Research Paper No. 974

Stochastic Production-Exchange Equilibria

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Preliminary Draft **May, 1986**

ABSTRACT

This paper examines the role of production and stock markets in a continuous–time stochastic economy. The results include sufficient conditions for the existence of general equilibria: spot price processes and security price processes under which there exist preference maximal consumption and portfolio choices for agents and share value maximizing production choices for firms that clear markets for commodities and securities at all dates and states. Specific stochastic growth and stochastic input–output production technologies satisfying the stated production conditions are illustrated. We also study traditional issues concerning the financial and production policy of the firm.

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We thank Bill Zame for helpful suggestions.

1. Introduction

This paper examines the role of production in a continuous–time stochastic economy. We take a general equilibrium approach, demonstrating price processes for securities and spot commodities at which individuals' optimal consumption and security trading strategies and firms' share–value–maximizing production strategies clear markets at all dates.

The overall choice space for the economy is the set of consumption processes adapted to the available information. Although this space is typically infinite–dimensional a finite number of securities, with continuous trading possibilities, allows any consumption process to be financed by some security trading strategy. In developing the role of production in a stochastic economy, we chose to model information as generated by diffusion state–variables, this being particularly well suited to our purposes and allowing us to draw a connection with earlier work on stochastic equilibria with production. Cox, Ingersoll, and Ross [7,8], for example, provided necessary conditions for a single–agent stochastic production–exchange equilibrium satisfying certain "smoothness" conditions. Breeden [4] has done related work. Merton [20] has the seminal model of necessary conditions on continuous–time stochastic equilibria. One of the goals here is to actually demonstrate an equilibrium in a multi–agent production–exchange economy with the same "diffusion" information structure. The other major goal is to incorporate and illustrate stochastic extensions of classical production models within equilibrium.

We begin in the next section by laying out the primitives of an economy that fits within our general scope for demonstrating stochastic equilibria. In Section 3 we provide an equilibrium existence theorem for an economy with general production technologies. Section 4 examines the production and financial policies of the firm, reconfirming the Modigliani–Miller Invariance Principle as well as unanimous shareholder support for share value maximization by firms. Section 5 studies particular classes of production technologies. The principal examples here are : (i) production functions modeled as operators that map production input stochastic processes to production output stochastic processes, and (ii) capital stock accumulation, in the framework of stochastic growth models such as that modeled by Cox, Ingersoll, and Ross [7,8].

2. The Economy

Uncertainty

For simplicity we choose a finite time–interval $[0,1]$. Basic uncertainty is represented by a complete probability space $(\Omega, \mathcal{F}, P)$. Here Ω represents the set of states of the world, and $\mathcal{F}$ denotes the *tribe* (or, in some vocabularies, σ–algebra) distinguishing the *events*, or subsets of Ω that can be assigned a probability. A *reference probability measure* P is given. Agents need not agree on probability assessments stated by P, but must have bounds on their disagreements. More precisely, given a finite set $\mathcal{I} = \{1, \ldots, I\}$ of agents, each agent $i \in \mathcal{I}$ must have a probability measure P_i on $(\Omega, \mathcal{F})$ uniformly equivalent[1] with respect to the reference measure P.

Information Structure

Agents receive identical information represented by a filtration $\mathbf{F} = \{\mathcal{F}_t : t \in [0,1]\}$ of sub–tribes of $\mathcal{F} = \mathcal{F}_1$. The sub–tribe $\mathcal{F}_t$, for any time t in $[0,1]$, represents the set of events revealed by all information received up to and including time t. In particular, if $A \in \mathcal{F}_t$ then one will know at time t whether or not A "happens." The collection $(\Omega, \mathcal{F}, \mathbf{F}, P)$ of primitives is a *filtered probability space*, and is the fixed reference point for all probabilistic statements unless otherwise indicated.

Although we need not restrict ourselves to a particular filtration in order to demonstrate equilibria, we will do so for purposes of concreteness and for ease of comparison with the literature. We suppose that $B = (B^1, \ldots, B^N)^\top$ is an N–dimensional Standard Brownian Motion defined on $(\Omega, \mathcal{F}, P)$. An N–dimensional *state–variable process* Z is specified by the Itô integral equation:

$$Z(t) = Z(0) + \int_0^t \mu(Z(s), s)ds + \int_0^t \sigma(Z(s), s)dB(s) \quad \forall t \in [0,1] \quad a.s., \tag{2.1}$$

where $Z(0) \in \Re^N$, and where $\sigma : \Re^N \times [0,1] \to \Re^{N \times N}$ and $\mu : \Re^N \times [0,1] \to \Re^N$ are continuous and satisfy: for some constant K, all t in $[0,1]$, and all y and $\bar{y}$ in $\Re^N$;

$$| \mu(y,t) - \mu(\bar{y},t) | \leq K \mid y - \bar{y} \mid, \quad | \sigma(y,t) - \sigma(\bar{y},t) | \leq K \mid y - \bar{y} \mid,$$

[1] Two measures P and Q on $(\Omega, \mathcal{F})$ are *uniformly equivalent* provided there exist strictly positive scalars $\underline{K}$ and $\overline{K}$ such that $\underline{K}P(B) \leq Q(B) \leq \overline{K}P(B)$ for all events B in $\mathcal{F}$.

and

$$| \mu(y,t) |^2 \leq K(1+ | y |^2), \quad | \sigma(y,t) |^2 \leq K(1+ | y |^2),$$

referred to henceforth as a *Lipschitz condition* and a *growth condition*, respectively.[2] We assume that $\sigma(y,t)$ is nonsingular for each y and t. Theorem 9.3.1 of Arnold [1] ensures that (2.1) has a unique (strong) solution Z that is a diffusion process.

The agents' commonly endowed information structure is the augmented filtration $\mathbf{F} = \{\mathcal{F}_t : t \in [0,1]\}$ generated[3] by Z. By the assumption that $\sigma(y,t)$ is continuous and nonsingular, $\mathbf{F}$ is in fact the augmented filtration $\mathbf{F}^B$ generated by the underlying Brownian Motion (Harrison and Kreps [12]). In interpretation, agents observe a state process Z whose evolution over time depends upon B (is adapted to $\mathbf{F}^B$).[4] Observing Z provides agents with the same information that would be obtained by observing B directly.

Spot Commodities and Agents

The *spot commodity space* is $\Re^\ell$, for some integer number $\ell \geq 1$ of different commodities. An $\Re^\ell$–valued stochastic process $c = \{c(t) : t \in [0,1]\}$ representing consumption choices at each time t must be chosen on the basis of available information. We thus impose the restriction that $c : \Omega \times [0,1] \to \Re^\ell$ is *predictable*, meaning measurable with respect to the tribe $\mathcal{P}$ on $\Omega \times [0,1]$ generated by left–continuous[5] adapted processes. This may be interpreted as requiring that the consumption choice $c(t)$ at any time t must be based only on information obtained by observing the behavior of Z from time zero to time t.

In order to exploit continuity assumptions, we also require a consumption process c to be *square–integrable*, satisfying

$$E\left[\int_0^1 c(t)^\top c(t)dt\right] < \infty. \tag{2.2}$$

[2] We write $| \sigma |^2 = tr(\sigma\sigma^\top)$, where $^\top$ denotes *transpose* and *tr* denotes *trace*.

[3] The filtration $\mathbf{F}^Z = \{\mathcal{F}_t^Z : t \in [0,1]\}$ is generated by Z if $\mathcal{F}_t^Z$ is the smallest sub-tribe of $\mathcal{F}$ with respect to which $Z(s)$ is measurable for all s in $[0,t]$. The filtration is *augmented* by replacing $\mathcal{F}_t^Z$ with the tribe $\mathcal{F}_t$ generated by $\mathcal{F}_t^Z$ and all zero probability subsets of the complete tribe $\mathcal{F}$.

[4] A process $\{X(t) : t \in [0,1]\}$ is *adapted to* a filtration $\mathbf{F} = \{\mathcal{F}_t : t \in [0,1]\}$ if $X(t)$ is measurable with respect to $\mathcal{F}_t$ for all $t \in [0,1]$.

[5] An adapted process is *left–continuous* if its sample paths are left continuous almost surely. By Chung and Williams [6], the predictable and optional tribes corresponding to the filtration generated by any Hunt process such as Brownian Motion are the same tribe. Thus we are consistent with the consumption space of optional processes used in related papers [9].

Thus the consumption space is $L^2_\ell \equiv \times_{l=1}^{\ell} L^2(\Omega \times [0,1], \mathcal{P}, \nu)$, where ν is the product measure generated by P and Lebesgue measure. We henceforth abbreviate $\times_{l=1}^{n} L^q(\Omega \times [0,1], \mathcal{P}, \nu)$, for any $q \in [1, \infty)$ and any integer $n \geq 1$, as L^q_n. In summary, L^2_ℓ consists of any $\Re^\ell$–valued square–integrable predictable process. As usual, we identify any two consumption processes that are equal almost everywhere on $\Omega \times [0,1]$ (with respect to the product measure ν). The set of *positive consumption processes* is the positive cone $(L^2_\ell)_+$ of L^2_ℓ.

A *spot price process* is given by some $\psi = (\psi^1, \ldots, \psi^\ell) \in L^2_\ell$, where $\psi^l(\omega, t)$ is the unit price of the l–th commodity in state $\omega \in \Omega$ at time t.

Each agent $i \in \mathcal{I} = \{1, 2, \ldots, I\}$ has a non–zero *endowment* $e_i \in (L^2_\ell)_+$ and a *preference relation*[6] $\succeq_i$ on $(L^2_\ell)_+$. An example of a preference relation is given by the time–additive von Neumann Morgenstern form of utility representation "$E(\int_0^T u(c(t), t)dt)$," but that is vastly more restrictive than needed for the existence of equilibria.

Capital Assets, Financial Assets, and Trading Strategies

A set $\mathcal{J} = \{1, \ldots, J\}$ of firms is given, characterized by a *production set* $Y_j \subset L^2_\ell$ for each $j \in \mathcal{J}$. Any $y \in Y_j$ represents a feasible net production process for firm j. For example, the production set Y_j could be that represented by a *production output function* $f_j : (L^2_\ell)_+ \to (L^2_\ell)_+$ mapping production inputs to production outputs, a class of technologies studied in Section 5.

A *security* is identified with an *RCLL* integrable[7] process D defining its cumulative dividends. In other words, security D is a claim to cumulative dividends $D(t)$ up to any time t. If, for example, one share of D is bought at time s and sold at a later time t, a total of $D(t) - D(s)$ is received during the interim as dividends. The lump sum dividend paid at time t, if any, is the *jump* $\Delta D(t) \equiv D(t) - D(t_-)$. [By our convention, $\Delta D(0) = 0$.] As in Arrow's original model of 1953, dividends are denominated with respect to a numeraire that need not be any particular one of the ℓ commodities. One might think in terms of "dollars," although this is not a monetary economy.

[6] For our purposes, a *preference relation* $\succeq$ on a subset X of L^2_ℓ is merely a binary order on X. We interpret $x \succeq y$, for any x and y in X, as "x is at least as good as y". We do not require that $\succeq$ be complete or transitive, although this would automatically be the case if $\succeq$ is represented by a utility function $u : X \to R$, meaning $u(x) \geq u(y)$ whenever $x \succeq y$.

[7] An adapted process D is *integrable* if $\mid D(t) \mid$ has finite expectation for all t, and *RCLL* if its sample paths are right–continuous with left limits almost surely. The left limit of an *RCLL* process D at time t is $D(t_-) \equiv lim_{s \uparrow t} D(s)$.

Given a spot price process ψ, any element x of L^2_ℓ defines a security $D_{x\psi}$ by

$$D_{x\psi}(t) = \int_0^t \psi(s)^\top x(s)ds, \quad t \in [0,1]\, a.s.$$

(By the Cauchy-Schwarz inequality, $D_{x\psi}$ is integrable.) Each firm $j \in \mathcal{J}$ has one security D^j outstanding, called *common stock*, of the form $D^j = D_{y_j\psi}$, where $y_j \in Y_j$ is the production process chosen by firm j. In other words, firm j at time t sells its net output at the rate $y_j(t)$ on the spot market at prices $\psi(t)$, and pays all of the proceeds as dividends at the rate $\psi(t)^\top y_j(t)$, yielding the cumulative dividend process $\int \psi(t)^\top y_j(t)dt$. We could also allow firm j to issue, buy, or sell securities, for example debt financing, in order to adjust its dividend stream. The nature of the equilibrium we are about to demonstrate leaves all shareholders indifferent to such schemes, a Modigliani–Miller style invariance principle developed in Section 4. Thus we leave the financial policies of firms out of the model for the present.

Common stocks are *capital assets*, or claims to net sales of commodity production in strictly positive supply. A *financial asset*, on the other hand, is a security in zero net supply. In addition to capital assets, an economy includes some number $K - J$ of financial assets, defined by cumulative dividend processes $D^{J+1}, \ldots, D^K$.

Each agent $i \in \mathcal{I}$ is initially endowed with some share $\epsilon_i^j \geq 0$ of the common stock D^j of each firm $j \in \mathcal{J}$. Agent i's initial endowment ϵ_i^j of any financial asset $j > J$ is zero. By convention, $\sum_{i=1}^I \epsilon_i^j = 1$ for all $j \in \mathcal{J}$. We denote $(\epsilon_i^1, \ldots, \epsilon_i^K)$ by ϵ_i.

Each security D^j is assigned a *price process* S^j for its *ex–dividend market value*. That is, $S^j(t)$ is the random variable for the market value at time t of a claim to all future dividends to be paid by security D^j. [Since S^j is ex–dividend, $S^j(1) = 0$ almost surely, barring arbitrage.] A *gain operator* is a linear operator Π on the space of dividend processes into the space of *Itô processes*, which are reviewed in Appendix A. Under Π, for any dividend process D, the gain $G = \Pi(D)$ is the process defining the cumulative market value earned by holding one share of D, including both capital gain and dividend gain. That is, $\Pi(D) = D + S$, where S is the price process for D.

Agents take as given a vector $D = (D^1, \ldots, D^K)^\top$ of K securities, a spot price process ψ, and a gain operator Π. Let G denote the corresponding $\Re^K$–valued gain process for the

securities. As an Itô process, G has the stochastic differential form

$$dG(t) = dV(t) + \sigma(t)dB(t), \quad t \in [0,1], \tag{2.3}$$

where V is an $\Re^K$–valued bounded variation process and σ is an $K \times N$ matrix–valued predictable process, as described in more detail in Appendix A. Agents trade securities by holding portfolios prescribed by an $\Re^K$–valued predictable process $\theta = (\theta^1, \ldots, \theta^K)$, where $\theta^j(\omega, t)$ is the number of units of the j–th security held in state $\omega \in \Omega$ at time $t \in [0,1]$. For regularity we demand that

$$\int_0^1 \mid \theta(t)^k \mid |dV^k(t)| < \infty \quad a.s., \quad 1 \leq k \leq K, \tag{2.4}$$

and that

$$E\left[\int_0^1 |\theta(t)\sigma(t)|^2 \, dt\right] < \infty, \tag{2.5}$$

ensuring the existence of the stochastic integral $\int \theta dG$. The *total gain* of strategy θ is this integral $\int \theta dG$, representing the sum of the cumulative dividend gain $\int \theta dD$ and the cumulative capital gain $\int \theta dS$, if both integrals exist. The set of $\Re^K$–valued predictable processes θ satisfying (2.4) and (2.5) is the (linear) space $\Theta[G]$ of *trading strategies.*

Optimal Consumption and Trading Strategies

Given securities $D = (D^1, \ldots, D^K)^\top$, a gain operator Π, and a spot price process ψ, a consumption process c is *financed* by a trading strategy θ if

$$\theta(t)[S(t) + \Delta D(t)] = \theta(0)S(0) + \int_0^t \theta(s)dG(s) - \int_0^t \psi(s)^\top c(s)ds, \quad \forall t \in [0,1] \, a.s., \tag{2.6}$$

where $G = (\Pi(D^1), \ldots, \Pi(D^K))^\top$ and $S = (S^1, \ldots, S^K)^\top = G - D$, and if

$$\theta(T)[S(T) + \Delta D(T)] = 0. \tag{2.7}$$

The left–hand–side of (2.6) is the cum–dividend market value of the strategy θ at time t; on the right is the initial market value of θ, plus the cumulative capital and dividends gains from security trade up to and including time t, less the cumulative spot market cost of c. Given (D, Π, ψ), a pair $(c, \theta) \in (L^2_\ell)_+ \times \Theta(G)$ is a *budget feasible plan for agent i* if $\theta(0) = \epsilon_i$ and if θ finances the net trade $c - e_i$. A budget feasible plan (c, θ) is *optimal* for agent i if

there is no budget feasible plan (c', θ') such that $c' \succ_i c$. The equality constraint in (2.7) is without loss of generality given locally non–satiated preferences.

Share Value Maximizing Production Choices

Given a spot price process ψ and a gain operator Π, a production choice $y \in Y_j$ generates the share price process $S_{y\psi} \equiv \Pi(D_{y\psi}) - D_{y\psi}$ for firm j. Given a production choice $y \in Y_j$ and a stopping time T, a production plan z is a *continuation* of y at T if $z(t) = y(t)$ for all $t \leq T$ almost surely. The set of continuations of $y \in Y_j$ at T is denoted $Y_j(T, y)$. Given (ψ, Π), a production plan $y \in Y_j$ is *share value maximizing for firm j* if there is no stopping time T with $P(T < 1) > 0$ and continuation $z \in Y_j(T, y)$ such that $S_{z\psi}(T) > S_{y\psi}(T)$ almost surely. In other words, y is optimal in this sense if it impossible at any time to revise the firm's production plan in a consistent manner and improve the firm's current share price with non–zero probability. We should remark that, as the firm's cumulative dividend process is continuous, the cum and ex dividend values of the firm are equivalent. We would otherwise naturally describe market value maximization in terms of the cum–dividend value.

We define the *conditional expectation gain operator* $\overline{\Pi}$ by $\overline{\Pi}(D)_t = E[D(1)|\mathcal{F}_t]$, under which the current market value $S(t)$ of a security D is the current conditional expected value of its total future dividends, or $S(t) = E[D(1) - D(t)|\mathcal{F}_t]$. This gain operator is one of a large time–additive class with the property that maximizing initial share price implies share–value–maximization in the above sense.

PROPOSITION 2.1. *Given a spot price process ψ and the conditional expectation gain operator $\overline{\Pi}$, a production plan $y_j \in Y_j$ is share–value–maximizing for firm j if and only if, for all $y \in Y_j$,*

$$E\left[\int_0^1 \psi(t)^\top [y_j(t) - y(t)]dt\right] \geq 0.$$

PROOF: The *only if* assertion is trivial, taking the stopping time $T = 0$. For the reverse implication, suppose there exists a stopping time T with $P(T < 1) > 0$, and a continuation $y \in Y_j(T, y_j)$ such that $S_{y\psi}(T) > S_{y_j\psi}(T)$ almost surely, or equivalently, that

$$E\left[\int_T^1 \psi(s)^\top y(s)ds\,|\mathcal{F}_T\right] > E\left[\int_T^1 \psi(s)^\top y_j(s)ds|\mathcal{F}_T\right] \quad a.s.$$

Since y is a continuation of y_j at T, we have

$$\int_0^T \psi(t)^\top y(t)dt = \int_0^T \psi(t)^\top y_j(t)dt \quad a.s.$$

Taking the expectation of the sum of the two previous expressions inplies that

$$E\left[\int_0^1 \psi(t)^\top [y(t) - y_j(t)]dt\right] > 0,$$

a contradiction. ∎

Stochastic Equilibrium

A *stochastic production–exchange economy*, in summary, is a collection:

$$\mathcal{E} = \big((\succeq_i, e_i, \epsilon_i); (D^{J+1}, \ldots, D^K); (Y_j); i \in \mathcal{I}, j \in \mathcal{J}\big). \tag{2.8}$$

A collection $(\Pi, \psi, (y_j), (c_i, \theta_i), j \in \mathcal{J},\ i \in \mathcal{I})$, is an *equilibrium* for $\mathcal{E}$ if Π is a gain operator and ψ is a spot price process such that:

(*i*) given (ψ, Π), for all j in $\mathcal{J}$ the production choice $y_j \in Y_j$ is share value maximizing,

(*ii*) given $(\Pi, (D^1, \ldots, D^K), \psi)$, where $D^j = D_{y_j\psi}$, $1 \le j \le J$, for all i in $\mathcal{I}$ the plan (c_i, θ_i) is optimal for agent i, and

(*iii*) markets clear: $\sum_i c_i - e_i = \sum_j y_j$; $\sum_i \theta_i^j = 1$, $1 \le j \le J$; and $\sum_i \theta_i^j = 0$, $J+1 \le j \le K$.

3. Existence of Stochastic Equilibria

Our procedure will be: first demonstrate a static complete markets equilibrium; then demonstrate a stochastic equilibrium by construction, "implementing" the static equilibrium allocations by dynamic trading on security and spot markets.

Regularity Conditions

Recent advances by Zame [31], Mas–Colell [19], and Richard [31] in the theory of static (Arrow–Debreu) production economies provide weak conditions on production technologies that are particularly well–suited to the choice space at hand. Not yet having had the

opportunity to incorporate Richard's very recent extension of Mas–Colell's work, we will employ Zame's production conditions. The genesis for this recent spate of work is an early pure–exchange version of Mas–Colell's paper.

For topological conditions, we will sometimes use the norm $\|\cdot\|_1$ on L^2_ℓ defined (using Cauchy–Schwarz) by

$$\| c \|_1 = E\left[\int_0^1 \sum_{l=1}^{\ell} |c_l(t)|\, dt\right].$$

[This is the product $L^1(\nu)$–norm.] Continuity in this norm is more restrictive than continuity in the usual norm $\|\cdot\|_2$ defined by

$$\| c \|_2 = \left(E\left[\int_0^1 c(t)^\top c(t) dt\right]\right)^{1/2}.$$

We desire an equilibrium spot price process to be bounded for technical convenience, however, and so work with the finer $\|\cdot\|_1$ topology.

The appendix reviews Zame's *strongly bounded marginal efficiency* condition on production sets. We examine specific technologies that exhibit strongly bounded marginal efficiency in a later section. An appropriate condition on preferences over infinite–dimensional spaces was developed by Mas–Colell [19] and extended by Yannelis and Zame [30] to preference relations that need not be complete or transitive. A consumption choice $v \in (L^2_\ell)_+$ is *extremely desirable* for a preference relation $\succeq$ on $(L^2_\ell)_+$ if there is a scalar $\delta > 0$ with the following property. For any $c \in (L^2_\ell)_+$, if $z \in L^2_\ell$ and $\alpha \in (0,1)$ satisfy $z \leq c + \alpha v$ and $\| z \|_1 < \alpha\delta$, then $c + \alpha v - z \succ c$. Interpreting, the choice v is such a good direction to move in that one can compensate for the loss of z by gaining αv, provided $\| z \|_1$ is small enough. For complete transitive preferences, this is identical to Mas–Colell's *uniform properness*[8] condition. For a restrictive example, if $\succeq$ is represented by the time–additive form $E[\int_0^1 u(c_t)dt]$ where u is concave and monotonic with a finite right–derivative at zero, then $v(\omega, t) \equiv (1, 1, \ldots, 1) \in \Re^\ell$ is extremely desirable for $\succeq$.

[8] If the preference relation $\succeq$ is represented by a concave continuous monotonic utility function that can be extended to a $\|\cdot\|_1$–neighborhood of $(L^2_\ell)_+$ while preserving these properties, then $\succeq$ has extremely desirable choices, or in Mas–Colell's sense, is "uniformly proper". See Richard [26] and Richard and Zame [28]. For our purposes, we will sometimes place additional conditions on extremely desirable choices, so these results need not always apply.

The *total production set* for our economy is $Y = \sum_{j=1}^{J} Y_j$; any element of Y is a feasible addition to endowments in forming aggregate consumption. The *aggregate endowment* is $e = \sum_{i=1}^{I} e_i$.

Production Conditions. The production sets $Y_1, \ldots, Y_J$ are $\|\cdot\|_2$–closed, convex, include zero, and demonstrate strongly bounded marginal production efficiency. The set of feasible production choices $Y \cap ((L^2_\ell)_+ - \{e\})$ is $\|\cdot\|_2$–bounded.

Agent Conditions. The preference relations $\succeq_i$, $i \in \mathcal{I}$, are strictly increasing[9], convex,[10] $\|\cdot\|_1$–continuous,[11] and have an extremely desirable choice $v_i \in (L^2_\ell)_+$ such that $v_i \leq e$. The condition on extremely desirable choices is implied, for example, by strictly increasing uniformly proper preferences and an assumption that $\sum_{i=1}^{I} e_i$ is strictly positive almost everywhere. With production of intermediate goods, for example, we would prefer not to make the strictly positive aggregate endowment assumption. Aside from bounded marginal production efficiency, the other regularity conditions are fairly standard.

Static Equilibria

Underlying our stochastic economy is the static (Arrow–Debreu) economy

$$\mathcal{E}_S = \left(((L^2_\ell)_+, \succeq_i, e_i); (\epsilon_i^j); (Y_j);\ i \in \mathcal{I}, j \in \mathcal{J} \right).$$

A (static) equilibrium for $\mathcal{E}_S$ is a collection $(c_1, \ldots, c_I, y_1, \ldots, y_J, \phi)$ where $(c_1, \ldots, c_I)$ is a consumption allocation, $(y_1, \ldots, y_J)$ is a production allocation, and ϕ is a linear price functional on L^2_ℓ, such that

$$\sum_{i=1}^{I} c_i - e_i = \sum_{j=1}^{J} y_j \tag{3.1}$$

$$\phi(c_i) \leq \phi(e_i + \sum_{j=1}^{J} \epsilon_i^j y_j), \quad i \in \mathcal{I}, \tag{3.2}$$

$$z \succ_i c_i \Longrightarrow \phi(z) > \phi(c_i) \quad \forall z \in (L^2_\ell)_+, \quad i \in \mathcal{I}, \tag{3.3}$$

$$\phi(z) \leq \phi(y_j) \quad \forall z \in Y_j, \quad j \in \mathcal{J}. \tag{3.4}$$

[9] A preference relation $\succeq$ is *strictly increasing* if $c + y \succ_i c$ for all c and $y \in (L^2_\ell)_+$ with $y \neq 0$.

[10] A preference relation $\succeq$ on a subset X of a vector space is *convex* provided $\alpha x + (1 - \alpha) z \succeq w$ whenever $x \succeq w$ and $z \succeq w$, for any x, z, and w in X and any α in [0,1].

[11] A preference relation $\succeq$ on $(L^2_\ell)_+$ is $\|\cdot\|_1$–continuous provided the graph of $\succ$ is a relatively $\|\cdot\|_1$–open set. This can be weakened; see for example, Zame [31].

These are the usual conditions: feasibility (3.1), budget feasibility (3.2), optimality (3.3), and maximization of production value (3.4).

THEOREM 3.1 (EXISTENCE OF STATIC EQUILIBRIA). *Provided the Production Conditions and Agent Conditions are satisfied, the economy $\mathcal{E}_S$ has a static equilibrium. Moreover, an equilibrium price functional ϕ can be represented by a strictly positive*[12] *bounded spot price process ψ as*

$$\phi(x) = E\left[\int_0^1 \psi(t)^\top x(t)dt\right], \qquad x \in L^2_\ell. \tag{3.5}$$

The proof may be found in Appendix B.

Dynamic Spanning

In order to implement a static equilibrium as a stochastic equilibrium, we will make use of the theory of martingale generators. As the details have been extensively developed in the exchange case, we will be brief. A vector $m = (m^1, \ldots, m^H)$ of Itô integrals

$$m(t) = m(0) + \int_0^t \widehat{\sigma}(s)dB(s),$$

where $\widehat{\sigma}$ is an $H \times N$–matrix valued predictable process, is a *martingale generator* provided any square–integrable martingale[13] M has a representation of the form

$$M(t) = M(0) + \int_0^t \theta(s)dm(s) \quad \forall t \in [0,1], \, a.s., \tag{3.6}$$

where θ is an $\Re^H$–valued predictable processes satisfying $E\left[\int_0^1 |\theta(t)\widehat{\sigma}(t)|^2\, dt\right] < \infty$. For example, the underlying Brownian Motion $B = (B^1, \ldots, B^N)^\top$ is itself a martingale generator (Kunita and Watanabe [17]). More generally, we have:

LEMMA 3.1. *Suppose m is vector martingale of the form $m(t) = m(0) + \int_0^t \widehat{\sigma}(s)dB(s)$, $\forall t \in [0,1]$ almost surely, where $\widehat{\sigma}$ is an $H \times N$ matrix–valued predictable process with*

[12] An element ψ of L^2_ℓ is strictly positive if $\nu\{(\omega,t) \in \Omega \times [0,1] : \psi(\omega,t) \gg 0\} = 1$.

[13] A *martingale* is an integrable adapted process M satisfying $E[M(t) \mid \mathcal{F}_s] = M(s)$ whenever $0 \le s \le t \le 1$. On the Brownian filtration we can assume without loss of generality that all martingales have continuous sample paths. A martingale M is *square–integrable* if $E[M(1)^2] < \infty$.

rank $(\widehat{\sigma}(t)) = N$ $\nu - a.e.$, *and where* $\int_0^1 tr(\widehat{\sigma}(t)\widehat{\sigma}(t)^\top)dt < \infty$ *a.s. Then* m *is a martingale generator.*

This result is contained in a more general form in Chapter 4 of Jacod [16]; for concreteness a proof is given in Appendix A.

A security D is a *riskless bond* if $D(t) = 0$, $t \in [0,1)$ and $D(1) = 1$. A vector $D = (D^1, \ldots, D^K)^\top$ of securities is *fundamental* if D^K is a riskless bond and if there exists a martingale generator $m = (m^1, \ldots, m^{K-1})$ such that, for all t in $[0,1]$,

$$E\left[D^k(1) \mid \mathcal{F}_t\right] = m^k(t), \quad 1 \leq k \leq K-1.$$

Given a vector D of securities and a gain operator Π, markets are *dynamically complete* if, given any spot price process ψ, every $c \in L^2_\ell$ is financed by some trading strategy $\theta \in \Theta[G]$.

PROPOSITION 3.1. *If D is a fundamental vector of securities and $\overline{\Pi}$ is the conditional expectation gain operator, then markets are dynamically complete.*

A proof and partial converse may be found in Duffie [1985].

Stochastic Equilibria

Under the conditional expectation gain operator $\overline{\Pi}$, the initial investment required to finance a consumption plan is merely the total expected spot market cost of the plan.

PROPOSITION 3.2. *Given the conditional expectation gain operator $\overline{\Pi}$, any vector of securities D, and any spot price process ψ, if θ is a trading strategy financing a consumption process c, then*

$$\theta(0)S(0) = E\left[\int_0^1 \psi(t)^\top c(t)\, dt\right].$$

PROOF: The proof is immediate from (2.6)–(2.7) and from the fact that the gain $G = \overline{\Pi}(D)$ is a martingale, implying that $\int \theta dG$ is a martingale for any trading strategy $\theta \in \Theta[G]$. ∎

Harrison and Kreps [12] indicate that, barring arbitrage, any gain operator is a conditional expectation operator under some numeraire and probability measure. [For extensions, see Huang [15].] We have the luxury here of picking the probability measure P and numeraire in advance. Of course, if prices are denominated with respect to one of the ℓ commodities, then this form of nominally risk–neutral pricing is only possible in equilibrium with a risk–neutral

agent. With any equilibrium satisfying regularity conditions, however, one can normalize prices relative to one of the securities and construct a new probability measure under which nominally risk–neutral pricing applies. This does not require a fundamental "spanning" set of securities such as constructed in the following theorem, which states sufficient rather than necessary conditions for a stochastic equlibrium.

THEOREM 3.2 (EXISTENCE OF STOCHASTIC EQUILIBRIA). *Suppose that*

$$\mathcal{E}_S = \Big(((L^2_\ell)_+, \succeq_i, e_i); (\epsilon^j_i); (Y_j);\ i \in \mathcal{I}, j \in \mathcal{J}\Big)$$

satisfies the Production Conditions and the Agent Conditions. Then there exist financial assets $(D^{J+1}, \ldots, D^K)$ *such that the stochastic production–exchange economy*

$$\big((\succeq_i, e_i, \epsilon_i); (D^{J+1}, \ldots, D^K); (Y_j);\ i \in \mathcal{I}, j \in \mathcal{J}\big),$$

has a stochastic equilibrium. Moreover, the equilibrium allocation is Pareto optimal if $\succeq_i$ *is complete and transitive for all* $i \in \mathcal{I}$.

PROOF: By Theorem 3.1, there exists a static equilibrium $(c_1, \ldots, c_I, y_1, \ldots, y_J, \phi)$ for $\mathcal{E}_S$, where the price functional ϕ can be represented as in (3.5) by a strictly positive bounded spot price process ψ. We take this spot price process ψ and the Arrow–Debreu production plans (y_j), generating the common shares $D^j = D_{y_j\psi}$, $1 \le j \le J$. We take the conditional expectation gain operator $\overline{\Pi}$, defining a gain process G^j for the common share of each firm j. Since ψ is bounded, G^j is a square–integrable martingale, and is thus represented by an $\Re^N$–valued predictable process $\widehat{\sigma}_j$ with $E\left[\int_0^1 \widehat{\sigma}_j(t)\widehat{\sigma}_j(t)^\top dt\right] < \infty$ by

$$G^j(t) = \int_0^t \widehat{\sigma}_j(s)dB(s), \quad t \in [0,1], \ a.s.$$

Let η_1 denote the $J \times N$ matrix–valued process with j–th row $\widehat{\sigma}_j$, $j = 1, 2, \ldots, J$. Let $\widehat{J}$ satisfy $\text{rank}(\eta_1(t)) \ge \widehat{J} \quad \nu - a.e.$ and $\nu\{\text{rank}(\eta_1(t)) = \widehat{J}\} > 0$. Let $K = N - \widehat{J} + J + 1$. We define financial securities $D^{J+1}, \ldots, D^K$ as follows. Let D^K be a riskless bond. For $J + 1 \le j \le K - 1$, let

$$D^j(t) = \int_0^t \widehat{\sigma}_j(s)dB(s), \quad t \in [0,1],$$

where $\{\widehat{\sigma}_{J+1}, \ldots, \widehat{\sigma}_{K-1}\}$ are $\Re^N$–valued predictable processes such that the $N \times (K-1)$ matrix process $\widehat{\sigma}^\top = (\widehat{\sigma}_1^\top, \ldots, \widehat{\sigma}_{K-1}^\top)$ has rank N ν-a.e.[14] and satisfies $E\left[\int_0^1 tr(\widehat{\sigma}(t)\widehat{\sigma}(t)^\top)dt\right]$

[14] A measurable selection argument of the Aumann variety may be used here. See, for example, Hildenbrand [13], p. 54.

$< \infty$. By Lemma 3.1, D is a fundamental vector of securities. By Proposition 3.2, there exists for each agent $i \in \mathcal{I}$ a trading strategy $\theta_i \in \Theta[G]$ financing the net trade $c_i - e_i$. We take a trading strategy $\theta_i \in \Theta[G]$ with this property for each agent $i \in \{1, \ldots, I-1\}$. Let $\theta_I = \sum_{i=1}^{I} \epsilon_i - \sum_{i=1}^{I-1} \theta_i$. Since $\Theta[G]$ is a linear space, $\theta_I \in \Theta[G]$. It is then easily verified, using market clearing in the static economy $\mathcal{E}_S$, that (c_I, θ_I) is a budget feasible plan for I. For all i in $\mathcal{I}$, (c_i, θ_i) is an optimal plan given $(\overline{\Pi}, D^1, \ldots, D^K, \psi)$ by arguments given in Duffie [1985]. By Proposition 2.1, y_j is share value maximizing for each firm j in $\mathcal{J}$, given $(\overline{\Pi}, \psi)$. By construction we have market clearing. Pareto optimality follows under complete transitive preferences by a standard argument. ∎

This proof gives us stochastic equilibrium with the minimum number of financial securities required for dynamically complete markets. These financial securities are endogenous, depending on the particular static Arrow–Debreu equilibrium chosen for implementation. Of course, one has the existence of stochastic equilibrium for a large class of exogenously specified financial securities. For example, we could fix $(\overline{D}^{J+1}, \ldots, \overline{D}^{K-1}) = B$ and choose $\overline{D}^K$ to be a riskless bond.

THEOREM 3.3. *If $\mathcal{E} = ((\succeq_i, e_i, \epsilon_i); (\overline{D}^{J+1}, \ldots, \overline{D}^K); (Y_j); i \in \mathcal{I}, j \in \mathcal{J})$ satisfies the Agent conditions and the Production Conditions, then $\mathcal{E}$ has stochastic equilibria.*

The proof is a simplification of the proof of Theorem 3.2, using the fact that B is itself a martingale generator. In this case, since financial markets are dynamically spanning in their own right, there is no particular hedging or spanning role for capital markets.

4. The Production and Financial Policies of the Firm

This section reconfirms in a continuous–time setting two well known dictums on the policy of the firm. First, positive shareholders unanimously support a production plan that maximizes the value of the firm, provided markets are complete, or in this setting, provided markets are dynamically complete. Second, the value of the firm is independent of any issuance of debt by the firm, the *Modigliani–Miller [21] Proposition I.* For the former implication, suppose y and z are candidate production plans for firm j with respective share price processes $S_{y\psi}$ and $S_{z\psi}$, with $S_{y\psi}(0) > S_{z\psi}(0)$. Then, given dynamically complete markets and the conditional expectation gain operator $\overline{\Pi}$, if (c, θ) is budget feasible given z (fixing all other constraining data), if $\epsilon_i^j > 0$, and if $\succeq_i$ is locally non–satiated, then there exists a

budget feasible plan (c', θ') given y (holding other data fixed) such that $c' \succ_i c$. Thus agent i strictly prefers the production plan y with the larger market value. This hardly warrants further formalization or attention.

For the Modigliani–Miller result in a continuous–time setting, we could treat the financial policy of the firm merely by allowing the firm to trade securities in its own right. We preclude a firm trading its own shares, and the simultaneous trading of securities by other firms, in order to ensure that security price processes are well defined. (For the general case, see Duffie and Shafer [11].) Given a vector gain process G, any trading strategy $\theta \in \Theta[G]$ for firm j, with $\theta(0) = 0$, adds the dividend process $\int \theta dG$ to the original dividend process D^j of firm j. Under the conditional expectation gain operator $\overline{\Pi}$, the new share price of the firm at time zero is $E[D^j(1) + \int_0^1 \theta dG]$, but since G is a martingale, this is merely the original share price $E[D^j(1)]$. Barring arbitrage, the gain operator is always conditional expectation under a probability measure with respect to which the gain process G is a martingale [12,15]. Thus the financial policy of the firm has no effect on its initial share price. This has nothing to do with dynamically complete markets, requiring only that the gain operator be taken as given by firms. Furthermore, by the reasoning of the previous paragraph, if markets are dynamically complete, shareholders are indifferent to the financial policy of the firm.

More traditionally, we can study the *Modigliani–Miller Invariance Principle* by modeling the issuance of a *defaultable debt* security $\widehat{D}^j$ by a firm j whose production plan generates the dividend process D^j. Under a given gain operator Π, let $\widehat{S}^j = \Pi(\widehat{D}^j) - \widehat{D}^j$ denote the price process for $\widehat{D}^j$. By "defaultable", we mean that debtholders recognize that, if and when the market value of $\widehat{D}^j$ exceeds the market value of D^j, debtholders forego the claim to $\widehat{D}^j$ and receive instead the total dividends D^j generated by sale of the firm's output. In other words, the firm is placed in receivership. More formally, consider the stopping time

$$T = inf\,\{t \in [0,1] : \widehat{S}^j(t) + \Delta\widehat{D}^j(t) \geq S^j(t) + \Delta D^j(t)\}.$$

[For generality, we allow the dividends to have jumps.] Then the *effective debt security* of

firm j is the security $\tilde{D}^j$ defined by

$$\begin{aligned}\tilde{D}^j(t) &= \widehat{D}^j(t), \quad t < T, \\ &= \widehat{D}^j(T_-) + D^j(t) - D^j(T_-), \quad t \geq T.\end{aligned}$$

The *equity* of firm j is the security $C^j = D^j - \tilde{D}^j$ paying the residual dividends. Since Π is linear, the *total value* of the firm, that is, the value of the claim to $C^j + \tilde{D}^j$ is the same as the value of D^j. In traditional language, *the total value of a firm is independent of the firm's debt–equity mix.* In dynamically complete markets, shareholders are indifferent to the issuance of debt, as reasoned above. Of course, the span of incomplete markets can generally be changed by the issuance of debt, and shareholders may not be indifferent to debt policy in incomplete markets.

5. Production Technologies

We now illustrate production technologies satisfying our production conditions. We first examine a class of stochastic growth models of capital accumulation, including the model used by Cox, Ingersoll, and Ross [7,8]. A second class of technologies to be studied includes linear and non–linear stochastic input–output models.

Stochastic Growth Models of Capital Accumulation

First we outline a simple model of capital accumulation in a stochastic economy. Later we guarantee the existence of equilibria embedding such a technology. For simplicity, the consumption space here is L^2_1; there is a single commodity. An initial *capital stock* is given by a scalar $\kappa \geq 0$. The *growth rate of capital* is given exogenously by a real–valued Itô process X of the form

$$X_t = \int_0^t m(s)ds + \int_0^t v(s)dB_s, \quad t \in [0,1],$$

where m is a real–valued predictable process and v is an $\Re^N$–valued predictable process. A *capital stock process* is a positive Itô process K solving a stochastic integral equation of the form

$$K_t = \kappa + \int_0^t K_s dX_s - \int_0^t c_s ds, \quad t \in [0,1], \tag{5.1}$$

where $c = \{c_t : t \in [0,1]\}$ is a positive real–valued predictable process for consumption out of capital stock, or *depletion*. Suppose, for example, that X is the solution to the stochastic differential equation

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t, \quad t \in [0,1], \tag{5.2}$$

for a drift function $\mu : \Re \times [0,1] \to \Re$ and a diffusion coefficient $\sigma : \Re \times [0,1] \to \Re^N$ that are measurable and satisfy a Lipschitz and a growth condition. Then (5.1) is of the familiar stochastic differential form:

$$dK_t = [K_t\mu(X_t, t) - c_t]dt + K_t\sigma(X_t, t)dB_t, \tag{5.3}$$

which may be recognized as the Cox–Ingersoll–Ross growth model. Under weak regularity conditions on X and c, an application of Ito's Lemma guarantees the existence of a unique solution K to (5.1) as

$$K_t = e^{\widehat{X}_t}\kappa - e^{\widehat{X}_t}\int_0^t e^{-\widehat{X}_s}c_s ds, \quad t \in [0,1], \tag{5.4}$$

where $\widehat{X}$ is the Itô process given by

$$\widehat{X}_t = X_t - \frac{1}{2}\int_0^t v(s)^\top v(s)ds, \quad t \in [0,1].$$

A *feasible depletion rate* is any positive predictable process c with the property that the right–hand–side of (5.4) is well defined and positive. If c is positive and K is well–defined by (5.4), however, then K is positive if and only if

$$\int_0^1 e^{-\widehat{X}_t}c_t dt \leq \kappa \quad a.s. \tag{5.5}$$

Regularity conditions on the rate of return process X and the depletion rate c are required to suit our needs; the following conditions are unnecessarily restrictive.

PROPSITION 5.1. *Suppose the process $e^{-\widehat{X}_t}$ is square–integrable and a depletion process c must satisfy $\| c \|_2 \leq \beta$ for some scalar β. Then (5.4) is the unique solution to (5.1) for the capital stock process K, and the resulting set of feasible depletions*

$$Y^d = \{c \in (L^2_\ell)_+ : \| c \|_2 \leq \beta; \int_0^1 e^{-\widehat{X}_t}c_t dt \leq \kappa \quad a.s.\}$$

satisfies the Production Conditions.

PROOF: As claimed, (5.4) solves (5.1) by an application of Ito's Lemma. We must show that Y^d is $\|\cdot\|_2$–closed, convex, includes zero, demonstrates strongly $\|\cdot\|_1$–bounded marginal production efficiency, and that $Y^d \cap ((L^2_\ell)_+ - \{e\})$ is $\|\cdot\|_2$–bounded for any $e \in (L^2_\ell)_+$. Except for the closedness of Y^d, the proof is by simple inspection. For closedness, we note that the operator $f : L^2_1 \to L^1(\Omega, \mathcal{F}, P)$ given by $f(c) = \int_0^T e^{-\widehat{X}_t} c_t dt$ is well–defined and continuous by the Cauchy–Schwarz inequality. Thus, if $\{c_n\}$ is a sequence in Y^d converging in L, then $f(\lim c_n) = \lim f(c_n) \leq \kappa\, a.s.$ Of course, $\| \lim\{c_n\} \|_2 \leq \beta$ and $\lim\{c_n\} \in (L^2_\ell)_+$, since $\|\cdot\|_2$ is $\|\cdot\|_2$–continuous and $(L^2_\ell)_+$ is $\|\cdot\|_2$–closed. Thus Y^d is $\|\cdot\|_2$–closed. ∎

If there is a maximum depletion rate, then of course the set of feasible depletions is bounded in norm. It will be noted that the capital stock depletion technology is not assumed to be reversible. That is, we require any feasible depletion process to be positive; one cannot place endowments into the capital stock. We make this assumption mainly in order to guarantee strongly bounded marginal production efficiency, for if $Y^d \subset (L^2_\ell)_+$ then the Appendix B conditions (b.1)–(b.4) are trivially satisfied by $b = 0$ and $\widehat{y} = y$. Of course, if there are no endowments, or $e = 0$, then the positive depletion assumption is without loss of generality, but the assumed existence of an extremely desirable choice less than e is then impossible.[15]

As a corollary to the last proposition and Theorem 3.3, we have the following conditions for equilibrium in a stochastic growth economy.

PROPOSITION 5.2. *If* $\mathcal{E} = \big((\succeq_i, e_i, \epsilon_i); (D^{J+1}, \ldots, D^K); Y^d);\ i \in \mathcal{I}, j \in \mathcal{J}\big)$ *is a stochastic production exchange economy such that* $(D^{J+1}, \ldots, D^K)$ *is a fundamental vector of securities and* $(\succeq_i, e_i), i \in \mathcal{I}$, *satisfies the Agent Conditions, then* $\mathcal{E}$ *has stochastic equilibria.*

It is trivial to generalize to many spot goods and corresponding capital accumulation–depletion technologies of a similar nature. Since each corresponding production set retains the properties: closed, convex, bounded, positive, and including zero, the total production set inherits these properties and the Production Conditions apply.

In the above formulation, the current rate of growth of the capital stock depends entirely on current consumption and productivity. For a model with productivity lags,

[15] If one wishes to bar endowments, the choice space L^2_ℓ could be replaced by $\Re^\ell \times L^2_\ell$, where $\Re^\ell$ indicates the space of initial capital stocks. Individuals rather than firms could be allocated the initial capital stock κ as part of their endowment. The absence of endowments in L^2_ℓ is then not a problem.

goods in process, and so on, we can generalize to the case of a capital stock process K solving a stochastic Volterra equation of the form

$$K_t = \int_0^t f(t, s, K_{(\cdot)}) dX_s - \int_0^t c_s ds, \quad t \in [0, 1],$$

where $K_{(\cdot)} : \Omega \to C([0,1])$ denotes the sample path function for K and $f : [0,1] \times [0,1] \times \Omega \times C([0,1]) \to \Re$ is sufficiently well behaved. For sufficent conditions and many generalizations, see Protter [22] and Rao and Tsokos [24].

Extending to an infinite horizon model is also quite simple. This requires no change in the current conditions for a static equilibrium. That is, Theorem 3.1 applies as stated to the space of $\Re^\ell$–valued predictable processes on $\Omega \times [0, \infty)$ under the natural norm defined by

$$\| c \|_\alpha = \left[E \int_0^\infty e^{-\alpha t} c_t^\top c_t \, dt \right]^{1/2},$$

where α is a strictly positive scalar "discount." This allows us to include, for example, a non–zero constant consumption process. In order to extend a static equilibrium to a stochastic equilibrium, we would replace condition (2.7) by:

$$\theta(t)^\top [S(t) + \Delta D(t)] \geq E \left[\int_t^\infty e^{-\alpha t} \psi_t^\top c_t dt | \mathcal{F}_t \right] \quad t \in [0, \infty).$$

The proof of a stochastic equilibrium then goes through with only notational changes.

Stochastic Input–Output Technologies

Continuing to outline examples of production technologies that fit within our framework for demonstrating equilibria, we turn to a neoclassical model of a function mapping production inputs to production outputs. For simplicity, we take two spot goods to be called "labor" and "corn" and treat labor as a production input and corn as a production output. This can easily be generalized, although one may run into difficulties in guaranteeing bounded marginal efficiency of production if the same good can be used as both input and output.

A natural example of our "labor to corn" technology is given by a *stochastic Volterra kernel:*

$$K : \mathcal{T} \times \Omega \to R,$$

where $\mathcal{T}$ denotes the set of ordered pairs $(t,s) \in [0,1] \times [0,1]$ with $t \geq s$. We may think of $K(t,s,\omega)$ as the coefficient of productivity of inputs at time s for outputs at time t in state $\omega \in \Omega$. For simplicity, we assume that K is bounded, jointly measurable, and that $K(\cdot,s,\cdot)$ is predictable for each s. We can then define the output function $f : L_1^2 \to L_1^2$ by $f(y) = z$ where

$$z(\omega,t) = \int_0^t K(t,s,\omega)y(\omega,s)ds, \qquad (\omega,t) \in \Omega \times [0,1].$$

If K is a positive kernel then f is a positive[16] linear operator, and we can apply Corollary 5.1 (below) to guarantee that the resulting production technology satisfies the Production Conditions.

For a somewhat more general model, we could consider the case of a production input–output function, mapping an input process $y \in (L_1^2)_+$ to an output process z defined by

$$z(t) = \int_0^t g(t,s,y(s))ds + \int_0^t h(t,s,y(s))dB(s),$$

where $g : \mathcal{T} \times \Omega \times \Re_+ \to \Re$ and $h : \mathcal{T} \times \Omega \times \Re_+ \to \Re^N$ are measurable, satisfy a Lipschitz and a growth condition in y, and are predictable in t and s respectively. In this case, f maps into L_1^2. Of course, if g and h are linear in $y(s)$, then the input–output operator is linear. Further generaliztions are possible.

For the general case of a continuous linear operator $H : L_\ell^2 \to L_\ell^2$ mapping production inputs to production outputs, we can simplify the production choice of the firm to a myopic decision problem as follows. We have not provided proofs of the claims in this paragraph, so they must be treated as conjectures for the purposes of this draft of the paper. [There is nothing controversial here, in any case.] Let $(\cdot|\cdot)$ denote the inner product on L_ℓ^2 defined by

$$(x|\psi) = E\left[\int_0^1 x(t)^\top \psi(t)dt\right],$$

under which L_ℓ^2 is a Hilbert space. By Proposition 2.1 and the nature of the equilibria demonstrated by Theorem 3.2, the firm faces the maximization problem:

$$Max_{v \in V}(Hv - v|\psi), \tag{5.6}$$

[16] A function $g : L_\ell^2 \to L_\ell^2$ is *positive* if $g(x) \in (L_\ell^2)_+$ for all $x \in (L_\ell^2)_+$. If g is linear, then g is *bounded* or equivalently *continuous* if $sup\{\| g(x) \|; x \in L_\ell^2; \| x \| \leq 1\}$ is finite, where $\| \cdot \|$ denotes the relevant norm.

where $V \subset L^2_\ell$ denotes the *feasible input set*, generalizing from $V = (L^2_\ell)_+$. Let H^* denote the *adjoint* [25] of H, defined by $(Hx|\psi) = (x|H^*\psi)$ for all x and ψ in L^2_ℓ. It follows that problem (5.6) is equivalent to

$$Max_{v \in V}\,(v|\psi^* - \psi), \tag{5.7}$$

where $\psi^* = H^*\psi$ is the *adjoint price process.* That is, $\psi^*(t)$ is the vector of *shadow spot prices* at time t, indicating the effect on the share price of the firm at time t of production inputs at that time, exclusive of actual spot market costs of inputs charged at the rate $\psi(t)$. If the feasible input set is of the form

$$V = \{v \in L^2_\ell : v(\omega, t) \in Q(\omega, t)\ \nu - a.e.\},$$

where $Q : \Omega \times [0,1] \to 2^{\Re^\ell}$ is a (Hausdorff) predictable correspondence, then $y = Hv - v \in L^2_\ell$ is share–value–maximizing if and only if

$$v(\omega, t) \in \arg\, Max_{x \in \Re^\ell} \quad x^\top[\psi^*(\omega, t) - \psi(\omega, t)] \quad \nu - a.e. \tag{5.8}$$

In other words, the firm may solve for the optimal stochastic investment process merely by solving a simple static finite–dimensional profit maximization problem at each state and date based on local price information. Of course, this is only useful given a solution to the adjoint price process. If H is given by a stochastic kernel $K : \mathcal{T} \times \Omega \to \Re^{\ell \times \ell}$ as above, then the adjoint spot price process ψ^* is given by

$$\psi^*(t) = E\left[\int_t^1 K(s,t)^\top \psi(s) ds | \mathcal{F}_t\right], \quad t \in [0,1]. \tag{5.9}$$

We turn to conditions on a general (possibly nonlinear) production output function $f : (L^2_1)_+ \to (L^2_1)_+$ guaranteeing that the production set

$$Y^f = \{(-c, b) \in (L^2_1)_- \times (L^2_1)_+ : b \leq f(c)\} \subset L^2_2$$

satisfies the Production Conditions, where $(L^2_1)_- \equiv -(L^2_1)_+$ is the negative cone of (L^2_1). Here, c is a production input (say "labor"); b is a production output of a different commodity (say "corn"). We note that Y^f admits free disposal of the output; this ensures the convexity of Y^f in the following result.

PROPOSITION 5.3. *Suppose f is positive, monotonic, concave, $\|\cdot\|_1$–Frechet differentiable at zero, and continuous. Then Y^f satisfies the Production Conditions.*

A proof may be found in Appendix B.

COROLLARY 5.1. *Suppose f is a positive $\|\cdot\|_1$–continuous linear operator. Then Y^f satisfies the Production Conditions.*

PROOF: By assumption, f is positive. By positivity and linearity, f is monotonic. By linearity, f is concave. Any continuous linear operator is Frechet differentiable. ∎

Appendix A. Ito Processes

We define an *Itô process* to be a stochastic process X of the form:

$$X(t) = X(0) + V(t) + \int_0^t \sigma(s)dB(s), \quad t \in [0,1], \tag{a.1}$$

where $\sigma = (\sigma^1, \ldots, \sigma^N)$ is an R^N–valued predictable process satisfying $\int_0^1 \sigma(s)\sigma(s)^\top ds < \infty$ almost surely, and V is a real–valued adapted process having continuous bounded variation sample paths. The restriction on σ guarantees the existence of $\int \sigma dB$ as an Ito integral; see Liptser and Shiryayev [18]. In short, an Itô process is the sum of an Itô integral and an adapted process having continuous bounded variation sample paths.

Proof of Lemma 3.1

Let M be any square–integrable martingale. Since B is a martingale generator, M has a representation

$$M(t) = M(0) + \int_0^t \eta(s)^\top dB(s), \quad t \in [0,1],$$

for some process $\eta = (\eta^1, \ldots, \eta^N)^\top$ satisfying $E\left[\int_0^1 \eta(t)^\top \eta(t)dt\right] < \infty$ Since $\widehat{\sigma}$ is of rank N ν-a.e., there exists an $N \times H$ matrix valued process κ such that $\kappa(t)\widehat{\sigma}(t)$ is an $N \times N$ identity matrix.[17] Let $\varphi = \{\varphi(t) = \kappa(t)^\top \eta(t) : t \in [0,1]\}$. We have

$$\begin{aligned} M(0) + \int_0^t \varphi(s)^\top dm(s) &= M(0) + \int_0^t \eta(s)^\top \kappa(s)\widehat{\sigma}(s)dB(s) \\ &= M(0) + \int_0^t \eta(s)^\top dB(s) = M(t) \quad \forall t \in [0,1]\, a.s. \end{aligned}$$

[17] When $H = N$, we simply take $\kappa(t)$ to be the inverse of $\widehat{\sigma}$. Since the inverse is a measurable function, κ is predictable. If $H > N$, a standard measurable selection argument provides for a predictable pseudo–inverse process.

In addition,

$$E\left[\int_0^1 \varphi(t)^\top \widehat{\sigma}(t)\widehat{\sigma}(t)^\top \varphi(t)dt\right] = E\left[\int_0^1 \eta(t)^\top \eta(t)dt\right] < \infty.$$

Hence the assertion follows.

Appendix B – Strongly Bounded Marginal Production Efficiency

Zame [31] has recently described regularity conditions on production sets that can be exploited to demonstrate the existence of Arrow–Debreu equilibria in infinte–dimensional choice spaces. Here we specialize those conditions to our setting and provide proofs of examples presented in the body of the paper.

First we define Zame's condition. Any production process $y \in L^2_\ell$ can be split into its *positive part* $y^+ = \max(y, 0) \in (L^2_\ell)_+$ and its *negative part* $y^- = \max(-y, 0) \in (L^2_\ell)_+$ so that $y = y^+ - y^-$. A *production allocation* is a collection $(y_1, \ldots, y_J)$ of production processes with $y_j \in Y_j$, $j \in \mathcal{J}$. A production allocation $(y_1, \ldots, y_J)$ is *positively dominated* if there exists another production allocation $(\tilde{y}_1, \ldots, \tilde{y}_J)$ such that $\sum_{j=1}^J \tilde{y}_j - y_j$ is positive and not zero. The production sets $Y_1, \ldots, Y_J$ demonstrate *strongly bounded marginal production efficiency* if there exists a scalar *marginal efficiency bound* $\gamma > 0$ with the following property:

For any production allocation $(y_1, \ldots, y_J)$ not positively dominated totalling $y = \sum_{j=1}^J y_j$, and any $a \in (L^2_\ell)_+$ with $a \leq y^-$, there exists some scalar $\rho > 0$, some $b \in (L^2_\ell)_+$ with $b \leq y^+$, and some production allocation $(\widehat{y}_1, \ldots, \widehat{y}_J)$ such that

$$(y^+ - b) - (y^- - \rho a) = \sum_{j=1}^J \widehat{y}_j \tag{b.1}$$

$$\| b \|_1 \leq \gamma \| \rho a \|_1 \tag{b.2}$$

$$\widehat{y}_j^+ \leq y_j^+, \quad j \in \mathcal{J} \tag{b.3}$$

$$\widehat{y}_j^- \leq y_j^-, \quad j \in \mathcal{J}. \tag{b.4}$$

The condition appears complicated, but has a reasonably simple interpretation. The process ρa represents a reduction in the total production "input" y^-. The process b represents the loss in production that is sustained in the switch to a new production allocation

$(\widehat{y}_1, \ldots, \widehat{y}_J)$, as indicated by relation (b.1). The constant γ indicates that there is a sufficiently small ρ regulating the lost input such that the lost output is controlled in magnitude by $\gamma \parallel \rho a \parallel_1$. The condition is obviously much simpler when there is a single firm. Zame provides additional characterization. Zame does not have the "not positively dominated" weakening of strong marginal production efficiency, but does not assume strictly monotone preferences.

Proof of Theorem 3.1

The choice space L^2_ℓ is given the $\|\cdot\|_1$ (product $L^1(\nu)$) toplogy, and as such is a normed vector lattice. Thus Theorem 2 of Zame [31] applies with minor modification as follows. For Zame's toplogy $\mathcal{T}$ we take the weak $\|\cdot\|_2$ toplogy. Although Zame's Standard Assumption (1) calls for $(L^2_\ell)_+$ to be $\|\cdot\|_1$–closed (which it is not), it is the case that $(L^2_\ell)_+$ is $\mathcal{T}$–compact, and this is sufficient for Zame's proof. Zame's other Standard Assumptions (1)–(6) are satisfied since: $(L^2_\ell)_+$ is convex and includes $e_i \quad \forall i \in \mathcal{I}$ (1), $\succeq_i$ is $\|\cdot\|_1$–continuous by assumption (and $\|\cdot\|_2$–continuous by the Cauchy–Schwarz inequality) $\forall i \in \mathcal{I}$ (2), $\succeq_i$ is convex (3), $\succeq_i$ has a $\|\cdot\|_1$–extremely desirable choice $\forall i \in \mathcal{I}$ by assumption (4), $0 \leq \epsilon_i^j \leq 1$ and $\sum_{i=1}^I \epsilon_i^j = 1 \quad \forall j \in \mathcal{J}$ (5), and Y_j is $\|\cdot\|_2$–closed, convex, and includes zero $\forall j \in \mathcal{J}$ (6). [Again, Zame calls for Y_j to be $\|\cdot\|_1$–closed, but it is enough for Y_j to be $\|\cdot\|_2$–closed and convex.]

The conditions (1) through (8) of Zame's Theorem 2 are satisfied since: $(L^2_\ell)_+ \cap Y$ is a bounded and closed subset of a reflexive Banach space, and therefore $\mathcal{T}$–compact (Schaefer [29]) (1)–(2); $\|\cdot\|_2$–closed convex sets are $\mathcal{T}$–closed (Schaefer [29]) and thus $\succ_i$ has a relatively $\mathcal{T}\times \|\cdot\|_2$–open graph (3); order intervals in an L^2 space are weak compact (4); Y_j is $\|\cdot\|_2$–closed and convex and thus $\mathcal{T}$–closed (5); the consumption set of each agent is $(L^2_\ell)_+$ (6); each agent $i \in \mathcal{I}$ has a $\|\cdot\|_1$–extremely desirable choice $v_i \in [0, \sum_{i=1}^I e_i]$(7); and the production sets $Y_1, \ldots, Y_j$ demonstrate strongly $\|\cdot\|_1$–bounded marginal efficiency by assumption (8). Although our version of strongly bounded marginal efficiency is slightly weaker than Zame's, the assumption of strictly increasing preferences allows one to apply Zame's proof with only slight modification. [The topology $\mathcal{T}$ is Hausdorff and weaker than the $\|\cdot\|_1$–topology by Cauchy–Schwarz, meeting Zame's stipulations for these conditions.] Thus, by Zame's theorem, there exists a quasi–equilibrium $(c_1, \ldots, c_I, y_1, \ldots, y_J, \phi)$ with

$\phi(\sum_i e_i) > 0$. Thus, $\phi(e_i) > 0$ for some i. By a standard argument, strict monotonicity implies that ϕ is strictly positive, that $\phi(e_i) > 0$ for all i, and thus that the quasi–equilibrium is in fact an equilibrium. Since ϕ is in the toplogical dual of L^1_ℓ, which is L^∞_ℓ, the Riesz representation theorem implies that ϕ has the indicated representation (3.5) with ψ bounded. Since ϕ is strictly positive, ψ is strictly positive. This ends the proof.

Proof of Proposition 5.3

Let $C = L^2_1$ with the usual L^2 norm $\|\cdot\|_C$. Suppose $\{(-c_n, b_n)\}$ is a sequence in Y^f converging to $(-c, b) \in C \times C$. We must show that $b \in C_+$ and $b \leq f(c)$ to prove that Y^f is $\|\cdot\|_2$–closed, but this follows by the fact that C_+ is $\|\cdot\|_2$–closed, the $\|\cdot\|_1$–continuity of f, and Cauchy–Schwarz. In order to prove convexity, suppose $(c_1, b_1) \in Y^f$ and $(c_2, b_2) \in Y^f$. Then $b_1 \leq f(c_1)$ and $b_2 \leq f(c_2)$ implies $b \equiv \alpha b_1 + (1-\alpha)b_2 \leq \alpha f(c_1) + (1-\alpha)f(c_2) \leq f(\alpha c_1 + (1-\alpha)c_2)$ by concavity of f, so Y^f is convex. Of course $0 \in Y^f$. Let e_1 denote the projection of the aggregate endowment $e \in L^2_\ell = C \times C$ into the first ("labor") factor space, and e_2 the projection of e into the second factor ("corn") space. Since f is increasing, aggregate feasible consumption is $\|\cdot\|_2$–bounded by $\|(e_1, e_2 + f(e_1))\|_2$. Thus the feasible production set is $\|\cdot\|_2$–bounded. It remains to show that Y^f demonstrates strongly $\|\cdot\|_1$–bounded marginal efficiency of production.

Let γ denote the norm of the $\|\cdot\|_1$–Frechet derivative of f at zero, and choose any not positively dominated $y = (y_1, y_2) \in Y^f \subset C_- \times C_+$. Let $a = (a_1, a_2) \in C_+ \times C_+$ with $a \leq y^- = (y_1, 0)$ as suggested by the definition of strongly bounded marginal efficiency of production. Then $a_2 = 0$. Let $d = f(-y_1, -a_1)$ and $\widehat{y} = (\widehat{y}_1, \widehat{y}_2) = (y_1 + a_1, d)$. Finally, let $b = (b_1, b_2) = (0, y_2 - d)$. For $\rho = 1$, we have

$$(y^+ - b) - (y^- - \rho a) = ((0, y_2) - (0, y_2 - d)) - ((-y_1, 0) - (a_1, 0)) = (y_1 + a_1, d) = \widehat{y},$$

verifying (b.1). Next,

$$\begin{aligned} \|b\|_1 &= \|y_2 - d\|_1 = \|y_2 - f(-y_1 - a_1)\|_1 \\ &\leq \|f(-y_1) - f(-y_1 - a_1)\|_1 \\ &\leq \|f(a_1) - f(0)\|_1 \leq \gamma \|a_1\|_1 = \gamma \|\rho a\|_1 \end{aligned}$$

by concavity and monotonicity of f and the definition of γ. This verifies (b.2). Since y is not positively dominated, $f(-y_1) = y_2$. Then, since $a_1 \geq 0$,

$$\widehat{y}^+ = (0, f(-y_1 - a_1)) \leq y^+ = (0, f(-y_1)),$$

and (b.3) is satisfied. Because $\widehat{y}^- = (-y_1 - a_1, 0) \leq y^- = (-y_1, 0)$, condition (b.4) is satisfied. Thus strongly bounded marginal efficiency of production is verified, completing the proof.

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