Martingales, Arbitrage, and Portfolio Choice

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Abstract

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1. Introduction

This is a brief and informal presentation, for mathematicians not familiar with the topic of the connections in finance theory between the notions of arbitrage and martingales, with applications to the pricing of securities and to portfolio choice. The objective is not to survey, but rather to tempt the reader to further explore the area .

Since the advent of the celebrated Black–Scholes (1973) model of option pricing, the theory and everyday practice of security pricing and financial risk management has depended heavily on mathematical models of stochastic processes, particulary the stochastic calculus, and associated statistical and numerical methods. I hope to convey a sense of how the application of techniques from the theory of continuous-time martingales has greatly simplified, both conceptually and computationally, several important problems in finance.

An arbitrage is a financial investment strategy that costs nothing and generates positive profits with no risk of loss. A standard theoretical assumption is that arbitrage is impossible, for if it were, there could not be a balance between supply and demand in markets. It turns out that this minimal consistency assumption of no arbitrage is enough to yield precise and easy answers to problems that might have seemed rather difficult at first glance.

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A martingale is, roughly speaking, a stochastic process whose value is at any time an unbiased predictor of its value at any future time. An equivalent martingale measure is, again roughly speaking, a probability measure under which all security price processes are martingales. Section 2 presents the notion of an equivalent martingale measure, introduced in more or less its definitive form by Harrison and Kreps (1979), and shows the essential equivalence between the existence of an equivalent martingale measure and the absence of arbitrage. Sections 3, 4, and 5 present several applications of the existence of equivalent martingale measures: the Black–Scholes option pricing formula, the Merton problem of optimal investment, and the Cox–Ingersoll–Ross model of the term structure of interest rates. Section 6 has concluding remarks.

2. Equivalent martingale measures

This section presents the basics of equivalent martingale measures, a useful and elegant construct with many applications in the theory of market equilibrium, financial asset pricing, and investment theory.

The basic idea

We start with a one-period model here, and then extend in generality. A probability space $(\Omega, \mathcal{F}, P)$ is fixed. We let $\mathcal{L}$ denote the space of random variables, treating two elements W and Z of $\mathcal{L}$ as the same if $W = Z$ almost surely. The primitives of the model are:

(a) A vector $X_0 = (X_0^{(1)}, \ldots, X_0^{(N)}) \in I\!R^N$ of prices for trade at time 0 of N financial securities.

(b) A vector $X_T = (X_T^{(1)}, \cdots, X_T^{(N)}) \in \mathcal{L}^N$ describing the prices of the same securities at some later time T. That is, $X_T^{(i)}(\omega)$ is the price in state ω at time T of security i.

Given these primitives, a *portfolio* $\theta \in I\!R^N$ describes the respective numbers of units of the securities that can be purchased at time 0 for $\theta \cdot X_0$, with value $\theta \cdot X_T \equiv \theta_1 \cdot X_T^{(1)} + \cdots + \theta_N X_T^{(N)}$ at time T.

An *arbitrage* is a portfolio θ such that

$$\theta \cdot X_0 \leq 0 \qquad \text{and} \qquad \theta \cdot X_T > 0 \tag{1}$$

or

$$\theta \cdot X_0 < 0 \qquad \text{and} \qquad \theta \cdot X_T \geq 0. \tag{2}$$

(For a random variable W, we take "$W \geq 0$" to mean that $W \geq 0$ almost