Multi-factor term structure models

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This is a survey of multi-factor structure models, concentrating on models in which the term structure has a finite-dimensional (Markov diffusion) state-space representation. The special 'affine' case is shown to be tractable.

1. Introduction

Figure 1 shows the term structure of interest rates at a given point in time. Stochastic models for fluctuations in term structure over time are commonly used in the finance industry for at least the following purposes.

(i) The pricing of fixed-income derivative securities, such as options and mortgage-backed securities.

(ii) The analysis of the risk of fixed-income portfolio strategies.

(iii) Managing the interest-rate risk of fixed-income positions.

By 'fixed income', we mean assets whose pay-offs depend on the term structure itself. In a wide sense, this can include bonds, bond derivatives such as options, swaps, or caps, defaultable bonds, and even foreign bonds or derivatives based in sometimes complicated ways on domestic and foreign interest rates. There are many other reasons for understanding the process by which interest rates are determined and change over time, but our focus will be on models that are particularly useful for the above three purposes.

Although various classes of stochastic models are used, the most common language of term structure modellers in industry and universities is that of continuous-time stochastic calculus, which reached popularity following the impact of the Black & Scholes (1973) option pricing formula and the associated modelling ideas developed by Merton (1973) and others. We will review how such models are constructed and applied, with particular reference to Markov diffusions that represent the current term structure in a finite-dimensional state space. Within this class, one can make reasonable trade-offs between economic realism and computational tractability, bearing in mind that no tractable model can fully capture the complexity of unexpected changes in interest rates.

2. Setup

We begin with a probability space $(\Omega, \mathscr{F}, P)$ and the augmented filtration $\{\mathscr{F}_t : t \in [0, \infty)\}$ generated by a standard brownian motion W^* in $\mathbb{R}^n$, for some $n \geqslant 1$. (For technical details, see, for example, Karatzas & Shreve (1988), Protter (1990) or other standard references.)

Given is a progressively measurable 'short rate' process r such that $\int_0^T |r_t| \, \mathrm{d}t < \infty$ almost surely for all $T > 0$. We may think of r_t as the interest rate at time t on loans

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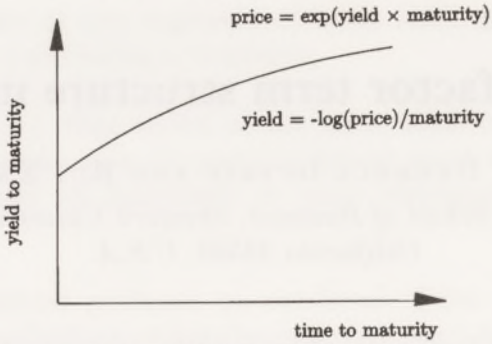

Figure 1. The yield curve.

of infinitesimal maturity. More properly, it is possible to invest one unit of account at any time t in deposits and receive at any $s \geqslant t$ the pay-off $\exp(\int_t^s r_u \, du)$.

For purposes of this survey, a security is a financial claim promising, for some time T, a pay-off defined by some $\mathscr{F}_T$-measurable random variable u. According to a model of Harrison & Kreps (1979), as subsequently developed by many (see, for example, the references cited in Duffie 1992, especially Ansel & Stricker 1992), under technical conditions there is not arbitrage if and only if there is a probability measure Q, equivalent to P, under which the price of a security paying u at time $T \geqslant t$ is given by

$$E\left[\exp\left(-\int_t^T r_s \, ds\right) u \,\middle|\, \mathscr{F}_t\right]. \tag{1}$$

Here, and throughout, E denotes expectation under such a probability measure Q, which is fixed. The obvious example is to take $u = 1$, defining the price $p_{t,T}$ of a zero-coupon bond maturing at T. The continuously compounding yield of a bond of maturity τ is then defined as

$$y_{t,\tau} = (-1/\tau) \log p_{t,t+\tau}, \quad (t, \tau) \in \mathbb{R}_+^2. \tag{2}$$

For practical applications, there remains the basic issue of how to model the probabilistic behaviour of the short rate process r under Q. One wants a model for r that is sufficiently rich to capture the essential nature of the actual market, while at the same time sufficiently tractable for purposes of econometric estimation and for computation of the prices of contingent claims as in (1), for a range of commonly traded securities whose pay-offs are represented by u. There are also many theoretically interesting questions regarding the equilibrium determination of the short rate process r and the equivalent 'martingale' measure Q. It is known that, under weak technical conditions, any short rate process r can be supported in a simple general equilibrium setting with easily specified utility functions and consumption endowments (see, for example, Heston 1991; Duffie 1992, exercise 9.3). In any case, we will be focusing here only on practical issues, and disregarding other aspects of the general equilibrium problem. From this point, we will review some basic classes of models for the behaviour of the short rate process r under the equivalent martingale measure Q. We begin with 'single-factor' models, move to 'multi-factor' models, and finally describe 'infinite-factor' models in the framework of Heath *et al.* (1992). For many applications, it will also be useful to model the distribution of processes under the original probability measure P. Conversion from P to Q and back will not be dealt with here, but is an important issue, particularly

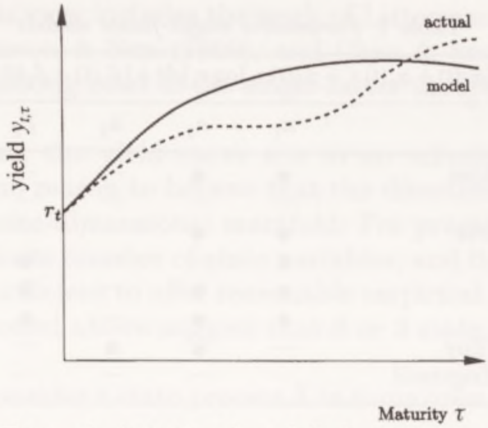

Figure 2. Actual and modelled yield curves.

from the point of view of statistical fitting of the models as well as the measurement of risk.

3. Single-factor models

The simplest class of models that we consider takes the short rate process to be the solution of a stochastic differential equation of the form

$$\mathrm{d}r_t = \mu(r_t)\,\mathrm{d}t + \sigma(r_t)\,\mathrm{d}W_t, \tag{3}$$

where W is a standard brownian motion under Q and where $\mu: \mathbb{R} \rightarrow \mathbb{R}$ and $\sigma: \mathbb{R} \rightarrow \mathbb{R}$ have enough regularity to ensure the existence of a unique solution to (3) (see, for example, Ikeda & Watanabe 1981). Since r is a strong Markov process under Q, we have $p_{t,T} = F(r_t, t)$, for some measurable function $F: \mathbb{R} \times [0, T] \rightarrow \mathbb{R}$, and we can therefore view the entire yield curve $y_t = \{y_{t,\tau}: \tau \geqslant 0\}$ defined by (2) as measurable with respect to r_t. Hence the label 'single-factor model' applies, since a single state variable, in this case the short rate r_t, is a sufficient statistic for all future yield curves.

Although simple and, as it turns out, quite tractable, the single-factor class of models given by (3) is (like any theoretical model) at variance with reality. Consequently, on a given day, the yield curve associated with the model differs from that observed in the market-place, as depicted in figure 2.

If significant, this discrepancy may suggest the development of a new theoretical model. In the finance industry, however, one needs to use some particular model, even if it is imperfect. In practice, the discrepancy between the actual and theoretical yield curves depicted in figure 2 is eliminated by introducing at the current time t, time dependence in the functions μ and σ, to arrive at a 'calibrated model' $\mu^t: \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$ and $\sigma^t: \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}$, of the form

$$\mathrm{d}r_s = \mu^t(r_s, s)\,\mathrm{d}s + \sigma^t(r_s, s)\,\mathrm{d}W_s, \quad s \geqslant t. \tag{4}$$

This calibrated model (μ^t, σ^t) is computed numerically from the original model (μ, σ) using algorithms that are described, for example, in Black *et al.* (1990). With proper calibration, the result is an exact match between the actual and modelled yield curves. Indeed, it is common to calibrate not only with the current yield curve, but also with certain volatility-related information available in the market through the prices of options.

At the next time period $t+1$, of course, there is again a discrepancy between the observed market yield curve and the yield curve computed at the new short rate r_{t+1}

Table 1. *Parametric single-factor models*

$dr_t = [\alpha_1(t) + \alpha_2(t)\, r_t + \alpha_3(t)\, r_t \log r_t]\, dt + [\beta_1(t) + \beta_2(t)\, r_t]^{\gamma}\, dB_t$						
	α_1	α_2	α_3	β_1	β_2	γ
Cox–Ingersoll–Ross	•	•	—	—	•	0.5
Dothan	—	—	—	—	•	1.0
Brennan–Schwartz	•	•	—	—	•	1.0
Merton	•	—	—	•	—	1.0
Vasicek	•	•	—	•	—	1.0
Pearson–Sun	•	•	—	•	•	0.5
Black–Derman–Toy	—	•	•	—	•	1.0
Constantinides–Ingersoll	—	—	—	—	•	1.5

with the previous calibration (μ^t, σ^t) of the model. It is common in practice to re-calibrate to a new model $(\mu^{t+1}, \sigma^{t+1})$. Since the necessity for re-calibration was not considered when using the previous version of the model for pricing purposes, this suggests a theoretical inconsistency in the application of the model. The compromise involved seems reasonable under the circumstances. It has sometimes been said that one can avoid this compromise with the modelling approach of Heath *et al.* (1992), since that framework essentially admits an arbitrary initial yield curve without the need for calibration. In effect, the state variable for the Heath–Jarrow–Morton (HJM) model is the entire yield curve itself. In fact, the HJM model admits movements in the yield curve generated only by a finite-dimensional brownian motion and therefore limits the sorts of movements of the yield curve that can be considered without calibration. Recent work by Kennedy (1992), however, extends the HJM model to allow for an infinite-dimensional brownian motion (in the framework of stochastic flows).

Most, if not all, of the parametric single-factor models appearing in the literature or in industry practice, are of the form

$$dr_t = [\alpha_1(t) + \alpha_2(t)\, r_t + \alpha_3(t)\, r_t \log r_t]\, dt + [\beta_1(t) + \beta_2(t)\, r_t]^{\gamma}\, dW_t, \tag{5}$$

for time-dependent deterministic coefficients α_1, α_2, α_3, β_1, and β_2, and for some exponent $\gamma \geqslant 0.5$. (For existence and uniqueness of solutions, additional coefficient restrictions apply.) Table 1 lists the origins of various special cases of this parametric class, indicating with • the coefficients that are non-zero (sometimes constant) for each special case, and indicating the choice of power γ. (By offering extensions with time-varying coefficients, Ho & Lee (1986) and Hull & White (1990) have popularized the constant coefficients models of Merton (1973) and Vasicek (1977).)

Even for this simple parametric class (5), there are clearly degrees of freedom in calibrating the model to the observed yield curve. It is also common in practice to calibrate the model to market prices for derivative securities, such as bond options or 'caps', both of which provide useful volatility-related information that can be used to obtain more realistic model behaviour. For a description, see Black & Karasinski (1992), for example.

4. Multi-factor models

Although single-factor models offer tractability, there is compelling reason to believe that a single state variable, such as the short rate r_t, is insufficient to capture reasonably well the distribution of future yield curve changes. The econometric evidence in favour of this view includes the work of Litterman & Scheinkman (1988), Stambaugh (1988), Pearson & Sun (1990), and Chen & Scott (1992*b*, 1993). (For empirical comparisons among most of the single-factor models considered in table 1, see Chan *et al.* (1992).)

In principle, of course, the yield curve sits in an infinite-dimensional space of functions, and there is no reason to believe that the direction of its movements will be restricted to some finite-dimensional manifold. For practical purposes, however, tractability suggests a finite number of state variables, and it is an empirical issue as to how many might be sufficient to offer reasonable empirical properties. Some of the empirical studies mentioned above suggest that 2 or 3 state variables might suffice for practical purposes.

In any case, we will consider a state process X in some open subset D of $\mathbb{R}^n$, defined as the solution to

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \tag{6}$$

where W is a standard brownian motion in $\mathbb{R}^n$ under Q, and where $\mu: D \to \mathbb{R}^n$ and $\sigma: D \to \mathbb{R}^{n\times n}$ satisfy sufficient regularity for existence and uniqueness of solutions. In what follows, we could add time dependence to μ and σ without changing the major ideas.

We also suppose that the short rate process r is given by $r_t = R(X_t)$, for some $R: D^n \to \mathbb{R}$. Thus the zero-coupon bond maturing at T has a price at $t \leqslant T$ given from (1) by

$$F(X_t, t) = E\left[\exp\left(-\int_t^T R(X_s)\,\mathrm{d}s\right)\middle| X_t\right]. \tag{7}$$

One could imagine that the state vector X_t might include various economic indices that would affect interest rates such as economic activity, monetary supply variables, central bank policy objectives, and so on. In order to facilitate the pricing and hedging of fixed-income derivatives, however, it is convenient to assume that one can find a change of variables under which we may view X_t as yield-related variables. This will be one of our objectives. We also desire a model that has some measure of numerical and econometric tractability. For both of these reasons, it may turn out to be convenient to take μ, $\sigma\sigma^{\mathrm{T}}$ and R to be affine functions on D into their respective ranges. (An affine function is a constant plus a linear function.) In this case, we say that the primitive model $(\mu, \sigma\sigma^{\mathrm{T}}, R)$ is affine.

Likewise, we say that the term structure is itself affine if there are C^1 functions $c: [0,\infty) \to \mathbb{R}$ and $C: [0,\infty) \to \mathbb{R}^n$ such that

$$y_{t,\tau} = c(\tau) + C(\tau)\cdot X_t, \quad t \geqslant 0, \quad \tau \geqslant 0, \tag{8}$$

so that yields are affine in the state variables.

Indeed, in Duffie & Kan (1992) it is shown that, under technical conditions, the basic model $(\mu, \sigma\sigma^{\mathrm{T}}, R)$ is affine if and only if the term structure is affine. This extends the same result for $n = 1$ given by Brown & Schaefer (1991). For an affine model, Duffie & Kan (1992) show that the coefficient functions c and C of (8) solve an ordinary differential equation of the form

$$C_i'(\tau) = k_i + K_i\cdot C(\tau) + C(\tau)^{\mathrm{T}} Q_i\, C(\tau), \quad i \in \{1, \dots, n\}, \tag{9}$$

$$c'(\tau) = k_0 + K_0\cdot C(\tau) + C(\tau)^{\mathrm{T}} Q_0\, C(\tau), \tag{10}$$

with boundary conditions

$$c(0) = C_i(0) = 0, \quad i \in \{1, \dots, n\}, \tag{11}$$

where $\{k_0, \ldots, k_n\} \subset \mathbb{R}$, $\{K_0, \ldots, K_n\} \subset \mathbb{R}^n$, and $\{Q_0, \ldots, Q_n\} \subset \mathbb{R}^{n \times n}$ are constant coefficients given in terms of the coefficients defining the underlying affine functions μ, $\sigma\sigma^{\mathrm{T}}$, and R. The Ricatti equation (9)–(11) can easily be solved numerically, for example by a Runge–Kutta method.

Given the solution (c, C) of (9)–(11), relation (8) provides an affine change of variables under which the state may be taken to be an n-dimensional 'yield-factor' process Y, where for some fixed maturities $\tau(1), \ldots, \tau(n)$, we take

$$Y_{ti} = y_{t,\tau(i)} = c(\tau(i)) + C(\tau(i)) \cdot X_t, \quad i \in \{1, \ldots, n\}. \tag{12}$$

We need only ensure that the 'basis maturities' $\tau(1), \ldots, \tau(n)$ are chosen so that the matrix K in $\mathbb{R}^{n \times n}$, defined by $K_{ij} = C_j(\tau(i))$, is non-singular. In that case, we have $Y_t = k + KX_t$, where $k_i = c(\tau(i))$, and the new state dynamics are given by

$$\mathrm{d}Y_t = \mu^*(Y_t)\,\mathrm{d}t + \sigma^*(Y_t)\,\mathrm{d}Wt, \tag{13}$$

where

$$\mu^*(y) = k\mu(K^{-1}y - k),$$

$$\sigma^*(y) = K\sigma(K^{-1}y - k),$$

for $y \in D^* = \{Kx + k : x \in D\}$.

If σ is constant, X and Y are Gauss–Markov processes of the Ornstein–Uhlenbeck form. For abstract factors, this gaussian model was developed by Langetieg (1980) and Jamshidian (1990, 1991). A Gauss–Markov yield-factor model was developed by El Karoui & Lacoste (1992) in the forward-rate setting of Heath *et al.* (1992), and in the current state-space setting, was developed as a special case of stochastic volatility models by Duffie & Kan (1992).

A simple example of non-constant σ is the multivariate Cox–Ingersoll–Ross model:

$$\mathrm{d}X_{it} = (a_i - b_i X_{it})\,\mathrm{d}t + c_i \sqrt{X_{it}}\,\mathrm{d}W_{it}, \quad i \in \{1, \ldots, n\}, \tag{14}$$

for positive constants a_i, b_i, c_i, appearing in Feller (1951), and developed for interest-rate modelling by Cox *et al.* (1985), Richard (1978), Heston (1991), Longstaff & Schwartz (1992), and Chen & Scott (1992*a*). Restrictions apply. For all i, we want

$$a_i > c_i^2/2. \tag{15}$$

As shown by Ikeda & Watanabe (1981), the latter restriction is necessary and sufficient to ensure that X will remain in the obvious open state space $D = \operatorname{int}(\mathbb{R}^n_+)$. Duffie & Kan (1992) study the general case, under which we can without loss of generality take

$$\mu(x) = ax + b; \quad \sigma_{ij}(x) = \gamma_{ij}\sqrt{(\alpha_{ij} + \beta_{ij} \cdot x)}, \tag{16}$$

for some $\gamma_{ij} \in \mathbb{R}$, $\alpha_{ij} \in \mathbb{R}$, $\beta_{ij} \in \mathbb{R}^n$, $a \in \mathbb{R}^{n \times n}$, and $b \in \mathbb{R}^n$. In this case, the state space is

$$D = \{x \in \mathbb{R}^n : \alpha_{ij} + \beta_{ij} \cdot x > 0, \quad i, j \in \{1, \ldots, n\}\}. \tag{17}$$

Strong restrictions on the coefficients (a, b, γ, α, β), analogous to (15) but more complicated, are shown by Duffie & Kan (1992) to imply the affine form and to guarantee the existence and uniqueness of solutions to $\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, x_0 \in D$, for (16)–(17).

Aside from the affine case, multivariate term-structure models appear in Brennan & Schwartz (1979), Chan (1992), El Karoui *et al.* (1992), Constantinides (1992), Beaglehole & Tenney (1991) and Jamshidian (1993). Most of these non-affine multifactor models do not allow direct observation of the state from the yield curve.

If one does not observe the state-vector directly, in principle one can filter the state variable from yield-curve data. There are debates concerning how much this limited observation property detracts from the practical application of the models. It can be said, for example, that we do not observe the yield curve in any case, but merely the prices of coupon bonds, from which one infers statistically (and with noise) the zero-coupon curve by some curve-fitting method such as splines or nonlinear least squares. In any case, it seems to be of at least some value to have state variables that can be observed in terms of the yield curve, as in the affine models described above.

5. Derivative pricing

Given a term structure model $(\mu, \sigma\sigma^{\mathrm{T}}, R)$, affine or not, one is interested in the pricing of derivative securities. Recall that the price of a security with pay-off u at time T is given at time t by

$$E\left[\exp\left(-\int_t^T R(X_s)\,\mathrm{d}s\right)u \,\middle|\, \mathscr{F}_t\right].$$

If u is measurable with respect to the yield curve at time T, as are bond options and other 'path-independent' derivatives, we may take $u = g(X_T)$ for some $g: D^n \rightarrow \mathbb{R}$, since the yield curve y_T is itself X_T-measurable. In this case, the Markov property of X implies that we can write the derivative price as

$$F(X_t, t) = E\left[\exp\left(-\int_t^T R(X_s)\,\mathrm{d}s\right)g(X_t)\,\middle|\, X_t\right], \tag{18}$$

for some $F: D \times [0, T] \rightarrow \mathbb{R}$. Under the technical regularity given, for example, in Friedman (1975), we also know that F is the unique solution in $C^{2,1}(D^n \times [0, T))$, under technical growth conditions, to the parabolic partial differential equation

$$\mathscr{D}F(x,t) - R(x)F(x,t) = 0, \quad (x,t) \in D \times [0, T), \tag{19}$$

$$F(x,t) = g(x), \quad x \in D, \tag{20}$$

where

$$\mathscr{D}f(x,t) = F_x(x,t)\,\mu(x) + F_t(x,t) + \tfrac{1}{2}\,\mathrm{trace}\,(\sigma(x)\,\sigma(x)^{\mathrm{T}} F_{xx}(x,t)).$$

One can then solve for path-independent derivative prices via a numerical solution of the partial differential equation (19)–(20), say by finite-difference methods. (For finite-difference algorithms, see for example Ames (1977).) For affine multi-factor models, fully worked examples are given by Duffie & Kan (1992) for the case $n = 2$. For large n, say more than 3, currently available algorithms and hardware are not up to the task, and Monte Carlo simulation may be applied (see, for example, Duffie & Glynn 1992; Kloeden & Platen 1992). For the path-dependent case, unless there is a simple way to augment the state space so as to capture the path dependence with an additional state variable, it may also be advisable to resort to Monte Carlo simulation. There are only rare cases, such as Jamshidian's (1991) solution for bond options in the gaussian setting, for which one can obtain explicit solution for derivative prices (see, also, El Karoui & Rochet 1989).

6. Where do we go from here?

A great deal of work remains to be done. First, we have discussed only the case of single-currency yield curves with no default risk. International models, which consistently include random exchange rate fluctuations, are difficult to model in a tractable way. An example, in the same affine state space setting emphasized here, is offered by Nielson & Saá-Requejo (1992). Modelling default risk in a consistent way, while maintaining tractability, is also challenging. Madan & Unal (1992) and Jarrow & Turnbull (1992) offer examples of models that push in this direction.

Econometric modelling of the term structure, particularly in a multi-factor setting, has stayed within a relatively narrow framework. Recent work by Gibbons & Ramaswamy (1992), Pearson & Sun (1990), and Chen & Scott (1992*b*, 1993), for example, stays strictly within the CIR single-factor or multi-factor cases of the affine model emphasized here. For the constant-volatility Gauss–Markov (affine) case, Frachot *et al*. (1992) together with Frachot & Lesne (1993) have done some empirical work in the Heath–Jarrow–Morton setting. Much remains to be done in integrating the use of statistical models within the practical applications of term structure models mentioned in the introduction.

Judging from the literature on term structure modelling, much also remains to be done in the development and application of numerical methods, such as finite-difference or finite-element algorithms for multidimensional Cauchy problems such as (19)–(20), to the particular sorts of applications that are found in fixed-income markets.

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