Research Paper No. 812

MULTIPERIOD SECURITY MARKETS WITH DIFFERENTIAL INFORMATION: Martingales and Resolution Times

Darrell Duffie∗ and Chi-fu Huang†

January 1985 — Revised: June 1986
Forthcoming in the Journal of Mathematical Economics

ABSTRACT

We model multiperiod securities markets with differential information. A price system that admits no free lunches is related to martingales when agents have rational expectations. We introduce the concept of *resolution time*, and show that a better informed agent and a less informed agent must agree on the resolution times of commonly marketed events if they have rational expectations and if there are no free lunches. It then follows that if all elementary events are marketed for a less informed agent then any price system that admits no free lunches to a better informed agent must eliminate any private information asymmetry between the two. We provide an example of a dynamically fully revealing price system that is arbitrage free and yields elementarily complete markets.

---

* Graduate School of Business, Stanford University, Stanford California 94305. †Alfred P. Sloan School of Management, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139. We acknowledge conversations with Michael Harrison and Andreu Mas–Colell, and the helpful comments of two referees. Tech-

Address for proofs:

Darrell Duffie

Graduate School of Business,

Stanford University,

Stanford, California, 94305-2391,

U.S.A.

nical help from Kai–Ching Lin of the University of Chicago is appreciated.

1. Introduction and Summary

This paper addresses differential information in a multiperiod model of security markets. Focusing on asset price processes that preclude "free lunches," we first extend the famous Harrison–Kreps connection between price processes and martingales. We go on to study the information that must be revealed by a better informed agent through price processes to a worse informed agent if free lunches are to be precluded. Finally, we formalize the connection between completely revealing price processes and dynamically complete markets for elementary contingent claims. The details and additional results are summarized below. The results of Harrison and Kreps [1979] connecting the behavior of price processes with martingales have opened up a theory of stochastic equilibrium with symmetric information. The results here are a limited step toward stochastic equilibrium with differential information.

The paper is summarized as follows. Section 2 motivates the concepts of differential information in a continuous–time setting. The less demanding discrete–time details can easily be surmised by readers. Section 3 presents the formal model.

A contingent claim is *marketed* if financed by some security trading strategy. Section 4 shows some of the advantages of better information: a larger space of admissible trading strategies and thus a larger space of marketed claims. Section 5 shows that the absence of *simple free lunches* (Kreps [1981]) implies a unique implicit price process for any marketed claim. A better informed agent cannot, in the absence of simple free lunches, attain a consumption claim at a smaller initial investment than a worse informed agent. Kreps [1981] shows a need to strengthen the *simple free lunch* concept of "no arbitrage" to the *free lunch* in infinite–dimensional cases. Barring free lunches for a particular agent, we demonstrate the existence of a *martingale measure* for that agent: a probability measure, absolutely continuous with respect to the agent's endowed probability measure, under which security price processes are martingales. This is an extension of a result by Harrison and Kreps [1979].

By assuming that the consumption space is separable and thereby extending a result of Kreps [1981], we are able to demonstrate an *equivalent martingale measure*, barring free lunches. This is a martingale measure assigning non–zero probability to precisely those events assigned non–zero probability by the agent's endowed probability measure. Harrison and Kreps [1979] demonstrated an equivalent martingale measure on the basis of viability: the existence of an optimal trading strategy for some agent with strictly monotonic, convex, and continuous preferences. Viability is in general a stronger assumption than the absence of free lunches, as shown by Kreps [1981]. Our results here should be of interest even in settings of symmetric information.

In Section 6 we introduce the *resolution time* of an event, the first time one learns that an event is to happen or not to happen with probability one. The resolution time of an event is random, and of course may be different for differently informed agents. A better informed agent has no later resolution times than a worse informed agent. An event A is *marketed* for an agent if there exists a marketed elementary contingent claim for this event, or equivalently, if there exists a trading strategy paying one unit of consumption in event A and nothing otherwise. We show that the absence of free lunches for a better informed agent implies that the resolution times of the better and worse informed agents for commonly marketed events must be equal (almost surely). It follows that, if all events are marketed for the poorly informed agent and if the better informed agent has no free lunches, then the price system has symmetrized the information, neutralizing the better informed agent's informational advantage.

Section 7 provides an example of a fully revealing arbitrage–free price system. Uncertainty is modeled by a Brownian Motion. Agent α observes the Brownian Motion directly. Agent β does not observe the Brownian Motion as it evolves through time but knows even at the beginning of time the final value of the Brownian Motion. Two long–lived securities are traded, one risky and one riskless. The price process of the single risky security symmetrizes the original infinite–dimensional informational assymetry between agents α and β! Although there is only one price process from which agents infer information, they have infinitely many observations. This is analagous to the option pricing theory, wherein two long–lived securities can effectively complete markets if continuous trading is allowed.

Section 8 adds some discussion, generalizations, and a brief concluding remark.

2. Differential Information

The central primitive for a model of uncertainty is a set Ω of "states of the world," one of which is "correct," loosely speaking. An agent, say Fred, receives information through time refining Fred's knowledge concerning which of the states in Ω is correct. For this section the *time set* $\mathcal{T}$ is any ordered subset of R_+, allowing one to cover continuous and discrete time, as well as finite and infinite horizon settings, in a single pass. How Fred receives information is formalized by specifying a *filtration* $\mathbf{F} = \{\mathcal{F}_t : t \in \mathcal{T}\}$ of *tribes* (also termed σ–algebras) on Ω. If some subset A of Ω is an element of the tribe $\mathcal{F}_t$ then at time t Fred knows whether or not the correct state is an element of A, that is, whether A has "occurred". To formalize the notion that such an occurrence is never forgotten, we say that $\mathbf{F}$ is *increasing*, meaning that $\mathcal{F}_t \subset \mathcal{F}_s$ for all $s \geq t$. A further assumption that is often technically convenient when $\mathcal{T}$ is a continuum time set is that the filtration $\mathbf{F}$ is *right–continuous*, or

$$\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s \quad \forall t \in \mathcal{T}.$$

A different agent, say George, would generally receive information through time specified by a different filtration $\mathbf{G} = \{\mathcal{G}_t : t \in \mathcal{T}\}$ of tribes on Ω. Of course, if $\mathcal{G}_t \subset \mathcal{F}_t$ then George is at an informational disadvantage at time t, since Fred "knows" every event that George knows at time t, and perhaps more. If $\mathcal{G}_t \subset \mathcal{F}_t$ for all $t \in \mathcal{T}$, we write $\mathbf{G} \subset \mathbf{F}$ and say that Fred is *better informed* than George.

The *join* of $\mathcal{F}_t$ and $\mathcal{G}_t$, denoted $\mathcal{F}_t \vee \mathcal{G}_t$, is the tribe generated by the union of $\mathcal{F}_t$ and $\mathcal{G}_t$, and represents the total information held by the two agents at time t. In other words, if $A \in \mathcal{F}_t \vee \mathcal{G}_t$ then at time t, by pooling their information, Fred and George would know whether A has occurred. The *joined filtration*, denoted $\mathbf{F} \vee \mathbf{G}$, is of course $\{\mathcal{F}_t \vee \mathcal{G}_t : t \in \mathcal{T}\}$. The *meet* of $\mathcal{F}_t$ and $\mathcal{G}_t$, on the other hand, is the intersection of $\mathcal{F}_t$ and $\mathcal{G}_t$ (which is indeed a σ–algebra), and is denoted $\mathcal{F}_t \wedge \mathcal{G}_t$. If $A \in \mathcal{F}_t \wedge \mathcal{G}_t$ then Fred and George know independently at time t whether A has occurred. The *meet* of the filtrations is denoted $\mathbf{F} \wedge \mathbf{G} = \{\mathcal{F}_t \wedge \mathcal{G}_t : t \in \mathcal{T}\}$.

In the "rational expectations" genre it is common to assume that agents have private information and also learn information from "market observables", in particular, the market values of traded assets. A market observable can be represented as a *stochastic process*, which we take to be nothing more than a real–valued function Y on $\Omega \times \mathcal{T}$. The value of Y at time t is denoted $Y(t)$, a function on Ω. The information revealed by observing Y up to and including time t is modeled as the tribe $\sigma(Y)_t$ on Ω generated by the functions $\{Y(s) : s \in \mathcal{T}, s \leq t\}$. The corresponding filtration of tribes generated by Y is denoted $F^Y = \{\sigma(Y)_t : t \in \mathcal{T}\}$. To belabor the point, if $A \in \sigma(Y)_t$ then one will know at time t whether or not A has occurred by observing the process Y up to and including time t. If $Y = (Y_1, \ldots, Y_N)$ is a vector process, we write $\mathbf{F}^Y = \bigvee_{n=1}^N \mathbf{F}^{Y_n}$.

If an agent is endowed with private information corresponding to a filtration $\mathbf{G}$ and learns from a vector of market observables Y, then the agent's "total" information is of course $\mathbf{G} \vee \mathbf{F}^Y$.

Suppose there is a set of agents indexed by $\alpha \in \mathcal{A}$ with private information given by filtrations $\widehat{\mathbf{F}}_\alpha, \alpha \in \mathcal{A}$. The market observable process Y is *fully revealing* if

$$\bigvee_{\alpha \in \mathcal{A}} \widehat{\mathbf{F}}_\alpha \subset \widehat{\mathbf{F}}^\beta \vee \mathbf{F}^Y \qquad \forall \beta \in \mathcal{A},$$

which means that observing Y allows each agent to learn all privately held information not already known.

A stochastic process Y is *adapted* to a filtration $\mathbf{G} = \{\mathcal{G}_t : t \in \mathcal{T}\}$ if $Y(t)$ is measurable with respect to $\mathcal{G}_t$ for all $t \in \mathcal{T}$. Roughly, Y is adapted to $\mathbf{G}$ if Y only conveys information already implicit in $\mathbf{G}$. Indeed, Y is adapted to $\mathbf{G}$ if and only if $\mathbf{F}^Y \subset \mathbf{G}$.

The question arises: Are market observables such as prices naturally adapted to the join of all private information? This is essentially a question about the economic structure of price formation. If one believes that market values are a direct consequence of individual strategies, such as bidding for example, then it would be natural to treat market observables as adapted. One could also imagine, perhaps, that certain market structures add "noise" to price formation. This issue is also addressed in Kreps [1977]. For our purposes, we are not forced to take a stance.

3. The formulation

In this section we consider an intertemporal pure exchange economy under uncertainty with differential information. Taken as primitive is a complete probability space $(\Omega, \mathcal{F}, P)$, where each $\omega \in \Omega$ denotes a complete description of the exogenous uncertain environment, and where P indicates the common probability assessments held by agents in the economy. The time set $\mathcal{T}$ is a subset of the interval $[0,1]$, including both 0 and 1. This allows us to handle both discrete and continuous time in a single pass. A single good is available for consumption only at time 1. We take $V \equiv L^1(P)$ as the commodity space, where $L^1(P)$ denotes the space of integrable random variables on $(\Omega, \mathcal{F}, P)$. Thus a given claim $v \in V$ provides $v(\omega)$ units of consumption at time one in state $\omega \in \Omega$. We equip V with the L^1–norm topology. Let $\widehat{\mathbf{F}}$ denote a countable collection of information filtrations from which the agents' private information filtrations are drawn.

The economy is populated with a set $\mathcal{A}$ of agents with characteristics $\{(\succeq_\alpha, \mathcal{V}_\alpha, \mathbf{F}^\alpha) : \alpha \in \mathcal{A}\}$, where for each $\alpha \in \mathcal{A}$, $\mathcal{V}_\alpha \subset V$ is the consumption set of agent α, $\succeq_\alpha$ is a preference relation on $\mathcal{V}_\alpha$, and $\mathbf{F}^\alpha \in \widehat{\mathbf{F}}$. Let K denote the positive cone of V with the origin deleted, and let $\succ_\alpha$ denote the strict preference relation on $\mathcal{V}_\alpha$ induced by $\succeq_\alpha$. We assume that $\succeq_\alpha$ is strictly increasing for all α in $\mathcal{A}$, in the sense that $v \in \mathcal{V}_\alpha$ and $k \in K$ imply that $v + k \succ_\alpha v$. A finite number of long–lived securities in zero net supply are indexed by $n \in \{1, 2, \ldots, N\}$. Each security n is represented by its dividend $d_n \in K$. We assume that $\sum_{n=1}^N d_n = 1$ *a.s.*, a normalization. We denote the N–tuple $(d_1, \ldots, d_N)$ by d.

A *price system* for long–lived securities is an N–vector of non–negative stochastic processes $S = \{S_n(t) : n = 1, 2, \ldots, N : t \in \mathcal{T}\}$ satisfying: S is a vector of semimartingales with respect to some information filtration $\mathbf{F}$ such that $\mathbf{F}^\alpha \subset \mathbf{F}$ for all $\alpha \in \mathcal{A}$;[1]

$$S_n(t) < 1 \quad a.s. \text{ and } \sum_{n=1}^{N} S_n(t) = 1 \quad a.s. \quad \forall t \in \mathcal{T}; \tag{3.1}$$

and, for $n = 1, 2, \ldots, N$,

$$E\left[\left({}^{\mathbf{F}}[S_n, S_n]_1\right)^{\frac{1}{2}}\right] < \infty, \tag{3.2}$$

[1] For the definition of a semimartingale see, for example, Jacod [1979, pp.29].

where $\{^{\mathbf{F}}[S_n, S_m]_t\}$ is the *joint variation* process between S_n and S_m with respect to $\mathbf{F}$ (Dellacherie and Meyer [1982, VII.44].) We also assume that $\mathbf{F} = \{\mathcal{F}_t : t \in \mathcal{T}\}$ is *augmented*, in that $\mathcal{F}_t$ contains all of the P–null sets for all $t \in \mathcal{T}$. Requiring S to be a semimartingale with respect to some information structure $\mathbf{F}$ involves little loss of generality. Any discrete time process is a semimartingale with respect to its natural filtration. A definition of financial gains from trade in continuous–time requires that S is a semimartingale. Condition (3.1) is a normalization and condition (3.2) is a technical restriction. A sufficient condition for (3.2) is given by Dellacherie and Meyer [1982, VII.98].

Agents in $\mathcal{A}$ have *rational expectations* in that they learn from the price system to refine their information. The information filtration of agent α after observing a price system S is denoted $H^\alpha(S) = \{\mathcal{H}_t^\alpha(S) : t \in \mathcal{T}\} = \mathbf{F}^\alpha \bigvee \mathbf{F}^S$. We assume throughout that $H^\alpha(S)$ satisfies the *usual conditions*: *augmented* and *right–continuous*. For consistency, we also require that the dividend d be measurable with respect to $\mathcal{H}_1^\alpha(S)$ for all $\alpha \in \mathcal{A}$. We first record some technical results.

LEMMA 3.1. *A price system is an N–vector of $H^\alpha(S)$–semimartingales for all agents α in $\mathcal{A}$.*

PROOF: We first claim that S is $H^\alpha(S)$–optional. By construction, S is adapted to $H^\alpha(S)$ and right–continuous, and therefore $H^\alpha(S)$–optional (Chung and Williams [1983, Theorem 3.4]). Thus S is an $\mathbf{F}$–semimartingale and is $H^\alpha(S)$–optional. It follows from Theorem 9.19(a) of Jacod [1979] that S is an $H^\alpha(S)$–semimartingale since $H^\alpha(S)$ is a *sub–filtration* of $\mathbf{F}$. ■

In an intertemporal economy with trading over time, before anything interesting can be said an intertemporal budget constraint for agents must be formulated. With uncertainty, this budget constraint involves stochastic integration. Jacod [1979, p.278–279] has shown that in order for stochastic integrals to have reasonable properties, it is *necessary* that the integrators be semimartingales. Thus when a price system carries information that is not endowed to the agent, the agent must learn the information or have an ill–defined budget constraint. In simple terms, if one doesn't observe prices for securities, one's current wealth is a random variable and a budget constraint is somewhat meaningless. This calls for $\mathbf{F}^S \subset H^\alpha(S)$; that is, rational expectations is required for a mathematically consistent model.

Another lemma is needed.

LEMMA 3.2. *There exists a process $\{[S_n, S_m]_t : t \in \mathcal{T}\}$ that is a common version*[2] *of the processes $\{{}^{\mathbf{F}}[S_n, S_m]_t : t \in \mathcal{T}\}$ and $\{{}^{H^\alpha(S)}[S_n, S_m]_t : t \in \mathcal{T}\}$.*

PROOF: Since S_n and S_m are semimartingales with respect to $\mathbf{F}$ and $H^\alpha(S)$, the two processes $\{{}^{\mathbf{F}}[S_n, S_m]_t : t \in \mathcal{T}\}$ and $\{{}^{H^\alpha(S)}[S_n, S_m]_t : t \in \mathcal{T}\}$ are well defined (Dellacherie and Meyer [1982, VII.42]). We first note that

$$ {}^{\mathbf{F}}[S_n, S_m]_t = \frac{1}{4}\left\{{}^{\mathbf{F}}[S_n + S_m, S_n + S_m]_t - {}^{\mathbf{F}}[S_n - S_m, S_n - S_m]_t\right\}, $$

and

$$ {}^{H^\alpha(S)}[S_n, S_m]_t = \frac{1}{4}\left\{{}^{H^\alpha(S)}[S_n + S_m, S_n + S_m]_t - {}^{H^\alpha(S)}[S_n - S_m, S_n - S_m]_t\right\} $$

from Jacod [1979], Section 2.25. Theorem 9.19(b) of Jacod [1979] demonstrates a common version of $\{{}^{\mathbf{F}}[S_n + S_m]_t\}$ and $\{{}^{H^\alpha(S)}[S_n + S_m]_t\}$, and similarly a common version of $\{{}^{F}[S_n - S_m]_t\}$ and $\{{}^{H^\alpha(S)}[S_n - S_m]_t\}$. This implies the existence of a common version of $\{{}^{F}[S_n, S_m]_t\}$ and $\{{}^{H^\alpha(S)}[S_n, S_m]_t\}$. ■

We use $\{[S_n, S_m]_t\}$ to denote a common version of $\{{}^{\mathbf{F}}[S_n, S_m]_t\}$ and $\{{}^{H^\alpha(S)}[S_n, S_m]_t\}$ for all α. (This is possible because $\widehat{\mathbf{F}}$ is a countable set.) Note that $\{[S_n, S_m]_t\}$ is adapted to both $\mathbf{F}$ and $H^\alpha(S)$ since both filtrations are complete. Since joint variation processes are right continuous, $\{[S_n, S_m]_t\}$, $\{{}^{\mathbf{F}}[S_n, S_m]_t\}$, and $\{{}^{H^\alpha(S)}[S_n, S_m]_t\}$ are in fact *indistinguishable*[3] processes. Indistinguishable processes are indeed the same process for any practical purpose. The above lemma implies that agents with rational expectations agree on the joint variation processes of security prices. If price processes are continuous, joint variation processes are covariance processes (Jacod [1979]). Thus rational agents agree on the variance–covariance matrix process.

[2] A process $\{X(t)\}$ is a *version* of another process $\{Y(t)\}$ if $X(t) = Y(t)$ with probability one for all $t \in \mathcal{T}$.

[3] Two processes $\{X(t)\}$ and $\{Y(t)\}$ are *indistinguishable* if $X(t) = Y(t)$ for all $t \in \mathcal{T}$ with probability one.

Let $\mathcal{P}_\alpha(S)$ denote the predictable tribe of subsets of $\Omega \times [0,1]$, that generated by left continuous $H^\alpha(S)$–adapted processes. A real–valued process Y on $\Omega \times [0,1]$ is *$H^\alpha(S)$–predictable* if measurable with respect to $\mathcal{P}_\alpha(S)$. Given a price system S, an *admissible trading strategy for agent α* is an N–vector of $H^\alpha(S)$–predictable processes $\theta = (\theta_1, \ldots, \theta_N)$ such that: (i) the stochastic integral $\int \theta(s)^\top dS(s)$ is well–defined (with respect to $H^\alpha(S)$); (ii) the strategy θ is *self–financing*:

$$\theta(t)^\top S(t) = \theta(0)^\top S(0) + \int_0^t \theta(s)^\top dS(s) \qquad \forall t \in \mathcal{T} \quad a.s., \tag{3.3}$$

(iii) $\theta(1)^\top d \in V$; and (iv) for all $n = 1, 2, \ldots, N$,

$$E\left[\int_0^1 \left(\theta_n(t)^2 d[S_n, S_n]_t\right)\right]^{1/2} < \infty. \tag{3.4}$$

Condition (3.3) is a natural budget constraint; condition (3.4) is technical. For the definition of $\int \theta^\top dS$, see Jacod [1979].

Let $\Theta^\alpha[S]$ denote the space of admissible trading strategies for agent α when the price system is S. By the linearity of stochastic integration and an application of the Kunita–Watanabe inequality (Dellacherie and Meyer [1982, pp.277]), $\Theta^\alpha[S]$ is a linear space.

4. The advantages of better information

We show in this section that a better informed agent is better off in the sense that he or she has access to more admissible trading strategies and therefore enjoys a larger feasible net trade space. The converse of the above statement is not true. We show that if agent α has access to a bigger space of admissible strategies than agent β, then agent α's information as refined by a price system is finer than that of agent β, except possibly at date 1.

Following Harrison and Kreps [1979], a consumption claim $v \in V$ is *marketed* for agent α given a price system S if there exists $\theta \in \Theta^\alpha[S]$ such that $\theta(1)^\top d = v$ almost surely. In that case, θ *generates* v for agent α and $\theta(0)^\top S(0)$ is *an implicit market value* for v at time zero. Let M_α denote the space of marketed consumption claims for agent α. Since $\Theta^\alpha[S]$ is a linear space, M_α is a linear subspace of V.

THEOREM 4.1. *If agent α is better informed than agent β, then $\Theta^{\beta}[S] \subset \Theta^{\alpha}[S]$, and therefore $M_{\beta} \subset M_{\alpha}$.*

PROOF: Let $\theta \in \Theta^{\beta}[S]$. We must show that θ satisfies the defining properties of $\Theta^{\alpha}[S]$. That the stochastic integral $\int \theta dS$ is well–defined with respect to $H^{\alpha}(S)$ follows from the first assertion of Theorem 9.26 of Jacod [1979]. Also, $\theta(1)^{\top} d \in V$ since $\theta \in \Theta^{\beta}[S]$. That θ is self–financing with respect to $H^{\alpha}(S)$ follows from the second assertion of Theorem 9.26 of Jacod [1979]. That θ satisfies (3.4) is trivial. Therefore $\theta \in \Theta^{\alpha}[S]$ and $M_{\beta} \subset M_{\alpha}$. ■

We have just shown that a better informed agent can employ any strategy available to a less informed agent. The converse of the above proposition is not true. For example, let α and β be agents whose private information structures are not ordered. Suppose that the information generated by the price system is identical to $\mathbf{F}^{\beta}$. Then $H^{\beta}(S) \subset H^{\alpha}(S)$ and $\Theta^{\beta}[S] \subset \Theta^{\alpha}[S]$. The following proposition shows that $\Theta^{\beta}[S] \subset \Theta^{\alpha}[S]$ (more trading strategies) does not imply that $H^{\beta}[S] \subset H^{\alpha}[S]$ (more information).

PROPOSITION 4.1. *Suppose $\mathcal{H}_t^{\beta}(S) \subset \mathcal{H}_t^{\alpha}(S)$ $\forall t \in \mathcal{T} \setminus \{1\}$. Then $\mathcal{P}_{\beta}(S) \subset \mathcal{P}_{\alpha}(S)$ and $\Theta^{\beta}[S] \subset \Theta^{\alpha}[S]$.*

PROOF: The predictable tribe $\mathcal{P}_{\beta}(S)$ is generated by the collection of $H^{\beta}(S)$–*predictable rectangles* of the form $\{0\} \times B_0$ and $(s, t] \times B$ with $B_0 \in \mathcal{H}_0^{\beta}(S)$ and $B \in \mathcal{H}_s^{\beta}(S)$ for $s, t \in \mathcal{T}$, $s < t$. [See, for example, Chapter 2 of Chung and Williams [1983].] By assumption, every $\mathcal{P}_{\beta}(S)$–predictable rectangle is a $\mathcal{P}_{\alpha}(S)$–predictable rectangle, implying $\mathcal{P}_{\beta}(S) \subset \mathcal{P}_{\alpha}(S)$. The rest of the assertion follows from arguments such as those of Theorem 4.1. ■

The intuition behind this proposition is clear. In a finite–horizon single commodity economy agents do not take advantage of information revealed at the final date of the economy. There need be no trading at the final date! A more precise statement follows.

THEOREM 4.2. *Suppose $\mathcal{H}_1^{\beta}(S) \subset \mathcal{H}_1^{\alpha}(S)$. Then $\Theta^{\beta}[S] \subset \Theta^{\alpha}[S]$ implies $H^{\beta}(S) \subset H^{\alpha}(S)$.*

PROOF: Suppose that there exists $t \in \mathcal{T} \setminus \{1\}$ such that $\mathcal{H}_t^{\beta}(S) \not\subset \mathcal{H}_t^{\alpha}(S)$. Then there is a set $B \in \mathcal{H}_t^{\beta}(S)$ such that $B \notin \mathcal{H}_t^{\alpha}(S)$. The rectangle $(t, 1] \times B$ is an element of $\mathcal{P}_\beta(S)$ but not an element of $\mathcal{P}_\alpha(S)$. Define a self–financing trading strategy by

$$\theta_i(\omega, s) = 1_{(t,1]\times B}(\omega, s)\,(1 - S_i(\omega, t)) \text{ for some } i,$$

$$\theta_n(\omega, s) = -1_{(t,1]\times B}(\omega, s) S_i(\omega, t) \text{ for all } n \neq i.$$

It is quickly checked that θ as defined above is an element of $\Theta^\beta[S]$. But θ is not $\mathcal{P}_\alpha(S)$–predictable since $\theta_i^{-1}(\Re_+ \setminus \{0\}) = (t, 1] \times B \notin \mathcal{P}_\alpha(S)$. This contradicts the hypothesis that $\Theta^\beta[S] \subset \Theta^\alpha[S]$. Thus $H^\beta(S) \subset H^\alpha(S)$. ■

COROLLARY 4.1. *Suppose that $\mathcal{H}_1^{\alpha}(S) = \mathcal{H}_1^{\beta}(S)$ and that $\Theta^\alpha[S] = \Theta^\beta[S]$. Then $H^\alpha(S) = H^\beta(S)$.*

5. Free lunches and martingales

We now show how a price system precluding free lunches is connected with martingales when agents have rational expectations. The following definitions of a *simple free lunch* and a *free lunch* are due to Kreps [1981]. A *simple free lunch* for agent α is a strategy $\theta \in \Theta^\alpha[S]$ such that $\theta(0)^\top S(0) \leq 0$ and $\theta(0)^\top d \in K$. A *free lunch* for agent α is a net $\{(\theta^\lambda, v^\lambda) : \lambda \in \Lambda\} \subset \Theta^\alpha[S] \times V$ and a choice $k \in K$ such that $\theta^\lambda(1)^\top d - v^\lambda \in K \cup \{0\}$ for all λ with $v^\lambda \to k$ and $\lim\inf\{\theta^\lambda(0)^\top S(0)\} \leq 0$. Implicit in the definition of a free lunch is a sense of continuity of agent α's preferences. We refer interested readers to Kreps [1981] for a host of related issues. Some consequences of the simple free lunch definition follow. The first has a trivial proof.

PROPOSITION 5.1. *Suppose that a price system S admits no simple free lunches for some agent α. Then $S_n(1) = d_n$ a.s. for all $n \in \{1, \ldots, N\}$.*

PROPOSITION 5.2. *Suppose that S admits no simple free lunches for agent α. Then there exists a linear functional π_α on M_α such that $\pi_\alpha(v)$ is the implicit market value of v for all v in M_α.*

PROOF: If S admits no simple free lunches for agent α, there exists a unique implicit price for every marketed claim (for agent α). Let $\theta \in \Theta^\alpha[S]$ be a strategy that generates $m \in M_\alpha$. Let $\pi_\alpha(m) = \theta(0)^\top S(0)$. Defined as such for all $m \in M_\alpha, \pi_\alpha$ has the desired property. ■

For a given marketed claim $m \in M_\alpha$, there is some trading strategy $\theta \in \Theta^\alpha[S]$ with the property that agent α can finance m by investing $\mathcal{S}_m^\alpha(t) \equiv \theta(0)^\top S(0) + \int_0^t \theta^\top dS$ at time t and then proceeding with the trading strategy θ from time t onward. If the resulting process $\mathcal{S}_m^\alpha$ is unique in this regard (up to indistinguishability), we define $\mathcal{S}_m^\alpha$ to be the *implicit price process* of agent α for m.

PROPOSITION 5.3. *Let $m \in M_\alpha$ be marketed for agent α. If there are no simple free lunches, there exists an implicit price process $\{S_m^\alpha(t) : t \in \mathcal{T}\}$ for m for agent α.*

PROOF: This is a direct consequence of no simple free lunch and the fact that there exists a portfolio of securities, available for all agents, whose implicit price process is unity throughout. (That is, hold one share of each long–lived security.) ■

This proposition is actually valid in any economy allowing agents to transfer strictly positive values across time.

COROLLARY 5.1. *Suppose that agent α is better informed than agent β and that the price system admits no simple free lunches for agent α. Let $m \in M_\beta$. Then the implicit price processes $\{S_m^\alpha(t)\}$ and $\{S_m^\beta(t)\}$ are indistinguishable.*

PROOF: This a consequence of Theorem 4.1. ■

If a price system precludes simple free lunches for a better informed agent, that agent cannot dynamically finance a consumption claim at a lower initial portfolio value than available to a less informed agent. The advantage of better information, when there are no simple free lunches, is *not* in the sense that some consumption claims can be financed at lower cost.

When a price system admits no free lunches we are able to say a bit more. Let Φ denote the set of non–zero positive linear functionals on V. Any positive linear functional on V is continuous (Schaefer [1980]). By the Riesz representation theorem, we can thus identify Φ with the positive cone of $L^\infty(P)$ with the origin deleted, where $L^\infty(P)$ denotes the space of essentially bounded random variables on $(\Omega, \mathcal{F}, P)$. A *uniform martingale measure* for agent α is a probability measure Q_α on $(\Omega, \mathcal{F})$, uniformly absolutely continuous with respect to P (meaning $\frac{dQ}{dP} \in L^\infty$), under which the price system S is a vector of $H^\alpha(S)$–martingales. The following proposition is due to Kreps [1981].

PROPOSITION 5.4. *Suppose S admits no free lunches for agent α. Then π has a positive linear extension $\phi_\alpha \in \Phi$.*

PROOF: See Lemma 1 of Kreps [1981]. ■

The following theorem connects the no–free–lunch condition to martingale theory.

THEOREM 5.1. *There is a one–to–one correspondence between uniform martingale measures Q_α for agent α and extensions $\phi_\alpha \in \Phi$ of π_α. The correspondence is given by*

$$Q_\alpha(B) = \phi_\alpha(1_B) \quad \forall B \in \mathcal{F}, \text{ and } \phi_\alpha(v) = E_\alpha(v) \quad \forall v \in V,$$

where $E_\alpha(\cdot)$ denotes the expectation under Q_α.

PROOF: Let ϕ_α be an extension of π_α to all of V that lies in Φ. Then there exists a nonzero positive essentially bounded random variable y_α on $(\Omega, \mathcal{F})$ such that $\phi_\alpha(v) = E(vy_\alpha)\ \forall v \in V$. By our normalization of prices to the unit simplex, $\phi_\alpha(1_\Omega) = 1$, so $E(y_\alpha) = 1$. Define the probability measure

$$Q_\alpha(B) = \int_B y_\alpha(\omega) P(d\omega), \qquad B \in \mathcal{F}.$$

We must show that S is a vector of $H^\alpha(S)$–martingales under Q_α. Fix $i \in \{1, \ldots, N\}$. Let $0 \le t_1 \le t_2 \le 1$ with $\{t_1, t_2\} \subset \mathcal{T}$. For any $B \in \mathcal{H}^\alpha_{t_1}(S)$ consider the trading strategy:

$$\begin{aligned} \theta_i(\omega, t) &= 1 - S_i(\omega, t_1) \,\text{for}\, t \in (t_1, t_2] \,\text{and}\, \omega \in B \\ &= S_i(\omega, t_2) - S_i(\omega, t_1) \,\text{for}\, t \in (t_2, 1] \,\text{and}\, \omega \in B \\ &= 0 \,\text{otherwise;} \end{aligned}$$

and, for all $n \neq i$,

$$\begin{aligned}\theta_n(\omega, t) &= -S_i(\omega, t_1) \,\text{for}\, t \in (t_1, t_2] \,\text{and}\, \omega \in B \\ &= S_i(\omega, t_2) - S_i(\omega, t_1) \,\text{for}\, t \in (t_2, 1] \,\text{and}\, \omega \in B \\ &= 0 \,\text{otherwise}.\end{aligned}$$

We claim that $\theta \in \Theta^\alpha[S]$. First, the stochastic integral $\int \theta dS$ is well–defined with respect to $H^\alpha(S)$ since $B \in \mathcal{H}^\alpha_{t_1}(S)$ and since θ is a left–continuous (and therefore predictable) simple trading strategy. The fact that θ is self–financing follows from reasoning similar to that given in Theorem 3.1 of Huang [1985]. Thirdly, since $\mid \theta \mid$ is bounded by 1,

$$\left[E\left(\int_0^1 \theta_n(t)^2 d[S_n, S_n]\right)^{\frac{1}{2}}\right] \leq E\left[([S_n, S_n]_1)^{\frac{1}{2}}\right] < \infty.$$

Finally, $\theta(1)^\top d \in V$ since $\theta(1)^\top d = 1_B\left(S_i(t_2) - S_i(t_1)\right)$ is bounded.

The consumption claim $1_B(S_i(t_2) - S_i(t_1))$ is marketed and has a implicit price of zero at time zero. Since t_1, t_2 and i are arbitrary, it follows that S is an N–vector of $H^\alpha(S)$–martingales under Q_α by the definition of a martingale.

Conversely, let Q_α be a uniform martingale measure for agent α and let $y = dQ_\alpha/dP$. We note that

$$E_\alpha\left(\int_0^1 (\theta_n(t))^2 \, d[S_n, S_n]_t\right)^{\frac{1}{2}} \leq (\text{ess sup}\, y) E\left(\int_0^1 (\theta_n(t))^2 \, d[S_n, S_n]_t\right)^{\frac{1}{2}} < \infty,$$

using the fact that joint variation processes are invariant under substitution of an absolutely continuous probability measure. The assertion then follows from arguments similar to those in Theorem 3.1 of Huang [1985]. ■

This theorem generalizes the martingale results of Harrision and Kreps [1979]. If a price system S admits no free lunches for an agent α with rational expectations, then S is a vector $H^\alpha(S)$–martingale under some probability Q_α absolutely continuous with respect P. The converse is also true. A price system S is *arbitrage–free* if it admits no free lunches for *all* agents in $\mathcal{A}$. In that case, for any agent α there exists a martingale measure Q_α.

The driving force behind the martingale result is the existence of a portfolio whose market value is unity throughout. The assumption that $\sum_{n=1}^{N} S_n(t) = 1$ *a.s.* for all $t \in \mathcal{T}$ is not necessary. For example, if there exists a riskless asset with a zero interest rate, then a price system that admits no free lunches for agent α is a $H^\alpha(S)$–martingale with respect to some martingale measure.

6. Resolution Times

In this section we show that the absence of free lunches implies that a better informed agent and a less informed agent agree on the *resolution times* of a particular set of events. We go on to formalize the link between dynamic market completeness and a *dynamically fully revealing* price system. Under some regularity conditions a price system precluding free lunches for a better informed agent and allowing dynamically complete markets for a less informed agent must convey all of the information of the better informed agent to the less informed agent.

For any event B in $\mathcal{H}_1^\alpha(S)$, the *resolution time* of B is in the stopping time $T_B^\alpha : \Omega \to [0, \infty]$ defined by:

$$T_B^\alpha = \inf\{t \in [0,1] : E[1_B \mid \mathcal{H}_t^\alpha(S)] = 1 \,\text{or}\, 0\},$$

where, as usual, when the infimum does not exist T_B^α takes the value ∞. Given that $E(1_B \mid \mathcal{H}_1^\alpha) = 1_B$ *a.s.* we know $T_B^\alpha \in [0,1]$ P–almost surely. We can thus redefine T_B^α on a P–null set such that its range is $[0,1]$. Literally, T_B^α is the first time that agent α knows that event B is to happen or not to happen with P–probability one, after observing the price system. Intuition suggests that if agent α is better informed than agent β, then agent α's resolution time for any $B \in \mathcal{H}_1^\beta(S)$ is no later than that of agent β. The following proposition formalizes this notion. Some technical lemmas are first recorded. For any stopping time T let $\mathcal{H}_T^\alpha(S)$ denote the "stopped tribe" representing information known to agent α at time T. For the definition of a stopping time (or equivalently, *optional time*), see Chung and Williams [1983; Section 17]. For the definition of $\mathcal{H}_T^\alpha(S)$, see Dellacherie and Meyer [1982].

LEMMA 6.1. *Let B be an event in $\mathcal{H}_1^\alpha(S)$. Then $B \in \mathcal{H}_{T_B^\alpha}^\alpha(S)$.*

PROOF: Define $\widehat{T}^\alpha_B : \Omega \to [0,\infty]$ by

$$\widehat{T}^\alpha_B(\omega) = T^\alpha_B(\omega) \qquad \omega \in B$$
$$= \infty, \qquad \omega \notin B.$$

Then $\widehat{T}^\alpha_B = \inf\{t \in [0,1] : E(1_B \mid \mathcal{H}^\alpha_t(S)) = 1\}$. Thus $\widehat{T}^\alpha_B$ is an $H^\alpha(S)$–stopping time. It follows from Theorem IV.53 of Dellacherie and Meyer [1982] that $B \in \mathcal{H}^\alpha_{T^\alpha_B}(S)$. ■

LEMMA 6.2. *If $H^\beta(S) \subset H^\alpha(S)$, then any $H^\beta(S)$–stopping time is an $H^\alpha(S)$–stopping time. Furthermore, let T be an $H^\beta(S)$–stopping time. Then $\mathcal{H}^\beta_T(S) \subset \mathcal{H}^\alpha_T(S)$.*

PROOF: Let T be an $H^\beta(S)$–stopping time. Then we have $\{T \le t\} \in \mathcal{H}^\beta_t(S),\ t \in \mathcal{T}$, by the definition of stopping time. Since $\mathcal{H}^\beta_t(S) \subset \mathcal{H}^\alpha_t(S), t \in \mathcal{T}$, we know $\{T \le t\} \in \mathcal{H}^\alpha_t(S), \quad t \in \mathcal{T}$. This implies that T is an $H^\alpha(S)$–stopping time by definition.

By definition $\mathcal{H}^\beta_T(S)$ contains all sets $B \in \mathcal{H}^\beta_1(S)$, such that $B \bigcap \{T \le t\} \in \mathcal{H}^\beta_t(S)$ for all $t \in \mathcal{T}$. Let $B \in \mathcal{H}^\beta_T(S)$. Since $H^\beta(S) \subset H^\alpha(S)$, we know that $B \in \mathcal{H}^\alpha_1(S)$ and $B \bigcap \{T \le t\} \in \mathcal{H}^\alpha_t(S), t \in \mathcal{T}$. Thus $B \in \mathcal{H}^\alpha_T(S)$. ■

PROPOSITION 6.1. *Suppose $\mathbf{F}^\beta \subset \mathbf{F}^\alpha$ and $B \in \mathcal{H}^\beta_1(S)$. Then $T^\alpha_B \le T^\beta_B$ P–a.s.*

PROOF: From the hypothesis, $H^\beta(S) \subset H^\alpha(S)$. From Lemma 6.1, $B \in \mathcal{H}^\beta_{T^\beta_B}(S)$. Thus

$$E(1_B \mid \mathcal{H}^\beta_{T^\beta_B}(S)) = 1_B \quad P-a.s.$$

From Lemma 6.2, $\mathcal{H}^\beta_{T^\beta_B}(S) \subset \mathcal{H}^\alpha_{T^\beta_B}(S)$. Then by the law of iterative expectation we get

$$E\left(1_B \mid \mathcal{H}^\beta_{T^\beta_B}(S)\right) = E\left(E\left(1_B \mid \mathcal{H}^\alpha_{T^\beta_B}(S)\right) \mid \mathcal{H}^\beta_{T^\beta_B}(S)\right)$$
$$= 1_B \quad P-a.s.$$

Since $E(1_B \mid \mathcal{H}^\alpha_{T^\beta_B}(S)) \in [0,1]$ P–a.s. we have

$$E\left(1_B \mid \mathcal{H}^\alpha_{T^\beta_B}(S)\right) = 1_B \quad P-a.s.,$$

which implies that $T^\alpha_B(S) \le T^\beta_B(S)$ P–a.s. ■

The following proposition shows that the definition of resolution times is invariant under the substitution of an equivalent probability measure.

PROPOSITION 6.2. *Let $B \in \mathcal{H}_1^\alpha(S)$ and Q be a probability measure equivalent to P. Suppose T_B^α and $\widehat{T}_B^\alpha$ are resolution times for B with respect to P and Q, respectively. Then $T_B^\alpha = \widehat{T}_B^\alpha$ P–a.s. and therefore Q–a.s.*

PROOF: Let $\xi = dQ/dP$ and fix a right–continuous version of $\xi(t) = E(\xi \mid \mathcal{H}_t^\alpha(S)), t \in \mathcal{T}$. Since Q is absolutely continuous with respect to P, $\{\xi(t)\}$ is a strictly positive process except possibly on a P–null set. We have

$$E_Q(1_B \mid \mathcal{H}_{T_B^\alpha}^\alpha) = \frac{E(1_B \xi \mid \mathcal{H}_{T_B^\alpha}^\alpha(S))}{\xi(T_B^\alpha)}$$
$$= 1_B \qquad P - a.s.$$

Then $\widehat{T}_B^\alpha \leq T_B^\alpha$ P–a.s. and therefore Q–a.s., since Q is absolutely continuous with respect to P. Reversing the above argument, $T_B^\alpha \geq \widehat{T}_B^\alpha$ P–a.s. and Q–a.s. Hence $T_B^\alpha = \widehat{T}_B^\alpha$ P–a.s. and Q–a.s. ■

Before proceeding to the main theorems of this section, we first strengthen certain results of Section 5. We now assume that $(\Omega, \mathcal{F}, P)$ is separable. Let Ψ denote the space of strictly positive linear functionals on V. An *equivalent uniform martingale measure* for agent α is a uniform martingale measure Q_α for α that is equivalent to P. The following proposition applies an extension of Theorem 3 of Kreps [1981].

PROPOSITION 6.3. *Suppose that S admits no free lunches for agent α. Then the implicit price functional π_α has a strictly positive linear extension $\psi_\alpha \in \Psi$.*

PROOF: Since $(\Omega, \mathcal{F}, P)$ is a separable probability space, V is a separable normed space. The assertion then follows from an Appendix theorem. ■

A direct consequence of the above proposition and Theorem 5.1 is:

PROPOSITION 6.4. *There is a one–to–one correspondence between equivalent uniform martingale measures Q_α for agent α, and extensions $\psi_\alpha \in \Psi$ of π_α. The* *correspondence is given by*

$$Q_\alpha(B) = \psi_\alpha(1_B), \quad B \in \mathcal{F}, \text{ and } \psi_\alpha(v) = E_\alpha(v) \quad \forall v \in V,$$

where E_α denotes the expectation under Q_α.

In summary, if the underlying probability space is separable, no free lunches for agent α implies the existence of a strictly positive extension of π_α to all of V. This in turn implies the existence of an equivalent uniform martingale measure for agent α.

Since all of the probability measures to appear are equivalent we use *a.s.* to denote *almost surely* under any probability measure involved. We next show that commonly marketed events are revealed to differently informed agents at the same time.

THEOREM 6.1. *Suppose that agent α is better informed than agent β, and that the price system admits no free lunches for agent α. Let $B \in \mathcal{H}_1^\beta(S)$ be such that $1_B \in M_\beta$. Then*

$$T_B^\alpha = T_B^\beta \qquad a.s.$$

PROOF: Let $\{S_B^\alpha(t)\}$ and $\{S_B^\beta(t)\}$ denote the implicit price processes for 1_B for agents α and β, respectively. From Corollary 5.1, $\{S_B^\alpha(t)\}$ and $\{S_B^\beta(t)\}$ are indistinguishable. We therefore use $\{S_B(t)\}$ to denote either. From Propositions 4.1 and 5.4, as well as Theorem 5.1,

$$\begin{aligned} S_B(t) &= E_\alpha(1_B \mid \mathcal{H}_t^\alpha(S)) \quad \forall t \in \mathcal{T} \quad a.s. \\ &= E_\beta(1_B \mid \mathcal{H}_t^\beta(S)) \quad \forall t \in \mathcal{T} \quad a.s., \end{aligned}$$

where $E_\alpha(\cdot)$ and $E_\beta(\cdot)$ denote the expectations under equivalent martingale measures Q_α and Q_β for agents α and β, respectively. Thus the resolution times of B under $(H^\alpha(S), Q_\alpha)$ and under $(H^\beta(S), Q_\beta)$ are equal almost surely. It then follows from Proposition 6.2 that the resolution times of B under $(H^\alpha(S), P)$ and under $(H^\beta(S), P)$ are equal almost surely. ■

Markets are *elementarily complete* for agent α if $1_B \in M_\alpha$ for all events $B \in \mathcal{H}_1^\alpha(S)$, meaning that agent α has some trading strategy financing a elementary contingent claim for any given event.

THEOREM 6.2. *Suppose that agent α is better informed than agent β. If markets are elementarily complete for β, then a price system precluding free lunches for agent α satisfies*

$$\mathcal{H}_t^\beta(S) = \mathcal{H}_t^\alpha(S) \bigcap \mathcal{H}_1^\beta(S).$$

PROOF: Elementary market completeness implies that any event in $\mathcal{H}_1^\beta(S)$ is marketed. Theorem 6.1 then implies that the resolution times for any event in $\mathcal{H}_1^\beta(S)$ under $(H^\alpha(S), P)$ and under $(H^\beta(S), P)$ is equal. We claim that $\mathcal{H}_t^\beta(S) = \mathcal{H}_t^\alpha(S) \bigcap \mathcal{H}_1^\beta(S)$ for all $t \in \mathcal{T}$. Since α is better informed than β, this equality holds at time 1. Also, clearly $\mathcal{H}_t^\beta(S) \subset \mathcal{H}_t^\alpha(S) \bigcap \mathcal{H}_1^\beta(S)$ for all $t \in \mathcal{T} \setminus \{1\}$. Now, suppose there exists $t \in \mathcal{T} \setminus \{1\}$ such that $\mathcal{H}_1^\beta(S) \neq \mathcal{H}_t^\alpha(S) \bigcap \mathcal{H}_1^\beta(S)$. Let $B \in \mathcal{H}_t^\alpha(S) \bigcap \mathcal{H}_1^\beta(S)$ and $B \notin \mathcal{H}_t^\beta(S)$. We know $P(B) > 0$ since the filtrations are augmented. Note that $B \in \mathcal{H}_t^\alpha(S)$ implies that

$$E[1_B | \mathcal{H}_t^\alpha(S)] = 1_B \qquad a.s.$$

Also, since $B \notin \mathcal{H}_t^\beta(S)$, on some event of strictly positive probability we have

$$E[1_B | \mathcal{H}_t^\beta(S)] \neq 1_B.$$

By right–continuity of the filtrations and the fact that $B \in \mathcal{H}_1^\beta(S)$, we know that the resolution times for B under $H^\alpha(S)$ and under $H^\beta(S)$ are not equal, a contradiction of Thereom 6.1. ■

The basic notion of the theorem, under its hypotheses, is that any event that β eventually resolves is resolved by β and α at the same time.

COROLLARY 6.1. *Suppose that α is better informed than β, that $\mathcal{F}_1^\alpha = \mathcal{F}_1^\beta$, and that markets are elementarily complete for β. Then $H^\alpha(S) = H^\beta(S)$. Moreover, suppose that agent β is one of the least informed agents in $\mathcal{A}$, meaning $\mathbf{F}^\beta \subset \mathbf{F}^\alpha \, \forall \alpha \in \mathcal{A}$, that $\mathcal{F}_1^\alpha = \mathcal{F}_1^\beta$ for all agents α in $\mathcal{A}$, and that markets are elementarily complete for agent β. Then any arbitrage free price system is dynamically fully revealing in the sense that $H^\alpha(S) = H^\beta(S)$ for all agents α and β in $\mathcal{A}$.*

PROOF: If $\mathcal{F}_1^\alpha = \mathcal{F}_1^\beta$, then $\mathcal{H}_1^\alpha(S) = \mathcal{H}_1^\beta(S)$. The assertions then follow from Theorem 6.2. ■

The assumption $\mathcal{F}_1^\alpha = \mathcal{F}_1^\beta$ of course means that the two agents eventually learn the same events, but at perhaps different times. The assumption that all elementary contingent claims for events in $\mathcal{F}$ are marketed (for example, the Black–Scholes model with assymetric learning) certainly implies that $\mathcal{F}_1^\alpha = \mathcal{F}_1^\beta$. In an extension of our results to an infinite horizon with intermediate consumption, the analagous assumption is that $\mathcal{F}_\infty^\alpha = \mathcal{F}_\infty^\beta$, which is obviously unrestrictive.

We take this section to be the central contribution of the paper. Probability theorists have recognized that the way information is revealed over time is closely related to the behavior of martingales and optional times. In an economic context with common knowledge, Huang [1985] analyzes this connection. Here we introduced the concept of a resolution time and explored the relationship between resolution time and arbitrage–free pricing and differential information. the latter is introduced in this section. The theorem proved in the Appendix used to demonstrate Proposition 6.3 is of independent interest. The conditions of Theorem 3 of Kreps [1981] ensuring the existence of a strictly positive continuous extension of a linear functional are sometimes hard to verify in applications. It turns out that separability of the probability space simplifies matters considerably.

7. An example

There are two long–lived securities traded, one risky and one riskless with a zero interest rate. Agent α observes a Standard Brownian Motion in "real time." Agent β knows at time zero the value of the Brownian Motion at time one but cannot observe its sample paths over time. The two agents are differently informed and there is no ordering between the two endowed private information structures. Markets are elementarily complete for both agents and the price system is dynamically fully revealing between them. The risky security price process, although one–dimensional, symmetrizes an infinite–dimensional assymetry of information. The key is that agents are able to observe an infinite number or realizations of the risky price process.

Formally, there is defined on a separable probability space $(\Omega, \mathcal{F}, P)$ a Standard Brownian Motion, $W = \{W(t) : t \in [0,1]\}$. We denote by $\mathbf{F}^W = \{\mathcal{F}_t^W : t \in [0,1]\}$ the augmented filtration generated by W. The set of trading dates of the economy

$\mathcal{T}$ is taken to be $[0,1]$. We take $\mathbf{F}^\alpha = \mathbf{F}^W$ and, for all $t \in \mathcal{T}$, $\mathcal{F}_t^\beta = \sigma\{W(1)\}$, the tribe generated by $W(1)$. Although consumption occurs only at time 1, knowing the value of the Brownian Motion at time 1 at the very beginning does not make agent β better informed. The optimal net trade for agent β, if he or she could observe W directly, might well be path–dependent.

There are two long–lived securities traded, with payoff structures

$$\begin{aligned} d_1(\omega) &= 1_\Omega(\omega), \\ d_2(\omega) &= 2W(\omega,1) - \int_0^1 \left(\gamma + \frac{W(\omega,1) - W(\omega,s)}{1-s}\right) ds, \end{aligned}$$

where γ is a real number. The price system is

$$\begin{aligned} S_1(\omega,t) &= 1_\Omega(\omega) \\ S_2(\omega,t) &= W(\omega,1) + W(\omega,t) - \int_0^t \left(\gamma + \frac{W(\omega,1) - W(\omega,s)}{1-s}\right) ds. \end{aligned}$$

Corollary 1.1 of Jeulin and Yor [1979] implies that

$$W(t)+(1-t)W(1) = W(1)+(1-t)\int_0^t \frac{1}{1-s} dS_2(s) + \int_0^t \frac{\gamma}{1-s} ds, \quad t \in [0,1), \quad P\text{−}a.s.$$

Since γ is a constant, $\{W(t) + (1-t)W(1) : t \in [0,1)\}$ and $\{S_2(t) : t \in [0,1)\}$ generate the same augmented filtration, henceforth denoted $\mathbf{F}^S = \{\mathcal{F}_t^S : t \in \mathcal{T}\}$. Corollary 1.1(a) of Jeulin and Yor [1979] implies that

$$\mathcal{F}_t^S = \bigwedge_{\epsilon>0} \left\{ \mathcal{F}_{t+\epsilon}^W \bigvee \sigma\{W(1)\} \right\}.$$

Thus $\mathbf{F}^S$ is right–continuous and finer than either $\mathbf{F}^\alpha$ or $\mathbf{F}^\beta$. It follows that $H^\alpha(S) = H^\beta(S) = \mathbf{F}^S$. That is, the price system is dynamically fully revealing between α and β.

We next claim that S admits no free lunches. It suffices to demonstrate the existence of a martingale measure. Let

$$\mathcal{W}(t) = W(t) - \int_0^t \frac{W(1) - W(s)}{1-s} ds, \quad t \in \mathcal{T}.$$

Theorem 1 of Jeulin and Yor [1979] shows that $\{\mathcal{W}(t) : t \in \mathcal{T}\}$ is a Standard Brownian Motion with respect to H^S under P. We now define $\xi = \exp\{\gamma\mathcal{W}(1) - \frac{1}{2}\gamma^2\}$, as well as the measure Q defined by

$$Q(B) = \int_B \xi(\omega)P(d\omega), \quad B \in \mathcal{F}_1^S.$$

Theorem 6.1 and Lemma 6.5 of Liptser and Shiryayev [1977] ensure that Q is a probability measure on $(\Omega, \mathcal{F}_1^S)$ equivalent to P. It follows from Girsanov's Theorem (Liptser and Shiryayev [1977, pp. 225]) that

$$\mathcal{W}^*(t) = \mathcal{W}(t) - \gamma t = W(1) - \int_0^t \left(\gamma - \frac{W(1) - W(s)}{1 - s}\right) ds, \quad t \in \mathcal{T},$$

is a Standard Brownian Motion adapted to H^S under Q. Thus S_2 is a H^S–martingale under Q since $S_2(t) = \mathcal{W}^*(t) + W(1)$. Finally, the fact that markets are elementarily complete for both agents follows from Corollary 1.1(e) of Jeulin and Yor [1979].

8. Discussions, generalizations, and concluding remarks

The assumption that the price system has been normalized cannot be taken lightly. In a Walrasian economy, normalization of prices is economically neutral. In a rational expectations economy such a procedure is not economically neutral in general. The information content of a price system may be altered by changing numeraires. Our analysis can be generalized in the following direction, however. Take S to be a vector of $\mathbf{F}$–semimartingales satisfying all the earlier stated regularity conditions except that they sum to a process bounded away from zero. We can then normalize the price processes to sum to one, denoting the resulting process S^*, but still allow agents to have access to the information generated by S. Then the results of this paper apply to S^*.

The assumption that consumption only occurs at date one is made only for ease of exposition. Our results are readily extended to an economy whose consumption space is the space of bounded variation processes, representing agents' accumulated net trades, equipped with a norm introduced by Huang and Kreps [1985]. For consumption processes as rates of time with a naturally defined norm, we can also extend to the setting of Duffie [1984].

Other than the sensitivity of information to price normalization referred to above, the results presented in this paper are robust. No arbitrage is a weak requirement to place on a price system. We have demonstrated that this requirement has pervasive implications. In a common information economy, the connection between martingales and an arbitrage free price system observed by Harrison and Kreps [1979] makes a dynamic equilibrium theory possible (Duffie and Huang [1985]). In a differential information context, no arbitrage is a minimum condition for a rational expectations equilibrium price system, and requires that a price system be a martingale under some probability measure. Readers should convince themselves that putting this paper and Duffie and Huang [1985] together leaves it trivial to prove the existence of a dynamic rational expectations equilibrium with a fully revealing price system. Under what general conditions there exists a partially revealing dynamic rational expectations equilibrium is an open question.

Appendix

Let X be a separable normed space and let K be a convex cone in X with the origin deleted. Let x_1 and x_2 be elements of X. We write $x_1 \geq x_2$ if $x_1 - x_2 \in K \bigcup \{0\}$. A securities markets model is a pair (M, π): a vector subspace M of X and a linear functional π on M. A *free lunch* in the securities markets model is a sequence $\{(m_n, x_n) : n = 1, 2, \ldots\} \subset M \times X$ such that $m_n \geq x_n \, x_n \to k \in K$, and $\liminf_n \pi(m_n) \leq 0$.

THEOREM A.1. *Suppose that (M, π) admits no free lunches. Then π has an extension $\psi \in \Psi$ where Ψ is the space of strictly positive continuous linear functionals on X.*

PROOF: From Lemma 5 of Kreps [1981] there exists a collection $\Gamma = \{\psi_k : k \in K\}$ of equicontinuous positive linear functionals on X such that $\psi_k(k) > 0$ and ψ_k is an extension of π. It follows that Γ is a separable metric space in the relative weak* topology (Theorem 5.4.7 of Schaefer [1980]). Let $\{\psi_n : n = 1, 2, \ldots\}$ be a countable dense subset of Γ in the relative weak* topology and let $\{\lambda_n : n = 1, 2, \ldots\}$ be a sequence of strictly positive real numbers such that $\sum_{n=1}^{\infty} = 1$. Since the convex hull of $\{\psi_n\}$ is also equicontinuous, it is relatively weak* compact, implying that the sequence $\sum_{n=1}^{N} \lambda_n \psi_n$ converges weak* as $N \to \infty$. Let the limit be denoted ψ. For any $k \in K$ there exists n such that $\mid \psi_n(k) - \psi_k(k) \mid \leq \psi_k(k)/2 > 0$, implying that $\psi_n(k) > 0$. Thus $\psi(k) > 0$. ■

References

Chung, K.L. and R. Williams, 1983, An Introduction to Stochastic Integration, (Boston: Birkhäuser).

Dellacherie, C. and P. Meyer, 1978, Probabilities and Potential, (North-Holland, New York).

———————, 1982, Probabilities and Potential B: Theory of Martingales, (North-Holl-and Publishing Company, New York).

Duffie, D., 1984, Stochastic Equilibria: Existence, Spanning Number, and The 'No Expected Financial Gain From Trade' Hypothesis, Research Paper 762, Graduate School of Business, forthcoming: *Econometrica*.

Duffie, D. and C. Huang, 1985, Implementing Arrow–Debreu equilibria by continuous trading of few long lived securities, Econometrica 53, 1337–1356.

Harrison, J.M. and D. Kreps, 1979, Martingales and arbitrage in multiperiod securities markets, Journal of Economic Theory 20, 381–408.

Huang, C., 1985, Information structure and equilibrium asset prices, Journal of Economic Theory 31, 33–71.

———————, 1986, Information Structures and Viable Price Systems, Unpublished, Massachusetts Institute of Technology, forthcoming:*Journal of Mathematical Economics*.

Huang, C. and D. Kreps, 1985, Intertemporal preferences with a continuous time dimension: An exploratory study, MIT mimio.

Jacod, J., 1979, Calcul Stochastique et Problémes de Martingales, Lecture Notes in Mathematics, No. 714, (Berlin: Springer Verlag.).

Jeulin, T. and M. Yor, 1978, Grossissement d'une Filtration et Semimartingale. Formules Explicites., in , Séminaires de Probabilités XII (lecture Notes in Mathematics No. 649), (Springer Verlag, Berlin-Heidelberg-New York).

Kreps, D., 1977, A Note on "Fulfilled Expectations" Equilibria, Journal of Economic Theory 14, 32–43.

———————, 1981, Arbitrage and equilibrium in economies with infinitely many commodities, Journal of Mathematical Economics 8, 15–35.

Liptser, R. and A. Shiryayev, 1977, Statistics of Random Processes I: General Theory, (Springer-Verlag, New York).

Memin, J., 1980, Espaces de semi Martingales et Changement de Probabilité, Zeitschrift für Wahrscheinlichkeitstheorie 52, 9-39.

Schaefer, H., 1971, Topological Vector Spaces, (New York: Springer–Verlag).