Credit Swap Valuation

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Abstract: This[1] article explains the basics of pricing credit swaps.

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1 Introduction

This paper is a relatively non-technical review of the pricing of credit swaps, a form of derivative security that can be viewed as default insurance on loans or bonds. Credit swaps pay the buyer of protection a given contingent amount at the time of a given credit event, such as default. The contingent amount is often the difference between the face value of a bond and its market value, paid at the default time of the underlying bond. This special case defines a "default swap." In return, the buyer of protection pays an annuity premium until the time of the credit event, or the maturity date of the credit swap, whichever is first.

Other key credit derivatives include

- Total-return swaps, which pay the net return of one asset class over another. If the two asset classes differ mainly in terms of credit risk, such as a treasury bond versus a corporate bond of similar duration, then the total-return swap is a credit derivative.

- Collateralized debt obligations, which are typically tranches of a structure collateralized by a pool of debt, whose cash flows are allocated, according to a specified proritized schedule, to the individual tranches of the structure.

- Spread options, which typically convey the right to trade bonds at given spreads over a reference yield, such as a treasury yield.

Credit swaps are perhaps the most popular, currently, of the above types of credit derivatives. As opposed to many other forms of credit derivatives, payment to the buyer of protection in a credit swap is triggered by a contractually defined event, that must be documented.

2 The Basic Credit Swap

The basic credit-swap contract is as follows. Parties A and B enter into a contract terminating at the time of a given credit event, or at a stated maturity, whichever is first. A commonly stipulated credit event is default by a named issuer, say Entity C, which could be a corporation or a sovereign issuer. There are interesting applications, however, in which credit events may be defined instead in terms of downgrades, events that may instigate (with some uncertainty perhaps) the default of one or more counterparties, or other credit-related occurrences.[2] There is some risk of disagreement over whether the event has in fact occurred. In valuing the credit swap, we will ignore this documentation or enforceability risk.

In the event of termination at the designated credit event, Party A pays Party B a stipulated termination amount, possibly contingent. For example, with the most common form of credit swap, called a *default swap*, A would pay B, if termination is triggered by the default of Entity C, an amount that is, in effect, the difference between the face value and the market value of a designated note issued by C.

In compensation for what it may receive in the event of termination by a credit event, Party B pays Party A an annuity, at a rate called the *credit-swap spread*, or sometimes the *credit-swap premium*, until the maturity of the credit swap, or termination by the designated credit event.

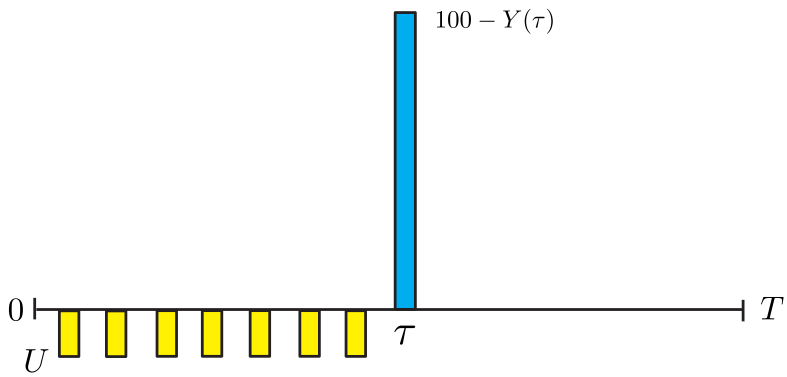
The cash flows of a credit swap are illustrated in Figure 2. The payment at the default time τ, if before maturity T, is the difference $D = 100 - Y(\tau)$ between the face value, say 100, and the market value $Y(\tau)$ of the designated underlying note at τ. The credit-swap annuity coupon rate is denoted U.

In some cases,[3] the compensating annuity may be paid as a spread over the usual plain-vanilla (non-credit) swap rate. For example, if the 5-year fixed-for-floating interest-rate swap rate is 6 percent versus LIBOR, and if B is the fixed-rate payer in the default swap, then B would pay a fixed rate higher than the usual 6 percent. If, for example, B pays 7.5 percent fixed versus LIBOR, and if the C-issued note underlying the default swap is of the same notional amount as the interest rate swap, then in this case we would say that the default-swap spread is 150 basis points. If B is the floating-rate payer on the interest rate swap, then B would pay floating plus

[2] At a presentation at the March, 1998 ISDA conference in Rome, Daniel Cunnigham of Cravath, Swaine, and Moore reviewed the documentation of credit swaps, including the specification of credit event types such as "bankruptcy, credit event upon merger, cross acceleration, cross default, downgrade, failure to pay, repudiation, or restructuring." The credit event is to be documented with a notice, supported with evidence of public announcement of the event, for example in the international press. The amount to be paid at the time of the credit event is determined by one or more third parties, and based on physical or cash settlement, as indicated in the confirmation form of the OTC credit swap transaction, a standard contract form with indicated alternatives.

[3] Discussions with a global bank indicated that of over 200 hundred default swaps, approximately 10 percent were combined with an interest-rate swap.

Credit Swap



- **Annuity paid at rate U to $\min(T, \tau)$ for credit event time τ.**
- **Receive par less market value $Y(\tau)$ of underlying at τ if $\tau \leq T$.**

Figure 1: Credit Swap Cash Flows

a spread in return for the usual market fixed rate on swaps, or in effect would receive fixed less a spread. It is not necessarily the case that the theoretical default-swap spread would be the same in the case of B paying fixed as in B paying floating. In general, combining the credit swap with an interest rate swap affects the quoted credit swap spread, because an interest-rate swap whose fixed rate is the at-market swap rate for maturity T, but with random early termination, does not have a market value of zero. For example, if the term structure of forward rates is steeply upward sloping, then an at-market interest rate swap to maturity T or credit event time, whichever is first, has a lower fixed rate than does a plain-vanilla at-market interest-rate swap to the maturity T. A credit spread of 150 basis points over the at-market plain vanilla swap rate to maturity T therefore represents a larger credit spread than does a credit swap, without an interest rate swap, paying a premium of 150 basis points.

Apparently, for corporate names underlying, it is not unusual to see default swaps in which the payment at default is reduced by the accrued portion of the credit-swap premium. We will briefly consider this variation below.

In short, the classic credit swap can be thought of as an insurance contract, under which the insured agent pays an insurance premium in return for coverage against a loss that occurs at a credit event.

There are in effect two pricing problems:

1. At origination, the standard credit swap involves no exchange of cash flows, and therefore (ignoring dealer margins and transactions costs), has a market value of zero. One must, however, determine the at-market annuity premium rate U, that for which the market value of the credit swap is indeed zero. This at-market rate is sometimes called the market credit-swap spread, or simply the credit-swap premium.

2. After origination, changes in market interest rates and in the credit quality of the reference entity C, as well as the passage of time, typically change the market value of the default swap. For a given credit swap, with a stated annuity rate U, one must determine the current market value, which is not generally zero.

When making markets, the former pricing problem is the more critical. When hedging or conducting a mark-to-market, the latter is relevant. Solution methods for the two problems may call for similar capabilities. The latter problem is more challenging, generally, as there is less liquidity for off-market default swaps, and pricing references, such as bond spreads, are of relatively less use.

The next section considers simple credit swaps. The following sections consider extensions. We will not consider more "exotic" forms of credit swaps, such as *first-to-default swaps*, for which the credit event time τ is the first of the default times of a given list of underlying notes or bonds, with a payment at the credit event time that depends on the identity of the first of the underlying bonds to default. For example, the payment could be the loss relative to face value of the first bond to default.

We will assume throughout that the credit swap counterparties A and B are default free, so as to avoid here the pricing impact of default by counterparties A and B, which can be treated by the first-to-default results in Duffie (1998b).

3 Simple Credit-Swap Spreads

For this section, we assume that the contingent payment amount specifed in the credit swap is the difference $D = 100 - Y(\tau)$ between the face value and the market value $Y(\tau)$ at the credit event time τ of a note issued by Entity C, at the time of the credit event.

We will explain the pricing in stages, adding complications as we go.

3.1 Credit-Swap Spreads: Starter Case

Our assumptions for this starter case are as follows.

1. There is no embedded interest-rate swap. That is, the default swap is an exchange of a constant coupon rate U paid by Party B until termination at maturity or by the stated credit event (which may or may not be default of the underlying C-issued note.) This eliminates the need to consider the value of an interest-rate swap with early termination at a credit event.

2. There is no payment of the accured credit-swap premium at default.

3. The underlying note issued by C is a par floating rate note (FRN) with the maturity of the credit swap. This important restriction will be relaxed.

4. It is costless to short the underlying FRN, for example through a reverse repurchase agreement, or "repo."[4] That is, we suppose for now that one can enter into a reverse repo, receiving the general collateral rate (GCR), repeatedly rolling the open repo position overnight to any desired date.

5. There exists a default-free FRN, with floating rate R_t at date t. The coupon payments on the FRN issued by C (C-FRN) are contractually specified to be $R_t + S$, where the spread S is fixed. In practice, FRN

[4]Under the repo contract, for a given term and repo rate, one receives the C-issued FRN as collateral on a loan with the repo counterparty, and returns the collateral at the end of the term of the repo. At term, one also receives the principal of the loan plus interest at the specified repo rate. The specific collateral is on special if the associated repo rate is below the general collateral rate, which can be thought of as the riskless interest for the specified term. See, for example, Duffie (1996).

spreads are usually relative to LIBOR, or some other benchmark floating rate that need not be a pure default-free rate. There is no difficulty for our analysis if the pure default-free floating rate R_t and the reference, say LIBOR, rate L_t differ by a constant. One should bear in mind that the short-term treasury rate is not a pure default-free interest rate, because of repo specials and the "moneyness" of treasuries[5] or tax advantages of treasuries. A better benchmark for risk-free borrowing is the term general collateral rate, which is close to default-free, and has typically been close to LIBOR, with a slowly varying spread to LIBOR, in US markets. For example, suppose the C-FRN is at a spread of 100 basis points to LIBOR, which in turn is at a spread to the general collateral rate which, while varying over time, is approximately 5 basis points. Then, for our purposes, an approximation of the spread of the C-FRN to the default-free floating rate would be 105 basis points.

6. There are no transactions costs, such as bid-ask spreads, in cash markets for the default-free or C-issued FRN. In particular, at the initiation of the credit swap, one can sell, at its market value, the underlying C-FRN. At termination, one can buy, at market value, the C-FRN.

7. The termination payment given a credit event is made at the immediately following coupon date on the underlying C-issued note. If not, there is a question of accrued interest, which can be accomodated by standard time-value calculations shown below.

8. The credit swap is settled, if terminated by the stated credit event, by the physical delivery of the C-FRN in exchange for cash in the amount of its face value. (Many credit-swaps are cash-settled, and as yet neither physical nor cash settlement seems to be a predominant standard.)

9. Tax effects can be ignored. If not, the calculations to follow apply after tax, using the tax rate of investors that are indifferent to purchasing the default swap at its market price.

With these assumptions in place, one can "price" the credit swap, that is, compute the at-market credit-swap spread U, by the following arbitrage

[5] Treasuries are often useful as a medium of exchange, for example in securities transactions that are conducted by federal funds wire, or for margining services. This conveys extra value.

argument, based on a synthesis of Party B's cash flows on the credit swap.

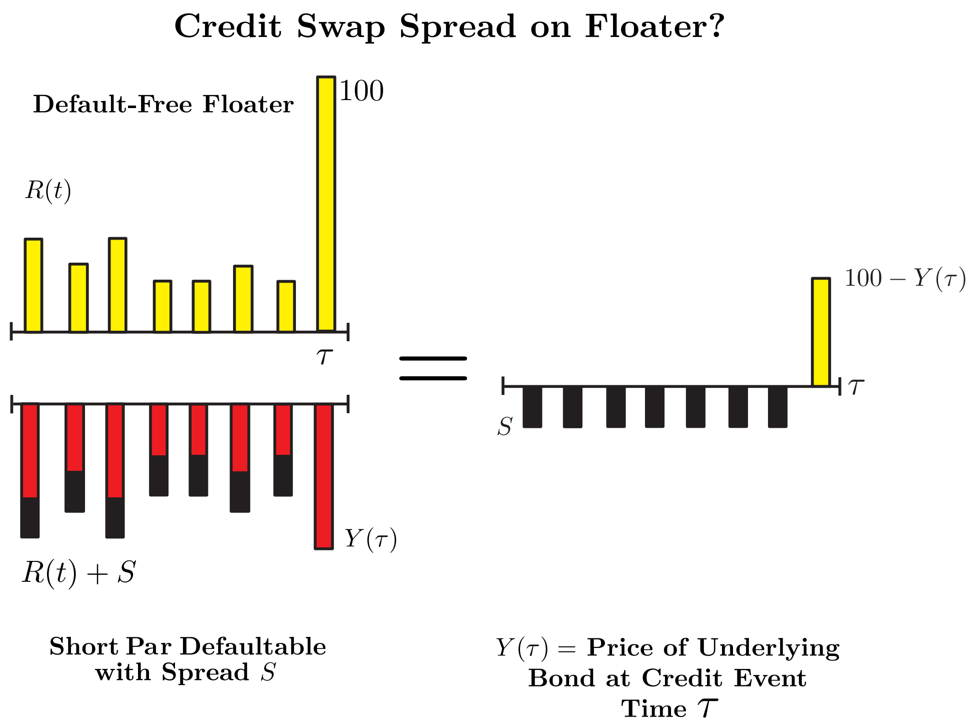


Figure 2: Synthetic Credit Swap Cash Flows

One can short the par C-issued FRN for an initial cash receivable of, say, 100 units of account, and invest this 100 in a par default-free FRN. One holds this portfolio through maturity or the stated credit event, whichever is earlier. In the meantime, one pays the coupons on the C-FRN and receives the coupons on the default-free FRN. The net paid is the spread S over the default-free floating rate on par C-FRNs.

If the credit event does not occur before maturity, then both notes mature at par value, and there is no net cash flow at termination.

If the credit event does occur before maturity, then one liquidates the portfolio at the coupon date immediately after the event time, collecting the difference $D = 100 - Y(\tau)$ between the market value of the default-free FRN (which is par on a coupon date) and the market value of the C-FRN. (Liquidation calls for termination of the short position in the C-FRN, which involves buying the C-FRN in the market for delivery against the short, say through the completion of a repo contract.)

As this contingent amount D is the same as that specified in the credit swap contract, the absence of arbitrage implies that the unique arbitrage-free at-market credit-swap spread U is S, the spread over risk-free on the underlying floating-rate notes issued by C. (Combining the above strategy with A's cash flows as the seller of the credit swap results in a net constant annuity cash flow of $U - S$, until maturity or termination. We must therefore have $U = S$ if there is no arbitrage for A or B, and no other costs.)

This arbitrage, under its ideal assumptions, is illustrated in Figure 2.

3.2 The Reference Par Spread for Default Swaps

We can relax the restrictive assumption that the underlying note has the same maturity as the credit swap, provided the credit swap is in fact a default swap. In that case, the relevant par spread S for fixing the credit swap spread is that of a possibly different C-issued FRN which is of the same maturity as the credit swap, and of the same priority as the underlying note. We call this the "reference C-FRN." Assuming absolute priority applies at default, so that the underlying note and the reference note have the same recovery value at default, the previous arbitrage pricing argument applies. This argument works, under the stated assumptions, even if the underlying note is a fixed-rate note of the same seniority as the reference C-FRN.

There are some cautions here. First, there may often be no reference C-FRN. Second, absolute priority need not apply in practice. For example, a senior short-maturity FRN and a senior long-maturity fixed-rate note may represent significantly different bargaining power, especially in a reorganization scenario at default.

3.3 Adding Repo Specials and Transactions Costs

Another important and common violation of the assumptions in the previous "starter case" is the ability to freely short the reference FRN issued by C ("C-FRN"). A typical method of shorting securities is via a reverse repo combined with a cash sale. That is, through a reverse repo, one can arrange to receive the reference note as collateral on a loan to a given term. Rather than holding the note as collateral, one can immediately sell the note. In effect, one has then created a short position in the reference note through the term of the repo.

In many cases, one cannot arrange a reverse repo at the general collateral rate (GCR). Instead, if the reference note is "scarce," one may be forced to offer a repo rate that is below the GCR in order to reverse in the C-FRN as collateral. This is called a repo "special." (See, for example, Duffie (1996).) Moreover, one may expend resources in arranging the repo (especially if the C-FRN is rare or otherwise difficult to obtain). In addition, particularly with risky FRNs, there may be a substantial bid-ask spread in the market for the reference FRN at initiation of the repo (when one sells) and at termination (when one buys).

Suppose that one can arrange a term reverse repo collateralized by the C-FRN, with maturity equal to the maturity date of the credit swap. We also suppose that default of the collateral triggers early termination of the repo at the originally agreed repo rate. (This is the case in many jurisdictions.)

The term repo special is the difference Y between the term general collateral rate (GCR) and the term specific collateral rate for the C-FRN. In order to short the C-FRN, one would then effectively pay an extra annuity of Y, and the default-swap spread would be approximately $S + Y$.

If the term repo does not necessarily terminate at the credit event, this is not an exact arbitrage-based spread. Because the probability of a credit event well before maturity is typically small, however, and because term repo specials are often small, the difference may not be large in practice.[6]

For the synthesis of a short position in the credit swap, one purchases the C-FRN and places it into a term repo in order to capture the term repo special.

If there are also transactions costs in the cash market, then the credit-swap broker-dealer may incur risk from uncovered credit-swap positions, or transactions costs, or some of each, and in principle may charge an additional premium. With two-sided market making and diversification, it is not clear how quickly these costs and risks build up over a portfolio of positions. We will not consider these effects directly here. In practice, for illiquid entities, the credit-swap spread can vary substantially from the reference par FRN spread, according to discussions with traders.

We emphasize the difference between a transactions cost and a repo special. The former simply widens the bid-ask spread on a default swap, increasing the default-swap spread quoted by a broker-dealer who sells a credit

[6]If the term repo rate applies to the credit-swap maturity, then $S + Y$ is a lower bound on the theoretical credit-swap premium.

swap, and reducing the quoted default-swap when a broker-dealer is asked by a customer to buy a default swap from the customer. A repo special, however, is not itself a transaction cost; but rather can be thought of as an extra source of interest income on the underlying C-FRN, effectively changing its spread relative to the default free rate. The existence of substantial specials, which raise the cost of providing the credit swap, do not necessarily increase the bid-ask spread. For example, in synthesizing a short position in a default swap, one can place the associated long position in the C-FRN into a repo position and profit from the repo special.

In summary, under our assumptions to this point, a dealer can broker a default swap (that is, take the position of Party A) at a spread of approximately $S + Y$, with a bid-ask spread of K, where

1. S is the par spread on a "reference" floating-rate note issued by the named entity C, of the same maturity as the default swap, and of the same seniority as the underlying note.

2. Y is the term repo special on par floating-rate notes issued by C, or otherwise an estimate of the annuity rate paid, through the term of the default swap, for maintaining a short position in the reference note to the termination of the credit swap.

3. K reflects any annuitized transactions costs (such as cash-market bid-ask spreads) for hedging, any risk premium for un-hedged portions of the risk (which would apply in imperfect capital markets), overhead, and a profit margin.

In practice, it is usually difficult to estimate the effective term repo special, as default swaps are normally of much longer term than repo positions. There have apparently been cases in which liquidity in a credit-swap has been sufficient to allow some traders to quote term repo rates for the underlying collateral by reference to the credit-swap spread!

3.4 Payment of Accrued Credit-Swap Premium

Some credit swaps, more frequently on corporate underlying bonds, specify that the buyer of protection must, at default, pay the credit-swap premium that has accrued since the last coupon date. For example, with a credit swap spread of 300 basis points and default one third of the way though the current semi-annual (say) coupon period, the buyer of protection would receive face value less recovery value of the underlying, less one third of the semi-annual annuity payment, which is 0.5% of the underlying face value.

For reasonablly small default probabilities and inter-coupon periods, the expected difference in time between the credit event and the previous coupon date is approximately one half of the length of an inter-coupon period. Thus, for pricing purposes in all but extreme cases, one can think of the credit swap as equivalent to payment at default of face value less recovery value less one half of the regular default swap premium payment.

For example, suppose there is some $h > 0$ that is the risk-neutral Poisson mean arrival rate of the credit event. Then one estimates a reduction in the at-market credit-swap spread for accrued premium, below that spread S appropriate without the accrued-premium feature, of approximately $hS/(2n)$, where and n is the number of coupons per year of the underlying bond. For a pure default swap, S is smaller than h because of partial recovery, so this correction is smaller than $h^2/(2n)$, which is negligible for small h. For example, at semi-annual credit swap coupon intervals and a risk-neutral mean arrival rate of the credit event of 2% per year (or 200 basis points), we have a correction of under 1 basis point for this accrued premium effect.

3.5 Accrued Interest on the Underlying Notes

For purposes of the synthetic arbitrage calculation described above, there is a question of accrued interest payment on the default-free floating rate note.

The typical credit swap specifies payment of the difference between face value *without accrued interest* and market value of the underlying note. The above described arbitrage portfolio (long default-free floater, short defaultable floater), however, is worth face value, *plus accrued interest on the default-free note*, less recovery on the underlying defaultable note. If the credit event involves default of the underlying note, then the previous arbitrage argument is not quite right.

Consider, for example, a one-year default swap with semi-annual coupons. Suppose the LIBOR rate is 8 percent. The expected value of the accrued interest on a default-free note at default is thus approximately 2 percent of face value, for small default probabilities. Suppose the annualized risk-neutral mean arrival rate of the credit event is 4 percent. Then there is a reduction in market value of the credit swap to the buyer of protection of roughly 8 basis points of face value, and therefore a reduction of the at-market credit-swap spread of roughly 8 basis points.

More generally, for credit swaps of any maturity, and with relatively small and constant risk-neutral default probabilities and relatively flat term structures of default-free rates, the reduction in the at-market credit-swap spread for the accrued-interest effect, below the par floating rate-spread plus effective repo special, is approximately $hr/(2n)$, where h is the risk-neutral Poisson arrival rate of the credit event, r is the average of the default-free forward rates through credit-swap maturity, and n is the number of coupons per year of the underlying bond. Of course, one could work out the effect more precisely with a term structure model, as mentioned below.

3.6 Approximating the Reference Floating-Rate Spread

If there is no available par floating rate note of the same credit quality, whose maturity is that of the default swap, then one could attempt to "back out" the reference par spread S from other spreads. For example, suppose that there is an FRN issued by C of the swap maturity, and the same seniority as the underlying, that is trading at a price p, which is not necessarily par, and paying a spread of $\hat{S}$ over the default-free floating rate. Let A denote the associated annuity price, that is, the market value of an annuity paid at a rate of 1 until the credit-swap termination (its maturity, or default of the underlying note, whichever is first).

For reasonably small credit risk and interest rates, A is close to the default-free annuity price, as most of the market value of the credit risk of a FRN is associated in this case with potential loss of principal. We will return below to consider a more precise computation of A.

As the difference between a par and a non-par FRN with the same maturity is the coupon spread (assuming the same recovery at default), we have

$$p - 1 = A(\hat{S} - S),$$

where S is the implied par spread. Solving for the implied par spreads, we have

$$S = \hat{S} + \frac{1-p}{A}.$$

With this, we can estimate the reference par spread S.

If the relevant price information is for a fixed-rate note issued by C of the reference maturity and seniority, then one can again resort to the assumption that its recovery of face value at default is the same as that of a par-floater of

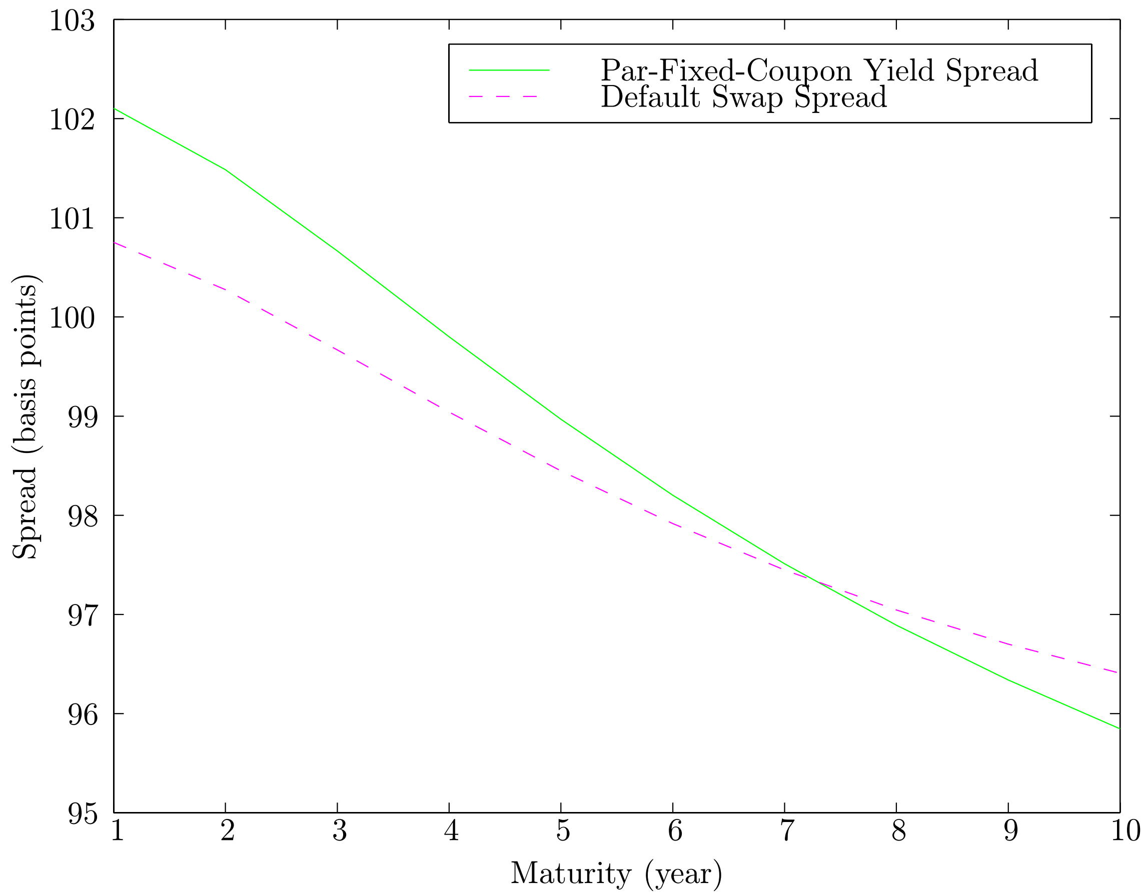


Figure 3: Term structures of bond and default-swap spreads.

the same seniority. (This is again reasonable on legal grounds in a liquidation scenario.) One can again attempt to "back out" there reference par floating rate spread S.

Spreads over default-free on par-fixed-rate and par-floating-rate notes are approximately equal.[7] Thus, if the only reference spread is a par fixed spread F, then it would be reasonably safe to use F in place of S in estimating the default-swap spread.

[7]The floating-rate spread is known theoretically to be slightly higher with the typical upward-sloping term structure, but the difference is typically on the order of 1 basis point or less on a 5-year note per 100 basis points of yield spread to the default-free rate. Again, see Duffie and Liu (1997) for details.

For example, Figure 3 shows a close relationship between the term structures of default-swap spreads and par fixed-coupon yield spreads for the same credit quality.[8] Some of the difference shown in Figure 3 between the default default-swap spread and the par fixed-coupon yield spread is in fact the accrued-interest effect discussed in the previous sub-section.

If the reference pricing information is for a non-par fixed-rate note, then we can proceed as before. Let p denote the price of the available fixed-rate note, with spread $\hat{F}$ over default free. We have

$$p - 1 = A(\hat{F} - F),$$

where A is again the annuity price to maturity or default, whichever is first. With an estimate of A, we thus obtain an estimate of the par fixed spread F, which in turn is a close approximation of the par floating-rate spread S, which is the quantity needed to compute the default-swap spread.

It is sometimes said that if the underlying is a fixed-rate bond, then the reference par floating-rate spread may be taken to be the asset-swap spread. The usefulness of this assumption is considered in the last section of this note.

4 Estimating Hazard Rates and Defaultable Annuity Prices

The hazard rate for the credit event is the arrival rate, in the sense of Poisson processes. For example, a constant hazard rate of 400 basis points represents a mean arrival rate of 4 times per 100 years. The mean time to arrival, conditional on to arrival to date t, remains 25 years, for any t.

This section contains some intermediate calculations that can be used to estimate implied hazard rates and the annuity price A described above. For small hazard rate h, the probability of defaulting during a time period small

[8]This figure is based on an illustrative correlated multi-factor Cox-Ingersoll-Ross model of default free-short rates and default arrival intensities. The short-rate model is a 3-factor CIR model calibrated to recent behavior in the term structure of LIBOR swap rates. The risk-neutral default-arrival intensity model is set for an initial arrival intensity of 200 basis points, with 100% initial volatility in intensity, mean-reverting in risk-neutral expectation at 25% per year to 200 basis points until default. Recovery at default is assumed to be 50% of face value. For details, see Duffie (1998b). The results depend on the degree of correlation, mean reversion, and volatility among short rates and default arrival intensities.

length Δ, conditional on survival to the beginng of the period, is approximately $h\Delta$.

4.1 The Case of Constant Default Hazard Rate

We begin by supposing that default occurs at a risk-neutral constant hazard rate of h. This implies that default occurs at a stopping time that, under "risk-neutral probabilities," occurs at the first jump time of a Poisson process with intensity h. We let

- $a_i(h)$ be the value at time 0 of receiving one unit of account at the i-th coupon date in the event that default is after that date.
- $b_i(h)$ is the value at time 0 of receiving one unit of account at the i-th coupon date in the event that default is between the $(i-1)$-th and the i-th coupon date.

Then

$$a_i(h) = e^{-(h+y(i))T(i)},$$

where $T(i)$ is the maturity of the i-th coupon date and $y(i)$ is the continuously compounding default-free zero-coupon yield to the i-th coupon date. Likewise, under these assumptions,

$$b_i(h) = e^{-y(i)T(i)}(e^{-hT(i-1)} - e^{-hT(i)}).$$

The price A of an annuity of one unit of account paid at each coupon date until default or maturity $T = T(n)$, whichever comes first, is

$$A(h, T) = a_1(h) + \cdots + a_n(h).$$

The value of a payment of one unit of account at the first coupon date after default, provided the default date is before the maturity date $T = T(n)$, is

$$B(h, T) = b_1(h) + \cdots + b_n(h).$$

Now, we consider a classic default swap:

- Party B pays Party A a constant annuity U until maturity T or the default time τ of the C-FRN.

- If $\tau \leq T$, Party A pays Party B, at τ, 1 unit of account minus the value at τ of the C-FRN.

Now, suppose that the loss of face value at default carries no risk premium[9] and has an expected value of f.

Then, given the parameters (T, U) of the default-swap contract, and given the default-risk-free term structure, we can compute the market value $V(h, f, T, U)$ of the classic default swap as a function of any assumed default parameters h and f, to be

$$V(h, f, T, U) = B(h, T)f - A(h, T)U.$$

The at-market default swap spread $U(h, T, f)$ is obtained by solving $V(h, f, T, U) = 0$ for U, leaving

$$U(h, T, f) = \frac{B(h, T)f}{A(h, T)}.$$

For more accuracy, one can easily account for the difference in time between the credit event and the subsequent coupon date. At small hazard rates, this difference is just slightly more than one half of the inter-coupon period of the credit swap, and can be treated analytically in a direct manner. Alterntatively, a simple approximate adjustment can be made by noting that the effect is equivalent to the accrued-interest effect in adjusting the par floating rate spread to the credit-swap spread. As mentioned previously, this causes an increase in the implied default-swap spread that is on the order of $hr/(2n)$, where r is the average of the inter-coupon default-free forward rates through maturity. (For a steeply slope forward-rate curve, one can obtain a better approximation.)

Estimates of the expected loss f at default and the risk-neutral hazard rate h can be obtained from the prices of bonds or notes issued by the entity C, from risk-free rates, and from recovery data[10] for bonds or notes of the same seniority.

For example, suppose that some, possibly different, C-FRN sells at a price of p, has a maturity of $\hat{T}$, and has a spread of $\hat{S}$. Suppose the expected

[9] Recovery risk is sometimes viewed as reasonably diversifiable, and relatively unrelated to the business cycle. No rigorous test of these hypotheses are available.

[10] Sources include annual reports by Moody's and Standard and Poors for bonds, and Carey (1998) and sources cited therein for loans. The averages reported are typically by seniority.

default loss, relative to face value, is $\hat{f}$. Under the above assumptions, by purchase of a risk-free floater and shorting a C-FRN (with no repo specials), we have a portfolio with market value

$$1 - p = A(h, \hat{T})\hat{S} + B(h, \hat{T})\hat{f}.$$

This equation can be solved for the implied risk-neutral hazard rate h.

Provided the reference prices of notes used for this purpose are near par, there is a certain robustness here associated with uncertainty about recovery, as an upward bias in f results in a downward bias in h, and these errors (for small h) approximately cancel each other when estimating the mark-to-market value $V(h, f, T, U)$ of the default swap. For this robustness, it is better to use a reference note of approximately the same maturity as that of the default swap.

If the note issued by C that is chosen for price reference is a fixed-rate note, with price p, coupon rate c, expected loss at default relative to face value of $\hat{f}$, and maturity $\hat{T}$, then we would use the relationship

$$p = A(h, T)c + B(h, \hat{T})(1 - \hat{f}),$$

in order to estimate the risk-neutral hazard rate h.

In order to check the sensitivity of the model to relative choice of intensity and expected recovery, one can use the intuition that the coupon yield spread S of a fixed-rate bond is roughly the product of the mean default intensity and the fractional loss of value at default. This intuition can be given a formal justification in certain settings, as explained in Duffie and Singleton (1997). In the same multi-factor CIR setting for interest rates and default intensities that we considered in Figure 3, we plot in Figure 4, for various par 10-year coupon spreads (S), at each assumed level $w = (1 - f)$ of expected recovery of face value at default, the risk-neutral mean (set equal to initial) default intensity $\overline{h}$ implied by the term-structure model, and that mean intensity implied by the approximation $S = f\overline{h}$.

As one can see, up to a high level of fractional recovery spread S, the effects of varying h and f are more or less offsetting in the fashion suggested. (That is, if one over-estimates f by a factor of 2, then for a given reference coupon rate, one will underestimate h by a factor of roughly 2 using even a crude term structure model, and the implied par-coupon spread will be relatively unaffected, meaning that the default swap-spread is also relatively unaffected.) This approximation is more accurate for maturities of less than 10 years. The degree to which the approximation works poorly at high spreads is mainly due to the fact that par spreads are measured on a bond-equivalent yield (semi-annual compounding) basis, while the mean intensity is measured on a continuously compounding basis.

If there are multiple reference notes with maturities similar to that of the underlying default swap, then one could, for example, average their implied hazard rates, or discard outliers and then average, or use non-linear least-squares fitting, or conduct some similar pragmatic estimation procedure. There may, however, be important differences based on institutional effects that affect relative recovery. For example, in negotiated workouts, one investor group may be favored over another for bargaining reasons.

Default swaps seem, at least currently, to be a benchmark for credit pricing. For example, it is sometimes the case that the at-market default-swap quote U^* is available, and one wishes to estimate the implied risk-neutral hazard rate h. This is obtained from solving $U(h, T, f) = U^*$ for h. This market-implied risk-neutral hazard rate is denoted $H(U^*, f, T)$ for future purposes. As suggested above, the model result depends more or less linearly on the modeling assumption for the expected fractional loss at default. Sensitivity analysis is suggested if the objective is to apply the intensity estimate to price an issue that has substantially different cash-flow features than that of the reference default swap.

4.2 The Term Structure of Hazard Rates

If the reference credit pricing information is for maturities different than that of the credit swap, it is advisable to estimate the term structure of hazard rates. For example, one could assume that the hazard rate between coupon dates $T(i-1)$ and $T(i)$ is $h(i)$. In this case, given the vector $h = (h(1), \ldots, h(n))$, we have (assuming equal inter-coupon time intervals) the more general calculations

$$a_i(h) = e^{-(H(i)+y(i))T(i)},$$

where

$$H(i) = \frac{h_1 + \cdots + h_i}{i},$$

and

$$b_i(h) = e^{-y(i)T(i)}(e^{-H(i-1)T(i-1)} - e^{-H(i)T(i)}).$$

With these changes in place, all of our previous results apply. As there is a well established dependence of credit spreads on maturity, it is wise to consider the term structure when valuing credit swaps or inferring default probabilities from credit swap spreads.

When information regarding the shape of the term structure of hazard rates for the reference entity C is critical but not available, it may be pragmatic to assume that the shape is that of comparable issuers. For example, one might use the shape implied by Bloomberg par yield spreads for firms of the same credit rating and sector, and then to scale the implied hazard rates to match the pricing available for the reference entity. This is *ad hoc*, and subject to the modeler's judgement.

A more sophisticated approach would be to build a term-structure model for a stochastically varying risk-neutral intensity process h, as in Duffie (1998a), Duffie and Singleton (1995), Jarrow and Turnbull (1995), or Lando (1998). Default-swap pricing is reasonably robust, however, to the model of intensities, calibrated to given spread correlations and volatilities, according to tests conducted by the author. For example, Figure 5 shows that default swap spreads do not depend significantly on how much the default arrival intensity h is assumed to change with each 100 basis-point change in the short rates. The effect of volatility of default risk on default-swap spreads becomes pronounced only at relatively high levels of volatility, as indicated in Figure 7. For this figure, the volatility is measured in the usual sense (percentage standard deviation), but at initial conditions, for a CIR style intensity model. (These figures are based on the same illustrative model used as a basis for Figure 3.) The effect of volatility arises essentially from Jensen's inequality.[11]

Even the general structure of the defaultable term-structure model may not be critical for determining default-swap spreads. For example, Figure 6 shows par-coupon yield spreads for two term-structure models. One ("RMV") assumes recovery of 50% market value at default, based on Duffie and Singleton (1995). The other ("RFV") assumes recovery of 50% of face value at default. They have the identical CIR-model for short rates and

[11]The risk-neutral survival probability to term T for a risk-neutral intensity process h is, under standard regularity assumptions, given by $E^*\left[\exp\left(-\int_0^T h(t)\,dt\right)\right]$, where E^* denotes risk-neutral expectation. See, for example, Lando (1998) for a survey. Because $\exp(\,\cdot\,)$ is convex, more volatility of risk-neutral intensity causes a higher risk-neutral survival probability, other things equal, and thus narrower credit spreads.

intensities used in all of our previous illustrations, with the same parameters and initial conditions. Despite the difference in recovery assumptions, with no atempt to calibrate the two models to given prices, the implied term structures are rather similar. With calibration to a reference bond of maturity similar to that of the underlying bond, the match of credit-swap spreads implied by the two models would be closer. (This does not, however, address the relative pricing of callable or convertible bonds with these two classes of models.)

Some remarks follow.

- The risk-neutral hazard-rate h need not be the same as the hazard-rate under an objective probability measure. The "objective" (actual) hazard rate is never used here.

- Even if intensities are stochastic, the previous calculations apply if intensities are independent (risk-neutrally) of interest rates. In this case, we simply interpret h_i to be the rate of arrival of default during the i-th interval, conditional only on survival to the beinning of that interval. This "forward default rate" is by definition deterministic. This idea is based on the "forward default probability," introduced by Litterman and Iben (1991).

- If the notes used for pricing reference are on special in the repo market, an estimate of the "hidden" term repo-specialness Y should be included in the above calculations, as an add-on to the floating-rate spread $\hat{S}$ or the fixed rate coupon c, when estimating the implied risk-neutral hazard rate h.

- If necessary, one could use actuarial data on default incidence for comparison firms, and adjust the estimated actual default arrival rate $\hat{h}$ by a multiplicative corrective risk-premum factor, estimated cross-sectionally perhaps, to incorporate a risk premium.[12]

- If one assumes "instant" payment at default, rather than payment at the subsequent coupon date as assumed above, then the factor $b_i(h)$ is

[12]Multiplicative factors are preferred to additive factors, based on general economic considerations and the form of Girsanov's Theorem for point processes, as in Protter (1991). For information on the pricing of notes at actuarially implied default rates, see for example Fons (1994). Fons does not, however, provide an estimate of default arrival intensity.

replaced by

$$b_i^*(h) = e^{-(y(i-1)+H(i-1))T(i-1)}k_i(h(i)),$$

where

$$k_i(h_i) = \frac{h(i)}{h(i)+\varphi(i)}\left(1 - \exp\left[-(h(i)+\varphi(i))(T(i)-T(i-1))\right]\right),$$

is the price at time T_{i-1}, conditional on survival to that date, of a claim that pays one unit of account at the default time, provided the default time is before $T(i)$, and where φ_i is the instantaneous default-free forward interest rate, assumed constant between $T(i-1)$ and $T(i)$. This can be checked by noting that the conditional density of the time to default, given survival to $T(i-1)$, is $p(u) = e^{-h_i u}h_i$ over the interval $[T(i-1), T(i)]$. For reasonably small inter-coupon periods, default probabilities, and interest rates, the impact of assuming instant recovery, rather than recovery at the subsequent coupon date, is relatively small.

5 The Role of Asset Swaps

An asset swap is a derivative security that can be viewed, in its simplest version, as a portfolio consisting of a fixed-rate note and an interest-rate swap that pays fixed and receives floating, to the stated maturity of the underlying fixed-rate note. The fixed rate on the interest-rate swap is conventionally chosen so that the asset swap is valued at par when traded.

It should be noted that the net coupons of the interest-rate swap are exchanged through maturity even if the underlying note defaults and its coupon payments are thereby discontinued.

Recently, the markets for many fixed-rate notes have sometimes been less liquid than the markets for the associated asset swaps, whose spreads are thus often used as benchmarks for pricing default swaps. In fact, because of the mismatch in termination with default between interest-rate swap embedded in the asset swap and the underlying fixed-rate note, the asset-swap spread does not on its own provide precise information for default-swap pricing. For example, as illustrated in Figure 8, it is *not* the case that a synthetic credit swap can be created from a portfolio consisting of a default-free floater and a short asset swap.

The asset-swap spread and the term structure of default-free rates can, however, together be used to obtain an implied par floating-rate spread, from which the default-swap spread can be estimated.

For example, suppose an asset swap is at a quoted spread $\hat{S}$ to the default-free floating rate. For the following, we ignore repo specials and transactions costs, but these can easily be added. Suppose the stated underlying fixed rate on the note is C and the at-market default-free interest-rate swap rate is C^*. Then the interest-rate swap underlying the asset swap is an exchange of floating for $C - \hat{S}$. We can compute the desired par fixed-rate spread F over the default-free coupon rate of the same credit quality from the relationship implied by the price of a portfolio consisting of the asset swap and a short position in a portfolio consisting of a par fixed-rate note of the same credit quality as the underlying C-issued fixed rate note combined with an at-market interest rate swap. This portfolio is worth

$$1 - 1 = 0 = A(C - F + C^*) + A^*(C^* - C + \hat{S}), \tag{1}$$

where A is the defaultable annuity price described above, and A^* is the default-free annuity price to the same maturity. All of $C, C^*, \hat{S}$ and A^* are available from market quotes. Given the defaultable annuity price A, which can be estimated as above, we can therefore solve this equation for the implied par fixed-rate spread F. We have

$$F = C - C^* - \frac{A^*}{A}(C - \hat{S} - C^*).$$

This implied par rate F is approximately the same as the par floating-rate spread S, which is then the basis for setting the default-swap spread. For small default probabilities, under our other assumptions, the default-swap spread S and the par asset-swap spread are approximately the same.

To assume that the asset-swap spread is a reasonable proxy for the default-swap spread is dangerous for premium or discount bonds, as illustrated in Figure 9, which shows the divergence between the term structures of asset swap spreads for premium (coupon rate 400 basis points over the par rate), par, and discount (coupon rate 400 basis points under the par rate) bonds. This figure is based on the same defaultable term structure model that was used as a basis for Figure 3.

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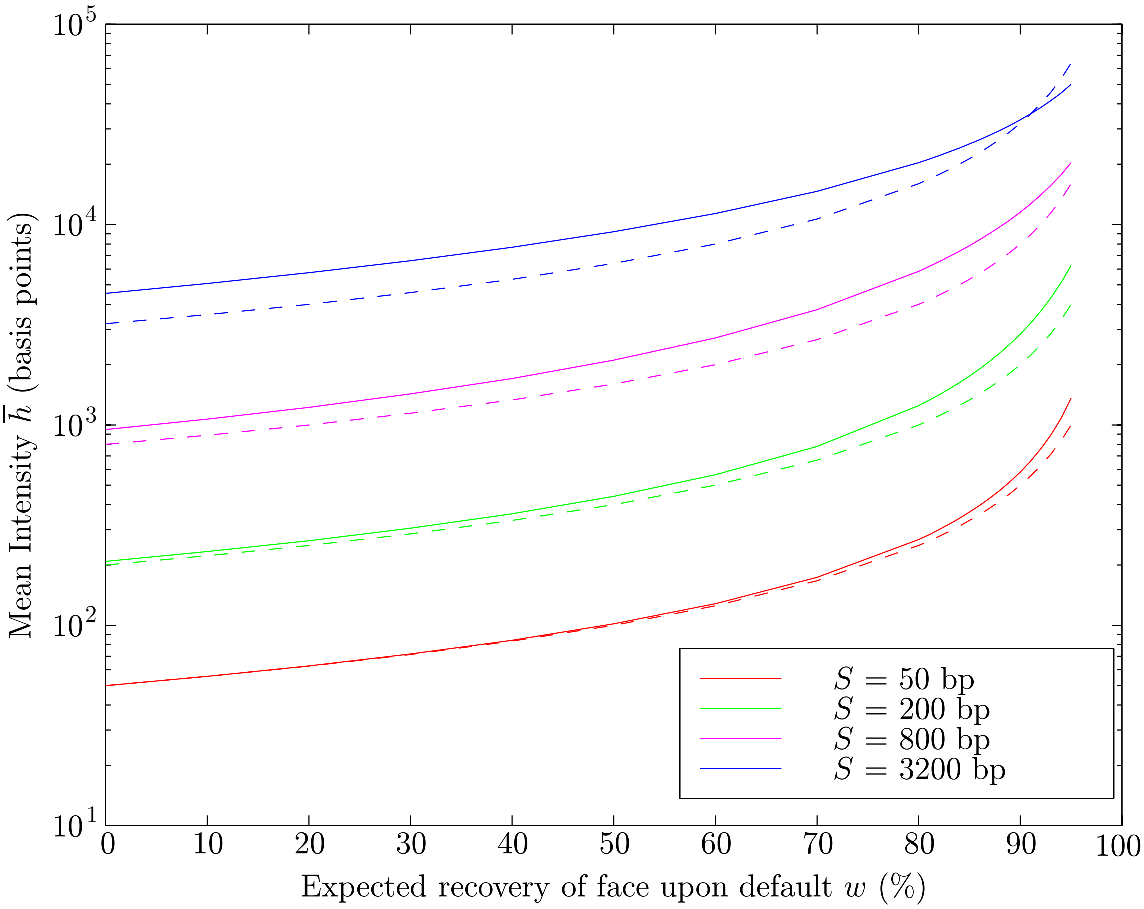


Figure 4: For fixed ten-year par-coupon spreads S, dependence of mean intensity $\overline{h}$ implied by assumed expected fractional recovery w of face value at default. The dashed lines are the approximation $\overline{h} = S/(1-w)$.

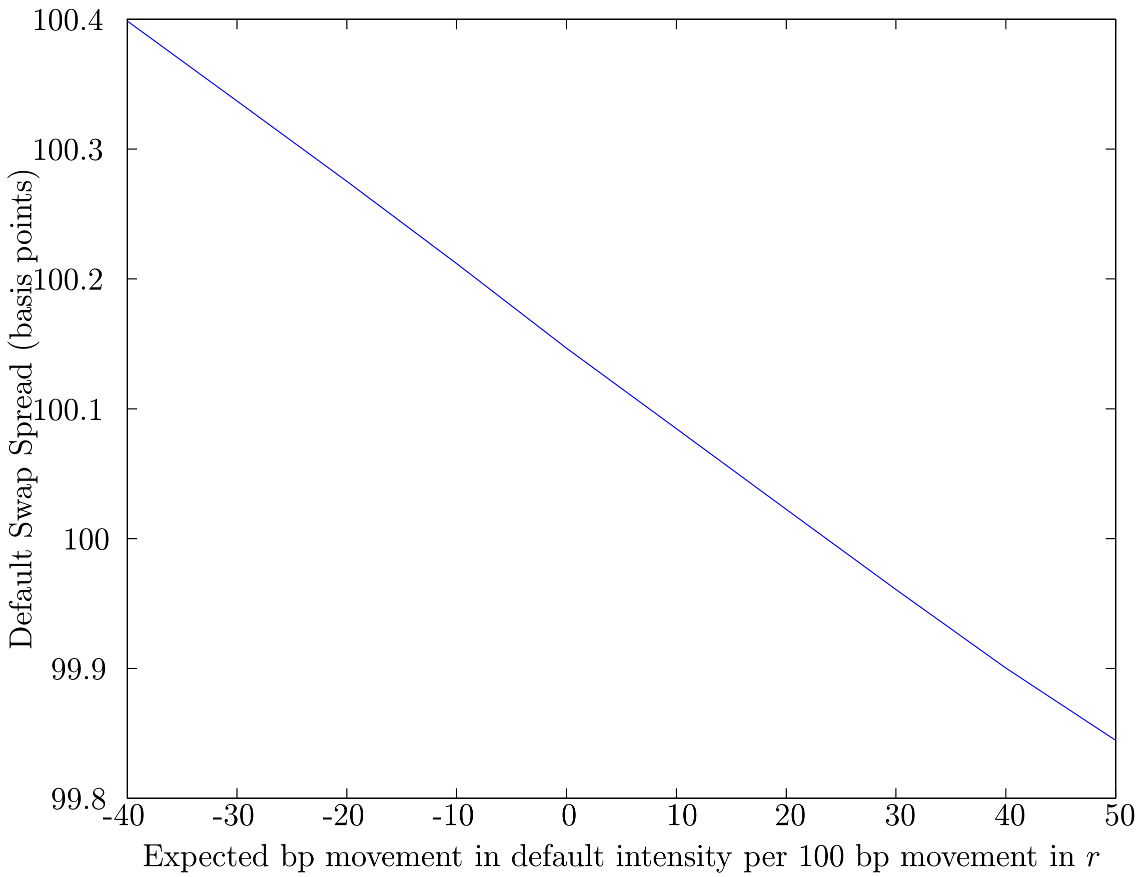


Figure 5: Default swap spread (2-year), varying expected response of default intensity to change in short default-free rate.

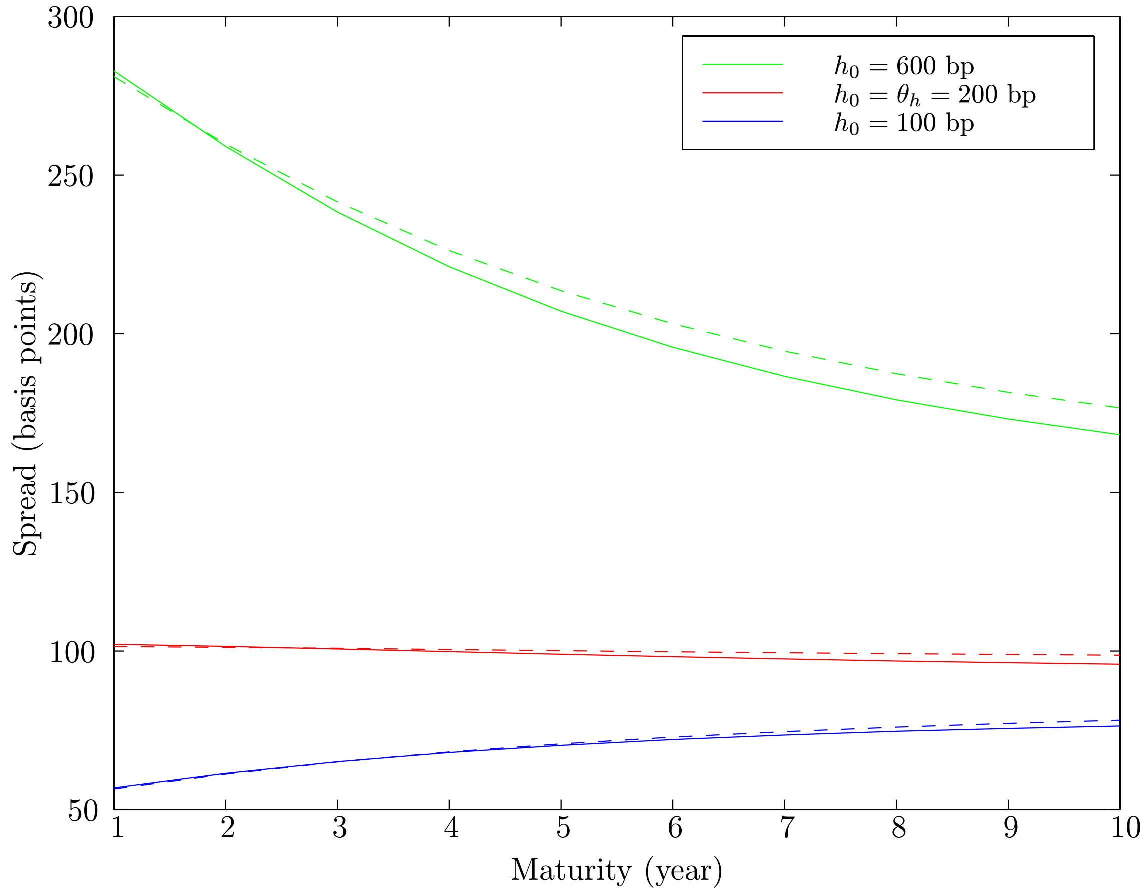


Figure 6: Term structures of par-coupon yield spreads for RMV (dashed lines) and RFV (solid lines), with 50% recovery upon default, a long-run mean intensity $\theta_h = 200$ bp, mean reversion rate of $\kappa = 0.25$, and an initial intensity volatility of 100%.

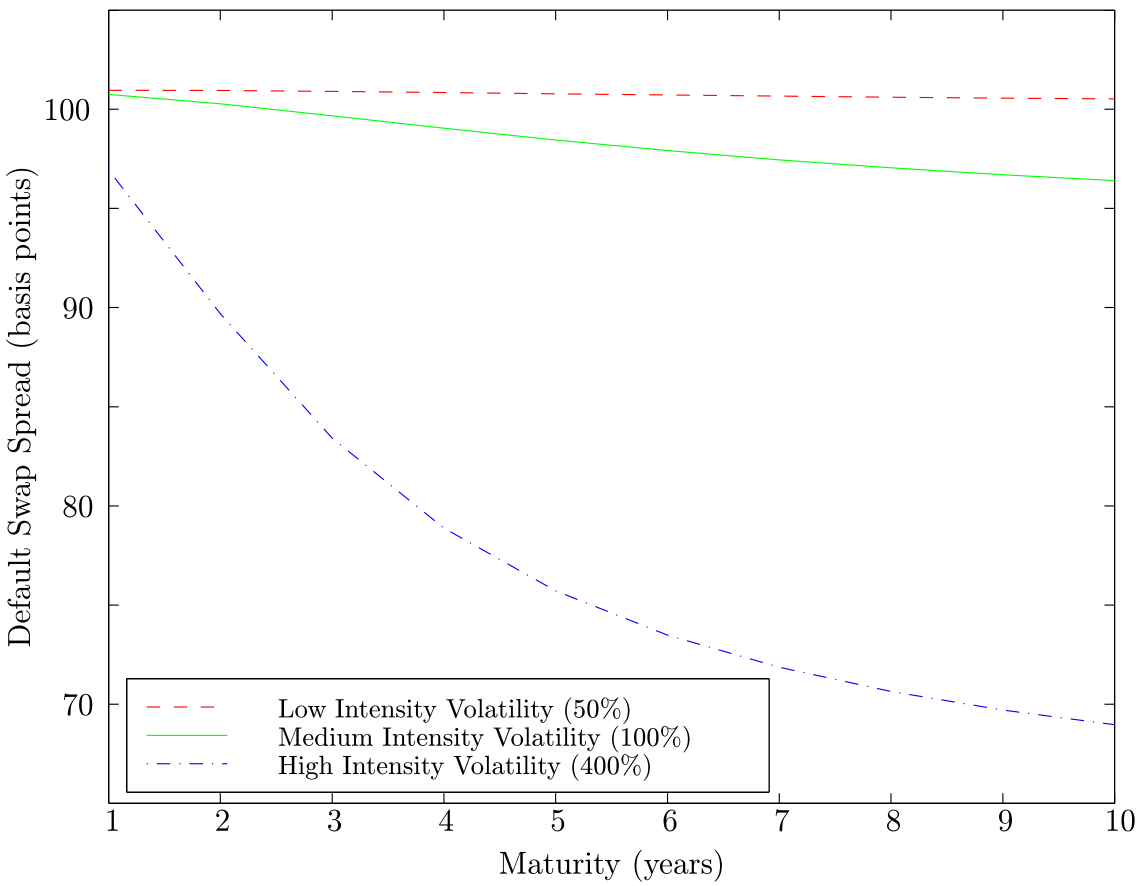
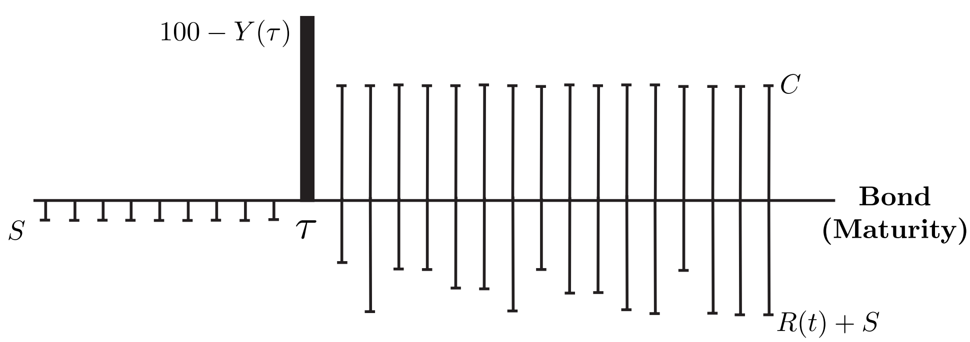


Figure 7: Term Structure of default-swap spreads, varying intensity volatility.

From Fixed-Rate Bond to Credit Swap?



Par Default-Free Floater with Short Asset Swap (Spread S)

Figure 8: Failed attempt to synthesize a credit swap from an asset swap.

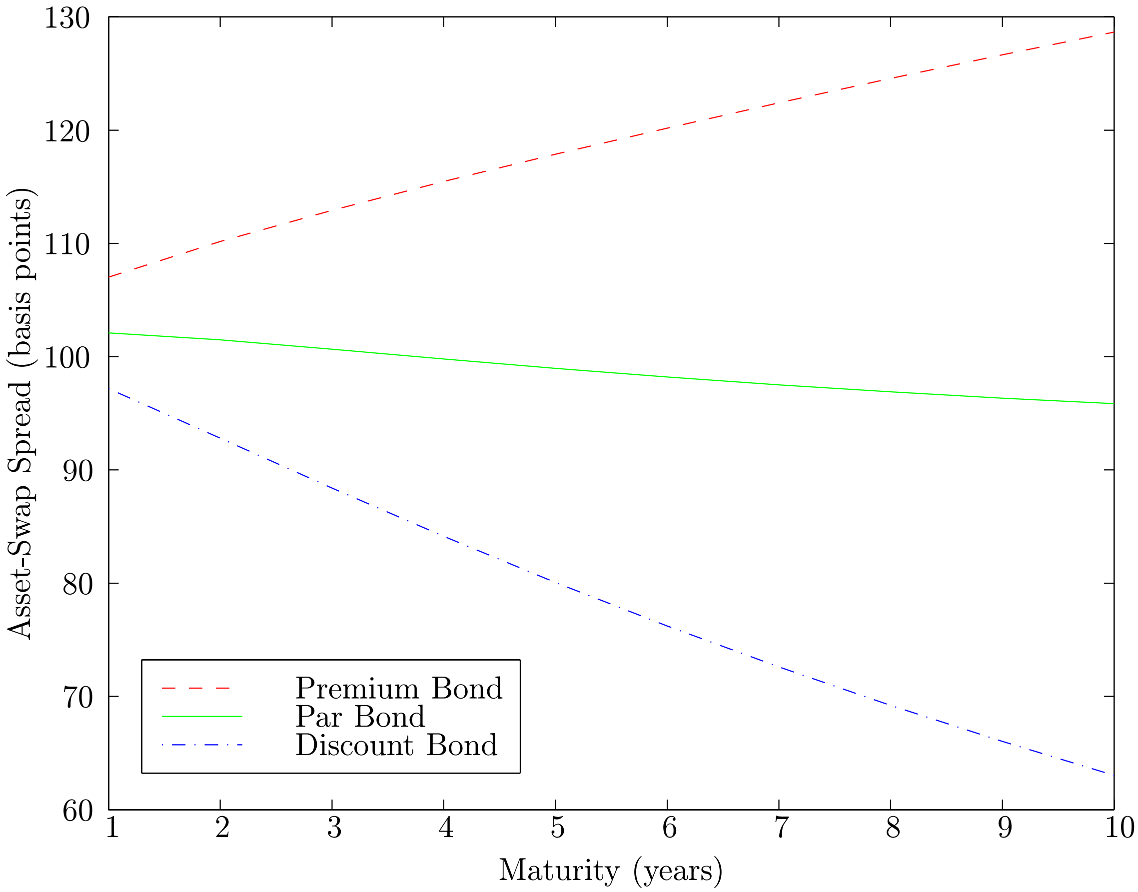


Figure 9: Term structures of asset-swap spreads. The premium coupon rates 400 bps above par, discount coupon rates 400 bp below par.