# Credit Swap Valuation

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*This review of the pricing of credit swaps, a form of derivative security that can be viewed as default insurance on loans or bonds, begins with a description of the credit swap contract, turns to pricing by reference to spreads over the risk-free rate of par floating-rate bonds of the same quality, and then considers model-based pricing. The role of asset swap spreads as a reference for pricing credit swaps is also considered.*

Credit swaps pay the buyer of protection a given contingent amount at the time of a given credit event, such as a default. The contingent amount is often the difference between the face value of a bond and its market value and is paid at the time the underlying bond defaults. The buyer of protection pays an annuity premium until the time of the credit event or the maturity date of the credit swap, whichever is first. The credit event must be documented with a notice, supported with evidence of public announcement of the event in, for example, the international press. The amount to be paid at the time of the credit event is determined by one or more third parties and based on physical or cash settlement, as indicated in the confirmation form of the OTC credit swap transaction, a standard contract form with indicated alternatives.

The term "swap" applies to credit swaps because they can be viewed, under certain ideal conditions to be explained in this article, as a swap of a default-free floating-rate note for a defaultable floating-rate note.

Credit swaps are currently perhaps the most popular of credit derivatives.[1] Unlike many other derivative forms, in a credit swap, payment to the buyer of protection is triggered by a contractually defined event that must be documented.

## The Basics

The basic credit swap contract is as follows. Parties A and B enter into a contract terminating at the time of a given credit event or at a stated maturity, whichever is first. A commonly stipulated credit event is default by a named issuer—say, Entity C, which could be a corporation or a sovereign issuer. Credit events may be defined in terms of downgrades, events that could instigate the default of one or more counterparties, or other credit-related occurrences.[2] Swaps involve some risk of disagreement about whether the event has, in fact, occurred, but in this discussion of valuing the credit swap, such risk of documentation or enforceability will be ignored.

In the event of termination at the designated credit event, Party A pays Party B a stipulated termination amount. For example, in the most common form of credit swap, called a "default swap," if the termination is triggered by the default of Entity C, A pays B an amount that is, in effect, the difference between the face value and the market value of the designated note issued by C.

In compensation for what it may receive in the event of termination by a credit event, until the maturity of the credit swap or termination by the designated credit event, Party B pays Party A an annuity at a rate called the "credit swap spread" or, sometimes, the "credit swap premium."

The cash flows of a credit swap are illustrated in **Figure 1**, where $U$ is the swap's annuity coupon rate, $\tau$ is the time of the default event, $Y(\tau)$ is the market value of the designated underlying note at time $\tau$, and $T$ is the maturity date. The payment at credit time $\tau$, if before maturity $T$, is the difference, $D$, between the underlying note's face value—100 units, for example—and $Y(\tau)$, or in this case, $D = 100 - Y(\tau)$.

For instance, in some cases, the compensating annuity may be paid as a spread over the usual plain-vanilla (noncredit) swap rate.[3] For example, if the five-year fixed-for-floating interest rate swap rate is 6 percent versus LIBOR and B is the fixed-rate payer in the default swap, then B pays a fixed rate higher than the usual 6 percent. If, for example, B pays 7.5 percent fixed versus LIBOR and if the C-issued note underlying the default swap is of the same notional amount as the interest rate swap, then

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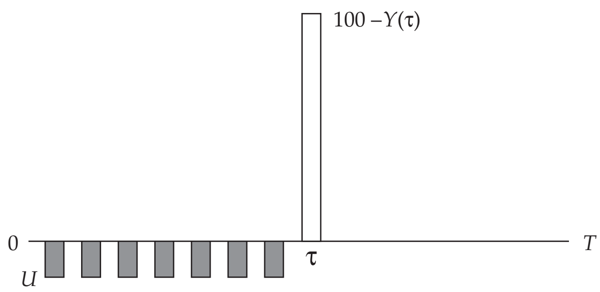
**Figure 1. Credit Swap Cash Flows**




*Note*: Receive par less market value $Y(\tau)$ of underlying note at $\tau$ if $\tau \le T$.

in this case, the default swap spread is 150 basis points (bps). If B is the floating-rate payer on the interest rate swap, then B pays floating plus a spread in return for the usual market fixed rate on swaps or, in effect, receives fixed less a spread. The theoretical default swap spread is not necessarily the same in the case of B paying fixed as in B paying floating.

In general, combining the credit swap with an interest rate swap affects the quoted credit swap spread because an interest rate swap whose fixed rate is the at-market swap rate for maturity $T$ but has a random early termination does not have a market value of zero. For example, if the term structure of forward rates is steeply upward sloping, then an at-market interest rate swap to maturity $T$ or the credit event time, whichever is first, has a lower fixed rate than a plain-vanilla at-market interest rate swap to maturity $T$. A credit spread of 150 bps over the at-market plain-vanilla swap rate to maturity $T$, therefore, represents a larger credit spread than does a credit swap without an interest rate swap that pays a premium of 150 bps.

Apparently, when corporate bonds are the underlying securities, default swaps in which the payment at default is reduced by the accrued portion of the credit swap premium are not unusual. This variation is briefly considered later.

In short, the classic credit swap can be thought of as an insurance contract in which the insured party pays an insurance premium in return for coverage against a loss that may occur because of a credit event.

The credit swap involves two pricing problems:

- At origination, the standard credit swap involves no exchange of cash flows and, therefore (ignoring dealer margins and transaction costs), has a market value of zero. One must, however, determine the at-market annuity premium rate, $U$, for which the market value of the credit swap is indeed zero. This at-market rate is the credit swap premium, sometimes called the "market credit swap spread."
- After origination, changes in market interest rates and in the credit quality of the issuing entity, as well as the passage of time, typically change the market value of the credit swap. For a given credit swap with stated annuity rate $U$, one must then determine the current market value, which is not generally zero.

When making markets, the first pricing problem is the more critical. When hedging or marking to market, the second problem is relevant. Methods for solving the two problems are similar. The second problem is generally the more challenging because off-market default swaps have less liquidity and because pricing references, such as bond spreads, are of relatively less use.

This article considers simple credit swaps and their extensions.[4] In all the following discussions, the credit swap counterparties A and B are assumed to be default free in order to avoid dealing here with the pricing impact of default by counterparties A and B, which can be treated by the first-to-default results in Duffie (1998b).

## Simple Credit Swap Spreads

For this section, the contingent-payment amount specified in the credit swap (the amount to be paid if the credit event occurs) is the difference between the face value of a note issued by Entity C and the note's market value $Y(\tau)$ at the credit event time, $\tau$—that is, the contingent-payment amount is $D = 100 - Y(\tau)$.

**Starter Case.** The assumptions for this starter case are as follows:

- The swap involves no embedded interest rate swap. That is, the default swap is an exchange of a constant coupon rate, $U$, paid by Party B until termination at maturity or at the stated credit event (which may or may not be default of the underlying C-issued note.) This constraint eliminates the need to consider the value of an interest rate swap with early termination at a credit event.
- There is no payment of the accrued credit swap premium at default.
- The underlying note issued by C is a par floating-rate note (FRN) with the maturity of the credit swap. This important restriction will be relaxed later.
- For this starter case, the assumption is that an investor can create a short position by selling today the underlying C-issued note for its current market value and can buy back the note on the date of the credit event, or on the credit swap maturity date, at its then-current market value, with no other cash flows.

- A default-free FRN exists with floating rate $R_t$ at date $t$. The coupon payments on the FRN issued by C (the C-FRN) are contractually specified to be $R_t + S$, the floating rate plus a fixed spread, $S$. In practice, FRN spreads are usually relative to LIBOR or some other benchmark floating rate that need not be a pure default-free rate. Having the pure default-free floating rate and reference rate (which might be LIBOR) differ by a constant poses no difficulties for this analysis. (Bear in mind that the short-term U.S. Treasury rate is not a pure default-free interest rate because of repo [repurchase agreement] "specials" [discussed later] and the "moneyness" or tax advantages of Treasuries.[5] A better benchmark for risk-free borrowing is the term general collateral rate, which is close to a default-free rate and has typically been close to LIBOR, with a slowly varying spread to LIBOR in U.S. markets.) For example, suppose the C-FRN is at a spread of 100 bps to LIBOR, which is at a spread to the general collateral rate that, although varying over time, is approximately 5 bps. Then, for purposes of this analysis, an approximation of the spread of the C-FRN to the default-free floating rate would be 105 bps.
- In cash markets for the default-free note and C-FRN, there are no transaction costs, such as bid–ask spreads. In particular, at the initiation of the credit swap, an investor can sell the underlying C-FRN at its market value. At termination, the assumption is that an investor can buy the C-FRN at market value.
- The termination payment if a credit event occurs is made at the immediately following coupon date on the underlying C-issued note. (If not, the question of accrued interest arises and can be accommodated by standard time value of money calculations, shown later.)
- If the credit swap is terminated by the stated credit event, the swap is settled by the physical delivery of the C-FRN in exchange for cash in the amount of its face value. (Many credit swaps are settled in cash and, so far, neither physical nor cash settlement seems to have gained predominance as the standard method.)
- Tax effects can be ignored. (If not, the calculations to be made are applied after tax and using the tax rate of investors that are indifferent to purchasing the default swap at its market price.)

With these assumptions, one can "price" the credit swap; that is, one can compute the at-market credit swap spread, on the basis of a synthesis of Party B's cash flows on the credit swap, by the following arbitrage argument:

An investor can short the par C-FRN for an initial cash receivable of, say, 100 units of account and invest the 100 units in a par default-free FRN. The investor holds this portfolio through maturity or the stated credit event. In the meantime, the investor pays the floating rate plus spread on the C-FRN and receives the floating rate on the default-free FRN. The net paid is the spread.

If the credit event does not occur before maturity, both notes mature at par value and no net cash flow occurs at termination.

If the credit event does occur before maturity, the investor liquidates the portfolio at the coupon date immediately following the event and collects the difference between the market value of the default-free FRN (which is par on a coupon date) and the market value of the C-FRN—in this example, the difference is $D = 100 - Y(\tau)$. (Liquidation calls for termination of the short position in the C-FRN, which involves buying the C-FRN in the market for delivery against the short sale through, for example, the completion of a repo contract.)

Because this contingent amount, the difference $D$, is the same as the amount specified in the credit swap contract, the absence of arbitrage implies that the unique arbitrage-free at-market credit swap spread, denoted $U$, is $S$, the spread over the risk-free rate on the underlying floating-rate notes issued by C. (That is, combining this strategy with Party A's cash flows as the seller of the credit swap results in a net constant annuity cash flow of $U - S$ until maturity or termination. Therefore, in the absence of other costs, for no arbitrage to exist, $U$ must equal $S$.)

This arbitrage under its ideal assumptions, is illustrated in **Figure 2**.

**Extension: The Reference Par Spread for Default Swaps.** Provided the credit swap is, in fact, a default swap, the restrictive assumption that the underlying note has the same maturity as the credit swap can be relaxed. In this case, the relevant par spread for fixing the credit swap spread is that of a (possibly different) C-issued FRN that is of the same maturity as the credit swap and of the same priority as the underlying note. This note is the "reference C-FRN." As long as absolute priority applies at default (so that the underlying note and the reference note have the same recovery value at default), the previous arbitrage pricing argument applies. This argument works, under the stated assumptions, even if the underlying note is a fixed-rate note of the same seniority as the reference C-FRN.

Some cautions are in order here. First, often no reference C-FRN exists. Second, absolute priority need not apply in practice. For example, a senior short-maturity FRN and a senior long-maturity

**Figure 2. Synthetic Credit Swap Cash Flows**

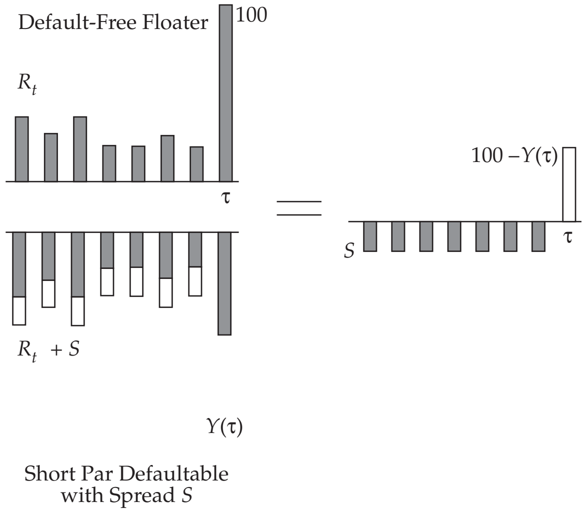
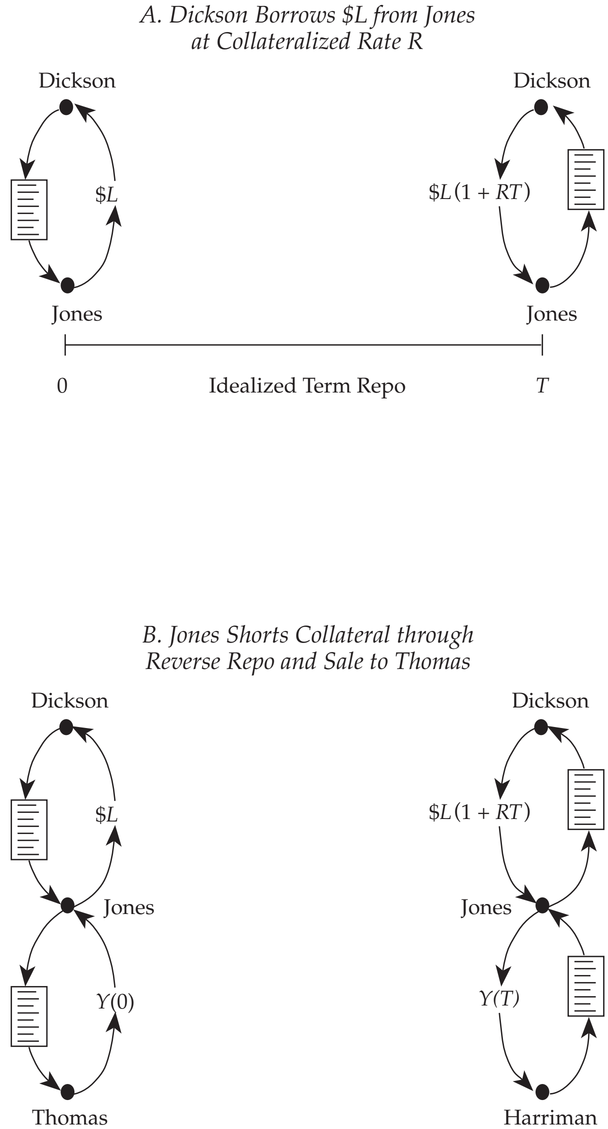



**Figure 3. Reverse Repo Combined with Cash Sale**




fixed-rate note may represent significantly different bargaining power, especially in a reorganization scenario precipitated by default.

**Extension: Adding Repo Specials and Transaction Costs.** Another important and common relaxation of the assumptions in the starter case involves the ability to freely short the reference C-FRN. A typical method of shorting securities is via a reverse repo combined with a cash sale. That is, through a reverse repo, an investor can arrange to receive the reference note as collateral on a loan to a given term. Rather than holding the note as collateral, the investor can immediately sell the note. In effect, the investor has then created a short position in the reference note through the term of the repo. As shown in the top part of **Figure 3** (with Dickson as the investor), each repo involves a collateralized interest rate, or repo rate, $R$. A loan of $L$ dollars at repo rate $R$ for a term of $T$ years results in a loan repayment of $L(1 + RT)$ at term. As shown in the bottom part of Figure 3, the repo counterparty—in this case, Jones—who is offering the loan and receiving the collateral may, at the initiation of the repo, sell the collateral at its market value, $Y(0)$. Then, at the maturity date of the repo contract, Jones may buy the note back at its market value, $Y(T)$, so as to return it to the original repo counterparty, in this case, Dickson. If the general prevailing interest rate, $r$, for such loans, called the "general collateral rate," is larger than the specific collateral rate $R$ for the loan collateralized by the C-issued note in question, Jones will have suffered costs in creating the short position in the underlying C-issued note.[6]

In many cases, one cannot arrange a reverse repo at the general collateral rate (GCR). If the reference note is "scarce," an investor may be forced to offer a repo rate that is below the GCR in order to reverse in the C-FRN as collateral. This situation is termed a repo special (see, e.g., Duffie 1996). In addition, particularly with risky FRNs, a substantial bid–ask spread may be present in the market for the reference FRN at initiation of the repo (when one sells) and at termination (when one buys).

Suppose that a term reverse repo collateralized by the C-FRN can be arranged, with maturity equal to the maturity date of the credit swap. Also suppose that default of the collateral triggers early termination of the repo at the originally agreed repo rate (which is the case in many jurisdictions). The term repo special, $Z$, is the difference between the term GCR and the term specific collateral rate for the C-FRN. Shorting the C-FRN, therefore, requires an extra annuity payment of $Z$. The arbitrage-based default swap spread would then be approximately $S + Z$. If the term repo does not necessarily terminate at the credit event, this spread is not an exact arbitrage-based spread. Because the probability of a credit event occurring well before maturity is typically small, however, and because term repo specials are often small, the difference may not be large in practice.[7]

Now consider the other side of the swap: For the synthesis of a short position in the credit swap, an investor purchases the C-FRN and places it into a term repo to capture the term repo special.

If transaction costs in the cash market are a factor, the credit swap broker/dealer may incur risk from uncovered credit swap positions, transaction costs, or some of each, and may, in principle, charge an additional premium. With two-sided market making and diversification, how quickly these costs and risks build up over a portfolio of positions is not clear.[8]

The difference between a transaction cost and a repo special is important. A transaction cost simply widens the bid–ask spread on a default swap, increasing the default swap spread quoted by the broker/dealer who sells the default swap and reducing the quoted default swap spread when the broker/dealer is asked by a customer to buy a default swap from the customer. A repo special, however, is not itself a transaction cost; it can be thought of as an extra source of interest income on the underlying C-FRN, a source that effectively changes the spread relative to the default-free rate. Substantial specials, which raise the cost of providing the credit swap, do not necessarily increase the bid–ask spread. For example, in synthesizing a short position in a default swap, an investor can place the associated long position in the C-FRN into a repo position and profit from the repo special.

In summary, under the assumptions stated up to this point, a dealer can broker a default swap (that is, take the position of Party A) at a spread of approximately $S + Z$ with a bid–ask spread of $K$, where

- $S$ is the par spread on a reference floating-rate note issued by a named entity, called here Entity C, of the same maturity as the default swap and of the same seniority as the underlying note;
- $Z$ is the term repo special on par floating-rate notes issued by C or else an estimate of the annuity rate paid, throughout the term of the default swap, for maintaining a short position in the reference note to the termination of the credit swap; and
- $K$ contains any annuitized transaction costs (such as cash market bid–ask spreads) for hedging, any risk premium for unhedged portions of the risk (which would apply in imperfect capital markets), overhead, and a profit margin.

In practice, estimating the effective term repo special is usually difficult because default swaps are normally of much longer term than repo positions. In some cases, liquidity in the credit swap market has apparently been sufficient to allow some traders to quote term repo rates for the underlying collateral by reference to the credit swap spread.

**Extension: Payment of Accrued Credit Swap Premium.** Some credit swaps, more frequently on underlying corporate rather than sovereign bonds, specify that, at default, the buyer of protection must pay the credit swap premium that has accrued since the last coupon date. For example, with a credit swap spread of 300 bps and default one-third of the way through a current semiannual coupon period, the buyer of protection would receive face value less recovery value of the underlying asset less one-third of the semiannual annuity payment, which would be 0.5 percent of the underlying face value.

For reasonably small default probabilities and intercoupon periods, the expected difference in time between the credit event and the previous coupon date is approximately half the length of an intercoupon period. Thus, for pricing purposes in all but extreme cases, one can think of the credit swap as equivalent to payment at default of face value less recovery value less one-half of the regular default swap premium payment.

For example, suppose there is some risk-neutral probability $h > 0$ per year for the credit event.[9] Then, one estimates a reduction in the at-market credit swap spread for the accrued premium that is below the spread that is appropriate without the accrued-premium feature—approximately $hS/2n$, where $n$ is the number of coupons a year of the underlying bond. For a pure default swap, spread $S$ is smaller than $h$ because of partial recovery, so this correction is smaller than $h^2/2n$, which is negligible for small $h$. For example, at semiannual credit swap coupon intervals and for a risk-neutral mean arrival rate of the credit event of 2 percent a year, the correction for the accrued-premium effect is less than 1 bp.

**Extension: Accrued Interest on the Underlying Notes.** For calculating the synthetic arbitrage described previously, the question of accrued interest payment on the default-free floating rate note arises. The typical credit swap specifies payment of the difference between face value *without* *accrued interest* and market value of the underlying note. However, the arbitrage portfolio described here (long a default-free floater, short a defaultable floater) is worth face value *plus accrued interest on the default-free note* less recovery on the underlying defaultable note. If the credit event involves default of the underlying note, the previous arbitrage argument is not quite right.

Consider, for example, a one-year default swap with semiannual coupons. Suppose the LIBOR rate is 8 percent. Then, the expected value of the accrued interest on the default-free note at default is approximately 2 percent of face value for small default probabilities. Suppose the risk-neutral probability of occurrence of the credit event is 4 percent a year. Then, the market value of the credit swap to the buyer of protection is reduced roughly 8 bps of face value and, therefore, the at-market credit swap spread is reduced roughly 8 bps.

Generally, for credit swaps of any maturity with relatively small and constant risk-neutral default probabilities and relatively flat term structures of default-free rates, the reduction in the at-market credit swap spread for the accrued-interest effect, below the par floating rate-spread plus effective repo special, is approximately $hr/2n$, where $h$ is the annual risk-neutral probability of occurrence of the credit, $r$ is the average of the default-free forward rates through credit swap maturity, and $n$ is the number of coupons per year of the underlying bond. Of course, one could work out the effect more precisely with a term-structure model, as described later.

**Extension: Approximating the Reference Floating-Rate Spread.** If no par floating-rate note of the same credit quality is available whose maturity is that of the default swap, then one can attempt to "back out" the reference par spread, $S$, from other spreads. For example, suppose C issues an FRN of the swap maturity and of the same seniority as the underlying note and it is trading at a price, $p$, that is not necessarily par and paying a spread of $\hat{S}$ over the default-free floating rate.

Let $AP$ denote the associated annuity price—that is, the present value of an annuity paid at a rate of 1 unit until the credit swap termination (default of the underlying note or maturity).

For reasonably small credit risks and interest rates, $AP$ is close to the default-free annuity price because most of the market value of the credit risk of an FRN is associated in this case with potential loss of principal. A more precise computation of $AP$ is considered later.

The difference between a par and a nonpar FRN with the same maturity is the coupon spread (assuming the same recovery at default); therefore,

$$p - 1 = AP(\hat{S} - S),$$

where $S$ is the implied reference par spread. Solving for the implied reference par spread produces

$$S = \hat{S} + \frac{1-p}{AP}.$$

With this formula, one can estimate the reference par spread, $S$.

If the relevant price information is for a fixed-rate note issued by C of the reference maturity and seniority, one can again resort to the assumption that its recovery of face value at default is the same as that of a par floater of the same seniority (which is again reasonable on legal grounds in a liquidation scenario). And one can again attempt to "back out" the reference par floating-rate spread.

Spreads over default-free rates on par fixed-rate notes and par floating-rate notes are approximately equal.[10] Thus, if the only reference spread is a par fixed-rate spread, $F$, using $F$ in place of $S$ in estimating the default swap spread is reasonably safe.

An example in **Figure 4** shows the close relationship between the term structures of default swap spreads and par fixed-coupon yield spreads for the same credit quality.[11] Some of the difference between the spreads shown in Figure 4 is, in fact, the accrued-interest effect discussed in the previous subsection.

If the reference pricing information is for a nonpar fixed-rate note, then one can proceed as before. Let $p$ denote the price of the available fixed-rate note, with spread $\hat{F}$ over the default-free rate. Then,

$$p - 1 = AP(\hat{F} - F),$$

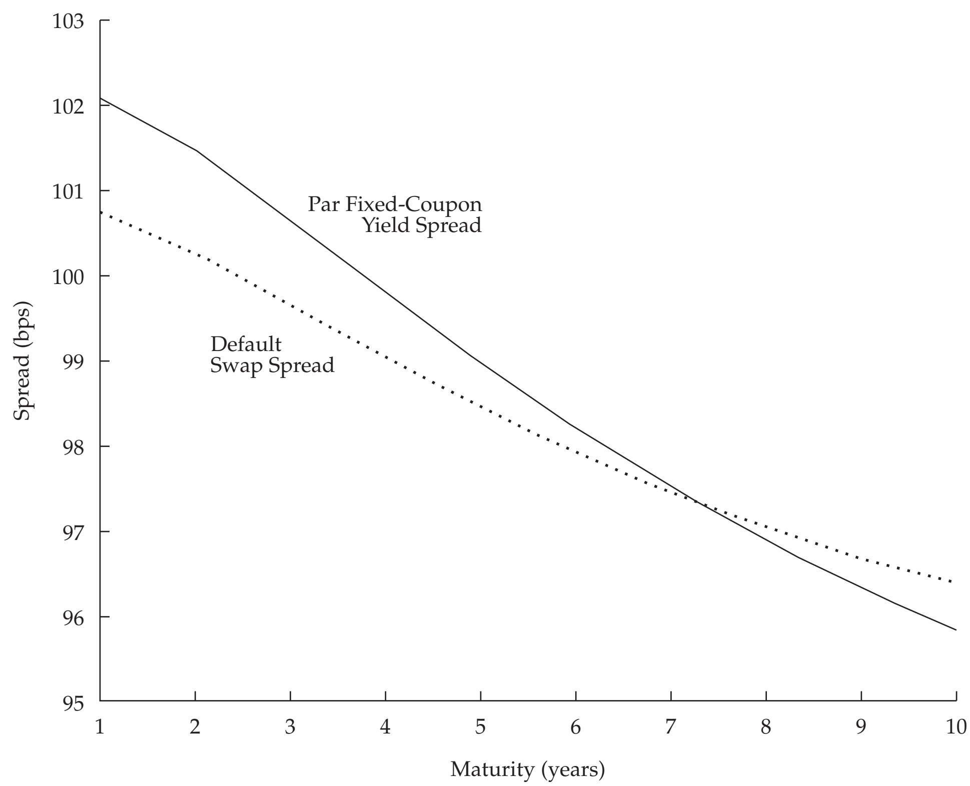
where $AP$ is again the annuity price to maturity or default. So, with an estimate of $AP$, one can obtain an estimate of the par fixed spread, $F$, which is a close approximation of the par floating-rate spread, $S$, the quantity needed to compute the default swap spread.[12]

## Estimating Hazard Rates and Defaultable Annuity Prices

The hazard rate for the credit event is the arrival rate of the credit event (in the sense of Poisson processes). For example, a constant hazard rate of 400 bps represents a mean arrival rate of 4 times per 100 years. The mean time to arrival, conditional on no event arrival date by $T$, remains 25 years after $T$ for any $T$. Begin by assuming a constant risk-neutral

**Figure 4. Term Structures of Bond and Default Swap Spreads**




hazard rate, $h$, for the event. In this simple model (to be generalized shortly), at any time, given that the credit event has not yet occurred, the amount of time until it does occur is risk-neutrally exponentially distributed with parameter $h$. For small $h$, the probability of defaulting during a time period of small length, $\Delta$, conditional on survival to the beginning of the period, is then approximately $h\Delta$. This section contains some intermediate calculations that can be used to estimate implied hazard rates and the annuity price.

**The Case of Constant Default Hazard Rate.** Suppose default by Entity C occurs at a risk-neutral constant hazard rate of $h$. In that case, default occurs at a time that, under "risk-neutral probabilities," is the first jump time of a Poisson process with intensity $h$. Let

- $a_i(h)$ be the value at time zero of receiving 1 unit of account at the $i$th coupon date in the event that default is after that date and
- $b_i(h)$ be the value at time zero of receiving 1 unit of account at the $i$th coupon date in the event default is between the $(i - 1)$th and the $i$th coupon date.

Then,

$$a_i(h) = \exp\{-[h + y(i)]T(i)\},$$

where $T(i)$ is time to maturity of the $i$th coupon date and $y(i)$ is the continuously compounding default-free zero-coupon yield to the $i$th coupon date. Similarly, under these assumptions,

$$b_i(h) = \exp[-y(i)T(i)]\{\exp[-hT(i-1)] - \exp[-hT(i)]\}.$$

The price of an annuity of 1 unit of account paid at each coupon date until default by C or maturity $T(n)$ is

$$A(h, T) = a_1(h) + \ldots + a_n(h).$$

The market value of a payment of 1 unit of account at the first coupon date after default by C, provided the default date is before maturity date $T(n)$, is

$$B(h, T) = b_1(h) + \ldots + b_n(h).$$

Now, consider a classic default swap:

- Party B pays Party A a constant annuity $U$ until maturity $T$ or the default time $\tau$ of the underlying note issued by C.
- If $\tau \leq T$, then at $\tau$, Party A pays Party B 1 unit of account minus the value at $\tau$ of the underlying note issued by C.

Suppose now that the loss of face value at default carries no risk premium and has an expected value of $f$.[13] Then, given the parameters ($T$, $U$) of the default swap contract and given the default-risk-free term structure, one can compute the market value of the classic default swap as a function of any assumed default parameters $h$ and $f$:

$$V(h, f, T, U) = B(h, T)f - A(h, T)U.$$

The at-market default swap spread, $U(h,T,f)$, is obtained by solving $V(h, f, T, U) = 0$ for $U$, leaving

$$U(h, T, f) = \frac{B(h, T)}{A(h, T)}.$$

For more accuracy, one can easily account for the difference in time between the credit event and the subsequent coupon date. At small hazard rates, this difference is slightly more than half the intercoupon period of the credit swap and can be treated analytically in a direct manner. Alternatively, one can make a simple approximating adjustment by noting that the effect is equivalent to the accrued-interest effect in adjusting the par floating-rate spread to the credit swap spread. As mentioned previously, this adjustment causes an increase in the implied default swap spread that is on the order of $hr/2n$, where $r$ is the average of the intercoupon default-free forward rates through maturity. (One can obtain a better approximation for a steeply sloped forward-rate curve.)

Estimates of the expected loss, $f$, at default and the risk-neutral hazard rate, $h$, can be obtained from the prices of bonds or notes issued by Entity C, from risk-free rates, and from data on recovery values for bonds or notes of the same seniority.[14] For example, suppose a C-issued FRN, which is possibly different from the note underlying the default swap, sells at price $p$, has maturity $\hat{T}$, and has spread $\hat{S}$. And suppose the expected default loss of this note, relative to face value, is $\hat{f}$. Under the assumptions stated here, a portfolio containing a risk-free floater and a short position in this C-issued FRN (with no repo specials) has a market value of

$$1 - p = A(h, \hat{T})\hat{S} + B(h, \hat{T})\hat{f}.$$

This equation can be solved for the implied risk-neutral hazard rate, $h$.

Provided the reference prices of notes used for this purpose are near par, a certain robustness is associated with uncertainty about recovery. For example, an upward bias in $f$ results in a downward bias in $h$ and these errors (for small $h$) approximately cancel each other out when the mark-to-market value of the default swap, $V(h, f, T, U)$, is being estimated. To obtain this robustness, it is best to use a reference note of approximately the same maturity as that of the default swap.

If the C-issued note that is chosen for price reference is a fixed-rate note with price $p$, coupon rate $c$, expected loss $\hat{f}$ at default relative to face value, and maturity $\hat{T}$, then $h$ can be estimated from the pricing formula

$$p = A(h, T)c + B(h, \hat{T})(1 - \hat{f}).$$

To check the sensitivity of the model to choice of risk-neutral default arrival rate and expected recovery, one can use the intuition that the coupon yield spread of a fixed-rate bond is roughly the product of the mean default intensity and the fractional loss of value at default. This intuition can be given a formal justification in certain settings, as explained in Duffie and Singleton (1997). For example, **Figure 5** contains plots of the risk-neutral mean (set equal to initial default) intensity $\bar{h}$ implied by the term-structure model and that mean intensity implied by the approximation $S = f\bar{h}$, for various par 10-year coupon spreads $S$ at each assumed level of expected recovery of face value at default, $w = (1 - f)$.

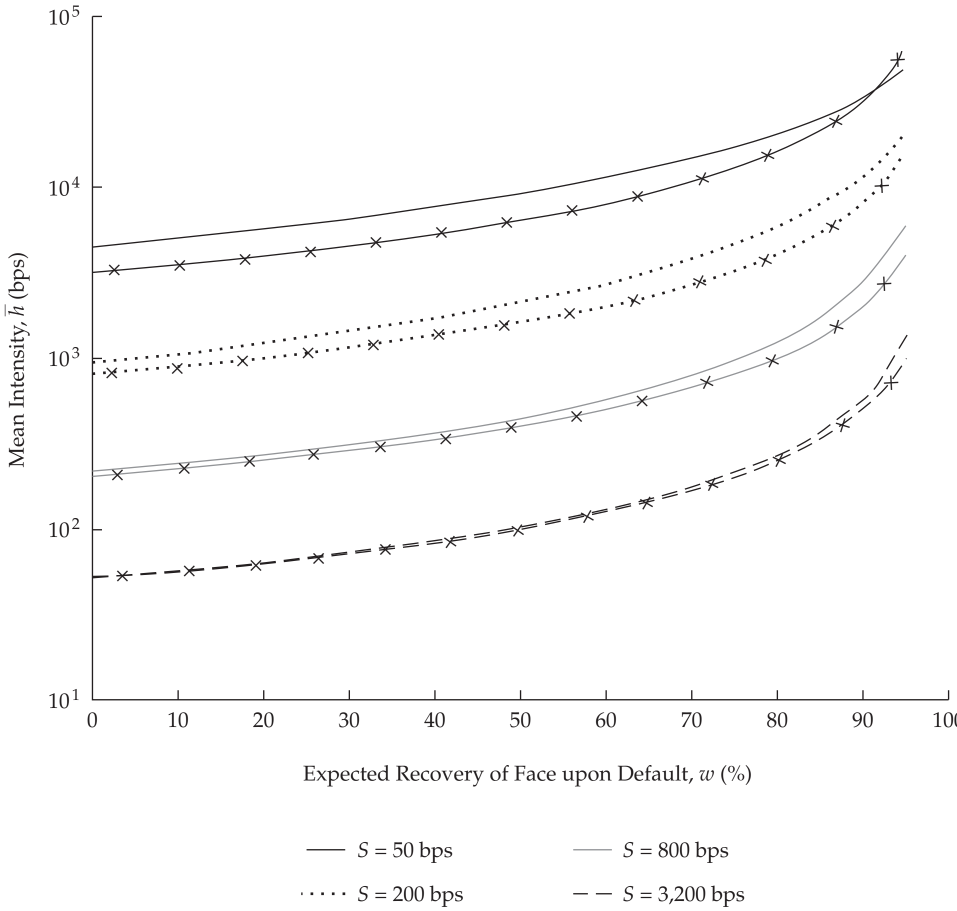
Figure 5 shows that, up to a high level of fractional recovery, the effects of varying $h$ and $f$ are more or less offsetting in the fashion previously suggested. (That is, if one overestimates $f$ by a factor of 2, even a crude term-structure model will underestimate $h$ by a factor of roughly 2 and the implied par-coupon spread will be relatively unaffected, which means that the default swap spread is also relatively unaffected.) This approximation is more accurate for shorter maturities. The fact that the approximation works poorly at high spreads is mainly because par spreads are measured on the basis of bond-equivalent yield (compounded semiannually) whereas the mean intensity is measured on a continuously compounded basis.

If multiple reference notes with maturities similar to that of the underlying default swap are available, an investor might average their implied hazard rates, after discarding outliers, and then average the rates. An alternative is to use nonlinear least-squares fitting or some similar pragmatic estimation procedure. The reference notes may, however, have important institutional differences that will affect relative recovery. For example, in negotiated workouts, one investor group may be favored over another for bargaining reasons.

Default swaps seem to serve, at least currently, as a benchmark for credit pricing. For example, if the at-market default swap quote, $U^*$, is available and an investor wishes to estimate the implied risk-neutral hazard rate, the process is to solve $U(h, T, f) = U^*$ for $h$. As suggested previously, the model result depends more or less linearly on the modeling assumption for the expected fractional loss at

**Figure 5. Hazard Rate Implied by Spread and Expected Recovery**




*Note*: Lines with cross marks are the approximations.

default. Sensitivity analysis is warranted if the objective is to apply the hazard-rate estimate to price an issue that has substantially different cash flow features from those of the reference default swap.

**The Term Structure of Hazard Rates.** If the reference credit's pricing information is for maturities different from the maturity of the credit swap, an investor is advised to estimate the term structure of hazard rates. For example, one could assume that the hazard rate between coupon dates $T(i-1)$ and $T(i)$ is $h(i)$. In this case, given the vector $h = [h(1), \ldots, h(n)]$, and assuming equal intercoupon time intervals, we have the more general calculations:

$$a_i(h) = \exp\{-[H(i) + y(i)]T(i)\},$$

where

$$H(i) = \frac{h_1 + \ldots + h_i}{i},$$

and

$$b_i(h) = \exp[-y(i)T(i)]\{\exp[-H(i-1)T(i-1)] - \exp[-H(i)T(i)]\}.$$

Following these changes, the previous results apply.

Because of the well-established dependence of credit spreads on maturity, the wise analyst will consider the term structure when valuing credit swaps or inferring default probabilities from credit swap spreads.

When information regarding the shape of the term structure of hazard rates for the reference entity C is critical but not available, a pragmatic approach is to assume that the shape is that of comparable issues. For example, one might use the shape implied by Bloomberg par yield spreads for issues of the same credit rating and sector and then scale the implied hazard rates to match the pricing available for the reference entity. This *ad hoc* approach is subject to the modeler's judgment.

A more sophisticated approach to estimating hazard rates is to build a term-structure model for a stochastically varying risk-neutral intensity process, as in Duffie (1998a), Duffie and Singleton (1997), Jarrow and Turnbull (1995), or Lando (1998). Default swap pricing is reasonably robust, however, to the model of intensities, calibrated to given spread correlations and volatilities. For example, **Figure 6** shows that default swap spreads do not depend significantly on how much the default arrival intensity is assumed to change with each 100 bp change in the short-term rates. The effect of default-risk volatility on default swap spreads becomes pronounced only at relatively high levels of volatility of $h$, as indicated in **Figure 7**. For this figure, volatility was measured as percentage standard deviation, at initial conditions, for an intensity model in the style of Cox–Ingersoll–Ross. The effect of volatility arises essentially from Jensen's inequality.[15]

Even the general structure of the defaultable term-structure model may not be critical for determining default swap spreads. For example, **Figure 8** shows par coupon yield spreads for two term-structure models. One, the RMV model, is based on Duffie and Singleton (1997) and assumes recovery of 50 percent of *market value* at default. The other, the RFV model, assumes recovery of 50 percent of *face value* at default. Despite the difference in recovery assumptions, with no attempt to calibrate the two models to given prices, the implied term structures are similar. With calibration to a reference bond of maturity similar to that of the underlying bond, the match of credit swap spreads implied by the two models would be even closer. (This discussion does not, however, address the relative pricing of callable or convertible bonds with these two classes of models.)

Some cautions or extensions are as follows:

- The risk-neutral hazard-rate need not be the same as the hazard rate under an objective probability measure. The "objective" (actual) hazard rate is never used here.
- Even if hazard rates are stochastic, the previous calculations apply as long as they are independent (risk-neutrally) of interest rates. In such a case, one simply interprets $h(i)$ to be the rate of arrival of default during the $i$th interval, conditional only on survival to the beginning of that interval. This "forward default rate" is by definition deterministic.[16]

**Figure 6. Two-Year Default Swap Spread by Expected Response of Default Intensity to Change in Short-Term Default-Free Rate**

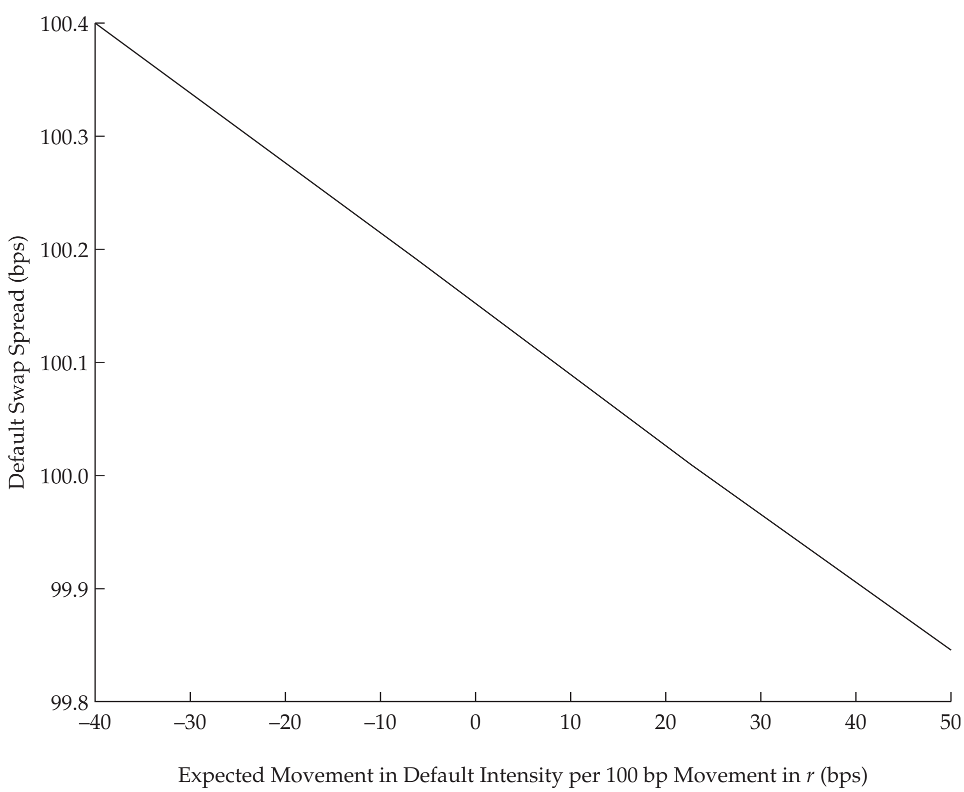
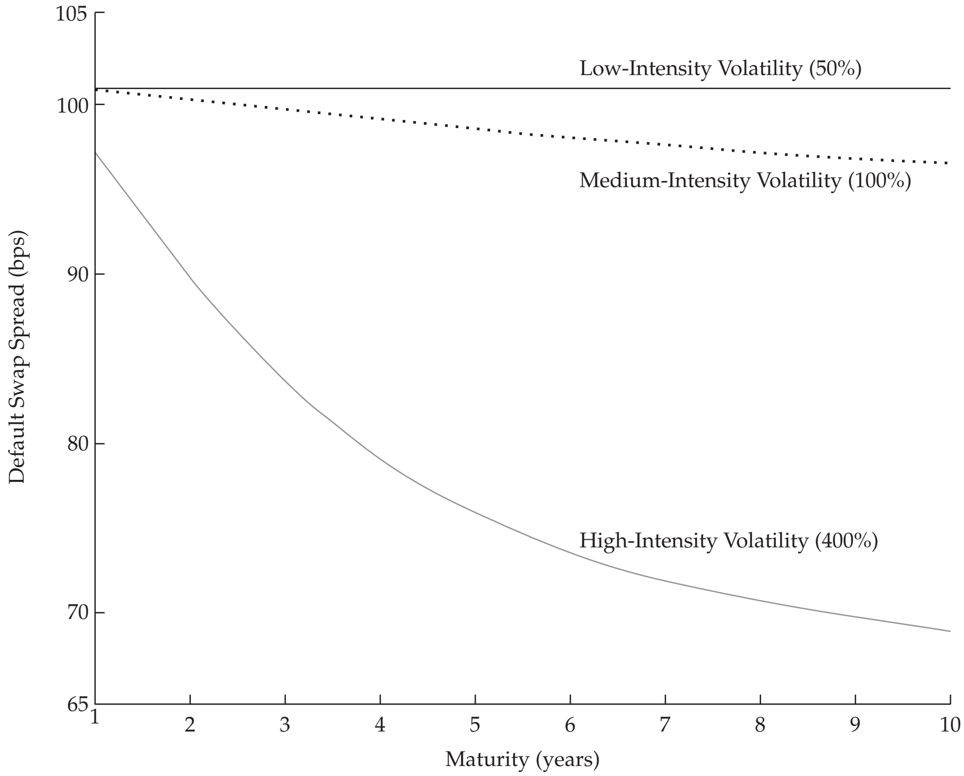


**Figure 7. Term Structure of Default Swap Spreads as Intensity Volatility Varies**




- If the notes used for pricing reference are on special in the repo market, an estimate of the "hidden" specialness, $Y$, should be included in the preceding calculations as an add-on to the floating-rate spread, $\hat{S}$, or the fixed-rate coupon, $c$, when estimating the implied risk-neutral hazard rate, $h$.
- If necessary, one can use actuarial data on default incidence for comparable companies and adjust the estimated actual default arrival rate by a multiplicative corrective risk-premium factor, estimated cross-sectionally perhaps, to incorporate a risk premium.[17]
- If one assumes "instant" payment at default, rather than payment at the subsequent coupon date, the factor $b_i(h)$ is replaced by

$$b_i^*(h) = \exp\{-[y(i-1) + H(i-1)]T(i-1)\, k_i[h(i)]\},$$

where

$$k_i[h(i)] = \frac{h(i)}{h(i)+\varphi(i)} \langle 1 - \exp\{-[h(i)+\varphi(i)][T(i)-T(i-1)]\}\rangle,$$

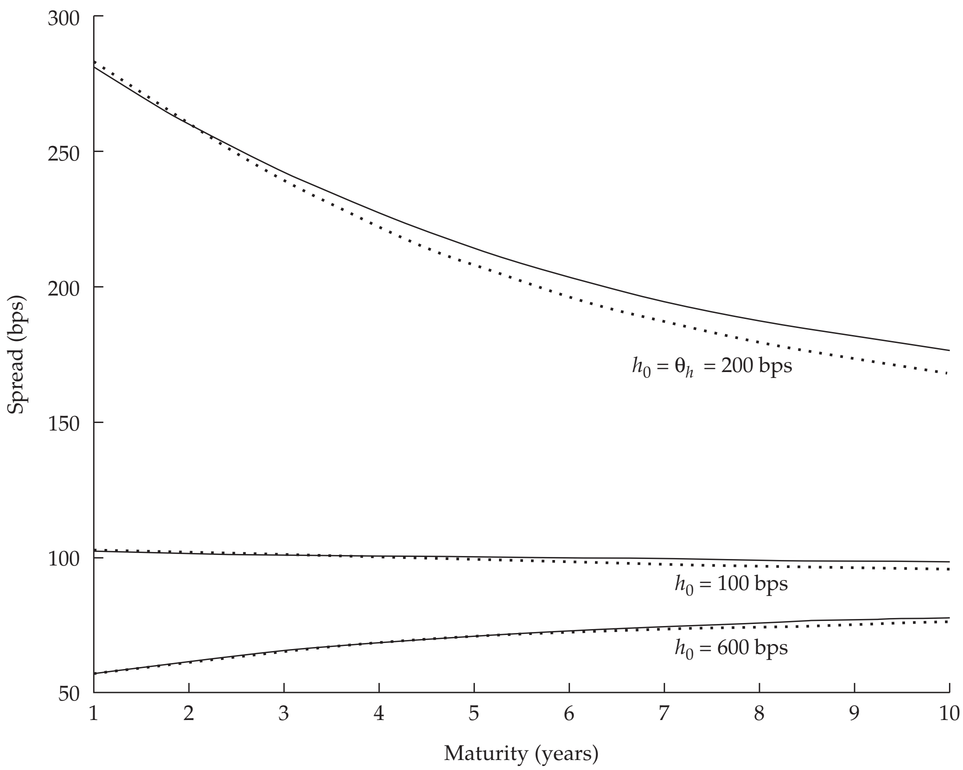
is the price at time $T(i-1)$, conditional on survival to that date, of a claim that pays 1 unit of account at the default time provided the default time is before $T(i)$ and where $\varphi_i$ is the instantaneous default-free forward interest rate, assumed constant between $T(i-1)$ and $T(i)$. This equation can be checked by noting that the conditional density of the time to default, given survival to $T(i-1)$, is over the interval $[T(i-1), T(i)]$. For reasonably small intercoupon periods, default probabilities, and interest rates, the impact of assuming instant recovery rather than recovery at the subsequent coupon date is relatively small.

## The Role of Asset Swaps

An asset swap is a derivative security that can be viewed in its simplest version as a portfolio consisting of a fixed-rate note and an interest rate swap that pays the fixed rate and receives the floating rate to the stated maturity of the underlying fixed-rate note. The fixed rate on the interest rate swap is conventionally chosen so that the asset swap is valued at par when traded. An important aspect is that the net coupons of the interest-rate swap are exchanged through maturity even if the underlying note defaults and its coupon payments are thereby discontinued.

Recently, the markets for many fixed-rate notes have sometimes been less liquid than the markets for associated asset swaps, whose spreads are thus often used as benchmarks for pricing default swaps. In fact, because of the mismatch in

**Figure 8. Par Coupon Yield Spreads for RMV and RFV Term-Structure Models**




*Notes*: The solid line in each pair is the RMV model, and the dotted line in each pair is the RFV model, with 50 percent recovery upon default; long-run mean intensity, $\theta_h$, of 200 bps; mean reversion rate, κ, of 0.25; and initial intensity volatility of 100 percent. All coefficients are risk neutral.

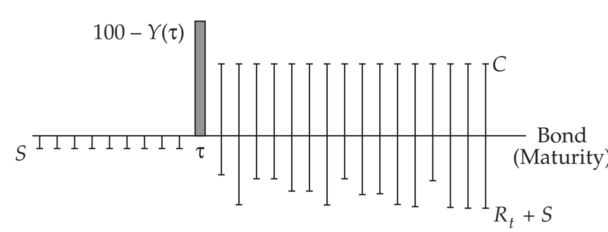
termination with default between the interest rate swap embedded in the asset swap and the underlying fixed-rate note, the asset swap spread does not on its own provide precise information for default swap pricing. For example, as illustrated in **Figure 9**, a synthetic credit swap *cannot* be created from a portfolio consisting of a default-free floater and a short asset swap.

The asset swap spread and the term structure of default-free rates together, however, can be used to obtain an implied par floating-rate spread from which the default swap spread can be estimated. For example, suppose an asset swap is at quoted spread $\hat{S}$ to the default-free floating rate. (In the following, repo specials and transaction costs are ignored, but they can be easily added.) Suppose the stated underlying fixed rate on the note is $c$ and the at-market default-free interest-rate swap rate is $c^*$. Then, the interest rate swap underlying the asset swap is an exchange of the floating rate for $c - \hat{S}$. An analyst can compute the desired par fixed-rate spread, $F$, over the default-free coupon rate of the same credit quality from the relationship implied by the price of a portfolio consisting of the asset swap and a short position in a portfolio consisting of a par fixed-rate note of the same credit quality as the underlying C-issued fixed-rate note combined with an at-market interest rate swap. This portfolio is worth

$$\begin{aligned} 1 - 1 &= 0 \\ &= AP(c - F) + AP^*(c^* - c + \hat{S}), \end{aligned}$$

where $AP$ is the defaultable annuity price described previously and $AP^*$ is the default-free annuity price to the same maturity. All the variables $c$, $c^*$, $\hat{S}$, and $AP^*$ are available from market quotes. Given the defaultable annuity price $AP$, which can be estimated as discussed previously, an analyst can thus solve this equation for the implied par fixed-rate spread:

**Figure 9. Failed Attempt to Synthesize a Credit Swap from an Asset Swap**




*Note*: Par default-free floater with short asset swap.

$$F = c - \frac{AP^*}{AP}(c - \hat{S} - c^*).$$

The implied par rate $F$ is approximately the same as the par floating-rate spread, $S$, which is then the basis for setting the default swap spread. For small default probabilities, under the other assumptions given here, the default swap spread $S$ and the par asset swap spread are approximately the same.

To assume that the asset swap spread is a reasonable proxy for the default swap spread is dangerous, however, for premium or discount bonds. **Figure 10** shows the divergence between the term structures of asset swap spreads for premium bonds (coupon rate 400 bps above the par rate), par bonds, and discount bonds (coupon rate 400 bps under the par rate).

## Concluding Remarks

This article has explained how the superficially simple arbitrage pricing (and synthesis) of a credit swap through a portfolio of default-free and defaultable floating-rate notes may, in fact, be difficult. Key concerns are (1) the ability to short the underlying note without incurring the cost of repo specials and (2) the valuation and recovery of reference notes used for pricing purposes in relation to the actual note underlying the swap. Model-based pricing may be useful because it adds discipline to the measurement and use of default probabilities and recoveries. For additional modeling of default swaps, see Davis and Mavroidis (1997).

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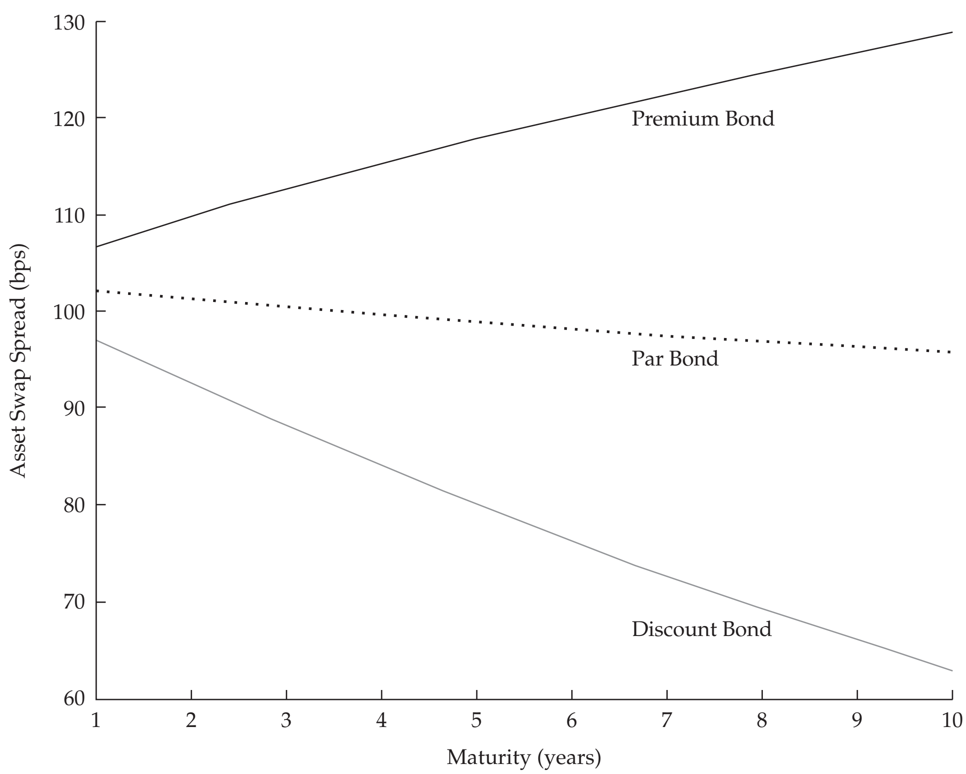
**Figure 10. Term Structure of Asset Swap Spreads**



# Notes

1. Key credit derivatives in addition to credit swaps include *total-return swaps*, which pay the net return of one asset class over another (if the two asset classes differ mainly in terms of credit risk, such as a U.S. Treasury bond versus a corporate bond of similar duration, then the total-return swap is a credit derivative); *collateralized debt obligations*, which are typically tranches of a structure collateralized by a pool of debt whose cash flows are allocated according to a specified priority schedule to the individual tranches of the structure; and *spread options*, which typically convey the right to trade bonds at given spreads over a reference yield, such as a Treasury yield.
2. At a presentation at the March 1998 International Swap Dealers Association conference in Rome, Daniel Cunningham of Cravath, Swaine, and Moore reviewed the documentation of credit swaps, including the specification of such credit event types as "bankruptcy, credit event upon merger, cross-acceleration, cross-default, downgrade, failure to pay, repudiation, or restructuring."
3. My discussions with a global bank indicate that of more than 200 default swaps, approximately 10 percent of the total were combined with an interest rate swap.
4. I do not consider here "exotic" forms of credit swaps, such as "first-to-default" swaps, for which credit event time $\tau$ is the first of the default times of a given list of underlying notes or bonds, with a payment at the credit event time that depends on the identity of the first of the underlying bonds to default. For example, the payment could be the loss relative to face value of the first bond to default.
5. The moneyness of Treasuries refers to their usefulness as a medium of exchange in, for example, securities transactions that are conducted by federal funds wire or for margin services. This usefulness conveys extra value to Treasury securities.
6. As to the costs, a haircut would normally apply. For example, at a haircut of 20 percent, a note trading at a market value of $100 would serve as collateral on a loan of $80. At a general collateral rate of 5 percent, a specific collateral rate of 1 percent, and a term of 0.5 year, Jones incurs an extra shorting cost of which the present value is $80 × (5 percent – 1 percent) × [0.5/(1 + 5 percent × 0.5)] = $1.56.
7. If the term repo rate applies to the credit swap maturity, then $S + Y$ is a lower bound on the theoretical credit swap premium.
8. This article does not consider these effects directly, but traders have noted that, in practice, the credit swap spread for illiquid entities can vary substantially from the reference par FRN spread.
9. This rate may be interpreted as a Poisson arrival rate, in the sense of hazard rates explained later in the article.
10. The floating-rate spread is known theoretically to be slightly higher than the fixed-rate spread in the case of the typical upward-sloping term structure, but the difference is typically on the order of 1 bp or less on a five-year note per 100 bps of yield spread to the default-free rate. See Duffie and Liu (1997) for details.
11. Figures 4–10 are based on an illustrative correlated multi-factor Cox–Ingersoll–Ross model of default-free short rates and default arrival intensities. The short-rate model is a three-factor Cox–Ingersoll–Ross model calibrated to recent behavior in the term structure of LIBOR swap rates. The model of risk-neutral default-arrival intensity is set for an initial arrival intensity of 200 bps, with 100 percent initial volatility in intensity, mean reverting in risk-neutral expectation at 25 percent a year to 200 bps until default. Recovery at default is assumed to be 50 percent of face value. For details, see Duffie (1998b). The results depend on the degree of correlation, mean reversion, and volatility among short-term rates and default-arrival intensities.
12. Sometimes the statement is made that if the underlying asset is a fixed-rate bond, the reference par floating-rate spread may be taken to be the asset swap spread. The usefulness of this assumption is considered in the last section of this article.
13. Recovery risk is sometimes viewed as reasonably diversifiable and relatively unrelated to the business cycle. No rigorous test of these hypotheses is available.
14. Sources of recovery data include annual reports by Moody's Investors Service and Standard & Poor's Corporation, Altman (1993) for bonds, and Carey (1998) and sources cited in it for loans. The averages reported are typically by seniority.
15. The risk-neutral survival probability to term $T$ for a risk-neutral intensity process $h$ under standard regularity assumptions is given by $E^*\left\{\exp\left[-\int_0^T h(t)dt\right]\right\}$, where $E^*$ denotes risk-neutral expectation. See Lando for a survey. Because $\exp(\cdot)$ is convex, more volatility of risk-neutral intensity causes, other things being equal, a higher risk-neutral survival probability and thus narrower credit spreads.
16. This idea is based on the "forward default probability" introduced by Litterman and Iben (1991).
17. Multiplicative factors are preferred to additive factors in light of general economic considerations and the form of Girsanov's Theorem for point processes, as in Protter (1990). Fons (1994) provides information on the pricing of notes at actuarially implied default rates, but Fons does not provide an estimate of default arrival intensity.

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