Arbitrage Pricing of Russian Options and Perpetual Lookback Options

J. Darrell Duffie; J. Michael Harrison

The Annals of Applied Probability, Volume 3, Issue 3 (Aug., 1993), 641-651.

Stable URL:
http://links.jstor.org/sici?sici=1050-5164%28199308%293%3A3%3C641%3AAPOROA%3E2.0.CO%3B2-1

The Annals of Applied Probability
1993, Vol. 3, No. 3, 641–651

ARBITRAGE PRICING OF RUSSIAN OPTIONS AND PERPETUAL LOOKBACK OPTIONS

BY J. DARRELL DUFFIE AND J. MICHAEL HARRISON

Stanford University

Let $X = \{X_t,\ t \geq 0\}$ be the price process for a stock, with $X_0 = x > 0$. Given a constant $s \geq x$, let $S_t = \max\{s, \sup_{0 \leq u \leq t} X_u\}$. Following the terminology of Shepp and Shiryaev, we consider a "Russian option," which pays S_τ dollars to its owner at whatever stopping time $\tau \in [0, \infty)$ the owner may select. As in the option pricing theory of Black and Scholes, we assume a frictionless market model in which the stock price process X is a geometric Brownian motion and investors can either borrow or lend at a known riskless interest rate $r > 0$. The stock pays dividends continuously at the rate δX_t, where $\delta \geq 0$.

Building on the optimal stopping analysis of Shepp and Shiryaev, we use arbitrage arguments to derive a rational economic value for the Russian option. That value is finite when the dividend payout rate δ is strictly positive, but is infinite when $\delta = 0$. Finally, the analysis is extended to perpetual lookback options.

The problems discussed here are rather exotic, involving infinite horizons, discretionary times of exercise and path-dependent payouts. They are also perfectly concrete, which allows an explicit, constructive treatment. Thus, although no new theory is developed, the paper may serve as a useful tutorial on option pricing concepts.

1. Introduction. Let $X = \{X_t,\ t \geq 0\}$ be a one-dimensional diffusion process satisfying the stochastic differential equation

$$dX = \mu X\,dt + \sigma X\,dW, \tag{1.1}$$

where $W = \{W_t,\ t \geq 0\}$ is a standard Brownian motion (or Wiener process). The state space of X is $(0, \infty)$ and its initial state is $X_0 = x$, so X can be represented in the form

$$X_t = x \exp\left\{\sigma W_t + \left(\mu - \tfrac{1}{2}\sigma^2\right)t\right\}, \qquad t \geq 0. \tag{1.2}$$

Now consider a market in which one can invest in a common stock or in a riskless bank account with instantaneous (continuously compounding) interest rate $r > 0$. The stock price dynamics are modeled by the geometric Brownian motion X. Let us further assume that the stock pays dividends continuously at the rate δX, where $\delta \geq 0$ is a given constant. Thus, if an investor owns a_t shares of stock and has b_t dollars in the bank at time t, the market value of the investor's portfolio at that time is $Z_t = a_t X_t + b_t$, and his wealth dynamics are given by

$$dZ = a\,dX + br\,dt + a\delta X\,dt = \left[a(\mu + \delta)X + br\right]dt + a\sigma X\,dW. \tag{1.3}$$

Received November 1992; revised December 1992.
AMS 1991 *subject classifications.* 90A09, 60H30.
Key words and phrases. Options, arbitage, Black-Scholes model, optimal stopping.

Finally, let us assume that investors can buy and sell (or sell short) unlimited amounts of stock, and can borrow or lend unlimited amounts at the interest rate r, without incurring brokerage fees or other transaction costs. A positive value for b_t indicates that the investor is lending at the riskless interest rate, whereas a negative value indicates riskless borrowing. Similarly, a positive value for a_t indicates a long position in stock (i.e., the investor owns stock), whereas a negative value indicates a short position (i.e., the investor has sold stock short). If the stock pays dividends, an investor who sells stock short must pay dividends so as to match the income generated by "real" shares.

What we have described in the previous paragraph is the frictionless market model made famous by Merton (1969, 1971, 1973) and Black and Scholes (1973), modified in two rather minor ways. First, we have allowed for the possibility that the stock pays dividends, restricting attention to proportional-rate dividend policies defined by a single constant $\delta \geq 0$. As Merton (1973) originally pointed out, this added feature complicates just slightly the arbitrage valuation of options and other contingent financial claims. (By a "contingent financial claim," we mean a financial asset whose value at any given time is determined by past and present values of the stock price.) The second nonstandard feature in our model formulation is its infinite time horizon. The infinite horizon is essential for our purposes, as will become clear shortly.

Our objective is to determine through arbitrage considerations a rational economic value for the "Russian option" analyzed by Shepp and Shiryaev (1993). To define the Russian option, let

$$S_t = \max\Big(s, \sup_{0 \leq u \leq t} X_u\Big), \qquad t \geq 0, \tag{1.4}$$

where $s \geq x$ is a constant. The Russian option is a piece of paper that entitles its owner to choose a stopping time $\tau \in [0, \infty)$ and be paid S_τ at that time. It is crucial that no bound is imposed on the stopping time τ.

PROPOSITION 1. *Suppose that $\delta > 0$, and let $V(x, s \mid \mu, \sigma, r)$ be defined by formula* (2.4) *of Shepp and Shiryaev* (1993) (*and repeated below for convenience*). *Then the rational economic value of the Russian option is $V(x, s \mid r - \delta, \sigma, r)$. That is, if the Russian option is offered for sale at any other price, then arbitrage profits can be made by means of the trading strategy described in Section* 3.

PROPOSITION 2. *If $\delta = 0$, then the Russian option has infinite rational value. That is, if $\delta = 0$ and the Russian option is offered for sale at any finite price, then arbitrage profits can be made by means of the trading strategy described in Section* 4.

In both cases (dividends or no dividends) the arbitrage value is independent of the average rate of return μ for the stock, a result familiar to students of option theory but surprising to the novice.

Given the results of Shepp and Shiryaev (1993), there is little in our paper that can really be called new, but the paper may serve as a useful tutorial on option pricing concepts. Proposition 1 almost follows from a general theorem of Karatzas (1988), but the latter result involves a technical restriction that is not obviously satisfied in our context. Perhaps more to the point, we deal with a concrete example in an explicit and constructive fashion, showing exactly how arbitrage strategies are executed, whereas Karatzas' more general treatment involves relatively abstract arguments. Also, given Proposition 1 and the Shepp–Shiryaev analysis, Proposition 2 is more or less obvious, although we are not aware of prior work that deals in a precise way with infinite valuations and the associated notion of arbitrage, so there is at least formal novelty in our treatment. The tutorial value of the paper, as a stress test of basic option pricing concepts, is enhanced by the resulting combination of features: the possibility of infinite valuation, the infinite time horizon, the discretionary (American) exercise time and the path dependency of the payoff.

In the next section we summarize briefly the essential mathematical results of Shepp and Shiryaev (1993). Propositions 1 and 2 are then proven in Sections 3 and 4, respectively. In each case we give explicit arbitrage strategies to back up our pricing results. The final section extends the pricing analysis to perpetual "lookback" options, those giving the right to sell at the high, or buy at the low.

2. Preliminaries. Shepp and Shiryaev (1993) solve the following mathematical problem. Given an interest rate $r > 0$, parameters μ and $\sigma > 0$ describing stock price dynamics, and positive initial values x and s for the processes X and S, respectively, find a stopping time $\tau \geq 0$ to maximize

$$E_{(x,s)}(e^{-r\tau}S_\tau). \tag{2.1}$$

For parameter combinations such that $\mu < r$, Shepp and Shiryaev define a stopping constant

$$\alpha = \alpha(\mu, \sigma, r) = \left(\frac{1-\gamma_1^{-1}}{1-\gamma_2^{-1}}\right)^{(\gamma_2-\gamma_1)^{-1}},$$

where

$$\gamma_1 = \frac{\sigma^2/2 - \mu - \sqrt{(\sigma^2/2-\mu)^2 + 2\sigma^2 r}}{\sigma^2},$$

$$\gamma_2 = \frac{\sigma^2/2 - \mu + \sqrt{(\sigma^2/2-\mu)^2 + 2\sigma^2 r}}{\sigma^2}.$$

Then they define

$$V(x,s) = V(x,s|\mu,\sigma,r) = \begin{cases} \dfrac{s}{\gamma_2-\gamma_1}\left(\gamma_2\left(\dfrac{\alpha x}{s}\right)^{\gamma_1} - \gamma_1\left(\dfrac{\alpha x}{s}\right)^{\gamma_2}\right), & \dfrac{s}{\alpha} \le x \le s, \\ s, & 0 < x \le \dfrac{s}{\alpha}. \end{cases}$$

They show that $V(x,s)$ is the supremum of (2.1) over all stopping times τ, and that this supremum is achieved by the optimal stopping time

$$\tau^* = \inf\{t \ge 0: X_t \le S_t/\alpha\}. \tag{2.2}$$

Consider again the situation described in Section 1, in which the stock with price process X pays dividends continuously at rate δX. Restricting attention initially to the case $\delta > 0$, let us define

$$\beta(\sigma,\delta,r) = \alpha(r-\delta,\sigma,r) \tag{2.3}$$

and

$$F(x,s|\sigma,\delta,r) = V(x,s|r-\delta,\sigma,r). \tag{2.4}$$

From (2.3), (2.4) and the foregoing explicit formulas for α and V, one has the following properties (all of them are either noted in Section 2 of the Shepp–Shiryaev paper or else are easy to verify):

$$\begin{aligned} &F \text{ is } C^1 \text{ on } \mathscr{D} = \{(x,s): 0 < x \le s < \infty\} \\ &\text{and is } C^2 \text{ on } \mathscr{D}_0 = \{(x,s): s/\beta < x \le s\}; \end{aligned} \tag{2.5}$$

$$F(x,s) \ge s \quad \text{on } \mathscr{D}; \tag{2.6}$$

$$F(x,s) = s \quad \text{for } 0 < x \le s/\beta; \tag{2.7}$$

$$(r-\delta)xF_x(x,s) + \tfrac{1}{2}\sigma^2x^2F_{xx}(x,s) = rF(x,s) \text{ for } (x,s) \in \mathscr{D}_0; \tag{2.8}$$

$$F_s(s,s) = 0 \quad \text{for } s > 0; \tag{2.9}$$

$$\text{for all } (x,s) \in \mathscr{D}, \qquad F(x,s|\sigma,\delta,r) \uparrow \infty \quad \text{as } \delta \downarrow 0; \tag{2.10}$$

$$\beta(\sigma,\delta,r) > 1 \quad \text{and} \quad \beta(\sigma,\delta,r) \uparrow \infty \quad \text{as } \delta \downarrow 0; \tag{2.11}$$

$$F_x(x,s) \ge 0 \quad \text{on } \mathscr{D}. \tag{2.12}$$

For the case of positive dividends ($\delta > 0$), we will show in Section 3 that if the Russian option is sold in the market for *any price other than* $F(x,s)$, then arbitrage profits can be made. That is, the optimal stopping analysis of Shepp and Shiryaev gives us the rational economic value of the option *if we first change the drift parameter for the stock price process from* μ *to* $r-\delta$. This valuation procedure is one that might be guessed from existing-but-not-precisely-applicable general theory; compare with Merton (1973), Harrison and Kreps (1979) and Karatzas (1988). In Section 3 we prove its validity by explicit construction, making full use of the penetrating analysis by Shepp and Shiryaev.

Given the valuation formula $F(x, s)$ for the case of positive dividends, one naturally anticipates from (2.10) that *any* finite price for the Russian option will create arbitrage opportunities when $\delta = 0$. In Section 4 we prove rigorously that this is the case, again relying heavily on the Shepp–Shiryaev analysis.

3. Proof of the valuation formula with positive dividends. To prove Proposition 1 we must construct an arbitrage if the price of the option is either higher than or lower than $F(x, s)$. In each of these two cases, our analysis involves the following "replicating strategy." Let

$$a_t = F_x(X_t, S_t), \qquad t \geq 0, \tag{3.1}$$

and

$$b_t = F(X_t, S_t) - a_t X_t, \qquad t \geq 0. \tag{3.2}$$

Consider the trading strategy that holds a_t shares of stock at time t and maintains a bank account of b_t at time t. The market value of this stock–bank portfolio at time t is

$$Z_t = a_t X_t + b_t = F(X_t, S_t), \qquad t \geq 0. \tag{3.3}$$

In particular, one must invest $Z_0 = F(x, s) > 0$ in order to establish the initial position (a_0, b_0).

The following proposition shows that trading strategy (a, b) is "self-financing," at least up until the stopping time:

$$\tau = \inf\{t \geq 0: X_t \leq S_t/\beta\}. \tag{3.4}$$

Lemma 1. *If $x > s/\beta$ (that is, $\tau > 0$), then for each stopping time $T \in (0, \tau)$,*

$$Z_T - Z_0 = \int_0^T (a_t\, dX_t + \delta a_t X_t\, dt + r b_t\, dt). \tag{3.5}$$

Remark. The first term on the right represents capital gains realized over the time interval $[0, T]$ on the investor's stock holdings; compare with Harrison and Pliska (1981). The second term represents dividend income earned over that interval and the third term represents interest income from the investor's bank account. Thus (3.5) says that all changes in the market value of the investor's portfolio are due to gains and losses on investments; no new cash is infused after time zero, nor is cash withdrawn after time zero, and hence the trading strategy (a, b) is said to be self-financing over the time interval $(0, \tau)$.

Proof. Observe that S is a continuous process of bounded variation (its sample paths are increasing). By (2.5), we can apply Itô's formula for $T < \tau$, giving us

$$dZ = F_x(X, S)\, dX + F_s(X, S)\, dS + \tfrac{1}{2} F_{xx}(X, S)(dX)^2; \tag{3.6}$$

the second-order terms involving $dS\, dX$ and $(dS)^2$ are both zero. Also, the sample path of S increases only at times t when $X_t = S_t$, so the second term on the right side of (3.6) can be rewritten as $F_s(S,S)\,dS$, and then (2.9) implies that this term is zero. Next, from (1.1) we have $(dX)^2 = \sigma^2 X^2\,dt$, so (3.6) reduces to

$$dZ = F_x(X,S)\,dX + \tfrac{1}{2}\sigma^2 X^2 F_{xx}(X,S)\,dt. \tag{3.7}$$

We now substitute (2.8), (3.1) and (3.2) into (3.7) to obtain

$$\begin{aligned} dZ &= F_x(X,S)\,dX + \left[rF(X,S) - (r-\delta)XF_x(X,S)\right]dt \\ &= a\,dX + aX\delta\,dt + rb\,dt. \end{aligned} \tag{3.8}$$

Integrating both sides of (3.8) over $[0,T]$ gives (3.5), so the lemma is proved. □

LEMMA 2. *At time τ the market value of the stock–bank portfolio (a,b) is $Z_\tau = S_\tau$.*

PROOF. This is immediate from (3.3), (3.4) and (2.7). □

Moving now to the proof of Proposition 1, let us first dispense with the trivial case in which $x \le s/\beta$ and moreover the Russian option is offered for sale at a price $p < F(x,s)$. When $x \le s/\beta$ we have $F(x,s) = s$ by (2.7), so an investor can simply buy the option, exercise it immediately to earn s, making an instantaneous profit of $s - p > 0$ with no risk whatever. This is the simplest example of an arbitrage.

Next consider the case in which $x > s/\beta$, implying $\tau > 0$ by (3.4), and moreover the Russian option is offered for sale at some price $p < F(x,s)$ at time zero. Consider a trading strategy which:

(3.9) buys a Russian option for p dollars at $t = 0$, holds it over the interval $(0,\tau)$ and exercises it at time τ, thus earning S_τ at time τ;

(3.10) sets up and maintains the stock–bank portfolio $(-a,-b)$ over the interval $[0,\tau)$, producing stock–bank holdings at time τ with market value $-Z_\tau = -S_\tau$; and

(3.11) uses time τ earnings from exercise of the Russian option to redeem stock–bank obligations at that time, and exits the market.

This trading strategy generates $Z_0 = F(x,s)$ dollars of cash at time zero through short sales, and only p of those dollars are needed to buy the Russian option, so the investor pockets a "bonus" of $F(x,s) - p$ dollars at $t = 0$. Thereafter, the strategy described by (3.9) through (3.11) is self-financing, so the bonus earned at time zero is not accompanied by any risk of subsequent loss. In other words, this trading strategy gives the investor an arbitrage profit of $F(x,s) - p$ dollars.

Finally, suppose that the Russian option is offered for sale at some price $p > F(x,s)$ at time zero. The idea of the arbitrage is to sell short the option and to dominate the exercise payment required by the buyer of the option with the value of the stock–bank strategy (a, b) previously described. Because the owner will not necessarily exercise at the "rational" time τ, the strategy (a, b) need not be self-financing, but the following extension of Lemma 1 shows that any cash flows generated by it are to the advantage of the arbitrageur. [In a slightly different setting, Karatzas (1988) shows that this can be done by exploiting the definition of the "Snell envelope." He does not go on to make the corresponding arbitrage argument.]

LEMMA 3. *For any stopping time* $T \in [0, \infty)$ *we have*

$$Z_T - Z_0 = \int_0^T (a_t\, dX_t + \delta a_t X_t\, dt + r b_t\, dt) - \int_0^T r S_t 1_{(X_t < S_t/\beta)}\, dt. \tag{3.12}$$

REMARK. Again one interprets the first term on the right as total investment earnings over the interval $(0, T)$. Thus (3.12) says that an investor maintaining the stock–bank portfolio (a, b) can continuously withdraw cash at rate $rS\,dt$ when $X < S/\beta$, but need not ever infuse new cash.

PROOF. Recall that $Z_t = F(X_t, S_t)$ by (3.3). If F were a C^2 function, we would proceed as follows based on the standard version of Itô's formula. (In fact, F is not C^2, but we address that issue later.) If $X_t > S_t/\beta$, formula (3.8) for dZ_t is obtained exactly as before. If $X_t < S_t/\beta$, on the other hand, then $Z_t = F(X_t, S_t) = S_t$ by (2.7). Moreover, $\beta > 1$ by (2.11), so S remains constant when $X_t < S_t/\beta$, implying that $dZ_t = dS_t = 0$. Integrating dZ from 0 to T gives

$$Z_T - Z_0 = \int_0^T (a_t\, dX_t + \delta a_t X_t\, dt + r b_t\, dt) 1_{(X_t \geq S_t/\beta)}. \tag{3.13}$$

From (2.7), (3.1) and (3.2) we have $a_t = 0$ and $b_t = F(X_t, S_t) = S_t$ when $X_t < S_t/\beta$, so (3.13) is equivalent to (3.12).

In fact, F is not C^2. Rather, as implied by (2.5) and (2.7), F is C^1 and is C^2 everywhere in $\mathscr{D}$ *except* on the ray $\{(x, s) \in \mathscr{D}: x = s/\beta\}$. Because $\beta > 1$, we can use the fact that S_t remains constant while (X_t, S_t) is in an open set containing this ray, say $\mathscr{D}_1 = \{(x, s) \in \mathscr{D}: 0 < x < s/\beta_1\}$, where $1 < \beta_1 < \beta$. Because F is C^2 off $\mathscr{D}_1$, this effectively reduces the problem to calculating, for each fixed s, an increment in Z of the form $F(X_\tau, s) - F(X_0, s)$, where $X_0 \in \mathscr{D}_1$ and $\tau = \inf\{t: X_t = s/\beta_1\}$. Because the interval $(0, s/\beta_1)$ contains only a single point at which $F(\cdot, s)$ is not C^2, we can apply Itô's rule for C^1 functions, that are C^2 except at isolated points, of a one-dimensional continuous semimartingale. For example, the formula given in the statement of Theorem 1.5 of Revuz and Yor [(1991), page 208] is sufficiently general. This application of Itô's formula gives exactly the same result asserted previously for the C^2 case, completing the proof of the lemma. □

With this preliminary, we claim that arbitrage profits can be made by:

(3.14) selling short one Russian option at time zero and paying the buyer S_T at whatever stopping time T he may choose to exercise;

(3.15) setting up and maintaining the stock–bank portfolio (a, b) over the time interval $[0, T)$; and

(3.16) using the market value Z_T of the stock–bank portfolio to meet the required payment S_T on exercise of the option.

To prove the arbitrage, three facts must be noted. First, because $p > Z_0 = F(x, s)$ by hypothesis, short sale of the option generates more than enough cash to establish the initial stock–bank position; the investor can pocket the difference $p - F(x, s)$ at time zero. Second, because $Z_T = F(X_T, S_T) \geq S_T$ by (2.6), liquidation of the stock–bank portfolio generates *at least* enough cash to meet the required payment S_T at the time of exercise. Finally, by Lemma 3, the arbitrageur need never infuse new cash during the interval $(0, T)$; he/she will even be able to garner additional arbitrage profits if the buyer of the option fails to exercise at time τ.

4. Infinite value in the case of no dividends. Moving now to the proof of Proposition 2, we consider the case in which the stock pays no dividends ($\delta = 0$). Suppose that a Russian option is offered for sale at any price $p < \infty$. By (2.10) we can choose a *fictional dividend rate* $\delta > 0$ small enough that

$$F(x, s) = V(x, s | r - \delta, \sigma, r) > p. \tag{4.1}$$

Fixing such a $\delta > 0$, set $\beta = \alpha(r - \delta, \sigma, r)$ and define $\tau = \inf\{t \geq 0\colon X_t \leq S_t/\beta\}$ as in (3.3). Now consider the stock–bank trading strategy (a, b) defined over $[0, \tau)$ by (3.1) and (3.2). As before, $Z_t = a_t X_t + b_t = F(X_t, S_t)$ represents the market value at time t of an investor's portfolio if strategy (a, b) is followed. As we will see shortly, however, this strategy is not self-financing. Arguing exactly as in Section 3, one finds that (3.5) remains valid, and we restate that relationship in the convenient form

$$Z_T - Z_0 = \int_0^T (a_t\, dX_t + r b_t\, dt) + \int_0^T a_t \delta X_t\, dt, \qquad T \leq \tau. \tag{4.2}$$

The first term on the right side of (4.2) represents cumulative investment gains or losses. Property (2.12) implies that $a_t \delta X_t \geq 0$, so the second term is increasing and represents a cumulative *infusion of new cash* that must be added continuously to maintain the stock–bank portfolio (a, b). Now consider a trading strategy which:

(4.3) buys a Russian option at price p at $t = 0$;

(4.4) establishes the stock–bank portfolio $(-a_0, -b_0)$ at $t = 0$, this generating cash in the amount $Z_0 = F(x, s) > p$;

(4.5) maintains the stock–bank portfolio $(-a, -b)$ over the interval $(0, \tau)$, which continuously generates cash at rate $aX\delta$; and

(4.6) exercises the Russian option at time τ, which pays S_τ at that time, allowing the investor to redeem his stock–bank obligation of $Z_\tau = F(S_\tau, X_\tau) = S_\tau$ dollars.

Under this strategy the investor pockets $F(x, s) - p > 0$ at $t = 0$ *and* has continuous earnings at rate $aX\delta$ up to time τ, with no risk of loss. This is an arbitrage profit, so Proposition 2 is proved.

If p is very large, then the fictional interest rate δ must be very small to satisfy (4.1), and it follows from (2.11) that β will be large. This means that τ is large with high probability, so an investor will need a lot of time to execute the arbitrage strategy outlined in this section. For example, for sufficiently large values of p, the probability of completing the strategy in one human lifetime will be negligible. Of course, the arbitrage strategy itself can be resold from generation to generation.

It is worth remarking that the rational economic value of the option is also infinite if dividends are paid at *any* nonpositive rate, whether or not of the form δX_t determined by a constant δ. The arguments are essentially the same as before. This case would include random storage costs as well as random depreciation rates.

5. Extensions to lookback options. We now extend to the pricing of lookback options, those giving the opportunity to sell the stock at the highest price it has reached (the "sell-at-the-max" put) or to buy the stock at the lowest price it has reached (the "buy-at-the-min" call). The European versions of these lookback options, which are traded in over-the-counter markets, were given arbitrage-free pricing formulas by Goldman, Sosin and Gatto (1979). We will give explicit formulas for the prices of perpetual American versions of these options in the case $\delta = 0$. For $\delta > 0$, readers may use our methods to deduce convenient bounds, but not explicit prices.

First, we take the perpetual sell-at-the-max put, with $\delta = 0$. This is a security that offers the right, but not the obligation, to sell the security at a stopping time τ chosen by its owner, for the maximum price that the stock has reached to date. Because that exercise price is always at least as great as the current market value, the option pays $S_\tau - X_\tau$ at the exercise date (stopping time) τ chosen by its owner. The rational price of the option is infinite. This follows from much the same reasoning used in Section 4. Suppose, for example, the put were offered for sale at some finite price p. Let $\delta > 0$ be chosen small enough, as a fictitious dividend coefficient, that $F(x, s|r - \delta, \sigma, r) - x > p$. The strategy of buying the call for p, buying the stock for x and selling the replicating strategy described in Section 4, generates an arbitrage. The initial profit is $F(x, s|r - \delta, \sigma, r) - x - p > 0$. The cash flows on the Russian replicating strategy are more than covered by exercise of the Russian option at the exercise time $\inf\{t\colon X_t = S_t/\beta(\sigma, \delta, r)\}$, as described in Section 4. Thus exercising the lookback at the same stopping time yields an arbitrage. We summarize as follows.

PROPOSITION 3. *If $\delta = 0$, then the sell-at-the-max put option has infinite rational value. That is, if this option is offered for sale at any finite price, then arbitrage profits can be made.*

Let us consider now a sell-at-the-max put option with *finite* maturity T (that is, an option whose owner may exercise at any stopping time $\tau \leq T$). Using Proposition 3, it can be shown that the value of this finite-maturity American lookback put increases without bound as $T \to \infty$. (Although this conclusion seems "obvious" from Proposition 3, we have not found an easy proof. To be more specific, the simplest proof that we have found uses special properties of Brownian motion.) On the other hand, Goldman, Sosin and Gatto (1979) derived an explicit formula for the analogous finite-maturity *European* put option (allowing exercise only at time T), and it is easy to show that their finite-maturity value converges to a *finite* limit as $T \to \infty$. Combining those two facts, one may conclude that: (a) a finite-maturity American lookback put is, in general, more valuable than its European counterpart, and consequently (b) it is sometimes optimal to exercise a finite-maturity lookback put early. Incidentally, both of these conclusions can be inferred under certain conditions from the demonstration by Goldman, Sosin and Gatto (1979) that finite-maturity European lookback put options may actually *decrease* in value with an increase in maturity.

A third conclusion that one can draw from this comparison between the asymptotic values of American and European lookbacks is that, in order to approach the full economic value of the perpetual put, one must exercise at *carefully chosen*, *path-dependent* stopping times that diverge to infinity, and that one cannot approach the full value simply by waiting sufficiently long. An appropriate sequence of exercise times is $\tau_n = \inf\{t\colon X_t = S_t/\beta_n\}$, where $\beta_n \downarrow 0$. Taking $\tau_n = n$ will not approximate the full (infinite) value.

Next, we consider the perpetual buy-at-the-min call. This is a security that offers the right, but not the obligation, to buy the stock at a stopping time τ chosen by its owner, at the minimum price the stock has reached to date. Because that exercise price is always less than or equal to the current market value, the option pays $X_\tau - Y_\tau$ at the stopping time τ chosen by its owner, where

$$Y_t \equiv \min\Big(y, \inf_{s \leq t} X_s\Big) \tag{5.1}$$

for $y \leq x$.

We claim that the unique arbitrage-free price of the perpetual buy-at-the-min call is simply x, the current stock price. (That is, owning such an option is essentially equivalent to owning a share of stock.) The argument goes as follows. First, suppose that the option is being traded at some price $x + \varepsilon$, where $\varepsilon > 0$. One may then sell such an option for $x + \varepsilon$, simultaneously buy a share of stock for x, pocket the net cash flow ε and simply hold the share of stock until the buyer decides to exercise his option. At that time (say, τ) the buyer will pay $Y_\tau > 0$ and will be given the share of stock bought at time zero, which generates another positive cash flow of Y_τ, all with no risk to the seller of the option.

On the other hand, suppose that the perpetual buy-at-the-min call option is traded at some price $x - \varepsilon$, where $\varepsilon > 0$. One may then buy such an option for $x - \varepsilon$, simultaneously sell short a share of stock to generate a positive cash flow of ε, pocket $\varepsilon/2$ as arbitrage profits, put the other $\varepsilon/2$ in the bank and wait. The purchase price Y associated with the call option can only decrease from its initial value Y_0, whereas the value of the bank account is $b_t = (\varepsilon/2)\exp(rt)$ after t time units. At the first time t that $b_t = Y_t$, one may exercise the call option, using the bank account to pay for the share of stock and using that share of stock to redeem the short position assumed at time zero. Thus, the income of $\varepsilon/2$ at time zero is earned without any associated risk. We summarize this reasoning as follows.

PROPOSITION 4. *Suppose that $\delta = 0$. Then the rational economic value of the perpetual buy-at-the-min option is x, the initial stock price. That is, if the buy-at-the-min option is offered for sale at any other price, then arbitrage profits can be made.*

Acknowledgments. We are grateful to Larry Shepp for early access to his work with A. N. Shiryaev and for helpful conversations. We are also grateful for comments from two referees and the Editor.

REFERENCES

BLACK, F. and SCHOLES, M. (1973). The pricing of options and corporate liabilities. *J. Political Economy* **81** 637–654.

GOLDMAN, B., SOSIN, H. and GATTO, M. (1979). Path-dependent options: Buy at the low, sell at the high. *J. Finance* **34** 1111–1127.

HARRISON, J. M. and KREPS, D. (1979). Martingales and arbitrage in multiperiod security markets. *J. Econom. Theory* **20** 381–408.

HARRISON, J. M. and PLISKA, S. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Process. Appl.* **11** 215–260.

KARATZAS, I. (1988). On the pricing of American options. *Appl. Math. Optim.* **17** 37–60.

MERTON, R. (1969). Lifetime portfolio selection under uncertainty: The continuous time case. *Rev. Econom. Statist.* **51** 247–257.

MERTON, R. (1971). Optimum consumption and portfolio rules in a continuous time model. *J. Econom. Theory* **3** 373–413.

MERTON, R. (1973). The theory of rational option pricing. *Bell J. Econom. Management Sci.* **4** 141–183.

REVUZ, D. and YOR, M. (1991). *Continuous Martingales and Brownian Motion.* Springer, New York.

SHEPP, L. and SHIRYAEV, A. N. (1993). The Russian option: Reduced regret. *Ann. Appl. Probab.* **3** 631–640.

GRADUATE SCHOOL OF BUSINESS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305