of the spot short term LIBOR rate, that is, $\eta_t = \bar{\eta}(\rho_t)$. Finally, we assume that the short term LIBOR rate is modeled in the same way that Cox, Ingersoll, and Ross (1985) modeled the short rate r. That is,

$$d\rho_t = \kappa(\mu - \rho_t)dt + \sigma\sqrt{\rho_t}dW_t, \qquad (3.1)$$

where κ, μ, and σ are positive constants and W is a standard Brownian motion and an **F**-martingale relative to the equivalent martingale measure Q.

B. Method of Calculation

Valuation can be done in a Markovian setting. With the typical case of payment in arrears, one would use two state variables, the LIBOR rate $\hat{\rho}_t$ on the last reset date and the current LIBOR rate ρ_t. Note that $\hat{\rho}_t$ is constant between a reset date and the associated payment date. For the case of symmetric default risk for the two counterparties (including default-free valuation as a special case), a standard trick avoids the use of $\hat{\rho}_t$ by having dividends paid at reset dates according to appropriately discounted amounts. The nonlinear nature of valuation in the case of counterparties with asymmetric default risk prohibits such a simplification.

In order to avoid the use of two state variables, we numerically study the simpler case without payment-in-arrears. For completeness, in Appendix B we outline a numerical valuation method for defaultable swaps with payment-in-arrears. We expect the difference in swap spreads to be negligible for our purposes, since the impact of swap maturity on credit spreads is on the order of only 0.2 basis points per year of maturity, for our numerical example. (See Section III.*I*.)

Under our assumptions, the predefault value process V is of the form $V_t = J(\rho_t, t)$, where, assuming technical conditions[4] apply, J solves the partial differential equation

$$\frac{1}{2}\sigma^2 y J_{yy} + \kappa(\mu - y)J_y + J_t - [y + \bar{\eta}(y)1_{\{J \geq 0\}}]J = 0, \quad y \geq 0,$$

$$t_n \leq t < t_{n+1}, \quad (3.2)$$

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[4] The technical conditions are to ensure the existence of a unique solution to (3.2)–(3.4), so that the Feynman-Kac representation implies that this solution uniquely solves equation (2.5) with $V_T = 0$. This is a first boundary value problem of a semilinear parabolic differential equation. Technical conditions can be found, for example, in Ivanov (1984), pp. 170–171. Special treatment of the CIR model is usually required because of the appearance of the square root in the diffusion.

where $0 < t_1 < \cdots < t_N = T$ define the coupon dates. The boundary conditions are

$$J(y, T) = 0, \quad y \in [0, \infty), \tag{3.3}$$

and

$$J(y, t_n-) = J(y, t_n) + \delta(y), \tag{3.4}$$

where $\delta{:}\mathbf{R}_+ \to \mathbf{R}$ defines the net payment to counterparty A. For a coupon swap with semiannual exchange of a fixed rate C with the market six-month LIBOR rate on payment date t_n, we have

$$\delta(y) = \frac{C}{2} - \left(\frac{1}{p(y, 0.5)} - 1\right), \tag{3.5}$$

where $p(y, t)$ denotes the price of a zero-coupon LIBOR bond with time to maturity t and current LIBOR rate y. From Cox, Ingersoll, and Ross (1985),

$$p(y, t) = \alpha(t)\exp[-\beta(t)y], \tag{3.6}$$

with

$$\alpha(t) = \left[\frac{2\gamma e^{(\gamma + \kappa)t/2}}{(\gamma + \kappa)(e^{\gamma t} - 1) + 2\gamma}\right]^{2\kappa\mu/\sigma^2} \tag{3.7}$$

$$\beta(t) = \frac{2(e^{\gamma t} - 1)}{(\gamma + \kappa)(e^{\gamma t} - 1) + 2\gamma},$$

for $\gamma = (\kappa^2 + 2\sigma^2)^{1/2}$.

One can solve equation (3.2) by any of several finite-difference methods. We use the Crank-Nicholson method for our calculation. (See, for example, Duffie (1992) for an illustration of this method applied to pricing contingent claims in a CIR model.) For a given initial LIBOR rate ρ_0, the term swap rate is then obtained by searching for the fixed rate $\bar{C}(\rho_0)$ that makes the initial swap value $J(\rho_0, 0)$ equal to zero.

Our goal in this section is to study the quantitative relationship between the swap rate credit spread and the corporate bond yield credit spread of two counterparties. This relationship depends on the functional form of $\bar{\eta}(\cdot)$. Here, we study the relationship for:

$$\bar{\eta}(\rho_t) = c + b\rho_t,$$

for constants c and b, which allows for correlation, positive or negative, between the fixed-rate payer's default risk and LIBOR rates. The affine form of $\bar{\eta}$ allows us to compute bond yield spreads between the two counterparties in closed form, using variations of the Cox-Ingersoll-Ross zero-coupon bond price formula (3.6).
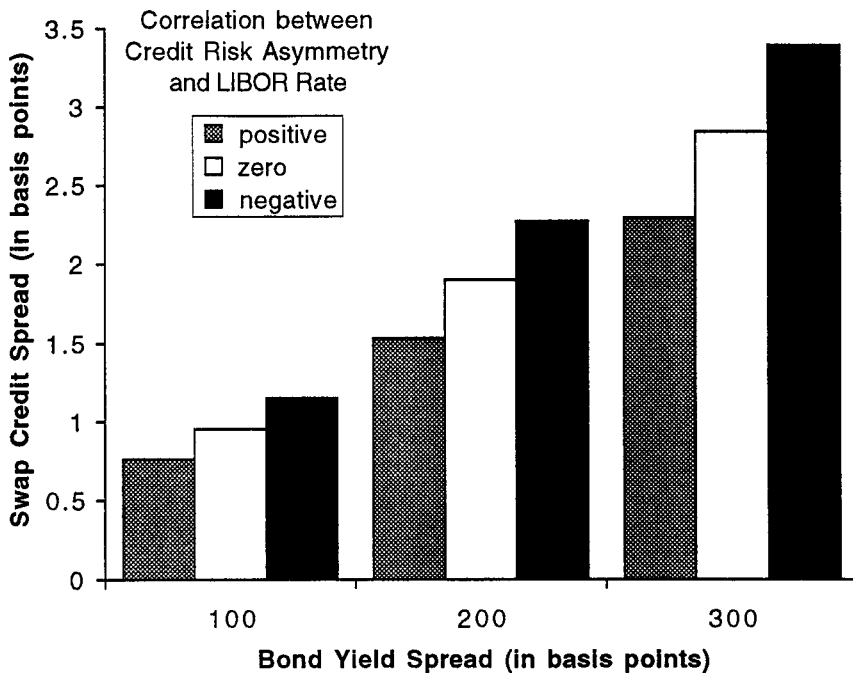
Figure 1. **The impact of credit risk on swap credit spreads.** Five-year coupon swap credit spreads for various levels of credit-risk asymmetry and covariance between the credit-risk asymmetry and the LIBOR rate. Positive, zero, and negative correlation are introduced by letting b be given by $+1$, 0, and -1, respectively, times the bond yield spread. The parameter c is adjusted to fit the bond yield spread.

C. Base-Case Results

Throughout this section, we assume that the CIR model parameters are $\kappa = 0.4$, $\mu = 0.1$, and $\sigma = 0.06$. These parameters are not empirical estimates, but are not atypical; see Pearson and Sun (1994), Gibbons and Ramaswamy (1993), Chen and Scott (1993), or Duffie and Singleton (1995).

We report the results in Figure 1 for a 5-year coupon swap. We calibrate the credit-spread parameters c and b to given 5-year zero-coupon bond yield spreads. The initial LIBOR rate is $\rho_0 = 10.18$ percent. The results are not sensitive to ρ_0 within reasonable ranges.

If one assumes a fractional recovery rate φ of 50 percent[5] for both counter-parties, our base case of a constant credit spread of 100 basis points (that is,

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[5] Moody's has reported an average recovery rate for defaulted bonds, covering the period 1974–1993, of 44.25 percent of par. One could calibrate our model so as to match the (not risk-neutralized) expected recovery rate as a fraction of par to a given level, such as 44.25 percent. In any case, the frational recovery rate of swaps is likely to be somewhat different from that of bonds, even for like firms, although there are no data on this issue, of which we are aware.

$c = 0.01$ and $b = 0$) would translate into an annual probability default by counterparty B that is roughly 2 percent more than that of counterparty A.

Figure 1 shows that the netting of fixed against floating payments in interest rate swaps significantly reduces the impact of default risk on swap rates relative to bond yields. With our parameter choices, a credit spread of 100 basis points in the bond market translates into a credit spread of less than one basis point in the coupon swap market.

A "pseudo" swap credit spread can be computed by treating the swap as an obligation by counterparty B to make all fixed rate coupon payments, regardless of default by counterparty A, and an obligation by counterparty A to make all floating rate payments, regardless of default by counterparty B. This form of agreement would have significantly larger credit spreads. For example, for the same parameters underlying the results of Figure 1, for the case in which credit spreads are uncorrelated with LIBOR rates, a "pseudo" swap spread of 26.4 basis points would apply for a bond yield spread of 100 basis points.

D. Off-Market Swap Rate Credit Spreads

Subsequent to the initiation of a swap, the credit spread of the swap may become higher (or lower) if market interest rate changes favor (or act against) the counterparty with higher quality. We will consider an interest-rate swap that is, with no credit spread, 100 basis points off the market in favor of the LIBOR party. Our setup is otherwise identical to that considered in Section III, C. For the case of credit spreads uncorrelated with LIBOR rates, a swap credit spread of 2.9 basis points is appropriate for a bond yield spread of 100 basis points. The swap credit spread becomes 0.2 basis points if the swap is off the market against the LIBOR party by 100 basis points while everything else remains the same.

E. 4-For-1 Swaps

In the above example of coupon swaps, we assumed that the fixed-rate payment dates match the floating-rate payment dates. Many coupon swaps, however, involve a fixed-rate payment once (or twice) a year, but a floating-rate payment four times per year. (We call these coupon swaps 4-for-1 (or 4-for-2).) Compared with the case of matched payment dates, the precedence of the floating-rate payments over the fixed-rate payments increases the exposure of the LIBOR payer, and generally results in a larger swap credit spread when the LIBOR payer has the higher credit quality. To illustrate this effect, we consider two coupon swaps between a LIBOR party and a second party with a default-adjusted short rate that is always 100 basis points higher than that of the LIBOR party. The LIBOR rate dynamics and the initial LIBOR rate are those underlying Figure 1, for the case of constant credit spread ($b = 0$). A 4-for-1 swap, in which the LIBOR party exchanges 4 quarterly payments of the 3-month LIBOR rate against a year-end fixed-rate payment by the other party, has a credit spread of 4.4 basis points. This higher credit spread reflects the

LIBOR party's additional exposure to default risk by the fixed-rate payer, for example at the point at which 3 quarterly payments by the LIBOR party have been made, while the offsetting annual fixed rate payment is yet to be made.

The above results are consistent with the recent practice of some investment banks that, as floating-rate payers, request that their floating-rate payments be compounded and paid on the fixed-rate payment dates. From our casual enquiries, these requests are apparently met without a change of swap rate. Our calculation indicates that this practice increases the value of the swap to the floating-rate payer, in addition to reducing the exposure to credit risk.

F. The Impact of Swap Credit Spreads of Slope of the Yield Curve

We show how the swap credit spread can depend on the slope[6] of the yield curve. We vary the initial spot rate ρ_0 and the long term mean μ of the CIR model (III.A), holding constant the swap rate for a LIBOR quality fixed rate payer. We then calculate the swap credit spread for a fixed-rate payer of $\eta = 100$ basis points credit spread against a LIBOR floating rate payer. Our setup is otherwise the same as that underlying Figure 1. Intuitively, we expect that, as the yield curve becomes less upward-sloping (or more downward-sloping), the expected exposure of the floating-rate payer should increase, since the median sample path of the market value of the swap becomes more upward sloping. This in turn causes the swap credit spread to increase. This is confirmed by the results shown in Figure 2.

G. Term Structure of Credit Spreads

In practice (Johnson (1967), Fons (1994)) and in theory (Merton (1974), Pitts and Selby (1979)), the term structure of yield spreads for medium and high quality issuers is upward sloping. We now assume that the default-spread asymmetry is linearly increasing in time, with $\eta(t) = ct$, for a constant $c > 0$, and that the model is otherwise the same as that underlying the results of Figure 1. If[7] we calibrate c so that there is a 100 basis points spread between the two counterparties in their 5-year zero coupon bond yields, then the swap credit spread is 0.84 basis points.
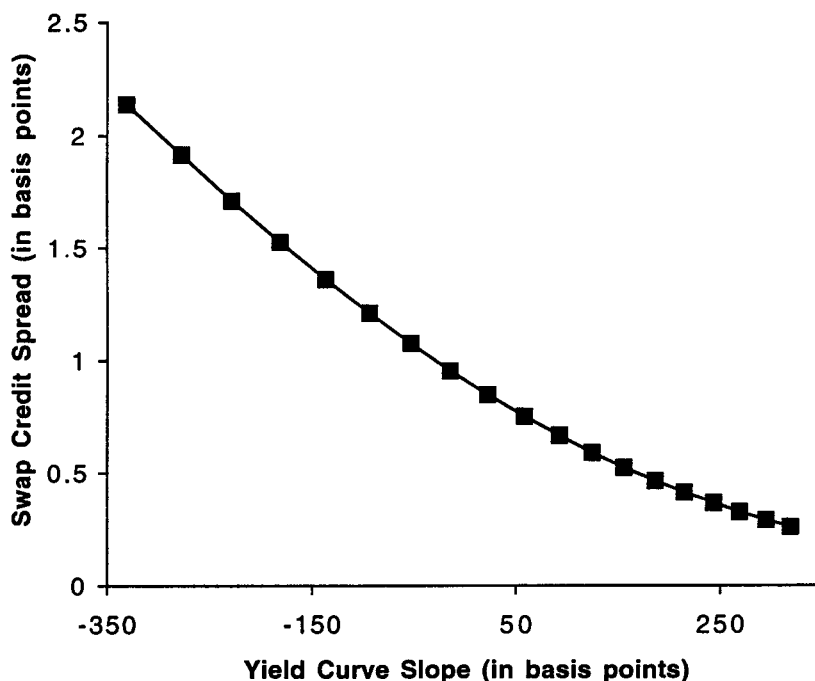
H. Low Quality Floating Rate Payer; Lower Quality Fixed Rate Payer

We consider the possibility that both parties have a positive default spread against the LIBOR rate. We assume that the effective instantaneous discount rates of the "LIBOR party" and other fixed-rate paying party are, respectively, 100 and 200 basis points higher than the spot LIBOR rate while keeping the model otherwise the same as that underlying the results of Figure 1. This leaves a bond yield spread of 100 basis points. We have computed the swap

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[6] We are grateful to Peter DeMarzo for suggesting this experiment.
[7] We are grateful to Philip Dybvig for suggesting this experiment.

Figure 2. The impact of yield curve shape on swap credit spreads. Five-year coupon swap credit spreads for a fixed-rate payer of $\eta = 100$ basis points credit spread against a LIBOR quality floating-rate payer, for various slopes of the yield curve. The initial spot rate ρ_0 and the long term mean μ of the CIR model are varied to give a constant swap rate for a LIBOR quality fixed-rate payer and various slopes of the yield curve.

credit spread to be 0.95 basis points, unchanged from the base case illustrated in Figure 1, which is otherwise identical.

I. The Maturity Effect on Swap Credit Spreads

Figure 3 shows the impact of variation of maturity on the swap spread. For the base case underlying Figure 1 (that is, for a 100 basis point bond yield spread and zero spread correlation ($b = 0$)), the effect of varying only the maturity of the swap is shown in Figure 3.

J. The Impact of Netting

As Proposition 2 shows, netting among swaps in a swap portfolio influences the value of the portfolio. We now use a simple example of two coupon swaps to illustrate the effect of a master swap agreement with netting on the valuation of swap portfolios and the setting of swap rates. We show that there is financial benefit of netting to the counterparty with higher credit quality, in addition to the risk-management benefits.
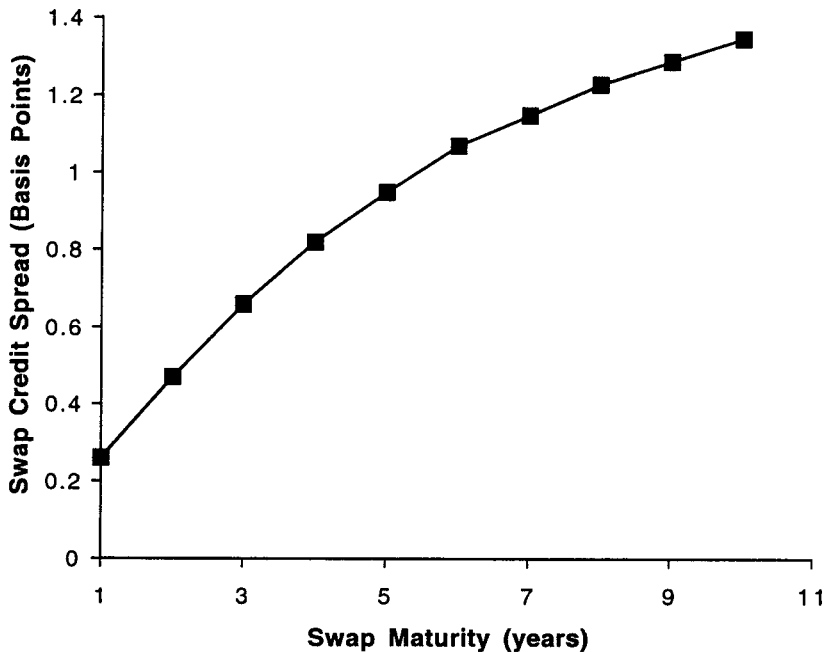
Figure 3. The maturity effect. The impact of variation of maturity on the swap credit spread for the base case underlying Figure 1 ($c = 100$ basis points and $b = 0$).

Suppose that counterparty A, the party with higher credit quality, is about to enter into a swap contract, called the *new swap*, with counterparty B. If the new swap is not to be netted with an existing swap portfolio, then the term rate of the new swap should be set such that the value of the new swap is zero at initiation. If, however, the new swap is to be netted with some existing swap portfolio between the two parties, then counterparty A may set a slightly lower rate for counterparty B because, for any given promised payment, the marginal value of the new swap with netting is higher than it is without netting. The amount of discount depends on the extent to which the credit exposure of the new swap offsets that of the existing swap portfolio. In practice, since swap rates are negotiated in the over-the-counter (OTC) market, there is an issue as to whether counterparty B is able to extract the full benefit of netting.

To illustrate this effect, we consider the following example. The setup for the new swap is the same as that underlying the results of Figure 1, except that there is a previous swap in place and a master swap agreement with full netting provisions. The terms of the previous swap are the same as those of the new swap, except that:

(i) Counterparty A pays fixed and counterparty B pays floating.

(ii) The notional amount of the previous swap is a multiple k of that of the new swap. We can think of k as a hedge ratio of the old swap against the

Hedging Coefficient of the Old Swap
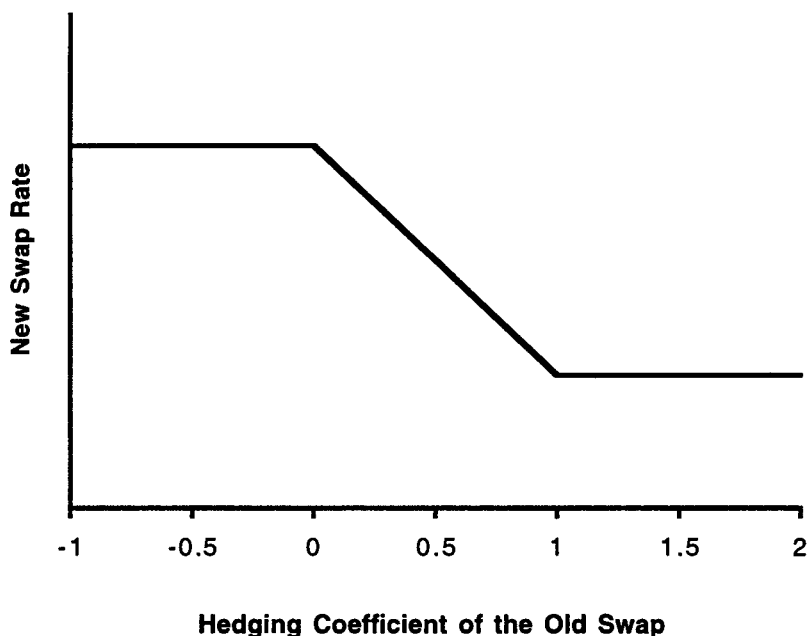
Figure 4. The impact of netting on swap rates. The swap rate, for a fixed-rate payer of $\eta =$ 100 basis points credit spread against a LIBOR quality floating-rate payer, of a 5-year coupon swap that is to be netted against a previous swap, of opposite promised cash flows, between the same two counterparties.

new. If $k = 1$, there is perfect hedging, since there is no net cash flow on the combined swap portfolio.

We take the case of a constant credit spread, $s^B - s^A$, of 100 basis points.

The term rate $C(k)$ of the new swap depends on the extent to which the payments in the old and new swaps offset each other, that is, the hedge coefficient k. If $k \leq 0$, the payoffs of the new swap and those of the old swap are linearly related and perfectly correlated. There is, therefore, no impact on swap rates of netting in this case. That is, $C(k) = C(0)$ for $k \leq 0$. For $0 < k < 1$, the payoff of the old swap partially offsets the that of the new swap, with the greatest offset occurring at $k = 1$. Consequently, the term rate $C(k)$ of the new swap is smaller than $C(0)$ and decreases with increasing k up to $k = 1$. For $k \geq 1$, the cash flows of the new swap are more than fully offset by the previous swap, and there are no additional marginal value benefits as k increases above 1. That is, $C(k) = C(1)$ for $k \geq 1$. This is illustrated in Figure 4. The linear dependence on k for $0 < k < 1$, shown in Figure 4, is demonstrated in Appendix C.

We provide the numerical result for one example. For $\rho_0 = 10.1818$ percent, we have

$$C(0) = 10.3017\%, \quad C(1) = 10.2835\%,$$

for a maximum impact of netting on swap rates of 1.8 basis points, which is, not surprisingly, about twice the 0.95 basis point spread indicated for the base case in Figure 1.

The netting effect of a master swap agreement should be quantitatively more significant for other forms of contracts, such as certain foreign exchange swaps or forwards, with larger credit risk exposure. This is illustrated in the next section.

IV. Credit Spreads on Currency Swaps with Exchange of Principals

Currency swaps that involve an exchange of principals in different currencies are typically subject to more exposure to default risks than are interest rate swaps. In this section, we calculate the impact of default risks on currency swap rates.

We use "dollar" and "yen" to denote, respectively, the units of the domestic and foreign currencies. Suppose that counterparties A and B are engaged in a fixed-for-fixed foreign currency swap with P_d and P_f denoting, respectively, the principal amounts of domestic currency (in dollars) and foreign currency (in yen). Counterparty A exchanges a fixed coupon payment of $\frac{1}{2} c_d P_d$ dollars for a fixed coupon payment of $\frac{1}{2} c_f P_f$ yen with counterparty B, semiannually until maturity at time T, where c_d and c_f are constant coupon rates. At maturity, counterparty A exchanges P_d dollars for P_f yen with counterparty B.

Since the volatility of the market value of the above fixed-for-fixed currency swap depends mostly on the volatility of the currency exchange rate, we simplify by taking constant domestic and foreign interest rates, r_d and r_f, respectively. The foreign exchange rate process q, with q_t defined as the market value (in dollars) of one yen at time t, is taken to be a geometric Brownian motion under the equivalent martingale measure Q. That is,

$$dq_t = (r_d - r_f)q_t dt + \sigma_q q_t dW_t, \qquad (4.1)$$

where σ_q is a constant and W is a Brownian motion under Q. The drift term $(r_d - r_f)q_t$ ensures that the gain process associated with rolling over one yen in short term riskless lending, discounted by the domestic interest rate, is a martingale under Q.

One way to estimate the impact of default risk on currency swap rates is to apply a PDE analogous to (3.2)–(3.4), taking q as the state variable. In this section, however, we also use the Gateaux derivative $\nabla V_t(0; \eta)$, given in equation (2.8), to estimate the impact of default risks on swap rates of the default-spread asymmetry η between the two parties. Figure 5 shows that the resulting first-order approximation provides sufficient accuracy for most practical applications. The advantage of this approximation method is that the Gateaux derivative $\nabla V_t(0; \eta)$ can be computed relatively explicitly.

Consistent with common practice, we assume for our example that P_d dollars have the same initial market value as P_f yen; that is, $P_d = q_0 P_f$. Second, we assume that the domestic and foreign interest rates are equal; that
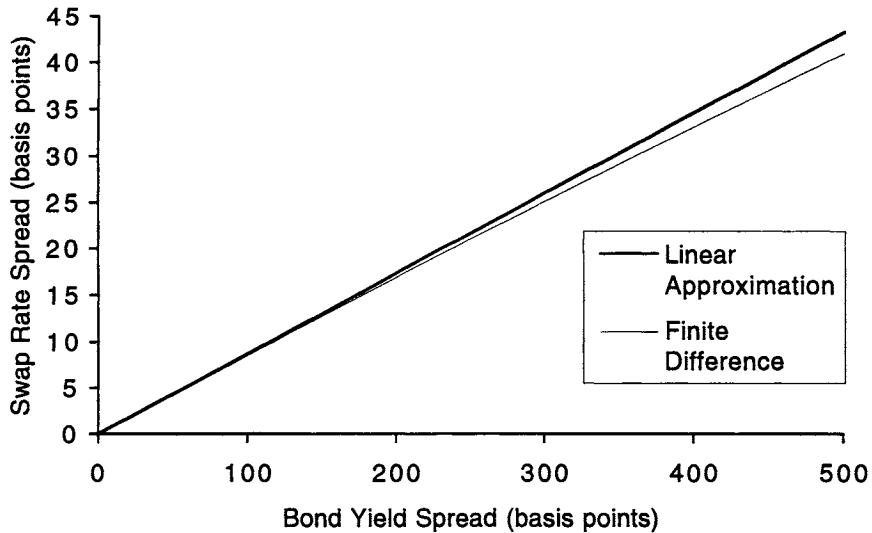
Figure 5. The impact of credit risk on currency swap spreads. The credit spread of a five-year currency swap calculated by both a finite difference method and by the given analytic formula for the first-order (linear) approximation. The default-spread asymmetry η is assumed to be constant. The dollar payer has a constant effective discount rate of $R^A = 6$ percent. Both dollar and yen coupons are 5 percent. The exchange rate is assumed to be a geometric Brownian motion with a volatility coefficient of 15 percent.

is, $r_d = r_f$. Third, we assume that the default spread s^A of counterparty A is a constant, so that $R^A = r_d + s^A$ is a constant. Finally, we assume that the default-spread asymmetry η is a constant c. These assumptions are made for analytical tractability and should not heavily influence the numerical relationship between the swap credit spread and the default-spread asymmetry η, which depends essentially on the exchange rate volatility.

We compute the Gateaux derivative (2.8) from a reference point of $\eta = 0$ for credit-spread asymmetry. Then, as shown in Appendix D, we can compute the corresponding first-order approximation of the swap rate as a function of the credit spread asymmetry. For example, with

$$\sigma_q = 15\%; \quad R^A = 6\%; \quad T = 5 \text{ years}; \quad c_d = c_f = 5\%,$$

we arrive at

$$C_f(c) - C_f(0) \approx 0.087c.$$

For a constant default-spread asymmetry of $c = 100$ basis points, this translates into a bond yield spread of 100 basis points, and a currency swap credit spread of about 8.7 basis points.

As with interest rate swaps, a major determinant of currency swap spreads is market volatility. For our currency swap example, doubling the volatility

parameter σ_q from 15 to 30 percent increases the currency swap spread from approximately 8.7 basis points to approximately 17.2 basis points. These estimates are roughly consistent with those obtained by Hull and White (1992, 1995).

V. A Concluding Remark

In practice, the impact of credit risk on swap rates may be somewhat less than indicated by our results given industry practices designed to reduce default risk. Among these are marks-to-market, collaterization, and options for early termination. Our model can be extended in a straightforward manner to measure the price impact of these credit-risk mitigation techniques.

Appendix A: Technical Lemmas and Proofs

This appendix contains some technical lemmas and proofs of all propositions.

LEMMA 1. *For a given $f \in \Lambda$, an $\mathcal{F}_T$-measurable random variable Y, and a finite variation process $\{D_t : 0 \leq t \leq T\}$, suppose there is some $p \in [1, \infty)$ such that $\int_0^T |f(0, \omega, t)| \, dt$, Y, and $\int_0^T |dD_t|$ are all in L^p, and suppose that there is some constant $k > 0$ such that f is k-lipschitz in its v argument: $|f(x, \omega, t) - f(y, \omega, t)| \leq k|x - y|$ for all (ω, t) and all $(x, y) \in \mathbf{R}^B$. Then there exists a unique solution V to the recursive stochastic integral equation*

$$V_t = \mathrm{E}\left[\int_t^T f(V_s, \omega, s) \, ds + dD_s + Y \,\middle|\, \mathcal{F}_t \right], \quad t \leq T,$$

in the space ν^p of all RCLL adapted processes that satisfy $\mathrm{E}[\int_0^T |V_t| \, dt)^p] < \infty$.

PROOF: For $p > 1$, this theorem is a simple generalization of the theorem proved in Appendix A of Duffie and Epstein (1992), with an added $\int_t^T dD_s$ term here. The proof is almost identical. See Antonelli (1994) for the case of $p = 1$.   Q.E.D.

For completeness, we restate here a version of the Stochastic Gronwall-Bellman Inequality, due to Costis Skiadas, and as originally stated in Lemma B2 of Duffie and Epstein (1992). We also add a strict inequality result to the lemma. This lemma is used in the proofs of Propositions 2 and 3.

LEMMA 2. *Let $(\Omega, \mathcal{F}, P)$ be a filtered probability space whose filtration $\{\mathcal{F}_t : t \in [0, T]\}$ satisfies the usual conditions. Suppose that Y is an integrable optional process, α is a constant, and G is a measurable process. Suppose, for all t, that $s \mapsto Y_s$ is right continuous and $s \mapsto \mathrm{E}(Y_s | \mathcal{F}_t)$ is continuous almost surely.*

If $Y_T \geq 0$ a.s. and, for all t, $G_t \geq -\alpha|Y_t|$ a.s. and $Y_t = \mathrm{E}[\int_t^T G_s \, ds + Y_T | \mathcal{F}_t]$ a.s. Then, for all t, $Y_t \geq 0$ a.s. Furthermore, for given t, let

$$A = \{(\omega, u) : u \geq t, \, G_u(\omega) > -\alpha|Y_u(\omega)|\};$$

$$B = \left\{ \omega : \mathrm{E}_Q\left[\int_0^T 1_A(\cdot, u) \, du \Big| \mathcal{F}_t \right] > 0 \right\} \in \mathcal{F}_t.$$

Then $Y_t > 0$ on B.

PROOF: See Lemma B2 in Duffie and Epstein (1992). The added strict inequality part can be proved using a strict version of the Gronwall-Bellman inequality.   Q.E.D.

The following lemma is a simple generalization of Lemma 1 of Duffie, Schroder, and Skiadas (1995). We use it to simplify the proof of Proposition 1.

LEMMA 3. *Let V be a semimartingale satisfying $\mathrm{E}(\int_0^T |V_t| \, dt) < \infty$, let D be a semimartingale satisfying $\mathrm{E}(\int_0^T |dD_t|) < \infty$, and let G be a progressively measurable process such that $\mathrm{E}(\int_0^T |G_t| \, dt) < \infty$. There exists a martingale m such that $dV_t = -G_t dt + dD_t + dm_t$, $t \in [0, T]$, if and only if*

$$V_t = \mathrm{E}\left(\int_t^T G_u \, du + dD_u + V_T \Big| \mathcal{F}_t \right), \quad t \in [0, T].$$

PROOF: See Duffie, Schroder, and Skiadas (1995), Lemma 1.   Q.E.D.

Proof of Proposition 1: Let $Z_t^A = S_{t-}[1_{\{S_{t-}<0\}}\varphi_t^A + 1_{\{S_{t-}\geq 0\}}]$ and $Z_t^B = S_{t-}[1_{\{S_{t-}\geq 0\}}\varphi_t^B + 1_{\{S_{t-}<0\}}]$. Substituting (2.3) into (2.1) and making use of the decomposition $H^i = A^i + M^i$, we obtain

$$S_t e^{-\int_0^t r_u du} = \mathrm{E}_Q\left[\int_t^T e^{-\int_0^s r_u du}[1_{\{s<\tau\}}dD_s + 1_{\{s\leq\tau\}}(Z_s^A dH_s^A + Z_s^B dH_s^B)] \Big| \mathcal{F}_t \right]$$

$$= -\int_0^t e^{-\int_0^s r_u du}1_{\{s<\tau\}}[(Z_s^A h_s^A + Z_s^B h_s^B) \, ds + dD_s] + m_t,$$

for some Q-martingale m. Using integration by parts and noting that $S_t = S_t 1_{\{t<\tau\}}$, we have

$$dS_t = -(Z_t^A h_t^A + Z_t^B h_t^B - r_t S_t)1_{\{t<\tau\}}dt - 1_{\{t<\tau\}}dD_t + d\hat{m}_t \qquad \text{(A.1)}$$

$$= [R_t(S_t, \omega) - h_t^A(S_t, \omega) - h_t^B(S_t, \omega)]S_t dt$$

$$\qquad - 1_{\{t<\tau\}}dD_t + d\hat{m}_t,$$

for some Q-martingale $\hat{m}$. Lemma 3 then implies that

$$S_t = \mathrm{E}_Q\left[\int_t^T -[R_u(S_u,\ \omega) - h_u^A(S_u,\ \omega) - h_u^B(S_u,\ \omega)]S_u\ du + 1_{\{u<\tau\}}dD_u\ \middle|\ \mathscr{F}_t\right],$$
$$t \leq T. \quad \text{(A.2)}$$

Lemma 1 shows that there exists a unique solution S to this recursive integral equation. Next, Lemma 1 shows that there exists a unique solution V for (2.5). Lemma 3 implies that

$$dV_t = R_t(V_t,\ \omega)V_t dt - dD_t + dM_t,$$

for some Q-martingale M. Suppose, further, that $\Delta V_\tau = 0$ almost surely. Then, using integration by parts, we have

$$d(V_t 1_{\{t<\tau\}}) = d[V_t(1 - H_t^A)(1 - H_t^B)]$$

$$= (1 - H_{t-}^A)(1 - H_{t-}^B)dV_t - V_{t-}[(1 - H_{t-}^B)dH_t^A + (1 - H_{t-}^A)dH_t^B]$$

$$= 1_{\{t\leq\tau\}}[dV_t - (h_t^A(V_t,\ \omega) + h_t^B(V_t,\ \omega))V_t dt] + d\bar{m}_t$$

$$= [R_t(V_t,\ \omega) - h_t^A(V_t,\ \omega) - h_t^B(V_t,\ \omega)]1_{\{t<\tau\}}V_t dt \quad \text{(A.3)}$$

$$- 1_{\{t<\tau\}}dD_t + d\hat{M}_t, \quad a.s.,$$

for some Q-martingales $\bar{m}$ and $\hat{M}$. Lemma 3 then implies that $V_t 1_{\{t<\tau\}}$ satisfies equation (A.2) and must be indistinguishable from its unique solution S. Q.E.D.

Proof of Proposition 2: Define $\rho_t = r_t + s_t^A$ as the discount rate for counterparty 1, and let $\eta_t = s_t^B - s_t^A \geq 0$ represent the asymmetry of default spreads between the two parties. For convenience, we change the numeraire as follows. For any swap with cumulative dividend D and value process V, we define $\tilde{D}_t = \int_0^t e^{-\int_0^s \rho_u du}\ dD_s$ and $\tilde{V}_t = e^{-\int_0^t \rho_u du}V_t$. Then, with Ito's lemma, we can rewrite equation (2.5) as

$$\tilde{V}_t = \mathrm{E}_Q\left[\int_t^T -\eta_u 1_{\{\tilde{V}_u\geq 0\}}\tilde{V}_u du + d\tilde{D}_u\ \middle|\ \mathscr{F}_t\right], \quad t \leq T. \quad \text{(A.4)}$$

Applying (A.4) to $\tilde{V}^a$, $\tilde{V}^b$, and $\tilde{V}^{ab}$, we have

$$\tilde{V}_t^{ab} - \tilde{V}_t^a - \tilde{V}_t^b$$

$$= \mathrm{E}_Q\left[\int_t^T -\eta_u[\max(\tilde{V}_u^{ab},\ 0) - \max(\tilde{V}_u^a,\ 0) - \max(\tilde{V}_u^b,\ 0)]du\ \middle|\ \mathscr{F}_t\right].$$

Defining $Y = \tilde{V}^{ab} - \tilde{V}^a - \tilde{V}^b$, we have $Y_T = 0$. Let α denote an upper bound of $|\eta_t|$. Then, using $\eta \geq 0$, we have

$$G_t \equiv -\eta_t[\max(\tilde{V}_t^{ab}, 0) - \max(\tilde{V}_t^a, 0) - \max(\tilde{V}_t^b, 0)]$$

$$\geq -\eta_t[\max(\tilde{V}_t^{ab}, 0) - \max(\tilde{V}_t^a + \tilde{V}_t^b, 0)]$$

$$\geq -\eta_t\max(\tilde{V}_t^{ab} - \tilde{V}_t^a - \tilde{V}_t^b, 0) \qquad (A.5)$$

$$\geq -\alpha|\tilde{V}_t^{ab} - \tilde{V}_t^a - \tilde{V}_t^b|$$

$$= -\alpha|Y_t|.$$

Applying to Y and G a consequence of the Stochastic Gronwall-Bellman Inequality due to Costis Skiadas that is stated in Lemma 2B of Duffie and Epstein (1992) (and restated in this article with an added strict inequality result, as Lemma 2), we conclude that $Y \geq 0$ and thus that $\tilde{V}^{ab} \geq \tilde{V}^a + \tilde{V}^b$.

If, for given t, $(\omega, u) \in \Lambda$, that is, $u \geq t$, $\eta_u(\omega) > 0$, and $\tilde{V}_u^a(\omega)$ and $\tilde{V}_u^b(\omega)$ have opposite signs. Then we have $\max(\tilde{V}_u^a(\omega), 0) + \max(\tilde{V}_u^b(\omega), 0) > \max(\tilde{V}_u^a(\omega) + \tilde{V}_u^b(\omega), 0)$. Inequality (A.5) is then strict on Λ, implying $G_u > -\alpha|Y_u|$ on Λ. Applying the strict inequality part of Lemma 2, we have $Y_t > 0$ on Γ and thus $\tilde{V}_t^{ab} > \tilde{V}_t^b + \tilde{V}_t^b$ on Γ. Q.E.D.

Proof of Proposition 3: Define $\rho_t = r_t + s_t^A$ and, for any swap with cumulative dividend D and value process V, define $\tilde{D}_t = \int_0^t e^{-\int_0^s \rho_u du} dD_s$ and $\tilde{V}_t = e^{-\int_0^t \rho_u du} V_t$. Applying (A.4) to $\tilde{V}^F$ and $\tilde{V}^G$, we have

$$\tilde{V}_t^F = E_Q\left[\int_t^T - \eta_u\max(\tilde{V}_u^F, 0)du + d\tilde{D}_u \,\middle|\, \mathcal{F}_t\right], \quad t \leq T; \qquad (A.6)$$

$$\tilde{V}_t^G = E_Q\left[\int_t^T - \eta_u\max(\tilde{V}_u^G, 0)du + d\tilde{D}_u \,\middle|\, \mathcal{G}_t\right], \quad t \leq T. \qquad (A.7)$$

Define process U by $U_t = E_Q[\tilde{V}_t^G|\mathcal{F}_t]$ for all t. Since $-\eta_t \max(v, 0)$ is concave in v, applying the conditional version of Jensen's inequality and the conditional version of Fubini's theorem to equation (A.7), and noting that η_t is measurable with respect to $\mathcal{G}_t$, we have

$$U_t \leq E_Q\left[\int_t^T - \eta_u\max(U_u, 0)\, du + d\tilde{D}_u \,\middle|\, \mathcal{F}_t\right], \quad t \leq T.$$

Combining this equation with (A.6), we have $\tilde{V}^F_T - U_T = 0$, and

$$\tilde{V}^F_t - U_t = E_Q\left[\int_t^T - \eta_u[\max(\tilde{V}^F_u, 0) - \max(U_u, 0)]du \,\middle|\, \mathscr{F}_t\right]$$

$$\geq E_Q\left[\int_t^T - \eta_u|\tilde{V}^F_u - U_u|du \,\middle|\, \mathscr{F}_t\right], \quad t \leq T.$$

Lemma 2 then implies that $\tilde{V}^F_t \geq U_t$ for all t, and, at $t = 0$, $V^F_0 \geq V^G_0$. Q.E.D.

Appendix B: Markov Valuation for Swaps with Payments in Arrears

For a swap with payments in arrears, we use $\bar{t}_n$ to denote the reset date for the payment date t_n. Usually, the reset date is no earlier than the last payment date. So we assume that

$$0 \leq \bar{t}_1 \leq t_1 \leq \bar{t}_2 \leq t_2 \leq \cdots \leq \bar{t}_N \leq t_N = T. \tag{B.1}$$

To describe the Markov structure of the value process, we need to use two state variables, the short term LIBOR rate $\hat{\rho}_t$ on the reset date and the spot short term LIBOR rate ρ_t, to describe the value process: $V(\omega, t) = J(\hat{\rho}_t, \rho_t, t)$. Note that, for $t \in [\bar{t}_n, \bar{t}_{n+1})$, $\hat{\rho}_t = \rho(\bar{t}_n)$ and is constant. The general Markovian setting equations (2.11)–(2.13) for the value process J can now be written as

$$\frac{1}{2} \sigma^B y J_{yy} + \kappa(\mu - y)J_y + J_t - [y + \bar{\eta}(y)1_{\{J \geq 0\}}]J = 0, \quad y \geq 0, \quad t_{n-1} \leq t < \bar{t}_n.$$

The boundary conditions are given by

$$J(\hat{y}, y, T) = 0, \quad y \in [0, \infty),$$

and, for $1 \leq n \leq N$,

$$J(\hat{y}, y, t_n-) = J(\hat{y}, y, t_n) + \delta_n(\hat{y});$$

$$J(\hat{y}, y, \bar{t}_n-) = J(y, y, \bar{t}_n),$$

where δ_n describes the functional dependence between the net payment to counterparty 1 on payment date t_n and the short term LIBOR rate on the corresponding reset date $\bar{t}_n$ as given in equation (3.5). Note that, within each sub-interval of the form $[t_{n-1}, \bar{t}_n)$, $J(\hat{y}, y, t)$ does not depend on $\hat{y}$.

Appendix C: The Impact on Swap Rates of Netting Fixed-for-LIBOR Swaps against LIBOR-for-Fixed Swaps

We verify Figure 4 by calculating $C(k)$, which is set such that the netted portfolio composed of the new and old swaps is marked to market. We will repeatedly explore the fact that, if a swap has zero market value and each party's default spread does not depend on the value of the swap, then multiplying all promised payments of each party by a positive constant leaves the swap marked to market.

We first characterize the term rate of the old swap, denoted $\bar{C}_k$. Note that $\bar{C}_k = k\bar{C}_1$ for $k \geq 0$. For $k \leq 0$, the old swap is equivalent to the new swap with all payments scaled by $|k|$, so $\bar{C}_k = kC(0)$, with $C(0)$ equivalent to the term rate of the new swap without netting.

We next turn to the netted swap portfolio. Note that the portfolio is equivalent to the new swap scaled by $(1 - k)$ if $k \leq 1$, or to the old swap scaled by $(k - 1)/k$ if $k \geq 1$. So we have

$$C(k) - \bar{C}_k = \begin{cases} (1 - k)C(0), & \text{for } k < 1; \\ -\dfrac{k - 1}{k}\,\bar{C}_k, & \text{for } k \geq 1. \end{cases} \tag{C.1}$$

Combining this with the above results on $\bar{C}_k$, we have

$$C(k) = \begin{cases} C(0), & \text{for } k \leq 0; \\ C(0) - k[C(0) - \bar{C}_1], & \text{for } 0 < k < 1; \\ \bar{C}_1, & \text{for } k \geq 1. \end{cases} \tag{C.2}$$

Appendix D: Explicit Solution of Marginal Credit Spread

For the currency swap described in Section IV, we compute explicitly the marginal value of credit risk.

For $\eta = 0$, the pre-default value process $V(\eta)$ (to counterparty A) can be calculated using equation (2.5'), and is given by

$$V_t(0) = P_d\left[\left(\frac{q_t}{q_0} - 1\right)e^{-R^A(T-t)} + \left(c_f\frac{q_t}{q_0} - c_d\right)\sum_{t_n > t} e^{-R^A(t_n - t)}\right], \tag{D.1}$$

where T is the time of maturity and t_n is, for each n, a coupon date. If $c_d = c_f$, then $V_0(0) = 0$. Our first step is to calculate, for the case of $c_f = c_d$, the impact of a small constant default-spread asymmetry c on the market value of the swap.

According to equation (2.8), the Gateaux derivative of the initial market value of such a swap with respect to the default-spread asymmetry η at $\bar{\eta} = 0$ is

$$\nabla V_0(0; c) = -\mathrm{E}_Q\left[\int_0^T e^{-R^A t}\max(V_t(0),\, 0)c\; \mathrm{d}t\right]$$

$$= -cP_d\mathrm{E}_Q\left[\int_0^T e^{-R^A t}\left[e^{-R^A(T-t)}\right.\right.$$

$$\left.\left.+ c_d\sum_{t_n>t} e^{-R^A(t_n-t)}\right]\max\left(\frac{q_t}{q_0}-1,\,0\right)\mathrm{d}t\right]$$

$$= -cP_d\left(e^{-R^A T}\int_0^T \mathrm{E}_Q\left[\max\left(\frac{q_t}{q_0}-1,0\right)\right]\mathrm{d}t\right. \tag{D.2}$$

$$\left.+ c_d\sum_{t_n} e^{-R^A t_n}\int_0^{t_n} \mathrm{E}_Q\left[\max\left(\frac{q_t}{q_0}-1,\,0\right)\right]\mathrm{d}t\right).$$

Using the Black-Scholes formula and integration by parts, we have

$$\int_0^s \mathrm{E}_Q\left[\max\left(\frac{q_t}{q_0}-1,\,0\right)\right]\mathrm{d}t = \int_0^s\left[2N\left(\frac{1}{2}\,\sigma_q\sqrt{t}\right)-1\right]\mathrm{d}t$$

$$= \left(s-\frac{4}{\sigma_q^B}\right)\left[2N\left(\frac{1}{2}\,\sigma_q\sqrt{s}\right)-1\right] \tag{D.3}$$

$$+ \frac{4}{\sqrt{2\pi}}\frac{\sqrt{s}}{\sigma_q}\exp\left(-\frac{\sigma_q^2}{8}\,s\right),$$

where $N(\,\cdot\,)$ is the standard normal cumulative distribution function. Substituting equation (D.3) into (D.2), we have

$$\nabla V_0(0; c) = -cP_d e^{-R^A T}\left[\left(T-\frac{4}{\sigma_q^B}\right)\left[2N\left(\frac{1}{2}\,\sigma_q\sqrt{T}\right)-1\right]\right.$$

$$\left.+\frac{4}{\sqrt{2\pi}}\frac{\sqrt{T}}{\sigma_q}\exp\left(-\frac{\sigma_q^2}{8}\,T\right)\right]$$

$$-cc_d P_d\sum_{t_n} e^{-R^A t_n}\left[\left(t_n-\frac{4}{\sigma_q^2}\right)\left[2N\left(\frac{1}{2}\,\sigma_q\sqrt{t_n}\right)-1\right]\right. \tag{D.4}$$

$$\left.+\frac{4}{\sqrt{2\pi}}\frac{\sqrt{t_n}}{\sigma_q}\exp\left(-\frac{\sigma_q^2}{8}\,t_n\right)\right].$$

Equation (D.4) gives the relationship between the initial swap value and a small default-spread asymmetry for the case of $c_d = c_f$. This result can help us obtain the relationship between the swap credit spread and a small default-spread asymmetry. Let $V_0(\eta;\, c_d,\, c_f)$ denote the initial value of a swap with coupon rates c_d and c_f. For each constant default-spread asymmetry c, we fix counterparty A's coupon rate c_d and search the coupon rate $c_f = C_f(c)$ of counterparty B with the property that $V_0(\eta;\, c_d,\, c_f) = 0$. For $\eta = 0$, we have $C_f(0) = c_d$. The swap credit spread is given by $C_f(c) - C_f(0)$, which is determined, with accuracy to the first order of c, by

$$\left[\nabla V_0(0;\, c) + (C_f(c) - C_f(0))\frac{\partial V_0(0;\, c_d,\, c_f)}{\partial c_f} \right]\Bigg|_{c_f = c_d} \approx 0,$$

from which we have

$$C_f(c) - C_f(0) \approx -\frac{\nabla V_0(0;\, c)}{\dfrac{\partial V_0(0;\, c_d,\, c_f)}{\partial c_f}}\Bigg|_{c_f = c_d}. \tag{D.5}$$

Combining equation (D.5) with (D.1) and (D.4) gives the relationship between the currency swap credit spread and a small default-spread asymmetry. Figure 5 shows that this calculation is relatively accurate even for large credit spreads.

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