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Intertemporal Arbitrage and the Markov Valuation of Securities*

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Abstract. This paper explores the intertemporal nature of arbitrage and its connection to Markov processes. We suppose that an economy is in one of a fixed set Ω of possible states at each date. Assuming that prices and dividends are described by functions of the state of the economy at each date and are intertemporally abitrage free, we construct a corresponding Markov process under which the current market value of any security is the expected value of its future dividends, given the current state of the Markov process. An equilibrium example is worked out using z–transform analysis.

Keywords: Arbitrage, intertemporal, Markov process, sub–Markov, Securities valuation, z–transform.

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1. INTRODUCTION

This paper improves the Harrison-Kreps theory for state-space economies. In principle, Harrison and Kreps (1979) showed that the absence of intertemporal arbitrage implies the existence of a numeraire and a choice of probability assessments (called a "martingale measure") under which the price of any security at any time may be viewed as its conditional expected future payoff. In a state-space setting, that is, an economy in which the securities' dividends and prices at each time are functions of an underlying Markov state process X, substitution of a Harrison-Kreps martingale measure may destroy the Markov property of X. Indeed, so long as markets are incomplete and the setting is non-trivial, there exists an infinite collection of such martingale measures, many of which will destroy the Markov property. However, we shall show in this paper that there is always at least one martingale measure which preserves the Markov property. We also show that it is possible, and furthermore analytically convenient, to avoid the Harrison-Kreps intertemporal change of numeraire, by instead converting X to a sub-Markov process whose death probabilities account for time discounting.

Ross (1973)[1] has suggested a fundamental representation theorem for arbitrage-free markets. The basic formulation can be summarized in the standard one-period (that is, two-date) framework using notation as follows. Let

p_j = the price at date 0 of security j, $j = 1,2, \cdots ,J$, and

$a_{sj} = p_{sj} + d_{sj}$ = the payoff to security j at date 1 if the prevailing state then is s, $s = 1,2, \cdots ,S$, where p_{sj} and d_{sj} are the conditional price and dividend, respectively, at date 1.

One version of a no-arbitrage condition is:

No Arbitrage, One Period. There exists no portfolio x_j such that

$$\sum_j x_j p_j < 0$$

and

$$\sum_j x_j a_{sj} \geq 0 \text{ for all states } s = 1,2, \cdots ,S, \qquad \text{(NA-1P)}$$

where x_j is unconstrained.

[1]Ross appears to have been the first to recognize that the separating-hyperplane notion is the essential feature of the no-arbitrage characterization; but see also the precursor work of Beja (1967) and (1971). Ross (1978) provides a more well-developed exposition of the basic ideas.

Ross's representation result states that there exists a nonnegative set of numbers $\{k_s\}$, *independent of j,* such that

$$p_j = \sum_s k_s a_{sj} \tag{R-1P}$$

if and only if (NA-1P) holds. This conclusion essentially rationalizes the existence of an implicit (Arrow-Debreu-type[2]) price structure $\{k_s\}$ even when markets are incomplete, that is, when the matrix $[a_{sj}]$ is of rank less than S. (Of course, if $[a_{sj}]$ is of rank S, the quantities $\{k_s\}$ are uniquely determined by (exogenous) prices and payoffs; otherwise they are non-unique.)

Subsequently, Rubinstein (1976) and others[3] have employed a dividend-streams version of the no-arbitrage paradigm. In this approach, an infinite number of dates is assumed, and the state at date t is denoted as $s(t)$. Implicit in this notation is that the set of possible states is uniform for each date; with a slight loss of generality[4], this is always possible by simply including calendar time as another descriptor in the definition of the "state". The no-arbitrage condition in this context is then

No Arbitrage, Dividend Stream. There exists no portfolio x_j such that

$$\sum_j x_j p_j < 0$$

and

$$\sum_j x_j d_{s(t)j} \geq 0 \quad \textit{for all states s(t) at all dates } t = 1,2,... \tag{NA-DS}$$

with x_j unconstrained, where $d_{s(t)j}$ is the state-conditional dividend of security j at date t.

The corresponding representation theorem would therefore seem[5] to be given as

$$p_j = \sum_{t=1}^{\infty} \sum_{s(t)} k_{ts(t)} d_{s(t)j}, \quad k_{ts(t)} \textit{ nonnegative}, \tag{R-DS}$$

where the quantities $k_{ts(t)}$ are analogous to a "term structure" of implicit (Arrow-Debreu) prices. (However, they do not necessarily have all the properties a "term structure" should possess, as we shall show

[2] If securities markets are complete, then the implicit price structure will be identical to the Arrow-Debreu prices for corresponding state-contingent claims; if markets are incomplete, they will also include certain Lagrange multipliers reflecting opportunity costs associated with the inability to secure various payoff patterns.

[3] See, for example, Cox and Ross (1976).

[4] Examples may be constructed in which the dimensionality of the state space expands "too rapidly" as time increases; in these, it may not be possible to establish a state "superset" which applies suitably at each individual date.

[5] Unfortunately, Farkas' Lemma does not generally hold for an infinite set of constraints. Thus on the technical side, the Rubinstein representation theorem lacks sufficient conditions to establish the result given. But this fact may have few practical implications, since counterexamples to the theorem seem to be limited to set of "knife-edge" cases. For practical purposes, we might accept the representation result as following from some deeper definition of "arbitrage", such as that given by Kreps (1981).

later. See also Example 1.)

One critique of the above representation theorems is that no consideration is given to the arbitrage possibilities offered by future markets. Garman (1977) and others[6] therefore reformulate the arbitrage problem as

No Arbitrage, Future Markets. There exists no portfolio x_j and future date τ such that

$$\sum_j x_j p_j < 0,$$

$$\sum_j x_j d_{s(t)j} \geq 0 \quad \textit{for all states } s(t) \textit{ and dates } t = 1,2, \cdots, \tau,$$

and

$$\sum_j x_j \{p_{s(\tau)j} + d_{s(\tau)j}\} \geq 0 \quad \textit{for all states } s(\tau). \qquad \text{(NA-FM)}$$

(Note that a solution to (NA-FM), if one exists, would generally depend upon the future date τ.) The corresponding representation theorem is thus

$$p_j = \sum_{t=1}^{\tau-1} \sum_{s(t)} k_{ts(t)} d_{s(t)j} + \sum_{s(\tau)} k_{\tau s(\tau)} \{p_{s(\tau)j} + d_{s(\tau)j}\}, \quad \textit{for } \tau = 1,2, \cdots, \qquad \text{(R-FM)}$$

where again the quantities $\{k_{s(t)t}\}$ are nonnegative.

Although they might initially seem so, neither of the last two representation theorems is truly intertemporal in nature, since neither considers the absence of arbitrage as a *continuing* feature of the economy. In other words, the no-arbitrage statements must apply to all choices of present dates and states as well as future dates and states. This fact has additional implications, and it is our purpose here to develop these.

The remainder of the paper is organized along the following lines. In Section 2 we describe the setup of the economy and define the absence of intertemporal arbitrage. Barring intertemporal arbitrage, we construct a family of matrices satisfying the evolution (or Chapman-Kolmogorov) property that plays the role of implicit state prices between dates. In Section 3, we construct a Markov process whose transition operators are the implicit state price matrices of Section 2. Under this Markov process, the current market value of any security is the conditional expected value of its future dividends, given the current value of the Markov process. Section 4 applies z-transform theory to achieve simple

[6] Garman and Ohlson (1980).

closed-form solutions of security prices in the time-homogeneous case. Section 5 extends the analysis to the infinite-horizon case. Section 6 provides a complete equilibrium example along the lines of Lucas (1978). The extension to a general state space is handled in Section 7, while Section 8 adds concluding remarks.

2. INTERTEMPORAL ARBITRAGE IN FINITE ECONOMIES

In this section, we establish the appropriate intertemporal arbitrage problem and derive the corresponding representation results for finite economies. A finite economy will be understood to possess a finite number of securities, a finite number of discrete dates at which economic activity can take place, and a finite number of possible states at each date. As before, we make the simplifying assumption that the *set* of possible states is uniform, that is, the same state space applies at each date. In finite economies, there can be no loss of generality in this assumption.[7] Notation for this section will be as follows:

$t, \tau = 0, 1, \cdots, T$ = indices for dates, where $T < \infty$.

$\Omega = \{1, 2, ..., S\}$ = the (stationary) set of possible states at each date; S is assumed to be finite. (In other words, each distinct state at each date is individually labelled.)

$P_t = [p_{sj}]_t$ = the matrix of conditional prices at time t; that is, the entry in row s and column j gives the price of asset j in state $s \in \Omega$ at date t.

$D_t = [d_{sj}]_t$ = the conditional dividend matrix at time t; the rows again correspond to states, the columns to assets.

$x = [x_j]$ = a column vector representing a portfolio position, where the j element gives the amount of security j held in the portfolio.

${}_tK_\tau$ = a matrix of implicit prices between dates t and $\tau \geq t$, with the rows of the matrix representing states at date t, columns representing states at date τ. Because of the assumed uniformity of Ω, the matrix ${}_tK_\tau$ is square with S rows and columns.

[7] If a different set of states exists at each date, we simply form the union of these, which must again be finite.

Employing the above matrix-vector notation, the intertemporal version of the no-arbitrage assertion is as follows:[8]

DEFINITION 1 (No Arbitrage, Intertemporal): Given a finite economy, if for every (present) date t and (future) date $\tau > t$, there does not exist any portfolio vector x such that

$$P_t x < 0$$

and

$$D_{t+1} x \geq 0,$$

$$\cdots$$

$$D_{\tau-1} x \geq 0,$$

$$\{P_\tau + D_\tau\} x \geq 0, \qquad \text{(NA-IT)}$$

then the economy is *intertemporally arbitrage-free*.

(The conventions on matrix inequalities will be as follows: "$\gg 0$" means strictly positive, that is, every element of the given matrix or vector is positive; "> 0" means positive, meaning some elements might be zero while at least one is positive; and "≥ 0" means nonnegative, or all elements are nonnegative.)

From Farkas' Lemma we immediately conclude that there will exist nonnegative square matrices ${}_tK_\tau$ such that the representation result

$$P_t = \sum_{m=t+1}^{\tau} {}_tK_m D_m + {}_tK_\tau P_\tau \qquad (1)$$

holds for any particular choices of $\tau \geq t$. However, we now go beyond this by considering all $\tau \geq t$ simultaneously and asserting the existence of a *set* of implicit price matrices $\{{}_tK_\tau\}$ which additionally possesses the "evolution" property, via an alternative definition:

DEFINITION 2 (Law of One Price, Intertemporal): If there exists a set of nonnegative implicit price matrices $\{{}_tK_\tau\}$ for which

$$P_t = \sum_{m=t+1}^{\tau} {}_tK_m D_m + {}_tK_\tau P_\tau \qquad \text{(R-IT)}$$

and, in addition, the matrices ${}_tK_\tau$ possess the *evolution* property that

[8] For a slightly stronger definition of intertemporal no-arbitrage, see the Appendix.

${}_tK_\tau = {}_tK_v\, {}_vK_\tau$ for *all* $t \leq v \leq \tau$,

then the economy satisfies the *intertemporal law of one price.*

We state our central finite-economies result:

THEOREM 1 (Nonnegative Evolution System): A finite economy is intertemporally arbitrage-free if and only if it satisfies the intertemporal law of one price.

Proof that Definition 2 implies Definition 1. This follows directly from the fact that (R-FM) implies (NA-FM) for all choices of dates, by means of Farkas' Lemma applied to each individual combination of dates.

Proof that Definition 1 implies Definition 2. This proof may be best accomplished via the following lemma, which has intuitive appeal of its own:

LEMMA 1 (Local Arbitrage): If arbitrage is possible between two dates which lie before and after an intermediate trading date, then arbitrage is also possible between one of these dates and the intermediate trading date.

Proof of Lemma. Suppose the contrary: that arbitrage is unavailable in the successive subperiods that comprise a larger period, but that arbitrage is available in that larger period. Let $u < v < w$ be the initial, intermediate, and terminal trading dates involved, respectively. By the absence of arbitrage, it follows that (1) applies to each subperiod individually. Hence we have

$$P_u = \sum_{m=u+1}^{v} {}_uK_m\, D_m + {}_uK_v\, P_v \tag{2}$$

and

$$P_v = \sum_{m=v+1}^{w} {}_vK_m\, D_m + {}_vK_w\, P_w. \tag{3}$$

Substituting (3) into (2), we have

$$P_u = \sum_{m=u+1}^{v} {}_uK_m\, D_m + \sum_{m=v+1}^{w} {}_uK_v\, {}_vK_m\, D_m + {}_uK_v\, {}_vK_w\, P_w. \tag{4}$$

We let ${}_uK_m \equiv {}_uK_v\, {}_vK_m$, $m = v+1,\ldots,w$. Clearly these matrices are nonnegative, being the products of nonnegative matrices. Applying this definition, we see that (4) is a representation theorem of form (R-IT) between dates u and w. Therefore, applying Farkas' Lemma in reverse, there can be no arbitrage between dates u and w, contradicting our original assumption. This completes the proof of the Lemma. ■

Now we return to the final part of the proof of our original theorem: that intertemporal no-arbitrage implies the intertemporal law of one price. Here we need only note the constructive nature of the proof of the Local Arbitrage lemma. That is, suppose arbitrage is prohibited between all adjacent dates; then implicit price matrices ${}_0K_1$, ${}_1K_2, \ldots, {}_{T-1}K_T$, all exist and are nonnegative. From these we may build up all two-period implicit price matrices: ${}_0K_2 \equiv {}_0K_1\,{}_1K_2$, ${}_1K_3 \equiv {}_1K_2\,{}_2K_3$, and so on. These are nonnegative and were shown to preclude two-period arbitrage. Three-period matrices may then be constructed, and so on; the associative law of matrix products assures the uniqueness of the construction, given the one-period matrices. By definition, a set of matrices (or more generally, operators) $\{{}_uK_w\}$ such that ${}_uK_w = {}_uK_v\,{}_vK_w$ for all $u \leq v \leq w$ (and which includes ${}_uK_u = I$, the identity matrix) is termed an *evolution system.*

This completes the proof of Theorem 1. ■

At this point, two corollaries are immediate.

COROLLARY 1. (Myopic Arbitrageur) Define two trading dates to be "adjacent" if there is no other trading date between them. Then finite induction may be applied to Lemma 1 to show that if arbitrage is available over a number of trading dates, it is available between at least two adjacent trading within these; conversely, if arbitrage is ruled out between all adjacent trading dates, then it is ruled out completely. Thus an arbitrageur need only look to the next trading date to secure his arbitrage profits, if any.

Although simple, this corollary represents the essence of the Local Arbitrage Lemma: in a finite economy, a repeated "myopic" no-arbitrage restriction ensures a global no-arbitrage restriction. On a related note, we observe that the portfolio strategy "x" contemplated in (NA-IT) is fixed, representing a "buy-and-hold" strategy. Is it of any purpose to investigate dynamic (that is, path-contingent) portfolio strategies as possibly enriching the arbitrage opportunities? The answer is negative:

COROLLARY 2. (Buy-and-Hold Arbitrageur) Since local no-arbitrage implies global no-arbitrage, and since all strategies are necessarily "buy-and-hold" between adjacent trading dates, "dynamic" arbitrage strategies add no new arbitrage possibilities in a finite economy.

We now distinguish a specialized class of finite economies:

DEFINITION 3 (Stationary Economies): If an intertemporally arbitrage-free finite economy possesses an implicit price evolution system $\{{}_tK_\tau\}$ with ${}_tK_\tau = {}_{t'}K_{\tau'}$ whenever $(t-\tau) = (t'-\tau') \leq 0$ (that is, implicit prices are a function of only the time between two dates), then the economy is termed *stationary*.

In other words, the implicit price matrix for a stationary economy depends only on the time period elapsing between the date when the payoffs occur and the date at which they are valued. We then have the additional

COROLLARY 3 (Semigroup Property) In a stationary, intertemporally arbitrage-free finite economy, the implicit price matrices $\{{}_tK_\tau\}$ of Theorem 1 form a *semigroup* of operators.

This follows by the definition of a semigroup, which is merely an evolution system under conditions of stationarity. (The semigroup is not necessarily a group, because the implicit price matrices need not possesses inverses.) The stationarity assumption provides numerous simplifications, which are exploited more fully later.[9] For example, it is clear that the semigroup is fully specified by the implicit price structure between adjacent dates, that is, ${}_tK_{t+1} = K$ (K is sometimes said to *generate* the semigroup), and that ${}_tK_\tau = K^{\tau-t}$.

We conclude this section with some clarifying remarks, and then illustrate these with an example. First it should be noted that, generally speaking, the implicit price matrices perform for economies under uncertainty the same role that discount rates perform for economies under certainty. In economies with no uncertainty, the evolution property is well known under the rubric of "term struc-

[9] Also, for implications and conjectures regarding semigroup pricing in continuous time, see Garman (1985).

ture" properties. For instance, it is widely recognized that a five-period discount rate must be the product of successive two- and three-period discount rates in order to avoid arbitrage. In this section we have merely placed the implicit price matrices of finite economies with uncertainty on an equal footing: a five-period implicit price matrix follows from the product of the two-period and three-period ones. Next, it should be noted that our discussions of the implicit price matrices always involved existence, not uniqueness. Since markets may not be "complete", as discussed in the previous section, the corresponding evolution systems of implicit price matrices may not be unique; multiple evolution systems may solve (R-IT) when arbitrage is absent in the sense of (NA-IT). Finally, it should be noted that in incomplete securities markets, there may also be nonnegative solutions of (R-IT) which do not constitute an evolution system. But if this is so, arbitrage is precluded and so there must also exist an evolution system which solves (R-IT) as well. These points are illustrated by the first example.

Example 1. Suppose there are three dates, 0,1,2, with two states at each date. Assume further that only one security exists, and that its conditional price and dividend matrices are (exogenously) given as

$$P_0 = \begin{bmatrix} 300 \\ 240 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 250 \\ 210 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 200 \\ 200 \end{bmatrix}$$

and

$$D_1 = \begin{bmatrix} 200 \\ 90 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ 100 \end{bmatrix}.$$

It is quickly verified that the set of positive implicit price matrices

$$_0J_1 \equiv \begin{bmatrix} .20 & .70 \\ .40 & .20 \end{bmatrix}, \quad _1J_2 \equiv \begin{bmatrix} .80 & .30 \\ .60 & .30 \end{bmatrix}, \quad _0J_2 \equiv \begin{bmatrix} .31 & .45 \\ .11 & .40 \end{bmatrix}$$

will solve the three representation equations $P_0 = {}_0J_1 D_1 + {}_0J_1 P_1$, $P_0 = {}_0J_1 D_1 + {}_0J_2 D_2 + {}_0J_2 P_2$, and $P_1 = {}_1J_2 D_2 + {}_1J_2 P_2$. But it is also clear that $\{{}_0J_1, {}_1J_2, {}_0J_2\}$ cannot constitute an evolution system, since ${}_0J_2 \neq {}_0J_1\, {}_1J_2$. Nevertheless, the existence and positivity of these implicit price matrices assure the absence of arbitrage opportunities. Therefore it is also guaranteed, by Theorem 1, that at least one implicit price structure having the evolution property exists, for instance

$$_0K_1 = \begin{bmatrix} .40 & .40 \\ .20 & .50 \end{bmatrix}, \quad _1K_2 = \begin{bmatrix} .50 & .50 \\ .30 & .50 \end{bmatrix}, \quad _0K_2 = \begin{bmatrix} .32 & .40 \\ .25 & .35 \end{bmatrix}.$$

The latter matrices can be shown both to solve the representation equations and to possess the evolution system property.

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3. MARKOV VALUATION OF SECURITIES

In this section we characterize the arbitrage-free price of a security as the conditional expected value of its future payouts given the current value of a constructed Markov process. This provides a foundation for statistical testing procedures resting upon Markovian distributional assumptions, and for such theoretical results as the recent characterization of long-term interest rates by Dybvig, Ingersoll, and Ross (1985). As in previous sections we assume a finite economy, generalizing this later on.

We first briefly review the characterization of a Markov process $X = \{X_0, X_1, \cdots, X_T\}$ with state space $\Omega = \{1,2,...,S\}$ in terms of its transition operators $\{{}_t\Pi_\tau\}$, where, for times t and $\tau > t$, ${}_t\Pi_\tau$ is a positive $S \times S$ matrix with rows summing to unity. The (i,j) element of ${}_t\Pi_\tau$ is the probability that $X_\tau = j$ given that $X_t = i$. For X to be Markov, these matrices must satisfy the Chapman-Kolmogorov equation

$$ {}_t\Pi_\tau = {}_t\Pi_s \, {}_s\Pi_\tau \tag{5} $$

whenever $t \le s \le \tau$. If f is any function on Ω, say the payoff of a security at time τ, we can equally well treat f as a vector in $\mathbf{R}^S$, and obtain the relation

$$ E\left[f(X_\tau) \mid X_t = s\right] = \left[{}_t\Pi_\tau f\right]_s , $$

or in vector form,

$$ E\left[f(X_\tau) \mid X_t\right] = {}_t\Pi_\tau f. \tag{6} $$

We now suppose, as in the previous section, that $D_1, D_2, \cdots, D_T$ and $P_0, P_1, ..., P_T$ are $S \times J$ matrices representing the dividends and prices of J given securities at each date $t = 0,1,2,...,T$. Let $p_{jt}(s)$ denote the (s,j) element of P_t, or the price of security j at date t in state s, and likewise for $d_{jt}(s)$.

DEFINITION 4 (Asset Growth Condition). The price-dividend sequence $\{P_t, D_t\}$ satisfies the *asset growth condition* if, for each time t and security j,

$$ \max_s |p_{jt}(s)| \le \max_s |p_{jt+1}(s) + d_{jt+1}(s)|. \tag{7} $$

Interpreting, the asset growth condition means that whatever state obtains currently, one need never invest more in any security than the most one could receive from it the following date. We now add two more definitions, with an eye towards showing the relationship between the asset growth condition and nonnegative interest rates.

DEFINITION 5 (Riskless Asset). If for every date t and state s there exists a portfolio for which the next-date payoff is $1 \equiv (1,1,...,1) \in \mathbf{R}^S$, then we say that the economy is *inhabited by the riskless asset.*

DEFINITION 6 (Impatience). If the implicit price matrices $\{{}_tK_\tau\}$ of an economy can be chosen so that

$$1 \geq {}_tK_\tau 1 \quad \text{for } all\ t \text{ and } \tau \geq t, \tag{8}$$

then we say that the economy exhibits *impatience.*

LEMMA 2 (Nonnegative Interest Rates). If an intertemporally arbitrage-free economy exhibits impatience, then it satisfies the growth condition. If an intertemporally arbitrage-free economy is inhabited by the riskless asset and satisfies the growth condition, then it exhibits impatience.

Proof. We first show that (8) implies (7): From the absence of intertemporal arbitrage, we have nonnegative numbers ${}_tK_{t+1} = [k_t(s,s')]$ such that

$$p_{jt}(s) = \sum_{s'} k_t(s,s')\{p_{jt+1}(s') + d_{jt+1}(s')\}$$

for all j and s. Hence

$$\begin{aligned} |p_{jt}(s)| &= |\sum_{s'} k_t(s,s')\{p_{jt+1}(s') + d_{jt+1}(s')\}| \\ &= \sum_{s'} |\{p_{jt+1}(s') + d_{jt+1}(s')\}| k_t(s,s') \\ &\leq \max_{s'} |\{p_{jt+1}(s') + d_{jt+1}(s')\}| \sum_{s'} k_t(s,s') \\ &\leq \max_{s'} |\{p_{jt+1}(s') + d_{jt+1}(s')\}| \end{aligned}$$

Since this inequality holds for all s, it is seen that (8) implies (7).

Now suppose that the growth condition (7) obtains and that the economy is inhabited by the riskless asset. Then by (7)

$$\max_s \delta_t(s) \leq 1,$$

where the right hand side is the payoff to the riskless asset and the left hand side is the price of the riskless asset. In vector terms, this implies that

$$\delta_t \le 1,$$

since the inequality must then hold for all states s. By the absence of intertemporal arbitrage, we have implicit prices $\{{}_tK_{t+1}\}$ such that

$$\delta_t = {}_tK_{t+1}1.$$

Combining the last two equations then demonstrates (8). ■

Since $[1/({}_tK_\tau 1)_s]-1$ is the implicit $(\tau-t)$-period interest rate for riskless borrowing at date t in state s, the second part of the lemma is fairly interpreted as: impatience implies nonnegative interest rates whenever such rates are well-defined, that is, via the presence of the riskless asset. However, impatience is a somewhat stronger property insofar as it provides a "contraction" condition on implicit prices even when the riskless asset is absent.

Nonnegative matrices $\{{}_tK_\tau\}$ satisfying the evolution property (which in probability theory is referred to as the *Chapman-Kolmogorov condition*) as well as relationship (8) are known as *sub-Markov transition operators.* They can represent the transition probabilities of a Markov process X which can also "die" with non-zero probability. It is conventional to adjoin a new *cemetery state* † to the original state space Ω, and to suppose that such a Markov process is "captured" by the cemetery state when it dies. That is, let $\Omega^\dagger \equiv \Omega \cup \dagger$ and, for each t and $\tau \ge t$, let

$$_t\Pi_\tau \equiv \begin{bmatrix} 1 & 0 \\ 1-{}_tK_\tau 1 & {}_tK_\tau \end{bmatrix}. \tag{9}$$

We can then harmlessly extend the definition of $\{P_t, D_t\}$ to $\Omega^\dagger$ so that $p_{jt}(\dagger)=d_{jt}(\dagger)=0$ for all t and j, leaving the following representation theorem for intertemporally arbitrage-free security valuation.

THEOREM 2 (Markov Valuation). Suppose an intertemporally arbitrage-free economy exhibits impatience. Then there exists a Markov process $X \equiv (X_0, X_1, \cdots, X_T)$ (with state space $\Omega^\dagger$) such that, for any security j and dates $t, \tau \ge t$,

$$p_{jt}(X_t) = E\left[\textstyle\sum_{m=t+1}^{\tau} d_{jm}(X_m) + p_{j\tau}(X_\tau) \mid X_t\right]. \tag{10}$$

Proof. By Theorem 1 and Lemma 2, there exist nonnegative implicit price matrices $\{{}_tK_\tau\}$ satisfying (R-IT), the Chapman-Kolmogorov equations (5), as well as the sub-Markov property (8). By adjoining the cemetery state † as in the previous discussion, we can extend $\{{}_tK_\tau\}$ by (8) to proper Markov transition matrices $\{{}_t\Pi_\tau\}$ for some Markov process X. Then (10) is implied by

(R-IT) and (6). ■

Of course the Markov pricing formula (10) also holds without adjoining a cemetery state by treating X as a sub-Markov process. In that case, the discounting for time is implicit in the probability that X does not survive.

Example 2. We now extend Example 1 to the construction of a Markov valuation process. Recall that an implicit price structure having the evolution property was given, namely

$$ {}_0K_1 = \begin{bmatrix} .40 & .40 \\ .20 & .50 \end{bmatrix}, \quad {}_1K_2 = \begin{bmatrix} .50 & .50 \\ .30 & .50 \end{bmatrix}, \quad {}_0K_2 = \begin{bmatrix} .32 & .40 \\ .25 & .35 \end{bmatrix}. $$

The corresponding Markov transition operators are then

$$ {}_0\Pi_1 = \begin{bmatrix} 1.00 & .00 & .00 \\ .20 & .40 & .40 \\ .30 & .20 & .50 \end{bmatrix}, \quad {}_1\Pi_2 = \begin{bmatrix} 1.00 & .00 & .00 \\ .00 & .50 & .50 \\ .20 & .30 & .50 \end{bmatrix}, \quad {}_0\Pi_2 = \begin{bmatrix} 1.00 & .00 & .00 \\ .28 & .32 & .40 \\ .40 & .25 & .35 \end{bmatrix}. $$

It is readily verified that ${}_0\Pi_1\,{}_1\Pi_2 = {}_0\Pi_2$. Interest rates are determined by examining the nondiagonal elements of the first columns of the Π matrices, which are termed the "killing rates" of the Markov process. Then the interest rate in state and date is directly related to the corresponding killing rate; for example, the interest rate between state 1 at date 0 and date 2 is the killing rate divided by its difference from 1.0, namely .28/(1-.28) = .389 over the two periods.

It may be remarked that the Markov valuation formula (10) is less general than the Harrison-Kreps (1979) martingale measure valuation model, but offers a more specific pricing structure. In the present setting, the existence of a Harrison-Kreps martingale measure implies, for dates t and $\tau \geq t$ and some security j, that there exists an expectation operator E^* such that

$$ P_t = E^* \left[\textstyle\sum_{m=t+1}^{\tau} D_m + P_\tau | F_t \right], \tag{11} $$

where F_t is the information set (σ-algebra for information) at date t. There is nothing in (11) suggesting the existence of a Markov process X satisfying (10). Indeed, even if $p_{jt}(\,)$ and $d_{jt}(\,)$ are functions of an exogenously given Markov process $Y = \{Y_1, Y_2, Y_3, Y_4\}$, it is *not* generally the case that Y would retain the Markov property under the change to a martingale measure inherent within (11). For instance, take Y to be a three-state process with $T=4$ and two securities, one risky and one a riskless numeraire. Barring arbitrage, one can then construct a martingale measure such that (11) holds, but with the property that the transition probabilities at date 3 for Y_4 depend upon Y_2, contradicting the

Markov property. Of course one can also construct Markov transition probabilities giving risk-neutral pricing; that is the point of Theorem 2. It should also be pointed out that, for Theorem 2, one need not assume the existence of a riskless numeraire, or otherwise normalize prices. In the Harrison-Kreps model, the discounting for time is implicit in their normalization to a prescribed numeraire spanning all dates; by contrast, the Markov valuation model has time-discounting implicit in the survival probabilities of the sub-Markov "pricing" process. This generality is not without its cost: we must assume that the given security prices and dividends are path-independent, or functions only of the current state, and not (in addition) of the historical path of states. Whether this cost is severe depends upon the difficulty of re-defining states so as to transform a path-dependent structure into an equivalent path-independent one, a context-specific procedure.[10]

4. TRANSFORM ANALYSIS OF THE STATIONARY CASE

Suppose the implicit price matrices $\{{}_tK_\tau\}$ for some impatient, intertemporally arbitrage-free economy are generated by the one-period constant matrix K. We then have, for any dates t and $\tau \geq t$,

$$P_t = \sum_{m=1}^{\tau-t} K^m D_{t+m} + K^{\tau-t} P_\tau. \tag{12}$$

We now employ z-transforms for a convenient closed-form solution of (12). For any sub-geometric (say bounded) sequence $F \equiv (F_1, F_2, \cdots)$, let $\hat{F}$ denote the z-transform of F, defined by

$$\hat{F}(z) \equiv \sum_{n=0}^{\infty} z^n F_n, \quad 0 \leq z < 1.$$

There is a one-to-one correspondence between such sequences F and their z-transforms $\hat{F}$. We call F the *inverse transform* of $\hat{F}$. (One may consult standard tables for z-transform pairs.) After some manipulation, the z-transform definition tells us that the sequence $F_n \equiv K^n$, $n = 0,1,2,...$, has z-transform $\hat{F}(z) = (I - zK)^{-1}$. For a security paying the constant dividend vector $d \equiv d_{jt} = (d_1, d_2, \cdots, d_S)$, we will also want to compute the sum $\sum_{m=1}^{n} K^m d$, and can make use of the z-transform $\hat{G}$ of $G(n) = \sum_{m=0}^{n-1} K^n$. Some manipulation then yields

$$\hat{G}(z) = \frac{z}{1-z}(I - zK)^{-1}.$$

[10] On a historical note, the notion that security prices may be treated as their expected payoff under some probabilities goes back to a comment in Arrow (1953).

Example 3. Suppose stationarity holds in such a fashion that

$$ {}_tK_{t+1} = K = \begin{bmatrix} 1/2 & 1/6 \\ 1/3 & 1/3 \end{bmatrix}. $$

Then by factorization we find that

$$ (I - zK)^{-1} = \frac{1}{(1-z/6)(1-2z/3)} \begin{bmatrix} 1-z/3 & z/6 \\ z/3 & 1-z/2 \end{bmatrix}. $$

Applying a partial-fractions expansion then yields

$$ (I - zK)^{-1} = \frac{1}{(1-2z/3)} \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} + \frac{1}{(1-z/6)} \begin{bmatrix} 1/3 & -1/3 \\ -2/3 & 2/3 \end{bmatrix} $$

Let $\hat{F}(z) = (I - zK)^{-1}$. Noting that the inverse z-transform of $1/(1-\alpha z)$ is α^n, we have the inverse transform

$$ F(n) = (2/3)^n \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} + (1/6)^n \begin{bmatrix} 1/3 & -1/3 \\ -2/3 & 2/3 \end{bmatrix}. $$

As claimed, $K^n = F(n)$ for all $n \geq 1$. A portfolio paying $a \equiv (-3,9)'$ at a date n periods from the present is thus currently valued at

$$ (2/3)^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1/6)^n \begin{bmatrix} -4 \\ 8 \end{bmatrix}. $$

Let $G(n) = \sum_{m=0}^{n-1} K^m$. Then

$$ \hat{G}(z) = \frac{z}{1-z}(I - zK)^{-1} $$

$$ = \frac{z}{(1-z)(1-z/6)} \begin{bmatrix} 1/3 & -1/3 \\ -2/3 & 2/3 \end{bmatrix} + \frac{z}{(1-z)(1-2z/3)} \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}. $$

Again by partial-fraction expansion,

$$ \hat{G}(z) = \left[\frac{6/5}{1-z} - \frac{6/5}{1-z/6} \right] \begin{bmatrix} 1/3 & -1/3 \\ -2/3 & 2/3 \end{bmatrix} + \left[\frac{3}{1-z} - \frac{3}{1-z/6} \right] \begin{bmatrix} 2/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix}. $$

Taking inverse transforms and distributing the constants,

$$ G(n) = \left[1 - (1/6)^n \right] \begin{bmatrix} 2/5 & -2/5 \\ -4/5 & 4/5 \end{bmatrix} + \left[1 - (2/3)^n \right] \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}. $$

Suppose a given portfolio pays the dividend vector $d = (3,12)'$ for each of the next n periods, counting the current period. The *cum*-dividend market value of the portfolio is then given as

$$ G(n)d = \begin{bmatrix} 72/5 \\ 126/5 \end{bmatrix} + (1/6)^n \begin{bmatrix} 18/5 \\ -36/5 \end{bmatrix} - (2/3)^n \begin{bmatrix} 18 \\ 18 \end{bmatrix}. $$

5. THE INFINITE-HORIZON STATIONARY CASE

We again consider the stationary case ${}_tK_{t+1} = K$ for all t, yielding the pricing relation (12). We furthermore now assume *strict impatience*, in the sense that (8) holds with strict inequality. Then the implicit price matrix K can be chosen to be "strictly contractive," meaning that all of its rows sum to strictly less than unity. Strict impatience then implies that $K^m \to 0$ as $m \to \infty$. We now consider a security whose price and dividend vectors p and d are independent of time. (This may be thought of as a "perpetual" security which always pays the same dividend, given the state; of course, such dividends may be state-dependent.) Since (12) implies that

$$p = \sum_{m=1}^{n} K^m d + K^n p$$

for all $n \geq 1$, we can let $n \to \infty$, leaving

$$p = \sum_{m=1}^{\infty} K^m d = G\, d,$$

where $G \equiv \sum_{m=1}^{\infty} K^m$. By a well-known series calculation, $G = (I - K)^{-1} - I$ where I is the $S \times S$ identity matrix. Strict impatience implies the convergence of this series calculation, which is then seen to provide for the valuation of all "perpetual" securities.

Example 4. Consider again, from Example 3,

$$K = \begin{bmatrix} 1/2 & 1/6 \\ 1/3 & 1/3 \end{bmatrix}.$$

We calculate

$$(I - K)^{-1} - I = \begin{bmatrix} 7/5 & 3/5 \\ 6/5 & 4/5 \end{bmatrix}.$$

Suppose a security always pays a dividend of 1 in state 1 and 2 in state 2, or $d = (1,2)'$. Then

$$p = \left[(I - K)^{-1} - I\right] d = \begin{bmatrix} 7/5 & 3/5 \\ 6/5 & 4/5 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 13/5 \\ 14/5 \end{bmatrix}$$

If security prices are calculated *cum*-dividend rather than *ex*-dividend, we have $p = (I - K)^{-1} d$. The matrix $(I - K)^{-1}$ is known as the *potential operator* corresponding[11] to the semigroup $\{K^m\}$.

For the type of result explored by Dybvig, Ingersoll, and Ross (1985), wherein the Markov valuation process X must satisfy a "mixing" condition, we may wish the transition operators for X to be strictly positive. A sufficient "strict no arbitrage" condition is stated in the Appendix.

6. A COMPLETE EQUILIBRIUM EXAMPLE

Consider an economy with the following primitives. An exogenously given S-state Markov process Y is governed by the stationary transition matrix Π. There are J securities defined by the dividend

[11] For further results connecting potential theory to the valuation of securities, see Duffie (1985).

functions $d = (d_1, d_2, \cdots, d_J)$ where security j pays[12] $d_j(s) \geq 0$ at date t if Y_t is in state s, for every t. Aggregate consumption is thus $C(s) \equiv \sum_{j=1}^{J} d_j(s)$ in state s at any date. Subject to a short-sales restriction and a budget constraint, the representative agent is free to hold securities in arbitrary amounts, with the objective of maximizing lifetime expected utility for consumption. Assuming that this agent possesses an additively separable utility function, he or she faces the problem:

$$\max_{c,b} E\left[\sum_{\tau=t}^{T} \rho^{\tau-t} u(c_\tau) \mid Y_t\right]$$

subject to, for all t, the consumption, portfolio, and budget constraints

$$c_t \geq 0, \quad b_{t+1} \geq 0,$$

$$c_t + p'_t\, b_{t+1} \leq W_t,$$

$$W_t \equiv b'_t\, [p_t + d(Y_t)],$$

where $0 < \rho < 1$ is a time-impatience factor, u is a strictly concave increasing differentiable function, c_t is the current consumption choice, b_{t+1} is the current portfolio of securities chosen at date t to carry to date $t+1$, and p_t is the current vector of securities prices. An equilibrium for this (one-agent) economy is easily show to exist in the form: $c_t = C(Y_t)$, $b_t = (1,1,...,1)$, and for all t,

$$p_t = \frac{1}{u'(C(Y_t))} E\left[\sum_{\tau=t+1}^{T} \rho^{\tau-t} u'(C(Y_t)) d(Y_t) \mid Y_t\right].$$

Thus we have the implicit price matrices

$$_tK_\tau = A\, \Pi^{\tau-t} \rho^{\tau-t} A^{-1},$$

where A is the diagonal $S \times S$ matrix whose s-th diagonal element is $u'(C(s))$. This example yields stationary pricing, as in Example 3, with $_tK_{t+1} = K = \rho A \Pi A^{-1}$. We note, however, that in contrast to the infinite horizon version of Lucas (1978), p_t is not a constant function of Y_t, but depends on the amount of time remaining. We do, nonetheless, have what Lucas has called the "stochastic Euler equation":

$$p_t = A^{-1} \rho \Pi A\, (d + p_{t+1}),$$

where p_t and d are treated as $S \times J$ matrices in the obvious manner.

[12] The assumption of nonnegative dividends and nonnegative security portfolios may be weakened somewhat, but is convenient here.

In order to apply our z-transform methods for a closed-form solution, let

$$V(T-t) \equiv A(p_t+d), \tag{14}$$

$$q \equiv A\,d,$$

so that $V(n)$ is the product of marginal utility and (*cum*-dividend) prices with n periods left to go. We then have

$$V(n+1) = q + \rho\Pi V(n), \quad 1 \le n \le T. \tag{15}$$

Letting $\hat{V}$ denote the z-transform of V, we take the z-transform of both sides of (14) to obtain

$$\frac{1}{z}[\hat{V}(z) - V(0)] = \frac{1}{1-z}q + \rho\Pi\hat{V}(z),$$

which after some manipulation yields

$$\hat{V}(z) = \frac{z}{1-z}(I-\rho z\Pi)^{-1}q + (I-\rho z\Pi)^{-1}V(0).$$

We have also the boundary condition $V(0)=q$, since $p_T=0$. This leaves

$$\hat{V}(z) = \hat{G}(z)q$$

where $\hat{G}(z) \equiv z/(1-z)(I-\rho z\Pi)^{-1}$. Taking inverse z-transforms, we see that

$$V(n) = G(n)q,$$

where G is the inverse transform of $\hat{G}$. Finally, from (14),

$$p_t = A^{-1}V(T-t) - d = [A^{-1}G(T-t)A - I]d. \tag{16}$$

For a specific numerical example,[13] suppose that $\rho=.5$ and Y is a two-state process with transition matrix

$$\Pi = \begin{bmatrix} 1/2 & 1/2 \\ 2/5 & 3/5 \end{bmatrix}$$

Thus by factorization and partial fraction decomposition,

$$\hat{G}(z) = \frac{1}{1-z}\begin{bmatrix} 28/9 & 10/19 \\ 8/19 & 30/19 \end{bmatrix} + \frac{1}{1-z/2}\begin{bmatrix} -8/9 & -10/9 \\ -8/9 & -10/9 \end{bmatrix} + \frac{1}{1-z/20}\begin{bmatrix} -100/171 & 100/171 \\ 80/171 & -80/171 \end{bmatrix}.$$

Taking inverse transforms

$$G(n) = \begin{bmatrix} 28/19 & 10/19 \\ 8/19 & 30/19 \end{bmatrix} + \frac{1}{2^n}\begin{bmatrix} -8/9 & -10/9 \\ -8/9 & -10/9 \end{bmatrix} + \frac{1}{20^n}\begin{bmatrix} -100/171 & 100/171 \\ 80/171 & -80/171 \end{bmatrix}. \tag{17}$$

Suppose aggregate consumption is $C(1) = 4$ in state 1, $C(2) = 9$ in state 2, and that $u(c_t) = (c_t)^{.5}$.

[13] These numerical calculations are from a different application by Howard (1960), pp.78-79.

Then $u'(c_t) = .5(c_t)^{-.5}$ and

$$A = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/6 \end{bmatrix}; \quad A^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}.$$

Taking a security dividend to be, say, $d_j = (4,3)'$, we can use (16) to easily calculate a closed-form solution for p_{jt} for all dates t, namely

$$p_{jt} = \begin{bmatrix} 132/19 \\ 138/19 \end{bmatrix} + \left[\frac{1}{2}\right]^{T-t} \begin{bmatrix} -52/9 \\ -78/9 \end{bmatrix} + \left[\frac{1}{20}\right]^{T-t} \begin{bmatrix} -200/171 \\ 240/171 \end{bmatrix} - \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$

One may drop the last term for *cum*-dividend pricing, which we find to be a more tractable convention in this type of model. We note that, as $T-t \to \infty$, the limit $p_{jt} \to (56/19,81/19)'$ obtains. This is of course precisely the price vector prescribed in Lucas' infinite horizon model. In matrix form the Lucas model is then

$$P = (A^{-1}G_\rho A - I)D, \tag{18}$$

where

$$G_\rho \equiv \textstyle\sum_{t=0}^{\infty}\rho^t\Pi^t = (I - \rho\Pi)^{-1}$$

is termed the *resolvent operator*[14] For the given numerical example,

$$G_\rho = \begin{bmatrix} 252/171 & 90/171 \\ 72/171 & 270/171 \end{bmatrix}.$$

7. THE GENERAL STATE SPACE

In some cases, the state space cannot reasonably be deemed finite. The state space may include, for example, a continuum of capital stock levels in the economy. In this section we extend our results to a general state space (Ω,F,μ), where F is a σ-algebra of subsets of Ω and μ is a σ-finite measure on (Ω,F). A typical example would take Ω to be $\mathbf{R}^n$ (or a measurable subset of $\mathbf{R}^n$), with the usual (Borel) subsets F and the usual (Lebesgue) measure μ. This would handle the case of an n-dimensional state vector. We allow the portfolio space L_t at any date t to be a (possibly infinite-dimensional) subspace of $L^\infty = L^\infty(\Omega,F,\mu)$, the space of (equivalence classes) of essentially bounded, measurable functions on Ω. A given portfolio payoff a in L_{t+1} is thus treated as a (bounded) random variable representing a payoff of $a(\omega)$ in the state $\omega\in\Omega$ at date $t+1$. We assume the availability of riskless borrowing in the form of portfolios having payoff $1\in L_t$ for all t, where $1(\omega)\equiv 1$, for all $\omega\in\Omega$. The price

[14] See Duffie (1985) for the properties of the resolvent operator.

of the security $a \in L_{t+1}$ at date t is denoted as $V_t a$, and is also taken to be a random variable in L^∞ whose value $[V_t a](\omega)$ in state ω represents the price at date t in state ω of a claim to the payoff a at date $t+1$. By the definition of portfolio formation, L_{t+1} is a linear subspace of L^∞. By the linearity of prices of portfolios and an assumed absence of arbitrage, V_t is a linear operator (from L_{t+1} into L^∞). The absence of arbitrage additionally implies that V_t is a *positive* operator, or $V_t x \geq 0$ whenever $x \geq 0$, denoted $V_t \geq 0$. (Recall our conventions on matrix inequalities.) More formally, we shall say that $\{V_t, t = 0,1,...\}$ is *arbitrage-free* if $V_t \geq 0$ for all t.

The norm $||a||$ of a random variable a in L^∞ is defined to be the essential supremum of $\{|a(\omega)|, \omega \in \Omega\}$. The norm of a linear operator V on a linear subspace L of L^∞ into L^∞ is defined to be $||V|| \equiv sup\{||Va||; a \in L, ||a|| \leq 1\}$. If $||V|| < \infty$, we say V is continuous. The general-state-space analogue to the finite-state-space impatience condition (8) is then

$$||V_t|| \leq 1 \quad \textit{for all } t. \tag{19}$$

Our notion of an implicit price matrix ${}_tK_{t+1}$ from the finite-state-space case generalizes to the concept of an *implicit price operator* ${}_tK_{t+1}$: $L^\infty \to L^\infty$, a positive linear operator that extends V_t to L^∞, meaning that ${}_tK_{t+1}a = V_t a$ for all a in L_{t+1}. For a Markov valuation formula, we would also like ${}_tK_{t+1}$ to be a norm-preserving extension, that is, $||{}_tK_{t+1}|| = ||V_t||$. The existence of such an extension is established by the following result from Duffie (1985).

LEMMA 3 (Valuation Extension). Suppose L is a linear subspace of L^∞ and $V: L \to L^\infty$ is a positive continuous linear operator. If $1 \in L$, then V has a positive, linear, norm-preserving extension $K: L^\infty \to L^\infty$.

A sub-Markov process $X = \{X_0, X_1, \cdots\}$ with state space (Ω, F, μ) is characterized by a collection $\{\Pi_0, \Pi_1, \cdots\}$ of positive linear operators with $||\Pi_t|| \leq 1$ for all t, having the property:

$$[\Pi_t f](s) = E\left[f(X_{t+1}) | X_t = s\right] \tag{20}$$

for all t and all f in L^∞. Thus, given the previous Extension Lemma, we immediately recover the general-state-space analogue to the Markov Valuation Theorem 2:

THEOREM 3 (Markov Valuation in a General State Space). Suppose $\{V_0, V_1, \cdots\}$ is arbitrage-free and exhibits impatience (19). Then there exists a sub-Markov process

$X = \{X_0, X_1, \cdots\}$ such that, for any dates t and $\tau \geq t$, and any security j,

$$p_{jt}(s) = E\left[\sum_{m=t+1}^{\tau} d_{jm}(X_m) + p_{j\tau}(X_\tau) \mid X_t = s\right]. \tag{21}$$

Comment. As with the finite-state-space case, we can take X to be a proper (rather than a sub-) Markov process by adjoining a cemetery state $\dagger$, and carrying out a few measure-theoretic technicalities.

Proof. By the conditions on $\{V_t\}$ and Lemma 3, there exist positive, linear, norm-preserving extensions $\{\Pi_t\}$ of $\{V_t\}$. Let X be a sub-Markov process with one-step transition operators $\{\Pi_t\}$. Then (21) follows from (20) and iteration. ■

Relation (21) also holds in the infinite-horizon case. For convergence to the pricing formula

$$p_{jt}(s) = E\left[\sum_{m=t+1}^{\infty} d_{jm}(X_m) \mid X_t = s\right]$$

one needs an additional condition, for example *strict impatience:* $\|V_t\| \leq \varepsilon < 1$ for all t. In the stationary case with strict impatience, we have $p = Gd$, where G is the potential operator associated with a positive, linear, norm-preserving extension of $V = V_t$, $d \in L^\infty$ is the dividend function (of the state) paid by a security at each date, and p is the price of the security, again a function of the state. Further analysis of the stationary infinite-horizon case is provided in Duffie (1985).

8. CONCLUSIONS

This paper adopts an intertemporal viewpoint on arbitrage, which provides implications beyond the standard representation results. In particular, we find that in a finite-state, finite-date economy, no-arbitrage as a local condition (that is, between adjacent trading dates) implies the global absence of arbitrage. Moreover, such absence of arbitrage exists if and only if there exists a set of nonnegative implicit prices exhibiting the evolution property. Provided that the economy also exhibits impatience, these implicit price matrices represent sub-Markov transition operators under which the prices of securities are the expected values of future payoffs. When a cemetery state is adjoined to the state space, a proper Markov process results instead. That is, associated with every intertemporally arbitrage-free finite economy which exhibits impatience is a Markov process which values all securities as their expected future payoffs. The z-transform is introduced as a useful tool in computing valuation quantities. A complete equilibrium example along the lines of Lucas (1978) is presented. Finally, the analysis is extended to the infinite-horizon and general-state-space settings.

APPENDIX: Strict No-Arbitrage

By strengthening somewhat the definition of no-arbitrage, we could take the matrices $\{{}_tK_\tau\}$ to be *strictly* positive via

DEFINITION 1' (Strict No Arbitrage, Intertemporal): A finite economy is *strictly free of intertemporal arbitrage* if, for any portfolio $x \in \mathbf{R}^J$ satisfying, for some $\tau > t$,

$$D_{t+1}x \geq 0,$$

$$\dots$$

$$D_{\tau-1}x \geq 0,$$

$$\{P_\tau + D_\tau\}x \geq 0,$$

with *strict* inequality holding for some u with $t+1 \leq u \leq \tau$, it follows that $P_t x \gg 0$.

Restricted to a static setting, strict no-arbitrage is equivalent to the "no free lunch" condition of Kreps (1981) for finite economies. In our intertemporal setting, the corresponding counterpart of the law of one price is

DEFINITION 2' (Strict Law of One Price, Intertemporal): If there exists a set of strictly positive implicit price matrices $\{{}_tK_\tau\}$ for which

$$P_t = \sum_{m=t+1}^{\tau} {}_tK_m D_m + {}_tK_\tau P_\tau \qquad \text{(R-IT)}$$

and, in addition, the matrices ${}_tK_\tau$ possess the evolution property, in that

$${}_tK_\tau = {}_tK_v \, {}_vK_\tau \quad \text{for } all \quad t \leq v \leq \tau,$$

then the economy satisfies the *strict intertemporal law of one price*.

The following analogue to Theorem 1 asserts the equivalence between the two foregoing definitions, which is demonstrated via Stiemke's Lemma, a close relative of Farkas' Lemma. (Strict impatience may then be added to satisfy the need for convergence in the infinite-horizon case.)

THEOREM 4 (Positive Evolution System): A finite economy is strictly free of intertemporal arbitrage if and only if it satisfies the strict intertemporal law of one price. Moreover, there exists a sub-Markov process X with strictly positive transition operators satisfying (10) under either of the additional conditions:

(i) the economy is inhabited by a riskless asset and exhibits impatience (nonnegative interest rates); or

(ii) the economy exhibits strict impatience ((8) with strict inequality).

Proof. By Stiemke's Lemma (see e.g. Mangasarian (1969, p.34)), if the economy is strictly free of intertemporal arbitrage, the implicit price matrices $\{{}_t\overline{K}_{t+1}\}$ can be chosen to be strictly positive. This proves the first assertion. For the second assertion, suppose (i), i.e. inhabitation by a riskless asset and impatience. Then we have ${}_t\overline{K}_{t+1}1 \le 1$ by the proof of Lemma 2, and we are done. Alternatively, suppose (ii), that we can choose implicit price matrices $\{{}_t\hat{K}_{t+1}\}$ such that

$$ {}_t\hat{K}_{t+1}1 \ll 1 \quad \text{for } all \ t. \tag{22} $$

For sufficiently small $\alpha \in (0,1)$, we have ${}_t\tilde{K}_{t+1} \equiv \alpha\, {}_t\overline{K}_{t+1} + (1-\alpha)\, {}_t\hat{K}_{t+1}$ satisfying ${}_t\tilde{K}_{t+1}1 \ll 1$ for all t. Thus $\{{}_t\tilde{K}_\tau \equiv {}_t\tilde{K}_{t+1} \cdots {}_{\tau-1}\tilde{K}_\tau\}$ are satisfactory implicit price matrices, clearly possessing the evolution property. ■

In simple economic terms, Definition 1' admits arbitrage in the following case where Definition 1 does not: Suppose a portfolio may be purchased for nothing and yet pays off a positive amount in some future state, but that it is nonetheless impossible to resell any amount of this contingent claim for more than zero. In this case, the arbitrageur must reflect on the possibility that the economy may value the claim at zero because either the state is deemed impossible or catastrophic. (Consider the value of a claim that pays $1 if the world is totally destroyed; is buying this claim for nothing an arbitrage opportunity?) Under strict no-arbitrage, the modeler may be burdened with more careful selection of the appropriate state space, whereas this may be achieved with simpler forms of consensus assessment in the ordinary no-arbitrage case. In finite economies it appears unnecessary to theoretically confront the issue. However, it can be conjectured that, in analogy to the usual treatments of continuous-time stochastic processes, it will prove much more convenient in continuous-time economies to employ strictly positive valuation operators. For this reason, the above alternative definition of arbitrage bears serious consideration.

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