ELSEVIER

Journal of Mathematical Economics 26 (1996) 1–8

JOURNAL OF Mathematical ECONOMICS

Incomplete security markets with infinitely many states: An introduction

Darrell Duffie

Graduate School of Business, Stanford University, Stanford, CA 94305-5015, USA

Submitted March 1995; accepted August 1995

Abstract

This paper introduces the special issue of *The Journal of Mathematical Economics*, "Equilibrium within Incomplete Markets and an Infinite State Space".

JEL classification: D5

Keywords: General equilibrium; Security markets; Infinite state space

1. Introduction

This special issue deals with the existence and characterization of general equilibrium in security markets with uncertainty over an infinite number of states. This paper introduces the topic and the papers in the issue.

Consider, for instance, the following setting, which is typical of some of the papers in this issue.

- A consumption space L of $\mathbb{R}^{\ell}$-valued random variables, for integer $\ell \geq 1$ on a probability space $(\Omega, \mathcal{F}, P)$.
- A finite number m of agents; agent i is defined by the consumption set $X_i \subset L$, an endowment e_i in X_i, and a utility function $U_i: X_i \to \mathbb{R}$. For simplicity, we will take X_i to be the set of Y-valued random variables, for some $Y \subset \mathbb{R}^{\ell}$. (For example, one could have $Y = \mathbb{R}^{\ell}_+$ or $Y = \mathbb{R}^{\ell}$.)

SSDI 0304-4068(95)00740-7

- A finite number n of securities collectively defined by a consumption dividend payment D in L^n. We can identify D with a measurable function $D: \Omega \to \mathbb{R}^{\ell \times n}$ so that $D_{kj}(\omega)$ is the amount of commodity k paid by security j in state ω.

An economy is thus a collection:

$$\mathscr{E} = \left(L, (e_i, U_i)_{i=1}^m, D\right). \tag{1}$$

A price system for an economy is a vector $q \in \mathbb{R}^n$ of security prices and a state-contingent consumption price vector $p \in L$. An admissible portfolio is an element of some subset $\Theta \subset \mathbb{R}^n$ which, for example, can bound short sales in one or more securities. Given a price system (p, q), a consumption-portfolio choice $(c, \theta) \in X_i \times \Theta$ is budget-feasible for agent i if

$$\theta \cdot q \leq 0, \tag{2}$$

and

$$p(\omega) \cdot \left[c(\omega) - e_i(\omega) + D(\omega)\theta\right] \leq 0, \quad P - \text{almost surely } (\omega). \tag{3}$$

Given (p, q), a budget-feasible choice (c, θ) for agent i is optimal if there is no budget-feasible choice (c', θ') for agent i such that $U_i(c') > U_i(c)$. An equilibrium for $\mathscr{E}$ is a price system (p, q) and a collection $((c_i, \theta_i)_{i=1}^m)$ of optimal choices for the respective agents, given (p, q), such that markets clear:

$$\sum_{i=1}^m \theta_i = 0 \quad \text{and} \quad \sum_{i=1}^m (c_i - e_i) = 0 \text{ almost certainly.} \tag{4}$$

What distinguishes this setting from much of the earlier work on general equilibrium in incomplete markets is the cardinality of the set $\mathscr{F}$ of events. While there has been significant work on the case of a finite or countably infinite set of events, until the papers appearing in this issue there have been essentially no non-parametric models of general equilibrium with a 'continuum of states' (such as the unit interval with the Lebesgue measurable subsets) and with incomplete security markets. Mas-Colell (1992) has already shown that continuum-state incomplete-markets equilibria have an especially strong degree of indeterminacy.

A major technical problem associated with a continuum of states is the difficulty of finding a natural topology for equilibrium analysis under which the left-hand side of (3) varies continuously with (p, c). This problem is circumvented in several papers in this issue by decomposing the infinite-dimensional fixed point problems into a family of connected finite-dimensional fixed point problems, in a manner to be described shortly.

A distinct technical problem associated with incomplete markets, regardless of the cardinality of the set of events, is the lack of continuity of the span of securities payoffs as a function of state-contingent consumption prices. This is the focus of much of a previous special issue of this journal on incomplete security market equilibrium with a finite number of states. (See Geanakoplos, 1990.) (For other surveys, see Cass (1992), Duffie (1992), Magill and Quinzii (1996a), and

Magill and Shafer (1991).) Radner (1972) has already shown that this is not a problem for existence in finite-dimensional settings provided there are lower bounds on short sales of securities. Without such bounds, Hart (1975) has pointed out that equilibrium may fail to exist. This difficulty does not disappear as one moves to a 'large' set of states. Indeed, it will be the case throughout this special issue that portfolio lower bounds of some kind or another will be in place. The question is whether, even given the benefit of such a restriction on short sales, one can demonstrate existence.

2. The basic approach

What follows is a sketch of one basic 'recipe' for an existence proof with a continuum of states and incomplete markets. With some significant variations, this is the style of approach undertaken by several papers in this issue (Hellwig, 1996; Mas-Colell and Monteiro, 1996; Mas-Colell and Zame, 1996). Many details are omitted here. There are four basic steps in this recipe:

(1) Restrict oneself to state-dependent von Neumann–Morgenstern utility. That is, we let

$$U_i(c) = \int_\Omega u_i(c(\omega), \omega) \, \mathrm{d}P(\omega),$$

where $u_i : Y \times \Omega \to \mathbb{R}$ is measurable and satisfies technical regularity.

(2) For each fixed $\theta = (\theta_1, \ldots, \theta_m) \in \Theta^m$ defining portfolio choices for all agents, consider the random economy $\mathscr{W}_1(\theta, \omega)$ induced in the second (consumption) round of trading in state ω by initial portfolio choices $\theta \in \Theta^m$. The finite-dimensional complete markets Walrasian economy $\mathscr{W}_1(\theta, \omega)$ is defined by

- commodity space $\mathbb{R}^\ell$;
- utility for agent i given by $u_i(\cdot, \omega) : Y \to \mathbb{R}$;
- endowment for agent i given by $e_i(\omega) + D(\omega)\theta_i$.

Let $\Pi : \Theta^m \to L$ denote the correspondence defined by selections from the Walrasian equilibrium price correspondence for the random economy $\mathscr{W}_1(\theta, \cdot)$.

(3) Fixing a state-contingent consumption price p, examine the finite-dimensional complete markets economy $\mathscr{W}_0(p)$, in which securities are treated as commodities, with

- commodity space $\mathbb{R}^n$;
- zero endowment;
- utility $V_{ip} : \Theta \to \mathbb{R}$ for agent i defined by

$$V_{ip}(\theta) = \int_\Omega u_i[C_i(p(\omega), \theta, \omega), \omega] \, \mathrm{d}P(\omega),$$

where $C_i : \mathbb{R}^{\ell} \times \Theta \to \mathbb{R}^{\ell}$ is the demand function defined by

$$C_i(\bar{p}, \theta, \omega) = \arg\max_{y \in Y} u_i(y, \omega),$$

subject to

$$\bar{p} \cdot [y - e_i(\omega) + D(\omega)\theta] \le 0.$$

For this economy $\mathscr{W}_0(p)$, we let $Q(p)$ denote the set of equilibrium security portfolio demands. Thus $Q : L \to \Theta^m$ is the correspondence assigning equilibrium portfolio demands to conjectured consumption price vectors.

(4) For each fixed θ in Θ, let $Z : \Theta^m \to \Theta^m$ denote the composition of Q and Π. That is, Z is the equilibrium portfolio demand correspondence defined at each θ in Θ^m by letting $Z(\theta)$ be the set of equilibrium portfolio demands for any economy of the form $Q(p)$, for some $p \in \Pi(\theta)$. Consider the fixed point problem of finding some θ^* in Θ^m with $\theta^* \in Z(\theta^*)$. For such a θ^*, there is, by definition, some contingent consumption price p^* such that, for P-almost every ω, the price vector $p^*(\omega)$ is an equilibrium commodity price vector for economy $\mathscr{W}_1(\theta^*, \omega)$. There is also some security price vector q^* that is an equilibrium for the economy $\mathscr{W}_0(p^*)$. It follows that, given (p^*, q^*), the choice (θ_i^*, c_i^*) is optimal for agent i in the original economy $\mathscr{E}$ given (p^*, q^*), where $c_i^*(\omega) = C_i(p^*(\omega), \theta_i^*, \omega)$. By construction, $((\theta_i^*, c_i^*)_{i=1}^m)$ satisfies the market clearing condition (4). This means that $(p^*, q^*, ((\theta_i^*, c_i^*)_{i=1}^m)$ is an equilibrium for $\mathscr{E}$.

In effect, this four-step procedure finesses the topological problems created by an infinite-dimensional consumption space. One solves the finite-dimensional fixed point problem of equilibrium portfolio holdings in the initial round of trade given the rationally conjectured state-contingent consumption prices that will be induced by these portfolio holdings in a second round of trade (with agents not considering their potential influence over these consumption prices). Because of von Neumann–Morgenstern utility, there are no cross-state effects in the consumption round of trade, allowing it, too, to be treated in a finite-dimensional setting, state by state.

For reasons of continuity (or hemi-continuity) it may be convenient to have controls on the atoms of the probability space $(\Omega, \mathscr{F}, P)$. One will want to take advantage of Lyapunov's Theorem (on the convexity and compactness of an integral of a correspondence over a non-atomic space), and to treat atoms separately, assuming, for example, that there are finitely many.

I first learned of this general conceptual approach, exploiting a backward-recursive structure and Lyaponuv's Theorem, in a presentation by Martin Hellwig [1] at a NATO-sponsored meeting in San Mineato, Italy, in 1986. This issue contains a

[1] It will be noted that Hellwig's setting is a bit different, in that the consumption commodities are also exchanged in the initial round of trade, along with securities, and that his method of proof is not literally that described above.

paper by Hellwig (1996) and one independently completed by Mas-Colell and Monteiro (1996) that bring this approach to fruition in full detail.

The paper by Mas-Colell and Monteiro follows more or less the four-step recipe outlined above. The Hellwig paper, on the other hand, works with the correspondence assigning marginal valuations (or 'supporting prices') to given portfolio choices $\theta \in \Theta^m$, rather than dealing directly with the Walrasian price correspondence Π. For technical reasons, this may be more robust. (The Mas-Colell–Monteiro approach is based on enough regularity to allow pointwise continuous selections from the Walrasian price correspondence Π.) The Hellwig approach is adopted by Mas-Colell and Zame (1996), whose further contributions are outlined below. Monteiro (1996) takes a different approach altogether, based on finite-dimensional approximations of the economy $\mathscr{E}$, along the lines of Bewley's (1972) original work on infinite-dimensional complete markets economies. The final section of Monteiro's paper is a summary of some of the distinguishing technical features of these various papers.

There is an unusually strong assumption that arises in one form or another in the work presented here: no matter what state arises and what portfolio is chosen from the admissible set Θ, an agent will have a non-negative bundle of commodities after adding endowments to security dividends. This restriction on $(e_1, \ldots, e_m, D, \Theta)$ has not been necessary with models having a finite or even countably infinite set of states, for which the net amount of a given commodity remaining in a given state after payment of security dividends may be negative, provided, of course, that commodity prices imply a non-negative net income in that state. That is, with finitely-many states, the natural restriction of budget-feasibility does not need to be strengthened by an artificial requirement that agents have non-negative income in each state *even if their price conjectures are arbitrarily wrong*. Mas-Colell and Zame (1996) show that this artificial requirement – in fact, even a strengthened version of it – cannot be easily dispensed with; they show a counterexample to the existence of equilibrium in a two-agent model when this restriction on $(e_1, \ldots, e_m, D, \Theta)$ is not imposed. In a private communication based on the counterexample of Mas-Colell–Zame, Monteiro (1994) has shown a similar counterexample that illustrates the concrete properties of utilities that lead to such failures of existence. Recently, Araujo et al. (1994) have shown how this strong 'survivability' assumption can be replaced, at least in part, by a bankruptcy penalty.

3. Many periods

With finitely-many periods, one can simply add a subscript 't' to each of u_i, e_i, D, c_i, θ_i, p and q, for each period t in $\{0, 1, \ldots, T\}$, and add the requirement that for each period t, each such subscripted random variable is measurable with respect to a given sub-σ-algebra $\mathscr{F}_t$ of $\mathscr{F}$. (One assumes, as usual, that $\mathscr{F}_t \subset \mathscr{F}_s$

for $t \leq s$.) The definition of equilibrium for the resulting multi-period economy is then the obvious one, and we will not belabor the details here. They are treated in this issue by Mas-Collel and Zame.

Mas-Colell and Zame (1996) show how to extend the backward recursive arguments used for the case of $T=1$ to the case of any finite T. They take the natural route of applying these recursive arguments inductively from the last period T to the first period. They also develop a clever way to address the following special difficulty in multi-period settings. At each period t, given the possibility of retrade, a security is effectively a claim to a payoff in period $t+1$ consisting of its dividend in period $t+1$ *and its market value* in period $t+1$. This issue does not arise for the case of $T=1$ because, by implicit assumption, security prices at the end of trade in the last period are zero. [2] Rather than directly extending the above approach to consider both security prices and consumption prices (or 'support prices') in the Walrasian correspondence for the next period's random economy, which presents its own difficulties, Mas-Collel and Zame expand the consumption set artificially from $Y \subset \mathbb{R}^{\ell}$ to $Y \times \Theta$ (with no direct utility for the portfolio component). A security portfolio θ is then treated in period t as a claim to the 'expanded-commodity bundle' $(D_{t+1}\theta, \theta)$. The original arguments sketched out above can then be applied almost without change. Indeed, Mas-Colell and Zame allow for utility functions are time-non-separable, provided they are state-wise-separable.

4. Infinitely many periods

A backward recursive approach does not apply so easily if there is no last period! Levine and Zame (1996) as well as Magill and Quinzii (1994, 1996b) address the case of a countably infinite number of discrete periods and states, and apply substantially different arguments. To overcome the joint income continuity in (p, c) raised above, they restrict themselves to a countable set of states, so that the product topology is a natural one in which income continuity is maintained. That is, a sequence $\{y_n\}$ of income processes converges to an income process y if $y_{nt}(\omega)$ converges to $y_t(\omega)$ for all (ω, t). The advantages of discreteness and this topology have been exploited by Florenzano and Gourdel (1993), Green and Spear (1989), Hernandez and Santos (1988), Levine (1989) and Zame (1988).

A phenomenon explored by Magill and Quinzii (1996b) that is particular to the infinite horizon setting is a 'speculative bubble', which one may think of as an

[2] This implicit assumption of zero terminal security prices is, in fact, a consequence of Walras' Law if the securities have non-negative dividends and are in positive supply. The assumption does involve loss of generality if the securities are in zero net supply, unless one introduces an extra round of security trade at the end of the last period.

equilibrium in which asset prices are not equal to the 'present values' of their future dividends according to given Arrow–Debreu state prices. Santos and Woodford (1993) present a relatively comprehensive treatment of speculative bubbles. Magill and Quinzii characterize a property of 'sufficient impatience' that, in combination with the other features of their model, rules out speculative bubbles.

5. Final remark

Despite the significant advances represented in this special issue, there remain large gaps in our ability to guarantee the existence of equilibria (even in some generic sense) for certain standard infinite-dimensional settings. Certainly, the issue of short sales presents a problem. Perhaps recent work on bankruptcy (as in Araujo, et al., 1994) offers some help, as well as complications, in that direction. There is also the case of incomplete markets continuous-time financial models (such as that of Grossman and Shiller (1982), for example). The existence of equilibrium has been widely assumed in this literature, but there are, as yet, no supporting results, save for special parametric or one-agent examples.

Acknowledgements

I am happy to acknowledge Martin Hellwig, Michael Magill, Andreu Mas-Colell, Paolo Monteiro, Martine Quinzii, Wayne Shafer and Bill Zame for their help in putting together this special issue.

References

Araujo, A., P. Monteiro and M. Páscoa, 1994, Existence of equilibria with infinitely many goods, incomplete markets, and bankruptcy, Working Paper, IMPA, Brazil.

Bewley, T., 1972, Existence of equilibria in economies with infinitely many commodities, Journal of Economic Theory 4, 514–540.

Cass, D., 1992, Incomplete financial markets and indeterminacy of competitive equilibrium, in: J.-J. Laffont, ed., Advances in Economic Theory – Sixth World Congress, Vol. II, pp. 263–288 (Cambridge University Press, Cambridge).

Duffie, D., 1992, Incomplete security markets, in: J.-J. Laffont, ed., Advances in Economic Theory – Sixth World Congress, Vol. II, pp. 214–262 (Cambridge University Press, Cambridge).

Florenzano M. and P. Gourdel 1993, Incomplete markets in infinite horizon: Debt constraints versus node prices, Working Paper, CEPREMAP.

Geanakoplos J., 1990, An introduction to general equilibrium with incomplete markets, Journal of Mathematical Economics 19, 1–38.

Green, R. and S. Spear, 1989, Equilibria in large commodity spaces with incomplete financial markets, Working Paper, Carnegie-Mellon University.

Grossman, S. and R. Shiller, 1982, Consumption correlatedness and risk measurement in economies with non-traded assets and heterogeneous information, Journal of Financial Economics 10, 195–210.

Hart, O., 1975, On the optimality of equilibrium when the market structure is incomplete, Journal of Economic Theory 11, 418–443.

Hellwig, M., 1996, Rational expectations equilibria in sequence economies with symmetric information: The two-period case, Journal of Mathematical Economics, 26, 9–49.

Hernandez, A. and M. Santos, 1988, Incomplete financial markets in infinite horizon economies, Working Paper, Department of Economics, University of Wisconsin.

Levine, D., 1989, Infinite horizon equilibrium with incomplete markets, Journal of Mathematical Economics 18, 357–376.

Levine, D. and W. Zame, 1996, Debt constraints and equilibrium in infinite horizon economies with incomplete markets, Journal of Mathematical Economics 26, 103–131.

Magill, M. and M. Quinzii, 1994, Infinite horizon incomplete markets, Econometrica 62, 853–880.

Magill, M. and M. Quinzii, 1996a, The Theory of Incomplete Markets, Vol. 1 (MIT Press, Cambridge, MA).

Magill, M. and M. Quinzii, 1996b, Incomplete markets over an infinite horizon: Longlived securities and speculative bubbles, Journal of Mathematical Economics, 26, 133–170.

Magill, M. and W. Shafer, 1991, Incomplete markets, in: W. Hildenbrand and H. Sonnenschein, eds., Handbook of mathematical economics, Vol. IV (North-Holland, Amsterdam).

Mas-Colell, A., 1992, Infinite dimensional equilibrium theory: discussion of Jones, in: J.-J. Laffont, (ed.), Advances in Economic Theory – Sixth World Congress, Vol. II, pp. 444–450 (Cambridge University Press, Cambridge).

Mas-Colell, A. and P. Monteiro, 1996, Self-fulfilling equilibria: An existence theorem for a general state space, Journal of Mathematical Economics, 26, 51–62.

Mas-Colell, A. and W. Zame, 1996. The existence of security market equilibrium with a non-atomic state space, Journal of Mathematical Economics, 26, 63–84.

Mas-Colell, A. and W. Zame, 1991, Equilibrium theory in infinite dimensional spaces, in: W. Hildenbrand and H. Sonnenschein, eds., Handbook of mathematical economics, Vol. IV (North-Holland, Amsterdam).

Monteiro P., 1996, A new proof of the existence of equilibrium in incomplete market economies, Journal of Mathematical Economics, 26, 85–101.

Monteiro, P., 1994, Private communication, IMPA, Brazil.

Radner, R., 1972, Existence of equilibrium of plans, prices and price expectations in a sequence of markets, Econometrica 40, 289–303.

Santos, M. and M. Woodford, 1993, Rational speculative bubbles, Working Paper, Department of Economics, University of Chicago.

Zame, W., 1988, Asymptotic behavior of asset markets: Asymptotic efficiency, in: M. Boldrin et al, General equilibrium and growth: The legacy of Lionel McKenzie (Academic Press, New York).