Dynamic Directed Random Matching*

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Abstract

We demonstrate the existence of a continuum of agents conducting directed random searches for counterparties, and characterize the implications. Our results provide the first probabilistic foundation for static and dynamic directed random search (including the matching function approach) that is commonly used in the search-based models of financial markets, monetary theory, and labor economics. The agents' types are shown to be independent discrete-time Markov processes that incorporate the effects of random mutation, random matching with match-induced type changes, and with the potential for enduring partnerships that may have randomly timed break-ups. The multi-period cross-sectional distribution of types is shown to be deterministic via the exact law of large numbers.

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1 Introduction

The economics literature is replete with models that assume independent random matching among a continuum of agents.[1] Agents in these models are usually motivated to focus their searches toward those types of counterparties that offer greater gains from interaction, or toward those types that are less costly to find. However, there has been no demonstration of a model of independent[2] directed random matching supporting the common appeal to a law of large numbers, by which the realized quantity of matches of a given pair of types is supposed to be equal to the corresponding expected quantity.[3]

We fill this gap by providing rigorous foundations for independent random matching that is "directed," in the sense that the probability q_{kl} that an agent of type k is matched to an agent of type l can vary with the respective types k and l, from some type space S. We first show, in Theorem 1, the existence of directed random matching in which counterparty types are independent across agents. It follows from the exact law of large numbers that the proportion of type-k agents matched with type-l agents is almost surely $p_k q_{kl}$, where p_k is the proportion of type-k agents in the population. By allowing the matching probabilities $\{q_{kl}\}_{k,l \in S}$ to depend on the underlying cross-sectional type distribution p, we also encompass the "matching-function" approach that has frequently been applied in the labor literature, as surveyed by Petrongolo and Pissarides (2001) and Rogerson, Shimer and Wright (2005), as well as over-the-counter trading models, as in Maurin (2015).

In typical dynamic settings for random matching, once two agents are matched, their types change according to some deterministic or random rule. For example, when an unemployed worker meets a firm with one vacant job, the worker changes her type to "employed." Random mutation of agent types is also a common model feature, allowing for shocks to preferences, productivity, or endowments.[4]

In practice, and in an extensive part of the literature, once a pair of agents is matched, they may stay matched for some time. Typical examples include the relationships between employer and employee, or between two agents that take time to bargain over their terms of trade.[5] In this paper, we develop the first mathematical model for independent random

[1]Hellwig (1976) is the first, to our knowledge, to have relied on the effect of the exact law of large numbers for random pairwise matching in a market. Other examples include Binmore and Samuelson (1999), Currarini, Jackson and Pin (2009), Duffie, Gârleanu, and Pedersen (2005), Green and Zhou (2002), Kiyotaki and Wright (1989), Lagos and Rocheteau (2009), Vayanos and Weill (2008), and Weill (2007).

[2]In this context, independence is in general viewed as a behavioral assumption. That is, when agents conduct searches without explicit coordination, it is reasonable to assume independence.

[3]Previous work by Duffie and Sun (2007, 2012) considers only the case of "un-directed" search, in the sense that when a given agent is matched, the paired agent is drawn uniformly from the population of other agents to be matched.

[4]See, for example, Duffie, Gârleanu, and Pedersen (2005) and Lester, Postlewaite and Wright (2012).

[5]See, for example, Acemoglu and Wolitzky (2011), Andolfatto (1996), Diamond (1982),

matching that allows the potential for enduring partnerships by introducing a per-period type-dependent break-up probability for matched agents.

We present a model of independent dynamic directed random matching that incorporates the effects of random mutation, random matching with match-induced type changes, and with the potential for enduring partnerships. The agents' types are shown to be independent discrete-time Markov processes. By the exact law of large numbers, the multi-period cross-sectional distribution of agents' types is deterministic. For the special time-homogeneous case, we obtain a stationary joint cross-sectional distribution of agent types, incorporating both unmatched agent types and pairs of currently matched types. Many previously studied search-based models of money, over-the-counter financial markets, and labor markets have relied on the properties that we demonstrate for the first time.[6]

We illustrate the potential applications of our model of directed random matching with four examples taken, respectively, from Duffie, Malamud and Manso (2014) in financial economics; Kiyotaki and Wright (1989) and Matsuyama, Kiyotaki and Matsui (1993) in monetary economics; and Andolfatto (1996) in labor economics. These examples[7] show how our model can be used to provide rigorous foundations for matching models that are commonly used in the respective literatures.

The remainder of the paper is organized as follows. In Section 2, we describe an independent static directed random matching model. We present the corresponding existence result along with an application to a typical random-matching model used in finance. In order to capture the effect of enduring partnerships, we must consider the separate treatments of existing matched pairs of agents and newly formed matched pairs of agents. In other words, we need to keep track of agents and their matched partners at each step (mutation, matching and type changing), in every time period. This extension of dynamic directed random matching to the case of enduring partnerships is considerably more difficult to analyze than the case in which the matched agents break up immediately. Its exposition is therefore postponed to Appendix A. In Section 3, we treat the relatively simpler case of a dynamical system with random mutation, directed random matching, match-induced type changing, but without enduring partnerships. This section includes results covering the existence and exact law of large numbers for a dynamical system with Markov conditional independence as well as applications to some matching models in monetary economics.

The major part of the proofs for our results make extensive use of tools from nonstandard

Mortensen and Pissarides (1994), Tsoy (2014), and the references in the surveys of Petrongolo and Pissarides (2001) and Rogerson, Shimer and Wright (2005).

[6] The earlier results of Duffie and Sun (2007, 2012) address only the case of "un-directed" search with time-homogeneous parameters and without enduring partnerships.

[7] See Examples 1, 2, 3, and Subsection A.5 below.

analysis.[8] Proofs are located in Appendix B. Section 4 offers some concluding remarks.

2 Static Directed Random Matching

This section begins with some mathematical preliminaries. Then a static model of directed random matching is formally given in Subsection 2.2. Here, we present the exact law of large numbers and the existence of independent directed random matching. Finally, in Subsection 2.3, we show how to interpret our results so as to provide rigorous probabilistic foundations for the notion of a "matching function" that is commonly used in the search literature of labor economics.

2.1 Mathematical preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space. An element of Ω is a state of the world. A measurable subset B of Ω (that is, an element of $\mathcal{F}$) is an event, whose probability is $P(B)$. The agent space is an atomless probability space $(I, \mathcal{I}, \lambda)$. An element of I represents an agent. The mass of some measurable subset A of agents is $\lambda(A)$. Because the total mass of agents is 1, we can also treat $\lambda(A)$ as the fraction of the agents that are in A. As noted in Proposition 2, I could be taken to be the unit interval $[0,1]$, $\mathcal{I}$ an extension of the Lebesgue σ-algebra $\mathcal{L}$, and λ an extension of the Lebesgue measure.

While a continuum of independent random variables, one for each of a large population such as I, can be formalized as a mapping on $I \times \Omega$, such a function can never be measurable with respect to the completion of the usual product σ-algebra $\mathcal{I} \otimes \mathcal{F}$, except in the trivial case in which almost all of the random variables are constants.[9] As in Sun (2006), we shall therefore work with an extension of the usual product probability space that retain the crucial Fubini property

$$\int_I \int_\Omega f(i,\omega)\, dP(\omega)\, d\lambda(i) = \int_\Omega \int_I f(i,\omega)\, d\lambda(i)\, dP(\omega),$$

for any correspondingly integrable function f on the underlying extended product probability space. To reflect the fact that such an extended product probability space has $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ as its marginal spaces, as required by the Fubini property, it will be denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.[10]

The Fubini extension could include a sufficiently rich collection of measurable sets to allow applications of the exact law of large numbers that we shall need. An $\mathcal{I} \boxtimes \mathcal{F}$-measurable function f will be called a "process," each f_i will be called a random variable of this process, and each f_ω will be called a sample function of the process.

[8]The reader is referred to the first three chapters of Loeb and Wolff (2015) for basic nonstandard analysis.
[9]See, for example, Proposition 2.1 in Sun (2006).
[10]For a formal definition, see Definition 2.2 in Sun (2006).

2.2 Static directed random matching

We follow the notation in Subsection 2.1. Let $S = \{1, 2, \ldots, K\}$ be a finite space of agent types and $\alpha : I \to S$ be an $\mathcal{I}$-measurable type function, mapping individual agents to their types. For any k in S, we let $p_k = \lambda(\{i : \alpha(i) = k\})$ denote the fraction of agents of type k. We can view $p = (p_k)_{k \in S}$ as an element of the space Δ of probability measures on S. Because $(I, \mathcal{I}, \lambda)$ has no atoms, for any type distribution $p \in \Delta$, one can find an $\mathcal{I}$-measurable type function with distribution p.

A function $q : S \times S \to \mathbb{R}_+$ is a matching probability function for the type distribution p if, for any k and l in S,

$$p_k \, q_{kl} = p_l \, q_{lk}, \quad \sum_{r \in S} q_{kr} \leq 1. \tag{1}$$

The matching probability q_{kl} specifies the probability that an agent of type k is matched to an agent of type l. Thus, $\eta_k = 1 - \sum_{l \in S} q_{kl}$ is the associated no-matching probability for an agent of type k.

Definition 1 *Let α, p, and q be given as above, and J a special type representing no-matching.*

(i) *A full matching ϕ is a one-to-one mapping from I onto I such that, for each $i \in I$, $\phi(i) \neq i$ and $\phi(\phi(i)) = i$.*

(ii) *A (partial) matching ψ is a mapping from I to $I \cup \{J\}$ such that for some subset B of I, the restriction of ψ to B is a full matching on B, and $I \setminus B = \psi^{-1}(\{J\})$. This means that agent i is matched with agent $\psi(i)$ for $i \in B$, whereas any agent i not in B is unmatched, in that $\psi(i) = J$.*

(iii) *A random matching π is a mapping from $I \times \Omega$ to $I \cup \{J\}$ such that (a) π_ω is a matching for each $\omega \in \Omega$; (b) after extending the type function α to $I \cup \{J\}$ so that $\alpha(J) = J$, and letting $g = \alpha(\pi)$, the function g is measurable from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to $S \cup \{J\}$.*

(iv) *A random matching π from $I \times \Omega$ to $I \cup \{J\}$ is "directed," and has parameters (p, q), if for λ-almost every agent i of type k, $P(g_i = J) = \eta_k$ and $P(g_i = l) = q_{kl}$.*

(v) *A random matching π is said to be independent if the type process g is essentially pairwise independent.*

For an agent $i \in I$ who is matched, the random variable $g_i = g(i, \cdot)$ is the type of her matched partner. Part *(iv)* of the definition thus means that for λ-almost every agent i of type k, her probability of being matched with a type-l agent is q_{kl}, while her no-matching probability is η_k.

The following result is a direct application of the exact law of large numbers. In particular, letting $I_k = \{i \in I : \alpha(i) = k\}$, the result follows from Theorem 2.8 of Sun (2006) by working with the process $g^{I_k} = g|_{I_k \times \Omega}$ on the rescaled agent space I_k.

Proposition 1 *Let π be an independent directed random matching with parameters (p, q). Then, for P-almost every $\omega \in \Omega$, we have*

(i) For $k \in S$, $\lambda(\{i \in I : \alpha(i) = k, g_\omega(i) = J\}) = p_k \eta_k$.

(ii) For any $(k, l) \in S \times S$, $\lambda(\{i : \alpha(i) = k, g_\omega(i) = l\}) = p_k q_{kl}$.

Let κ be the probability measure on $S \times (S \cup \{J\})$ defined by letting $\kappa(k, l) = p_k q_{kl}$ for any $(k, l) \in S \times S$ and $\kappa(k, J) = p_k \eta_k$ for $k \in S$. Proposition 1 says that the cross-sectional joint type distribution of (α, g_ω) is κ with probability one.

Theorem 1 *For any type distribution p on S and any matching probability function q for p, there exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ on which is defined a type function α and an independent directed random matching π with parameters (p, q).*

The proof of Theorem 1 will be given in Subsection B.1 for the case of a Loeb measure space of agents via the method of nonstandard analysis.[11] Since the unit interval and the class of Lebesgue measurable sets with the Lebesgue measure provide the archetype for models of economies with a continuum of agents, the next proposition shows that one can take an extension of the classical Lebesgue unit interval as the agent space for the construction of an independent directed random matching.

Proposition 2 *For any type distribution p on S and any matching probability function q for p, there exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ such that:*

1. *The agent space $(I, \mathcal{I}, \lambda)$ is an extension of the Lebesgue unit interval $(L, \mathcal{L}, \chi)$.*

2. *There is defined on the Fubini extension a type function α and an independent directed random matching π with parameters (p, q).*

The following example provides an illustrative application of Theorem 1 and Proposition 1 to a model of the over-the-counter financial markets.

[11] A standard treatment of nonstandard analysis is given by the book Loeb and Wolff (2015). We note that the proof of Theorem 1 is substantially different from the corresponding existence result for the case of "undirected" search in Duffie and Sun (2007).

Example 1 In Duffie, Malamud and Manso (2014), the economy is populated by a continuum of risk-neutral agents. There are M different types of agents that differ according to the quality of their initial information, their preferences for the asset to be traded, and the likelihoods with which they meet each of other types of agents for trade. The proportion of type-l agents is m_l, where $l = 1, \ldots, M$. Any agent of type l is randomly matched with some other agent with probability $\lambda_l \in [0, 1)$. This counterparty is of type-r with probability κ_{lr}. Viewed in our model, we can take the matching probability $q_{lr} = \lambda_l \kappa_{lr}$ for any $l, r \in S$. Theorem 1 guarantees the existence of independent directed random matching with the given parameters m_l, κ_{lr}. Proposition 1 implies that the total quantity of matches of agents of a given type l with the agents of a given type r is almost surely $m_l \lambda_l \kappa_{lr} = m_r \lambda_r \kappa_{rl}$. (See page 7 in Duffie, Malamud and Manso (2014).)

2.3 Matching functions

Proposition 1 and Theorem 1 also provide a rigorous probabilistic foundation for the "matching-function" approach that is widely used in the literature of search-based labor markets. Matching functions allow the probabilities of matching to be directed and to depend on an endogenously determined cross-sectional distribution of types.

In models of search-based labor markets, it is typical to suppose that firms and workers are characterized by their types. A commonly used modeling device in this setting is a matching function $m_{kl} : [0, 1] \times [0, 1] \to [0, 1]$ that specifies the quantity of type-k agents that are matched with type-l agents, for any k and l in S. (See Petrongolo and Pissarides (2001) for a survey of the matching-function approach.) Clearly one must require that for any k and l in S and any p in Δ,

$$m_{kl}(p_k, p_l) = m_{lk}(p_l, p_k), \quad \sum_{r \in S} m_{kr}(p_k, p_r) \le p_k. \tag{2}$$

Let $q_{kl} = m_{kl}(p_k, p_l)/p_k$ for $p_k \neq 0$, and let $q_{kl} = 0$ for $p_k = 0$. Then the requirements for a matching probability function are satisfied by q. By Theorem 1, there exists an independent directed random matching π with parameters (p, q). It follows from Proposition 1 that the cross-sectional joint type distribution of (α, g_ω) is κ with probability one, where, for any k and l in S,

$$\kappa(k, l) = p_k q_{kl} = m_{kl}(p_k, p_l).$$

That is, the mass of type-k agents that are matched with type-l agents is indeed $m_{kl}(p_k, p_l)$ with probability one. This means that any matching function satisfying Equation (2) can be realized through independent directed random matching, almost surely. For the special case of only two types of agents (say, types 1 and 2), any nonnegative matching function $m(p_1, p_2)$ with $m(p_1, p_2) \le \min(p_1, p_2)$ can be realized through independent directed random matching.

For this, one can simply take $q_{12} = m(p_1, p_2)/p_1$ and $q_{21} = m(p_1, p_2)/p_2$. More general cases are considered in Footnote 17.

A common parametric specification is the Cobb-Douglas matching function, for which

$$m_{UV}(p_U, p_V) = A\, p_U^{\alpha}\, p_V^{\beta},$$

for parameters α and β in $(0, 1)$, and a non-negative scaling parameter A. We emphasize that for some parameters α, β, and A, the inequality $A\, p_U^{\alpha}\, p_V^{\beta} \leq \min(p_U, p_U)$ may fail for some $(p_U, p_V) \in \Delta$. In that case, one can let $m(p_U, p_V) = \min(A\, p_U^{\alpha} p_V^{\beta}, p_U, p_V)$.

3 Dynamic Directed Random Matching

In this section we show how to construct a dynamical system that incorporates the effects of random mutation, directed random matching, and match-induced type changes with time-dependent parameters. As in Section 2, we fix an atomless probability space $(I, \mathcal{I}, \lambda)$ representing the space of agents, a sample probability space $(\Omega, \mathcal{F}, P)$, and a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

We first define such a dynamical system in Subsection 3.1. The key condition of Markov conditional independence is formulated in Subsection 3.2. Based on that condition, we prove in Subsection 3.3 an exact law of large numbers for such a dynamical system. The section ends with the existence of Markov conditionally independent dynamic directed random matching.

3.1 Definition of dynamic directed random matching

As in Section 2, let $S = \{1, 2, \ldots, K\}$ be a finite set of types and let J be a special type representing no-matching. We shall define a discrete-time dynamical system $\mathbb{D}_0$ with the property that at each integer time period $n \geq 1$, agents first experience a random mutation and then random matching with directed probability. Finally, any pair of matched agents are randomly assigned new types whose probabilities may depend on the prior types of the two agents.

At period $n \geq 1$, each agent of type $k \in S$ first experiences a random mutation, becoming an agent of type l with a given probability b_{kl}^n, with $\sum_{r \in S} b_{kr}^n = 1$. At the second step, every agent conducts a directed search for counterparties. In particular, for each $(k, l) \in S \times S$, the directed matching probability is determined by a function q_{kl}^n on the space of type distributions Δ, with the property that, for all k and l in S, the function that maps the type distribution $p \in \Delta$ to $p_k q_{kl}^n(p)$ is continuous and satisfies, for all $p \in \Delta$,

$$p_k\, q_{kl}^n(p) = p_l\, q_{lk}^n(p) \quad \text{and} \quad \sum_{r \in S} q_{kr}^n(p) \leq 1. \tag{3}$$

The intention is that, if the population type distribution in the current period is p, then an agent of type k is matched to an agent of type l with probability $q_{kl}^n(p)$. Thus, $\eta_k^n(p) = 1 - \sum_{l \in S} q_{kl}^n(p)$

is the associated probability of no match. When an agent of type k is matched at time n to an agent of type l, the agent of type k becomes an agent of type r with probability $\nu^n_{kl}(r)$, where $\sum_{r \in S} \nu^n_{kl}(r) = 1$. The primitive model parameters are (b, q, ν).

Let α^0 be the initial S-valued type process on the Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. For each time period $n \geq 1$, the agents' types after the random mutation step are given by a process h^n from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to S. Then, a random matching is described by a function π^n from $I \times \Omega$ to $I \cup \{J\}$. The end-of-period types are given by a process α^n from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to S.

At period n, a type-k agent first mutates to an agent with type l with probability b^n_{kl}. The post-mutation type function h^n satisfies

$$P(h^n_i = l \,|\, \alpha^{n-1}_i = k) = b^n_{kl}. \tag{4}$$

For the directed random matching step, let g^n be an $\mathcal{I} \boxtimes \mathcal{F}$-measurable function defined by $g^n(i, \omega) = h^n(\pi^n(i, \omega), \omega)$, with the property that for any type $k \in S$, for λ-almost every i and P-almost every $\omega \in \Omega$,

$$P(g^n_i = l \,|\, h^n_i = k, \bar{p}^n) = q^n_{kl}(\bar{p}^n(\omega)), \tag{5}$$

where $\bar{p}^n(\omega) = \lambda(h^n(\omega))^{-1}$ is the post-mutation type distribution realized in state ω. The end-of-period agent type function α^n satisfies, for λ-almost every agent i,

$$P(\alpha^n_i = r \,|\, h^n_i = k, g^n_i = J) = \delta_k(r) \text{ and } P(\alpha^n_i = r \,|\, h^n_i = k, g^n_i = l) = \nu^n_{kl}(r). \tag{6}$$

Thus, we have inductively defined the properties of a dynamical system $\mathbb{D}_0$ incorporating the effects of random mutation, directed random matching, and match-induced type changes with given parameters (b, q, ν).

3.2 Markov conditional independence (MCI)

We now add independence conditions on the dynamical system $\mathbb{D}_0$, along the lines of those in Duffie and Sun (2007, 2012). The idea is that each of the just-described steps (mutation, random matching, match-induced type changes) are conditionally independent across almost all agents.

We say that the dynamical system $\mathbb{D}_0$ is Markov conditionally independent (MCI) if, for λ-almost every i and λ-almost every j, for every period $n \geq 1$, and for all types k and l in S, the following four properties apply:

- Initial independence: α^0_i and α^0_j are independent.

- Markov and independent mutation:

$$P(h_i^n = k, h_j^n = l \,|\, \alpha_i^0, \ldots, \alpha_i^{n-1}; \alpha_j^0, \ldots, \alpha_j^{n-1}) = P(h_i^n = k \,|\, \alpha_i^{n-1}) P(h_j^n = l \,|\, \alpha_l^{n-1}).$$

- Markov and independent random matching:

$$P(g_i^n = k, g_j^n = l \,|\, \alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n; \alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n) = P(g_i^n = k \,|\, h_i^n) P(g_j^n = l \,|\, h_j^n).$$

- Markov and independent matched-agent type changes:

$$\begin{aligned} & P(\alpha_i^n = k, \alpha_j^n = l \,|\, \alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n, g_i^n; \alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n, g_j^n) \\ = \;& P(\alpha_i^n = k \,|\, h_i^n, g_i^n) P(\alpha_j^n = l \,|\, h_j^n, g_j^n). \end{aligned}$$

3.3 The exact law of large numbers for MCI dynamical systems

For a p in Δ, we let $\bar{p}_k(p) = \sum_{l \in S} p_l b_{lk}^n$ for $k \in S$. We define a sequence Γ^n of mappings from Δ to Δ such that, for each $p \in \Delta$,

$$\Gamma_r^n(p_1, \ldots, p_k) = \bar{p}_r(p) \eta_r^n(\bar{p}(p)) + \sum_{k,l \in S} \bar{p}_k(p) q_{kl}^n(\bar{p}(p)) \nu_{kl}^n(r).$$

The following theorem presents an exact law of large numbers for the agent type processes at the end of each period, and gives a recursive calculation for the cross-sectional joint agent type distribution p^n at the end of period n.

Theorem 2 *A Markov conditionally independent dynamical system $\mathbb{D}_0$ with parameters (b, q, ν), for random mutation, directed random matching and match-induced type changes, satisfies the following properties.*

(1) For each time $n \geq 1$, let $p^n(\omega) = \lambda(\alpha_\omega^n)^{-1}$ be the realized cross-sectional type distribution at the end of the period n. The expectation $\mathbb{E}(p^n)$ is given by

$$\mathbb{E}(p_r^n) = \Gamma_r^n(\mathbb{E}(p^{n-1})) = \bar{p}_r^n \eta_r^n(\bar{p}^n) + \sum_{k,l \in S} \bar{p}_k^n q_{kl}^n(\bar{p}^n) \nu_{kl}^n(r),$$

where $\bar{p}_k^n = \sum_{l \in S} \mathbb{E}(p_l^{n-1}) b_{lk}^n$.

(2) For λ-almost every agent i, the type process $\{\alpha_i^n\}_{n=0}^{\infty}$ of agent i is a Markov chain with transition matrix z^n at time $n-1$ defined by

$$z_{kl}^n = \eta_l^n(\bar{p}^n) b_{kl}^n + \sum_{r,j \in S} b_{kr}^n q_{rj}^n(\bar{p}^n) \nu_{rj}^n(l).$$

(3) *For λ-almost every i and λ-almost every j, the Markov chains $\{\alpha_i^n\}_{n=0}^\infty$ and $\{\alpha_j^n\}_{n=0}^\infty$ are independent.*

(4) *For P-almost every state ω, the cross-sectional type process $\{\alpha_\omega^n\}_{n=0}^\infty$ is a Markov chain with transition matrix z^n at time $n-1$.*

(5) *For P-almost every state ω, at each time period $n \geq 1$, $p^n(\omega) = \lambda(\alpha_\omega^n)^{-1}$, and the realized cross-sectional type distribution after random mutation $\lambda(h_\omega^n)^{-1}$ is $\bar{p}^n$.*

(6) *If there is some fixed $\ddot{p}^0 \in \Delta$ that is the probability distribution of the initial type α_i^0 of agent i for λ-almost every i, then the probability distribution $\zeta = p^0 \otimes_{n=1}^\infty z^n$ on S^∞ is equal to the sample-path distribution of the Markov chain $\alpha_i = \{\alpha_i^n\}_{n=0}^\infty$ for λ-almost every agent i. For P-almost every state ω, ζ is also the cross-sectional distribution $\lambda(\alpha(\omega))^{-1}$ of the sample paths of agents' type processes.*

(7) *Suppose that the parameters (b, q, ν) are time independent. Then there exists a type distribution $p^* \in \Delta$ such that p^* is a stationary distribution for any Markov conditionally independent dynamical system $\mathbb{D}_0$ with parameters (b, q, ν), in the sense that for every period $n \geq 0$, the realized cross-sectional type distribution p^n at time n is p^* P-almost surely, and $P\left(\alpha_i^n\right)^{-1} = p^*$ for λ-almost every agent i. In addition, all of the relevant Markov chains are time homogeneous with a constant transition matrix z^1 having p^* as a fixed point.*

3.4 Existence of MCI dynamic directed random matching

The following theorem provides for the existence of a Markov conditionally independent (MCI) dynamical system with random mutation, random matching, and match-induced type changes.

Theorem 3 *For any primitive model parameters (b, q, ν) and for any type distribution $\ddot{p}^0 \in \Delta$, there exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ on which is defined a dynamical system $\mathbb{D}_0$ with random mutation, random matching, match-induced type changes, that is Markov conditionally independent with these parameters (b, q, ν), and with the initial type distribution p^0 that is $\ddot{p}^0$ with probability one. These properties can be achieved with an initial type process α^0 that is deterministic, or i.i.d. across agents.*[12]

3.5 Applications in monetary economics

This subsection illustrate two example applications of dynamic directed random matching. These examples provide a mathematical foundation for the dynamic matching models used in

[12]This means that the process α^0 is essentially pairwise independent, and α_i^0 has distribution $\ddot{p}^0$ for λ-almost all $i \in I$.

Kiyotaki and Wright (1989), Kehoe, Kiyotaki and Wright (1993) and Matsuyama, Kiyotaki and Matsui (1993) in monetary economics.

The first example in this subsection is from Kiyotaki and Wright (1989) and Kehoe, Kiyotaki and Wright (1993).

Example 2 As in Model A of Kiyotaki and Wright (1989), three indivisible goods are labeled 1, 2, and 3. There is a continuum of agents of unit total mass. A given type of agent consumes good k and can store one unit of good l, for some $l \neq k$. This type is denoted (k, l). The economy is thus populated by agents of 6 distinct types $(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)$, which form our type space S. In order to avoid confusion over differences in terminology[13] with Kiyotaki and Wright (1989), we say that an agent who consumes good k has "trait" k. There are equal proportions of agents with the three respective traits.

In each period n, every agent randomly matches with some other agent. When matched, two agents decide whether or not to trade. If there is no trade between the matched pair, they keep their goods. If there is a trade, and if the agent who consumes good k gets good k from the other, then that agent immediately consumes good k and produces one unit of good $k+1$ (modulo 3), so that his type becomes $(k, k+1)$ (modulo 3, as needed). If there is a trade and an agent with trait k gets good l for $l \neq k$, then his type becomes (k, l). Kiyotaki and Wright (1989) and Kehoe, Kiyotaki and Wright (1993) consider the given matching model in terms of stationary and non-stationary trading strategies respectively.

We can use our model of dynamic directed random matching to give a mathematical foundation for the matching models in Kiyotaki and Wright (1989) and and Kehoe, Kiyotaki and Wright (1993) by choosing suitable parameters (b, q, ν) governing random mutation, random matching and match-induced type changing. At period n, let $b^n_{(k_1,l_1)(k_2,l_2)} = \delta_{k_1}(k_2)\delta_{l_1}(l_2)$ be the mutation probabilities, and $q^n_{(k_1,l_1)(k_2,l_2)}(p) = p_{(k_2,l_2)}$ the matching probabilities for $p \in \Delta$. We will need to specify the match-induced type changing probabilities in both cases.

First, a stationary trading strategy in (Kiyotaki and Wright, 1989, p. 931) is described by some $\tau : \{1,2,3\} \times \{1,2,3\} \to \{0,1\}$ that implies a trade, $\tau_k(l, r) = 1$, if a trait-k agent actually wants to trade good l for good r, and results in no trade, $\tau_k(l, r) = 0$, otherwise. Thus τ determines determines the match-induced type changes. Because the consumption traits of agents do not change, the type of a matched agent cannot change to a type with a different trait. Thus, for the type changing probability ν^n of an agent with trait k_1, the probability for the target types is concentrated on only two types, (k_1, k_1+1) and (k_1, k_1+2). This means that it suffices to define the type changing probability for only the target type (k_1, k_1+1). Suppose

[13]In Kiyotaki and Wright (1989), agents have three types. However, the meaning of "type" in Kiyotaki and Wright (1989) is different from that in our present paper. Here, we use the word "trait" to mean what Kiyotaki and Wright (1989) call "type."

that an agent i of type $(k_1, k_1 + 1)$ is matched with an agent j of type (k_2, l_2). For $l_2 = k_1 + 1$, there is no need to trade. When $l_2 = k_1$ and there is a trade, agent i will consume good k_1, produces a unit of good $k_1 + 1$, and keeps the same type $(k_1, k_1 + 1)$. (This applies trivially for the no-trade case.) When $l_2 = k_1 + 2$, the probability $\nu_{(k_1,k_1+1)(k_2,l_2)}(k_1, k_1 + 1)$ that agent i has a type change is the probability of no trade between agents i and j. The probability of having a trade between agents i and j is $\tau_{k_1}(k_1 + 1, l_2)\tau_{k_2}(l_2, k_1 + 1)$. We therefore have

$$\nu^n_{(k_1,k_1+1)(k_2,l_2)}(k_1, k_1 + 1) = \begin{cases} 1 & \text{if } l_2 \neq k_1 + 2 \\ 1 - \tau_{k_1}(k_1 + 1, l_2)\tau_{k_2}(l_2, k_1 + 1) & \text{if } l_2 = k_1 + 2. \end{cases}$$

By similar arguments, we have

$$\nu^n_{(k_1,k_1+2)(k_2,l_2)}(k_1, k_1 + 1) = \begin{cases} 0 & \text{if } l_2 = k_1 + 2 \\ \tau_{k_1}(k_1 + 2, l_2)\tau_{k_2}(l_2, k_1 + 2) & \text{if } l_2 \neq k_1 + 2. \end{cases}$$

Next, we consider the case of non-stationary trading strategies as in Sections 3 and 6 of Kehoe, Kiyotaki and Wright (1993). Suppose that $(s_1(n), s_2(n), s_3(n))$ is a time-dependent mixed strategy at period n, where $s_k(n)$ is the probability that an agent with trait k trades good $k+1$ for $k+2$. Based on $(s_1(n), s_2(n), s_3(n))$, one can compute the probability $P^n_{(k_1,k_2)}(k_3)$ that an agent with type (k_1, k_2) trades for good k_3.

What we need is to define the match-induced type changing probabilities corresponding to the given time-dependent mixed strategy $(s_1(n), s_2(n), s_3(n))$. Suppose that an agent i of type (k_1, k_1+1) is matched with an agent j of type (k_2, l_2). For cases with $l_2 = k_1$ or $l_2 = k_1+1$, the arguments used in Section 2 imply the type changing probability $\nu^n_{(k_1,k_1+1)(k_2,l_2)}(k_1, k_1+1) = 1$. When $l_2 = k_1+2$, the probability of a trade between agents i and j is $P^n_{(k_1,k_1+1)}(l_2)P^n_{(k_2,l_2)}(k_1+1)$. We can therefore obtain that

$$\nu^n_{(k_1,k_1+1)(k_2,l_2)}(k_1, k_1 + 1) = \begin{cases} 1 & \text{if } l_2 \neq k_1 + 2 \\ 1 - P^n_{(k_1,k_1+1)}(l_2)P^n_{(k_2,l_2)}(k_1 + 1) & \text{if } l_2 = k_1 + 2. \end{cases}$$

Similarly,

$$\nu^n_{(k_1,k_1+2)(k_2,l_2)}(k_1, k_1 + 1) = \begin{cases} 0 & \text{if } l_2 = k_1 + 2 \\ P^n_{(k_1,k_1+2)}(l_2)P^n_{(k_2,l_2)}(k_1 + 2) & \text{if } l_2 \neq k_1 + 2. \end{cases}$$

Our next example is from Matsuyama, Kiyotaki and Matsui (1993). Here, agents are divided into two groups. Agents are more likely to be matched to a counterparty of their own group than to a counterparty of a different group.

Example 3 The economy is populated by a continuum of infinitely-lived agents of unit total mass. Agents are from two regions, Home and Foreign. Let $p \in (0, 1)$ be the size of the Home population. There are $K \geq 3$ kinds of indivisible commodities. Within each region, there are equal proportions of agents with the K respective traits. An agent with trait k derives utility only from consumption of commodity k. After he consumes commodity k, he is able to produce one and only one unit of commodity $k+1 \pmod K$ costlessly, and can also store up to one unit of his production good costlessly. He can neither produce nor store other types of goods.

In addition to the commodities described above, there are two distinguishable fiat monies, objects with zero intrinsic worth, which we call the Home currency and the Foreign currency. Each currency is indivisible and can be stored costlessly in amounts of up to one unit by any agent, provided that the agent does not carry his production good or the other currency. This implies that, at any date, the inventory of each agent consists of one unit of the Home currency, one unit of the Foreign currency, or one unit of his production good, but does not include more than one of these objects in total at any one time.

For some $\beta \in (0,1)$, in each period n, a Home agent is matched to a Home agent with probability p, and is matched to a Foreign agent with probability $\beta(1-p)$. The probability with which he is not matched is thus $(1-\beta)(1-p)$. Similarly, a Foreign agent is matched to a Home agent with probability βp, is matched to a Foreign agent with probability $(1-p)$, and is unmatched with probability $(1-\beta)p$.

The type space S is the set of ordered tuples of the form (a,b,c), where $a \in \{H,F\}$, $b \in \{1,\dots,K\}$, and $c \in \{g,h,f\}$. Here, H represents Home, F represents Foreign, g represents good, h represents Home currency, and f represents Foreign currency.

An agent chooses a trading strategy to maximize his expected discounted utility, taking as given the strategies of other agents and the distribution of inventories. Matsuyama, Kiyotaki and Matsui (1993) focused on pure strategies that depend only on an agent's nationality and the objects that he and his counterparty have as inventories. Thus, the Home agent's trading strategy can be described simply as

$$\tau^H_{ab} = \begin{cases} 1 & \text{if he agrees to trade object } a \text{ for object } b \\ 0 & \text{otherwise,} \end{cases}$$

where a and b are in $\{g,h,f\}$. The Foreign agent's trading strategy can similarly be described as

$$\tau^F_{ab} = \begin{cases} 1 & \text{if he agrees to trade object } a \text{ for object } b \\ 0 & \text{otherwise.} \end{cases}$$

For example, $\tau^H_{gf} = 0$ means that a Home agent does not agree to trade his production good for the Foreign currency, while $\tau^F_{hg} = 1$ means that a Foreign agent agrees to trade the Home currency for his consumption good.

We can apply our model of dynamic directed random matching with immediate break-up to give a mathematical foundation for the matching model in Matsuyama, Kiyotaki and Matsui

(1993) by choosing suitable time-independent parameters (b, q, ν) governing random mutation, random matching, and match-induced type changing. To this end, we take mutation probabilities

$$b_{(a_1,b_1,c_1)(a_2,b_2,c_2)} = \delta_{a_1}(a_2)\delta_{b_1}(b_2)\delta_{c_1}(c_2).$$

The directed search probabilities are given by

$$q_{(a_1,b_1,c_1)(a_2,b_2,c_2)}(p) = \begin{cases} p_{(a_2,b_2,c_2)} & \text{if } a_1 = a_2 \\ \beta \cdot p_{(a_2,b_2,c_2)} & \text{if } a_1 \neq a_2, \end{cases}$$

for a cross-sectional agent type distribution $p \in \Delta$. Because the nationality and consumption traits of agents do not change, a matched agent cannot change to a type with a different nationality or trait. Thus, for the type changing probability ν of an agent with nationality a_1 and trait b_1, search is directed to the three counterparty types (a_1, b_1, g), (a_1, b_1, f) and (a_1, b_1, h).

Suppose that agent i is of type (a_1, b_1, g) and is matched with agent j, who has type (a_2, b_2, c_2). The probability that agent i changes type to (a_1, b_1, h) is $\nu_{(a_1,b_1,g)(a_2,b_2,c_2)}(a_1, b_1, h)$. We note that the good carried by an agent of type (a_1, b_1, g) must be $b_1 + 1$. For $b_2 \neq b_1 + 1 \pmod K$, the good that agent i carries is not the consumption good of agent j, which means that there is no trade, so the probability $\nu_{(a_1,b_1,g)(a_2,b_2,c_2)}(a_1, b_1, h)$ is 0. When $c_2 \neq h$, agent i cannot get the Home currency from j, so $\nu_{(a_1,b_1,g)(a_2,b_2,c_2)}(a_1, b_1, h)$ is also 0. When $b_2 = b_1 + 1$ and $c_2 = h$, $\nu_{(a_1,b_1,g)(a_2,b_2,c_2)}(a_1, b_1, h)$ is the probability that agent i trades with an agent with the type of agent j, which is $\tau^{a_1}_{gh} \cdot \tau^{a_2}_{hg}$. We therefore have

$$\nu_{(a_1,b_1,g)(a_2,b_2,c_2)}(a_1, b_1, h) = \begin{cases} \tau^{a_1}_{gh} \cdot \tau^{a_2}_{hg} & \text{if } b_2 \equiv b_1 + 1 \pmod K \text{ and } c_2 = h \\ 0 & \text{otherwise.} \end{cases}$$

The following type-change probabilities can be obtained by similar arguments:

$$\nu_{(a_1,b_1,g)(a_2,b_2,c_2)}(a_1, b_1, f) = \begin{cases} \tau^{a_1}_{gf} \cdot \tau^{a_2}_{fg} & \text{if } b_2 \equiv b_1 + 1 \pmod K \text{ and } c_2 = f \\ 0 & \text{otherwise;} \end{cases}$$

$$\nu_{(a_1,b_1,g)(a_2,b_2,c_2)}(a_1, b_1, g) = 1 - \nu_{(a_1,b_1,g)(a_2,b_2,c_2)}(a_1, b_1, h) - \nu_{(a_1,b_1,g)(a_2,b_2,c_2)}(a_1, b_1, f);$$

$$\nu_{(a_1,b_1,h)(a_2,b_2,c_2)}(a_1, b_1, g) = \begin{cases} \tau^{a_1}_{hg} \cdot \tau^{a_2}_{gh} & \text{if } b_2 \equiv b_1 - 1 \pmod K \text{ and } c_2 = g \\ 0 & \text{otherwise;} \end{cases}$$

$$\nu_{(a_1,b_1,h)(a_2,b_2,c_2)}(a_1, b_1, f) = \begin{cases} \tau^{a_1}_{hf} \cdot \tau^{a_2}_{fh} & c_2 = f \\ 0 & \text{otherwise;} \end{cases}$$

$$\nu_{(a_1,b_1,h)(a_2,b_2,c_2)}(a_1, b_1, h) = 1 - \nu_{(a_1,b_1,h)(a_2,b_2,c_2)}(a_1, b_1, g) - \nu_{(a_1,b_1,h)(a_2,b_2,c_2)}(a_1, b_1, f);$$

$$\nu_{(a_1,b_1,f)(a_2,b_2,c_2)}(a_1, b_1, g) = \begin{cases} \tau^{a_1}_{fg} \cdot \tau^{a_2}_{gf} & \text{if } b_2 \equiv b_1 - 1 \pmod K \text{ and } c_2 = g \\ 0 & \text{otherwise;} \end{cases}$$

$$\nu_{(a_1,b_1,f)(a_2,b_2,c_2)}(a_1,b_1,h) = \begin{cases} \tau_{fh}^{a_1} \cdot \tau_{hf}^{a_2} & \text{if } c_2 = h \\ 0 & \text{otherwise;} \end{cases}$$

$$\nu_{(a_1,b_1,f)(a_2,b_2,c_2)}(a_1,b_1,f) = 1 - \nu_{(a_1,b_1,f)(a_2,b_2,c_2)}(a_1,b_1,h) - \nu_{(a_1,b_1,f)(a_2,b_2,c_2)}(a_1,b_1,g).$$

4 Concluding Discussion

Previous results concerning the existence and law of large numbers for independent random matching, such as those of Duffie and Sun (2007, 2012), were limited by the assumption that the partner of a matched agent is drawn uniformly from the population of matched agents.[14] The main purpose of this paper is to provide a suitable search-based model of markets in which agents direct their searches, causing relatively higher per-capita matching probabilities with specific types of counterparties. Although models with directed search are common in the literatures covering money, labor markets, and over-the-counter financial markets, prior work has simply assumed that the exact law of large numbers would lead to a deterministic cross-sectional distribution of agent types, and that this distribution would obey certain properties. We provide a model that justifies this assumed behavior, down to the basic level of random contacts between specific individual agents. We provide the resulting transition distribution for the Markov processes for individual agents' types, and for the aggregate cross-sectional distribution of types in the population, and show the close relationship between these two objects.

By incorporating directed search, we are also able to provide the first rigorous probabilistic foundation for the notion of a "matching function" that is heavily used in the search literature of labor economics.

A secondary objective is to allow for random matching with enduring partnerships. The durations of these partnerships can be random or deterministic, and can be type dependent. Earlier work providing mathematical foundations for random matching presumes that partnerships break up immediately after matching. Enduring partnerships are crucial for search-based labor-market search models, such as those cited in Footnote 5, in which there are episodes of employment resulting from a match between a worker and a firm, eventually followed by

[14] Footnote 4 of McLennan and Sonnenschein (1991) showed the non-existence of a type-free (static) random full matching that satisfies a number of desired conditions when the agent space is taken to be the unit interval with the Borel σ-algebra and Lebesgue measure. That problem is resolved through the construction of an independent type-free random full matching with a suitable agent space as in Duffie and Sun (2007) and Podczeck and Puzzello (2012); see also Duffie and Sun (2012) and Podczeck and Puzzello (2012) regarding implications for independent random full matching with general type spaces. Xiang Sun (2016) extended the results on independent (static) random partial matching in Duffie and Sun (2007) from finite type spaces to general type spaces. All of these cited papers address the case of "undirected" search. For a detailed discussion of the literature on "non-independent" random matching, see Section 6 of Duffie and Sun (2012).

a random separation.[15] In some of these models, separation is *iid* across periods of employment. This is the case, for example, in Cho and Matsui (2013), Merz (1999), Pissarides (1985), Shi and Wen (1999), Shimer (2005), and Yashiv (2000), among many other papers. In other cases, the separation probability depends on the vintage of the match, and can depend on the quality of the match between the worker and the firm. Since the separation probabilities in our general model depend on the types of the matched agents, our results can cover such cases by introducing new types.

We have verified that our results can be extended under mild revisions of the proofs to settings in which agents have countably many types, and can enter and exit (for example, through "birth" and "death"), allowing for a total population size that is changing over time without a fixed bound as in Yashiv (2000).[16] It is also straightforward to allow for a background Markov process that governs the parameters determining probabilities for mutation, matching, and type change (as well as enduring match break-ups). In this case, the background Markov state causes aggregate uncertainty, but conditional on the path of the background state, the cross-sectional distribution of population types evolves deterministically, almost surely.

References

Daron Acemoglu and Alexander Wolitzky, The economics of labor coercion, *Econometrica* **79** (2011), 555–600.

David Andolfatto, Business cycles and labor-market search, *American Economic Review* **86** (1996), 112–132.

Ken Binmore and Larry Samuelson, Evolutionary drift and equilibrium selection, *Review of Economic Studies* **66** (1999), 363–393.

In-Koo Cho and Akihiko Matsui, Search theory, competitive equilibrium, and the Nash bargaining solution, *Journal of Economic Theory* **148** (2013), 1659–1688.

Sergio Currarini, Matthew O. Jackson and Paolo Pin, An economic model of friendship: Homophily, minorities, and segregation, *Econometrica* **77** (2009), 1003–1045.

Peter Diamond, Aggregate demand management in search equilibrium, *Journal of Political Economy* **90** (1982), 881–894.

[15] Further such situations arise in the models of Cho and Matsui (2013), Flinn (2006), Haan, Ramey and Watson (2000), Hall (2005), Hosios (1990), Merz (1995), Merz (1999), Mortensen (1982), Pissarides (1985), Shimer (2005), Shi and Wen (1999), and Yashiv (2000).

[16] Our results cover cases in which there is a fixed bound for the total population size in all time periods. In such cases, one can simply introduce a new type to represent the inactive agents and re-scale the total population size.

Darrell Duffie, Nicolae Gârleanu, and Lasse H. Pedersen, Over-the-counter markets, *Econometrica* **73** (2005), 1815–1847.

Darrell Duffie, Semyon Malamud and Gustavo Manso, Information percolation in segmented markets, *Journal of Economic Theory* **153** (2014), 1–32.

Darrell Duffie and Yeneng Sun, Existence of independent random matching, *Annals of Applied Probability* **17** (2007), 386–419.

Darrell Duffie and Yeneng Sun, The exact law of large numbers for independent random matching, *Journal of Economic Theory* **147** (2012), 1105–1139.

Christopher Flinn, Minimum wage effects on labor market outcomes under search, matching, and endogenous contact rates, *Econometrica*, **74** (2006), 1013–1062.

Edward J. Green and Ruilin Zhou, Dynamic monetary equilibrium in a random matching economy, *Econometrica* **70** (2002), 929–970.

Wouter J. D. Haan, Garey Ramey and Joel Watson, Job destruction and propagation of shocks. *American Economic Review* **90** (2000), 482–498.

Robert E. Hall, Employment fluctuations with equilibrium wage stickiness, *American Economic Review* **95** (2005), 50–65.

Martin Hellwig, A model of monetary exchange, Econometric Research Program, Research Memorandum Number 202, Princeton University, 1976.

Arthur Hosios, On the efficiency of matching and related models of search and unemployment, *Review of Economic Studies* **57** (1990), 279–298.

Timothy J. Kehoe, Nobuhiro Kiyotaki and Randall Wright, More on money as a medium of exchange. *Economic Theory* **3** (1993), 297–314.

Nobuhiro Kiyotaki and Randall Wright, On money as a medium of exchange, *Journal of Political Economy* **97** (1989), 927–954.

Ricardo Lagos and Guillaume Rocheteau, Liquidity in asset markets with search frictions, *Econometrica* **77** (2009), 403–426.

Benjamin Lester, Andrew Postlewaite and Randall Wright, Information, liquidity, asset prices, and monetary policy, *Review of Economic Studies* **79** (2012), 1209–1238.

Peter A. Loeb and Manfred Wolff, eds. *Nonstandard Analysis for the Working Mathematician*, second edition, Springer, 2015.

Kiminori Matsuyama, Nobuhiro Kiyotaki and Akihiko Matsui, Toward a theory of international currency, *Review of Economic Studies* **60** (1993), 283–307.

Vincent Maurin, Liquidity fluctuations in over the counter markets, Working paper, European University Institute, 2015.

Andrew McLennan and Hugo Sonnenschein, Sequential bargaining as a noncooperative foundation for Walrasian equilibrium, *Econometrica* **59** (1991), 1395–1424.

Monika Merz, Search in the labor market and the real business cycle, *Journal of Monetary Economics* **36** (1995), 269–300.

Monika Merz, Heterogeneous job matches and the cyclical behavior of labor turnover, *Journal of Monetary Economics* **43** (1999), 91–124.

Dale Mortensen, Property rights and efficiency in mating, racing, and related games, *American Economic Review* **72** (1982), 968–979.

Dale Mortensen and Christopher Pissarides, Job creation and job destruction in the theory of unemployment, *Review of Economic Studies* **61** (1994), 397–415.

Barbara Petrongolo and Christopher Pissarides, Looking into the black box: A survey of the matching function, *Journal of Economic Literature* **39** (2001) 390–431.

Christopher Pissarides, Short-run equilibrium dynamics of unemployment, vacancies, and real wages, *American Economic Review* **75** (1985), 676–690.

Konrad Podczeck and Daniela Puzzello, Independent random matching, *Economic Theory* **50** (2012), 1–29.

Richard Rogerson, Robert Shimer and Randall Wright, Search-theoretic models of labor markets, *Journal of Economic Literature* **43** (2005), 959–988.

Shouyong Shi and Quan Wen, Labor market search and the dynamic effects of taxes and subsidies, *Journal of Monetary Economics* **43** (1999), 457–495.

Robert Shimer, The cyclical behavior of equilibrium unemployment and vacancies, *American Economic Review* **95** (2005), 25–49.

Xiang Sun, Independent random partial matching with general types, *Advances in Mathematics* **286** (2016), 683–702.

Yeneng Sun, The exact law of large numbers via Fubini extension and characterization of insurable risks, *Journal of Economic Theory* **126** (2006), 31–69.

Yeneng Sun and Yongchao Zhang, Individual risk and Lebesgue extension without aggregate uncertainty, *Journal of Economic Theory* **144** (2009), 432–443.

Anton Tsoy, Trade delay, liquidity, and asset prices in over-the-counter markets, Working paper, Department of Economics, MIT, October, 2014.

Dimitri Vayanos and Pierre-Olivier Weill, A search-based theory of the on-the-run phenomenon, *Journal of Finance* **63** (2008), 1361–1398.

Pierre-Olivier Weill, Leaning against the wind, *Review of Economic Studies* **74** (2007), 1329–1354.

Eran Yashiv, The determinants of equilibrium unemployment, *American Economic Review* **90** (2000), 1297–1322.

Appendices

Some of the following appendix material could ultimately be placed in a separate online document.

A Dynamic Directed Random Matching with Enduring Partnerships

This appendix extends the model of dynamic directed random matching found in Section 3 so as to allow for enduring partnerships and for correlated type changes of matched agents. Unlike the more basic model of Section 3, in order to capture the effect of enduring partnerships we now must consider the separate treatments of existing matched pairs of agents and newly formed matched pairs of agents.

We first define such a dynamical system in Subsection A.1. The key condition of Markov conditional independence is formulated in Subsection A.2. Based on that condition, Subsection A.3 presents an exact law of large numbers for such a dynamical system. Subsection A.4 provides results covering the existence of Markov conditionally independent dynamical system with directed random matching and with partnerships that have randomly time breakups. In the final subsection, we illustrate the random break-up of partnerships through an example drawn from labor economics.

Theorem 2 in Section 3 is a special case of Theorem 4 and Proposition 3 in Subsection A.3, while Theorem 5 in Subsection A.4 extends Theorem 3 in Section 3. Hence the proofs of Theorems 2 and 3 are omitted. We prove all the results stated in this section in the next section.

As in Sections 2 and 3, we fix an atomless probability space $(I, \mathcal{I}, \lambda)$ representing the space of agents, a sample probability space $(\Omega, \mathcal{F}, P)$, and a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. We will show that all of our results can be obtained for an agent space that is a Loeb measure space as constructed in nonstandard analysis, or is an extension of the classical Lebesgue unit interval. This section has self-contained notation. In particular, some of the notation used in this section may have a meaning that differs from its usage in Section 3.

A.1 Definition of dynamic directed random matching with enduring partnerships

As in Sections 2 and 3, let $S = \{1, 2, \ldots, K\}$ be a finite set of types and let J be a special type representing no-matching. The "extended type" space is $\hat{S} = S \times (S \cup \{J\})$. An agent with an extended type of the form (k, l) has underlying type $k \in S$ and is currently matched to another agent of type l in S. If the agent's extended type is instead of the form (k, J), then the agent is "unmatched." The space $\hat{\Delta}$ of extended type distributions is the set of probability distributions $\hat{p}$ on $\hat{S}$ satisfying $\hat{p}(k, l) = \hat{p}(l, k)$ for all k and l in S.

Each time period is divided into three steps: mutation, random matching, match-induced type changing with break-up. We now introduce the primitive parameters governing each of these steps.

At the first (mutation) step of time period $n \geq 1$, each agent of type $k \in S$ experiences a random mutation, becoming an agent of type l with a given probability b_{kl}^n, a parameter of the model. By definition, for each type k we must have $\sum_{l \in S} b_{kl}^n = 1$.

At the second step, any currently unmatched agent conducts a directed search for counterparties. For each $(k,l) \in S \times S$, let q_{kl}^n be a function on $\hat{\Delta}$ into $\mathbb{R}_+$ with the property that for all k and l in S, the function $\hat{p}_{kJ} q_{kl}^n(\hat{p})$ is continuous in $\hat{p} \in \hat{\Delta}$ and satisfies, for any $\hat{p}$ in $\hat{\Delta}$,

$$\hat{p}_{kJ}\, q_{kl}^n(\hat{p}) = \hat{p}_{lJ}\, q_{lk}^n(\hat{p}) \quad \text{and} \quad \sum_{r \in S} q_{kr}^n(\hat{p}) \leq 1. \tag{7}$$

Whenever the underlying extended type distribution is $\hat{p}$, the probability[17] that an unmatched agent of type k is matched to an unmatched agent of type l is $q_{kl}^n(\hat{p})$. Thus, $\eta_k^n(\hat{p}) = 1 - \sum_{l \in S} q_{kl}^n(\hat{p})$ is the no-matching probability for an unmatched agent of type k.

At the third step, each currently matched pair of agents of respective types k and l (including those who have just been paired at the matching step) breaks up with probability θ_{kl}^n, where

$$\theta_{kl}^n = \theta_{lk}^n. \tag{8}$$

If a matched pair of agents of respective types k and l stays in their partnership, they become a pair of agents of types r and s, respectively, with a specified probability $\sigma_{kl}^n(r,s)$, where

$$\sum_{r,s \in S} \sigma_{kl}^n(r,s) = 1 \text{ and } \sigma_{kl}^n(r,s) = \sigma_{lk}^n(s,r) \tag{9}$$

for any $k,l,r,s \in S$. The second identity is merely a labeling symmetry condition. If a matched pair of agents of respective types k and l breaks up, the agent of type k becomes an agent of type r with probability $\varsigma_{kl}^n(r)$, where

$$\sum_{r \in S} \varsigma_{kl}^n(r) = 1. \tag{10}$$

[17]Let φ be any continuous function from $\hat{\Delta}$ to itself. Assume that: (1) for any $k,l \in S$, $\varphi_{kl}(\hat{p}) \geq \hat{p}_{kl}$; (2) for any $\hat{p} \in \hat{\Delta}$, $\varphi(\hat{p})$ and $\hat{p}$ have the same marginal measure on S, that is, for any $k \in S$, $\sum_{r \in S \cup \{J\}} \varphi_{kr}(\hat{p}) = \sum_{r \in S \cup \{J\}} \hat{p}_{kr}$. For any $k,l \in S$, let $q_{kl}(\hat{p}) = (\varphi_{kl}(\hat{p}) - \hat{p}_{kl})/\hat{p}_{kJ}$ if $\hat{p}_{kJ} > 0$ and $q_{kl}(\hat{p}) = 0$ if $\hat{p}_{kJ} = 0$. Then, the function q satisfies the continuity condition as well as Equation (7), as required for a matching probability function. In fact, any matching probability function can be obtained in this way. For the special case that all of the matched agents break up at the end of each period, we need only consider continuous functions from Δ to $\hat{\Delta}$. Let ϕ be any such continuous function with the property that for any $p \in \Delta$, $\varphi(p)$ has the marginal measure p on S. That is, for any $k \in S$, $\sum_{l \in S \cup \{J\}} \varphi_{kl}(p) = p_k$. For any $k,l \in S$, let $q_{kl}(p) = \phi_{kl}(p)/p_k$ if $p_k > 0$ and $q_{kl}(p) = 0$ if $p_k = 0$. Then, the function q satisfies the continuity condition as well as Equation (3), as required for a matching probability function. Again, any matching probability function for this particular setting can be obtained in this way.

We now give an inductive definition of the properties defining a dynamical system $\mathbb{D}$ for the behavior of a continuum population of agents experiencing, at each time period: random mutations, matchings, and match-induced type changes with break-up. We later state conditions under which such a system exists. The state of the dynamical system $\mathbb{D}$ at the end of each integer period $n \geq 0$ is defined by a pair $\Pi^n = (\alpha^n, \pi^n)$ consisting of:

- An agent type function $\alpha^n : I \times \Omega \to S$ that is $\mathcal{I} \boxtimes \mathcal{F}$-measurable. The corresponding end-of-period type of agent i is $\alpha^n(i) \in S$. For technical convenience, we always augment the agent and type spaces by including the element J, with $\alpha^n(J) = J$ (that is, $\alpha^n(J, \omega) = J$ for all $\omega \in \Omega$).

- A random matching $\pi^n : I \times \Omega \to I \cup \{J\}$, describing the end-of-period agent $\pi^n(i)$ to whom agent i is currently matched, if agent i is currently matched. If agent i is not matched, then $\pi^n(i) = J$. The associated partner-type function $g^n : I \times \Omega \to S \cup J$ provides the type $g^n(i) = \alpha^n(\pi^n(i))$ of the agent to whom i is matched, if i is matched, and otherwise specifies $g^n(i) = J$. As a matter of definition, we require that g^n is $\mathcal{I} \boxtimes \mathcal{F}$-measurable.

We take the initial condition $\Pi^0 = (\alpha^0, \pi^0)$ of $\mathbb{D}$ as given. The initial condition may, if desired, be deterministic (constant across Ω). The joint cross-sectional extended type distribution $\hat{p}^n$ at the end of period n is $\lambda(\beta^n)^{-1}$. That is, $\hat{p}^n(k, l)$ is the fraction of the population at the end of period n that has type k and is matched to an agent of type l. Likewise, $\hat{p}^n(k, J)$ is the fraction of the population that is of type k and is not matched.

For the purpose of the inductive definition of the dynamical system $\mathbb{D}$, we suppose that $\Pi^{n-1} = (\alpha^{n-1}, \pi^{n-1})$ has been defined for some $n \geq 1$, and define $\Pi^n = (\alpha^n, \pi^n)$ as follows.

Mutation. The post-mutation type function $\bar{\alpha}^n$ is $\mathcal{I} \boxtimes \mathcal{F}$-measurable, and satisfies, for any k_1, k_2, l_1, and l_2 in S, for any $r \in S \cup \{J\}$, and for λ-almost-every agent i,

$$P(\bar{\alpha}_i^n = k_2, \bar{g}_i^n = l_2 \,|\, \alpha_i^{n-1} = k_1, g_i^{n-1} = l_1) = b_{k_1 k_2}^n b_{l_1 l_2}^n \tag{11}$$

$$P(\bar{\alpha}_i^n = k_2, \bar{g}_i^n = r \,|\, \alpha_i^{n-1} = k_1, g_i^{n-1} = J) = b_{k_1 k_2}^n \delta_J(r). \tag{12}$$

Equation (11) means that a paired agent and her partner mutate independently. The post-mutation partner-type function $\bar{g}^n$ is defined by $\bar{g}^n(i, \omega) = \bar{\alpha}^n(\pi^{n-1}(i, \omega), \omega)$, for any $\omega \in \Omega$. We assume that $\bar{g}^n$ is $\mathcal{I} \boxtimes \mathcal{F}$-measurable. The post-mutation extended-type function is $\bar{\beta}^n = (\bar{\alpha}^n, \bar{g}^n)$. The post-mutation extended type distribution that is realized in state $\omega \in \Omega$ is $\check{p}^n(\omega) = \lambda \left(\bar{\beta}_\omega^n\right)^{-1}$.

Matching. Let $\bar{\pi}^n : I \times \Omega \to I \cup \{J\}$ be a random matching with the following properties.

(i) For each state $\omega \in \Omega$, let $A^{\omega} = \{i : \pi^{n-1}(i, \omega) \neq J\}$ be the set of agents who are matched. For P-almost all $\omega \in \Omega$, we take

$$\bar{\pi}^{n}_{\omega}(i) = \pi^{n-1}_{\omega}(i) \text{ for } i \in A^{\omega}, \tag{13}$$

meaning that those agents who were already matched at the end of period $n-1$ remain matched (to the same partner) at this step, which implies that the post-matching partner-type function $\bar{\bar{g}}^{n}$, defined by $\bar{\bar{g}}^{n}(i,\omega) = \bar{\alpha}^{n}(\bar{\pi}^{n}(i,\omega),\omega)$, satisfies

$$P(\bar{\bar{g}}^{n}_{i} = r \,|\, \bar{\alpha}^{n}_{i} = k, \bar{g}^{n}_{i} = l) = \delta_{l}(r), \tag{14}$$

for any k and l in S and any $r \in S \cup \{J\}$, where $\delta_{c}(d)$ is zero if $c \neq d$ and is one if $c = d$.

(ii) $\bar{\bar{g}}^{n}$ is $\mathcal{I} \boxtimes \mathcal{F}$-measurable.

(iii) Given the post-mutation extended type distribution $\check{p}^{n}$, an unmatched agent of type k is matched to a unmatched agent of type l with conditional probability $q^{n}_{kl}(\check{p}^{n})$, in that, for λ-almost every agent i and P-almost every ω,

$$P(\bar{\bar{g}}^{n}_{i} = l \,|\, \bar{\alpha}^{n}_{i} = k, \bar{g}^{n}_{i} = J, \check{p}^{n}) = q^{n}_{kl}(\check{p}^{n}(\omega)), \tag{15}$$

which also implies that

$$P(\bar{\bar{g}}^{n}_{i} = J \,|\, \bar{\alpha}^{n}_{i} = k, \bar{g}^{n}_{i} = J, \check{p}^{n}) = \eta^{n}_{k}(\check{p}^{n}(\omega)). \tag{16}$$

The extended type of agent i after the random matching step is $\bar{\bar{\beta}}^{n}_{i} = (\bar{\alpha}^{n}_{i}, \bar{\bar{g}}^{n}_{i})$.

Type changes of matched agents with break-up. This step determines an end-of-period random matching π^{n}, an $\mathcal{I} \boxtimes \mathcal{F}$-measurable agent type function α^{n}, and an $\mathcal{I} \boxtimes \mathcal{F}$-measurable partner-type function g^{n} so that we have $g^{n}(i,\omega) = \alpha^{n}(\pi^{n}(i,\omega),\omega)$ for all $(i,\omega) \in I \times \Omega$, and so that, for λ-almost every agent i and for any $k_1, k_2, l_1, l_2 \in S$ and $r \in S \cup \{J\}$,

$$\pi^{n}(i) = \begin{cases} \bar{\pi}^{n}(i), & \text{if } g^{n}(i) \neq J \\ J, & \text{if } g^{n}(i) = J, \end{cases} \tag{17}$$

$$P(\alpha^{n}_{i} = l_1, g^{n}_{i} = r \,|\, \bar{\alpha}^{n}_{i} = k_1, \bar{\bar{g}}^{n}_{i} = J) = \delta_{k_1}(l_1)\, \delta_{J}(r), \tag{18}$$

$$P(\alpha^{n}_{i} = l_1, g^{n}_{i} = l_2 \,|\, \bar{\alpha}^{n}_{i} = k_1, \bar{\bar{g}}^{n}_{i} = k_2) = (1 - \theta^{n}_{k_1 k_2})\sigma^{n}_{k_1 k_2}(l_1, l_2), \tag{19}$$

$$P(\alpha^{n}_{i} = l_1, g^{n}_{i} = J \,|\, \bar{\alpha}^{n}_{i} = k_1, \bar{\bar{g}}^{n}_{i} = k_2) = \theta^{n}_{k_1 k_2}\varsigma^{n}_{k_1 k_2}(l_1). \tag{20}$$

Equations (17) and (18) mean that unmatched agents stay unmatched without changing types, while Equations (19) and (20) specify the type changing probabilities for a pair of matched agents who stay together or break up. The extended-type function at the end of the period is $\beta^n = (\alpha^n, g^n)$.

Thus, we have inductively defined the properties of a dynamical system $\mathbb{D} = (\Pi^n)_{n=1}^{\infty}$ incorporating the effects of random mutation, directed random matching, and match-induced type changes with break-up, consistent with given parameters $(b, q, \theta, \sigma, \varsigma)$. The initial condition Π^0 of $\mathbb{D}$ is unrestricted. We next turn to the key Markovian independence properties for such a system, and then to the exact law of large numbers and existence of a dynamical system with these properties.

A.2 Markov conditional independence

We now add independence conditions on the dynamical system $\mathbb{D} = (\Pi^n)_{n=0}^{\infty}$, along the lines of those in Duffie and Sun (2007), Duffie and Sun (2012), and Section 3. The idea is that each of the just-described steps (mutation, random matching, and match-induced type changes with break-up) are conditionally independent across almost all agents. In the following definition, we will refer to objects, such as the intermediate-step extended type functions $\bar{\beta}^n$ and $\bar{\bar{\beta}}^n$, that were constructed in the previous sub-section.

We say that the dynamical system $\mathbb{D}$ is Markov conditionally independent (MCI) if, for λ-almost every i and λ-almost every j, for every period $n \geq 1$, and for all $k_1, k_2 \in S$, and $l_1, l_2 \in S \cup \{J\}$, the following five properties apply:

- Initial independence: β_i^0 and β_j^0 are independent.
- Markov and independent mutation:

$$\begin{aligned} P\left(\bar{\beta}_i^n = (k_1, l_1), \bar{\beta}_j^n = (k_2, l_2) \mid (\beta_i^t)_{t=0}^{n-1}, (\beta_j^t)_{t=0}^{n-1}\right) \\ = P\left(\bar{\beta}_i^n = (k_1, l_1) \mid \beta_i^{n-1}\right) P\left(\bar{\beta}_j^n = (k_1, l_1) \mid \beta_j^{n-1}\right). \end{aligned} \tag{21}$$

- Markov and independent random matching:

$$\begin{aligned} P\left(\bar{\bar{\beta}}_i^n = (k_1, l_1), \bar{\bar{\beta}}_j^n = (k_2, l_2) \mid \bar{\beta}_i^n, \bar{\beta}_j^n, (\beta_i^t)_{t=0}^{n-1}, (\beta_j^t)_{t=0}^{n-1}\right) \\ = P\left(\bar{\bar{\beta}}_i^n = (k_1, l_1) \mid \bar{\beta}_i^n\right) P\left(\bar{\bar{\beta}}_j^n = (k_2, l_2) \mid \bar{\beta}_j^n\right). \end{aligned} \tag{22}$$

- Markov and independent matched-agent type changes with break-up:

$$\begin{aligned} P\left(\beta_i^n = (k_1, l_1), \beta_j^n = (k_2, l_2) \mid \bar{\bar{\beta}}_i^n, \bar{\bar{\beta}}_j^n, (\beta_i^t)_{t=0}^{n-1}, (\beta_j^t)_{t=0}^{n-1}\right) \\ = P\left(\beta_i^n = (k_1, l_1) \,|\, \bar{\bar{\beta}}_i^n\right) P\left(\beta_j^n = (k_2, l_2) \,|\, \bar{\bar{\beta}}_j^n\right). \end{aligned} \tag{23}$$

A.3 The exact law of large numbers for MCI dynamical systems with enduring partnerships

For each period $n \geq 1$, we define a mapping Γ^n from $\hat{\Delta}$ to $\hat{\Delta}$ by

$$\begin{aligned}
\Gamma^n_{kl}(\hat{p}) &= \sum_{(k_1,l_1)\in S^2} \tilde{p}^n_{k_1 l_1}(1-\theta^n_{k_1 l_1})\sigma^n_{k_1 l_1}(k,l) + \sum_{(k_1,l_1)\in S^2} \tilde{p}_{k_1 J}\, q^n_{k_1 l_1}(\tilde{p})(1-\theta^n_{k_1 l_1})\sigma^n_{k_1 l_1}(k,l) \\
\Gamma^n_{kJ}(\hat{p}) &= \tilde{p}_{kJ}\eta^n_k(\tilde{p}) + \sum_{(k_1,l_1)\in S^2} \tilde{p}^n_{k_1 l_1}\theta^n_{k_1 l_1}\varsigma^n_{k_1 l_1}(k) + \sum_{(k_1,l_1)\in S^2} \tilde{p}_{k_1 J}\, q^n_{k_1 l_1}(\tilde{p})\theta^n_{k_1 l_1}\varsigma^n_{k_1 l_1}(k),
\end{aligned}$$

where $\tilde{p}_{kl} = \sum_{(k_1,l_1)\in S^2} \hat{p}_{k_1 l_1} b^n_{k_1 k} b^n_{l_1 l}$ and $\tilde{p}_{kJ} = \sum_{l\in S} \hat{p}_{lJ} b^n_{lk}$.

The following theorem, which extends Theorem 3.3, presents an exact law of large numbers for the joint agent-partner type processes at the end of each period. The result also provides a recursive calculation of the cross-sectional joint agent-partner type distribution $\hat{p}^n$ at the end of period n.

Theorem 4 *Let $\mathbb{D}$ be a dynamical system with random mutation, random matching, and match-induced type changes with break-up whose parameters are $(b, q, \theta, \sigma, \varsigma)$. If $\mathbb{D}$ is Markov conditionally independent, then:*

(1) For each time period $n \geq 1$, the expected cross-sectional type distribution $\tilde{p}^n = \mathbb{E}(\check{p}^n)$ after the mutation step and $\mathbb{E}(\hat{p}^n)$ at the end of the period are given by, respectively, $\mathbb{E}(\check{p}^n_{kl}) = \sum_{k_1,l_1\in S} \mathbb{E}(\hat{p}^{n-1}_{k_1 l_1}) b^n_{k_1 k} b^n_{l_1 l}$ and $\mathbb{E}(\check{p}^n_{kJ}) = \sum_{l\in S} \mathbb{E}(\hat{p}^{n-1}_{lJ}) b^n_{lk}$, and by $\mathbb{E}(\hat{p}^n) = \Gamma^n(\mathbb{E}(\hat{p}^{n-1}))$.

(2) For λ-almost every agent i, the extended-type process $\{\beta^n_i\}^\infty_{n=0}$ is a Markov chain in $\hat{S}$ whose transition matrix z^n at time $n-1$ is given by

$$\begin{aligned}
z^n_{(kJ)(k'J)} &= b^n_{kk'}\eta^n_{k'}(\tilde{p}^n) + \sum_{k_1,l_1,\in S} b^n_{kk_1} q^n_{k_1 l_1}(\tilde{p}^n)\theta^n_{k_1 l_1}\varsigma^n_{k_1 l_1}(k'), \\
z^n_{(kl)(k'J)} &= \sum_{k_1,l_1\in S} b^n_{kk_1} b^n_{ll_1}\theta^n_{k_1 l_1}\varsigma^n_{k_1 l_1}(k'), \\
z^n_{(kJ)(k'l')} &= \sum_{k_1,l_1\in S} b^n_{kk_1} q^n_{k_1 l_1}(\tilde{p}^n)(1-\theta^n_{k_1 l_1})\sigma_{k_1 l_1}(k',l'), \\
z^n_{(kl)(k'l')} &= \sum_{k_1,l_1\in S} b^n_{kk_1} b^n_{ll_1}(1-\theta^n_{k_1 l_1})\sigma^n_{k_1 l_1}(k',l'). \qquad (24)
\end{aligned}$$

(3) For λ-almost every i and λ-almost every j, the Markov chains $\{\beta^n_i\}^\infty_{n=0}$ and $\{\beta^n_j\}^\infty_{n=0}$ are independent.

(4) For P-almost every $\omega \in \Omega$, the cross-sectional extended-type process $\{\beta^n_\omega\}^\infty_{n=0}$ is a Markov chain[18] with transition matrix z^n at time $n-1$.

[18] For a given sample realization $\omega \in \Omega$, $\{\beta^n_\omega\}^\infty_{n=0}$ is defined on the agent space $(I, \mathcal{I}, \lambda)$, which is a probability space itself. Thus, $\{\beta^n_\omega\}^\infty_{n=0}$ can be viewed as a discrete-time process.

(5) For P-almost all $\omega \in \Omega$, at each time period $n \geq 1$, the realized cross-sectional type distribution after random mutation $\lambda(\bar{\beta}^n_\omega)^{-1}$ is equal to its expectation $\tilde{p}^n$, and the realized cross-sectional type distribution at the end of the period n, $\hat{p}^n(\omega) = \lambda(\beta^n_\omega)^{-1}$, is equal to its expectation $\mathbb{E}(\hat{p}^n)$.

(6) If there is some fixed $\ddot{p}^0 \in \hat{\Delta}$ that is the probability distribution of the initial extended type β^0_i of agent i for λ-almost every i, then for λ-almost every i the Markov chain $\beta_i = \{\beta^n_i\}_{n=0}^\infty$ has the sample-path probability distribution $\xi = \ddot{p}^0 \otimes_{n=1}^\infty z^n$ on the space $\hat{S}^\infty$. Moreover, in this case, $\xi = \lambda(\beta_\omega)^{-1}$ for P-almost every ω. That is, for any subset $A = \prod_{n=0}^\infty A_n \subset \hat{S}^\infty$ of sample paths, the probability $\xi(A)$ of the event

$$\{\beta_i \in A\} = \{\beta^0_i \in A_0, \beta^1_i \in A_1, \ldots\}$$

is equal, for P-almost every $\omega \in \Omega$, to the fraction $\lambda(\{i : \beta_i(\omega) \in A\})$ of agents whose extended type process has a sample path in A in state ω.

For the time-homogenous case, in which the parameters $(b, q, \theta, \sigma, \varsigma)$ do not depend on the time period $n \geq 1$), the following proposition shows the existence of a stationary extended type distribution.

Proposition 3 *Suppose that the parameters $(b, q, \theta, \sigma, \varsigma)$ are time homogeneous. Then there exists an extended-type distribution $\hat{p}^* \in \hat{\Delta}$ that is a stationary distribution for any MCI dynamical system $\mathbb{D}$ with parameters $(b, q, \theta, \sigma, \varsigma)$, in the sense that:*

(1) For every $n \geq 0$, the realized cross-sectional extended-type distribution $\hat{p}^n$ at time n is $\hat{p}^*$ P-almost surely;

(2) All of the relevant Markov chains in Theorem 4 are time homogeneous with a constant transition matrix z^1 having $\hat{p}^*$ as a fixed point;

(3) If the initial extended type process β^0 is i.i.d. across agents, then, for λ-almost every i, the extended type distribution of agent i at any period $n \geq 0$ is $P\left(\beta^n_i\right)^{-1} = \hat{p}^*$.

A.4 Existence of MCI dynamic directed random matching with enduring partnerships

The following theorem provides for the existence of a Markov conditionally independent (MCI) dynamical system with random mutation, random matching, and match-induced type changes with break-up. Theorem 3 is a special case.

Theorem 5 *For any primitive model parameters* $(b, q, \theta, \sigma, \varsigma)$ *and for any extended type distribution* $\ddot{p}^0 \in \hat{\Delta}$, *there exists a Fubini extension* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ *on which is defined a dynamical system* $\mathbb{D} = (\Pi^n)_{n=0}^{\infty}$ *with random mutation, random matching, and match-induced type changes with break-up, that is Markov conditionally independent with these parameters* $(b, q, \theta, \sigma, \varsigma)$, *and with the initial extended type distribution* $\hat{p}^0$ *being* $\ddot{p}^0$ *with probability one. These properties can be achieved with an initial condition* Π^0 *that is deterministic, or alternatively with an initial extended type* β^0 *that is i.i.d. across agents.*[19]

In the next proposition, we show that the agent space $(I, \mathcal{I}, \lambda)$ in Theorem 5 can be an extension of the classical Lebesgue unit interval $(L, \mathcal{L}, \chi)$. That is, we can take $I = L = [0, 1]$ with a σ-algebra $\mathcal{I}$ that contains the Lebesgue σ-algebra $\mathcal{L}$, and so that the restriction of λ to $\mathcal{L}$ is the Lebesgue measure χ.

Proposition 4 *Fixing any model parameters* $(b, q, \theta, \sigma, \varsigma)$ *and any initial cross-sectional extended type distribution* $\ddot{p}^0 \in \hat{\Delta}$, *there exists a Fubini extension* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ *such that:*

(1) The agent space $(I, \mathcal{I}, \lambda)$ *is an extension of the Lebesgue unit interval* $(L, \mathcal{L}, \chi)$.

(2) There is defined on the Fubini extension a dynamical system $\mathbb{D} = (\Pi^n)_{n=0}^{\infty}$ *that is Markov conditionally independent with the parameters* $(b, q, \theta, \sigma, \varsigma)$, *where the initial extended type distribution* $\hat{p}^0$ *is* $\ddot{p}^0$ *with probability one.*

(3) These properties can be achieved with an initial condition Π^0 *that is deterministic, or alternatively with an initial extended type process* β^0 *that is i.i.d. across agents.*

A.5 Matching in labor markets with multi-period employment episodes

This example is taken from Andolfatto (1996), whose Section 1 considers a discrete-time labor-market-search model. The agents are workers and firms. Each firm has a single job position. Section 2 of Andolfatto (1996) works with stationary distributions. We can use the model of dynamic directed random matching with enduring partnership developed in this section to capture the search process leading to Equation (1) of Andolfatto (1996) in the stationary setting.

The agent type space is $S = \{E, U, A, V, D\}$. Here, E and U represent respectively employed workers and unemployed workers while A, V and D represent active, vacant and

[19]This means that the process β^0 is essentially pairwise independent, and that β_i^0 has distribution $\ddot{p}^0$ for λ-almost every agent i.

dormant jobs respectively. Dormant job positions are neither matched with a worker nor immediately open. The proportion of agents that are workers is $w > 0$.

At the beginning of each period, each vacant firm may mutate to a dormant job and each dormant job may mutate to a vacant job. Let $\check{p}_{UJ}$ and $\check{p}_{VJ}$ be the respective proportions of unemployed workers and vacant firms after the mutation step. In the stationary setting, the quantity $M(\check{p}_{VJ}, e \cdot \check{p}_{UJ})$ of new job matches in a given period is governed by a continuous aggregate matching function $M : [0,1] \times \mathbb{R}_+ \to [0,1]$ that incorporates[20] the search effort e applied by each worker seeking employment with $M(\check{p}_{VJ}, e \cdot \check{p}_{UJ}) \le \min\{\check{p}_{VJ}, \check{p}_{UJ}\}$. Job-worker pairs that have existed for at least one period are assumed to break up with probability $\bar{\theta}$ in each period. Newly formed pairs cannot break up in the current period. While a job-worker pair maintain their partnership, their current types (A, E) do not change. On the other hand, if they break up, the job becomes vacant and the worker becomes unemployed.

Equation (1) in Andolfatto (1996) in the stationary setting is

$$E^* = (1 - \bar{\theta})E^* + M(V^*, e \cdot (w - E^*)), \tag{25}$$

where E^* and V^* are the respective fractions of employed workers and vacant jobs in the particular case.[21]

Viewed in terms of our model, the corresponding time-independent parameters are given as follows. Vacant firms could mutate to dormant, and vice versa. Workers and active firms do not mutate. For any k and l in S, let

$$b_{kl} = \begin{cases} \frac{1-w-E^*-V^*}{1-w-E^*} & \text{if } k = V \text{ or } D \text{ and } l = D \\ \frac{V^*}{1-w-E^*} & \text{if } k = V \text{ or } D \text{ and } l = V \\ \delta_k(l) & \text{otherwise.} \end{cases}$$

Matching occurs only between unemployed workers and vacant jobs. The matching probabilities are defined as follows. For any k and l in S, define

$$q_{kl}(\check{p}) = \begin{cases} \frac{M(\check{p}_{VJ}, e \cdot \check{p}_{UJ})}{\check{p}_{UJ}} & \text{if } (k,l) = (U,V) \text{ and } \check{p}_{UJ} > 0 \\ \frac{M(\check{p}_{VJ}, e \cdot \check{p}_{UJ})}{\check{p}_{VJ}} & \text{if } (k,l) = (V,U) \text{ and } \check{p}_{UJ} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Next, we consider the step of type changing with break-up. For any $k, l, r, s \in S$, we have

$$\theta_{kl} = \begin{cases} \bar{\theta} & \text{if } (k,l) = (E,A) \text{ or } (A,E) \\ 0 & \text{otherwise;} \end{cases} \tag{26}$$

[20]The mass of workers is assumed to be one in Andolfatto (1996). Since the matching function in Andolfatto (1996) is assumed to have constant returns to scale, one can re-scale the total worker-firm population to be one, with a proportion w of agents being workers.

[21]See (P6′) on page 120 of Andolfatto (1996) for the steady state equation with the Cobb-Douglas matching function.

$$\sigma_{kl}(r,s)=\begin{cases}\delta_E(r)\delta_A(s) & \text{if } k=U \text{ and } l=V\\ \delta_A(r)\delta_E(s) & \text{if } k=V \text{ and } l=U\\ \delta_k(r)\delta_l(s) & \text{otherwise;}\end{cases} \tag{27}$$

$$\varsigma_{kl}(r)=\begin{cases}\delta_U(r) & \text{if } k=E \text{ and } l=A\\ \delta_V(r) & \text{if } k=A \text{ and } l=E\\ \delta_k(r) & \text{otherwise.}\end{cases} \tag{28}$$

Equation (26) means that an employed worker has probability $\bar{\theta}$ of losing her job. When two agents are newly matched in the current period, the worker-firm types change from (U,V) to (E,A). For those paired agents who were matched in a previous period, their types do not change while they stay together. Finally, the worker-firm pair of types from (E,A) to (U,V) when they break up. Equations (27) and (28) express these ideas.

Taking the equilibrium search effort e as given, Theorem 4 and Proposition 3 imply that any stationary type distribution satisfies

$$\hat{p}^*_{EA}=\Gamma(\hat{p}^*)_{EA}. \tag{29}$$

We take a stationary type distribution $\hat{p}^*$ corresponding to the given fractions of employed workers and vacant jobs E^* and V^* as in Equation (25), which means that $\hat{p}^*_{EA}=E^*$ and $\hat{p}^*_{VJ}=V^*$. By the formulas above the statement of Theorem 4, we obtain that

$$\Gamma_{EA}(\hat{p}^*)=\tilde{p}_{EA}(1-\bar{\theta})+\tilde{p}_{UJ}q_{UV}(\tilde{p}),$$

$$\tilde{p}_{UJ}=\hat{p}^*_{UJ}=w-\hat{p}^*_{EA}=w-E^*,$$

$$\tilde{p}_{VJ}=\hat{p}^*_{VJ}b_{VV}+\hat{p}^*_{DJ}b_{DV}=\hat{p}^*_{VJ}b_{VV}+(1-w-\hat{p}^*_{EA}-\hat{p}^*_{VJ})b_{DV}=V^*.$$

Substituting the above terms into Equation (29), we derive

$$E^*=\hat{p}^*_{EA}=(1-\bar{\theta})E^*+M\left(V^*,\, e\cdot(w-E^*)\right).$$

Thus the stationary distribution of employed workers and vacant jobs considered in Andolfatto (1996) can be derived from our model of dynamic directed random matching with enduring partnership with appropriate parameters.

B Proofs

The main existence results in this paper are Theorems 1 and 5, which are proved in Subsections B.1 and B.3 respectively. The proofs of Theorem 4 and Proposition 3 will be given in Subsection B.2. Subsection B.4 presents the proofs of Propositions 2 and 4.

In Subsections B.1 and B.3, nonstandard analysis is used extensively. In particular, the space of agents used in those two subsections will be based on a hyperfinite Loeb counting probability space $(I, \mathcal{I}, \lambda)$ that is the Loeb space (see Loeb and Wolff (2015)) of the internal probability space $(I, \mathcal{I}_0, \lambda_0)$, where $I = \{1, \ldots, \hat{M}\}$, $\hat{M}$ is an unlimited hyperfinite integer in ${}^*\mathbb{N}_\infty$, $\mathcal{I}_0$ its internal power set, and $\lambda_0(A) = |A|/|I|$ for any $A \in \mathcal{I}_0$ (that is, λ_0 is the internal counting probability measure on I).

We shall also need to work with some hyperfinite internal probability space as the sample space in Subsections B.1 and B.3. A general hyperfinite internal probability space is an ordered triple $(\Theta, \mathcal{A}_0, \tau_0)$, where $\Theta = \{\vartheta_1, \vartheta_2, \ldots, \vartheta_\gamma\}$, for some unlimited hyperfinite natural number γ, $\mathcal{A}_0$ is the internal power set on Θ, and $\tau_0(B) = \sum_{1 \le j \le \gamma, \vartheta_j \in B} \tau_0(\{\vartheta_j\})$ for any $B \in \mathcal{A}_0$. When the weights $\tau_0(\{\vartheta_j\}), 1 \le j \le \gamma$ are all infinitesimals, $(\Theta, \mathcal{A}_0, \tau_0)$ is said to be atomless, and its Loeb space $(\Theta, \mathcal{A}, \tau)$, as a standard probability space, is atomless in the usual sense. Keisler's Fubini Theorem (see (Loeb and Wolff, 2015, p. 214)) shows that the Loeb space $(I \times \Theta, \mathcal{I} \boxtimes \mathcal{A}, \lambda \boxtimes \tau)$ of the internal product probability space $(I \times \Theta, \mathcal{I}_0 \otimes \mathcal{A}_0, \lambda_0 \otimes \tau_0)$ is a Fubini extension.

B.1 Proof of Theorem 1

As mentioned in the beginning of Appendix B, let $I = \{1, \ldots, \hat{M}\}$ be a hyperfinite set with $\hat{M}$ an unlimited hyperfinite integer in ${}^*\mathbb{N}_\infty$, $\mathcal{I}_0$ the internal power set on I, λ_0 the internal counting probability measure on $\mathcal{I}_0$. The corresponding Loeb counting probability space $(I, \mathcal{I}, \lambda)$ will be our space of agents. In the setting of directed random matching as in Theorem 1, all agents are initially unmatched. On the other hand, when one considers dynamic directed random matching with enduring partnership as in Theorem 5, only the unmatched agents will conduct directed random searches for counterparties while those existing paired agents will not participate in the search process. The following lemma will be used to prove both Theorems 1 and 5.

Lemma 1 *As above, let $(I, \mathcal{I}_0, \lambda_0)$ be the hyperfinite internal counting probability space with its Loeb space $(I, \mathcal{I}, \lambda)$. Then, there exists a hyperfinite internal set Ω with its internal power set $\mathcal{F}_0$ such that for any initial internal type function α^0 from I to S and initial internal partial matching π^0 from I to $I \cup \{J\}$ with $g^0 = \alpha^0 \circ \pi^0$ internal extended type distribution $\hat{\rho} = \lambda_0 \left(\alpha^0, g^0\right)^{-1}$, and for any internal matching probability function q from $S \times S$ to ${}^*\mathbb{R}_+$ with $\sum_{r \in S} q_{kr} \le 1$ and $\hat{\rho}_{kJ} q_{kl} \simeq \hat{\rho}_{lJ} q_{lk}$ (i.e., $\hat{\rho}_{kJ} q_{kl} - \hat{\rho}_{lJ} q_{lk}$ is an infinitesimal) for any $k, l \in S$, there exists an internal random matching π from $I \times \Omega$ to $I \cup \{J\}$ and an internal probability measure P_0 on $(\Omega, \mathcal{F}_0)$ with the following properties.*

(i) Let $H = \{i : \pi^0(i) \neq J\}$. Then $P_0\left(\{\omega \in \Omega : \pi_\omega(i) = \pi^0(i) \text{ for any } i \in H\}\right) = 1$.

(ii) Let g be the internal mapping from $I \times \Omega$ to $S \cup \{J\}$, defined by $g(i, \omega) = \alpha^0(\pi(i, \omega))$ for

any $(i,\omega) \in I \times \Omega$. Then, for any $k, l \in S$, $P_0(g_i = l) \simeq q_{kl}$ for λ-almost every agent $i \in I$ satisfying $\alpha^0(i) = k$ and $\pi^0(i) = J$.

(iii) *Denote the corresponding Loeb probability spaces of the internal probability spaces $(\Omega, \mathcal{F}_0, P_0)$ and $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes P_0)$ respectively by $(\Omega, \mathcal{F}, P)$ and $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. The mapping g is an essentially pairwise independent process from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to S.*

To reflect their dependence on (α^0, π^0, q), π and P_0 will also be denoted by $\pi_{(\alpha^0,\pi^0,q)}$ and $P_{(\alpha^0,\pi^0,q)}$.

Proof. For each $k \in S$, let $\eta_k = 1 - \sum_{r \in S} q_{kr}$, and $I_k = \{i \in I : \alpha^0(i) = k, \pi^0(i) = J\}$. For each agent $i \in I_k$, define a probability ζ_i on $S \cup \{J\}$ such that $\zeta_i(l) = q_{kl}$ for $l \in S$ and $\zeta_i(J) = \eta_k$. For each agent $i \in I$ such that $\pi^0(i) \neq J$, define a probability ζ_i on $S \cup \{J\}$ such that $\zeta_i(l) = \delta_J(l)$ for $l \in S \cup \{J\}$. Let $\Omega_0 = (S \cup \{J\})^I$ be the internal set of all the internal functions from I to $S \cup \{J\}$, and μ_0 the internal product probability measure $\Pi_{i \in I} \zeta_i$ on $(\Omega_0, \mathcal{A}_0)$.

Let $\Omega_1 = \{A_1 \times \cdots \times A_{K^2} : A_k \subseteq I$ and A_k is internal, where $1 \leq k \leq K^2\}$. For each $\omega_0 \in \Omega_0$, $k, l \in S$, let $\bar{A}^{\omega_0}_{kl} = \{i \in I_k : \omega_0(i) = l\}$, and $\bar{B}^{\omega_0}_k = \{i \in I_k : \omega_0(i) = J\}$. For $k, l \in S$ with $k \neq l$, let

$$C^{\omega_0}_{kl} = \{A : A \subseteq \bar{A}^{\omega_0}_{kl}, A \text{ is internal and } |A| = min\{|\bar{A}^{\omega_0}_{kl}|, |\bar{A}^{\omega_0}_{lk}|\}\}.$$

For $k \in S$, let $C^{\omega_0}_{kk} = \left\{\bar{A}^{\omega_0}_{kk} \backslash \{i\} : i \in \bar{A}^{\omega_0}_{kk}\right\}$ if $|\bar{A}^{\omega_0}_{kk}|$ is odd and $C^{\omega_0}_{kk} = \left\{\bar{A}^{\omega_0}_{kk}\right\}$ if $|\bar{A}^{\omega_0}_{kk}|$ is even. Denote the product space $\prod_{k,l \in S} C^{\omega_0}_{kl}$ by C^{ω_0}. For any $A^{\omega_0} \in C^{\omega_0}$, let $B^{\omega_0}_k = I_k \backslash (\bigcup_{l \in S} A^{\omega_0}_{kl})$, which is equal to $\bar{B}^{\omega_0}_k \cup \bigcup_{l \in S} \left(\bar{A}^{\omega_0}_{kl} \setminus A^{\omega_0}_{kl}\right)$. Let $B^{\omega_0} = \cup_{k=1}^K B^{\omega_0}_k$. Define an internal probability measure μ_1 on $\Omega_0 \times \Omega_1$ by letting $\mu_1(\omega_0, A) = \mu_0(\omega_0) \times \mu^{\omega_0}(A)$ for $\omega_0 \in \Omega_0$ and $A \in \Omega_1$, where μ^{ω_0} is the internal counting probability on C^{ω_0}, and $\mu^{\omega_0}(A) = 0$ for $A \notin C^{\omega_0}$. Note that we also use ω to represent a singleton set $\{\omega\}$.

Fix any $\omega_0 \in \Omega_0$ and $A^{\omega_0} \in C^{\omega_0}$. For each $k \in S$, let $\Omega^{\omega_0, A^{\omega_0}}_{kk}$ be the internal set of all the internal full matchings on $A^{\omega_0}_{kk}$, and $\mu^{\omega_0, A^{\omega_0}}_{kk}$ the internal counting probability measure on $\Omega^{\omega_0, A^{\omega_0}}_{kk}$. For $k, l \in S$ with $k < l$, let $\Omega^{\omega_0, A^{\omega_0}}_{kl}$ be the internal set of all the internal bijections from $A^{\omega_0}_{kl}$ to $A^{\omega_0}_{lk}$, and $\mu^{\omega_0, A^{\omega_0}}_{kl}$ the internal counting probability on $A^{\omega_0}_{kl}$. Let Ω_2 be the internal set of all the internal partial matching from I to $I \cup \{J\}$, and $\Omega^{\omega_0, A^{\omega_0}}_2$ the set of $\phi \in \Omega_2$, with

(i) the restriction $\phi|_H = \pi^0|_H$, where $H = \{i : \pi^0(i) \neq J\}$;

(ii) $\{i \in I_k : \phi(i) = J\} = B^{\omega_0}_k$ for each $k \in S$;

(iii) the restriction $\phi|_{A^{\omega_0}_{kk}} \in \Omega^{\omega_0, A^{\omega_0}}_{kk}$ for $k \in S$;

(iv) for $k, l \in S$ with $k < l$, $\phi|_{A^{\omega_0}_{kl}} \in \Omega^{\omega_0, A^{\omega_0}}_{kl}$.

Define an internal probability measure $\mu_2^{\omega_0, A^{\omega_0}}$ on Ω_2 such that

(i) for $\phi \in \Omega_2^{\omega_0, A^{\omega_0}}$, $\mu_2^{\omega_0, A^{\omega_0}}(\phi) = \prod_{1 \leq k \leq l \leq K} \mu_{kl}^{\omega_0, A^{\omega_0}}(\phi|_{A^{\omega_0}_{kl}})$;

(ii) $\phi \notin \Omega_2^{\omega_0, A^{\omega_0}}$, $\mu_2^{\omega_0, A^{\omega_0}}(\phi) = 0$.

Define an internal probability measure P_0 on $\Omega = \Omega_0 \times \Omega_1 \times \Omega_2$ with the internal power set $\mathcal{F}_0$ by letting

$$P_0\left((\omega_0, F, \omega_2)\right) = \begin{cases} \mu_1(\omega_0, F) \times \mu_2^{\omega_0, F}(\omega_2) & \text{if } F \in C^{\omega_0} \\ 0 & \text{otherwise.} \end{cases}$$

For $(i, \omega) \in I \times \Omega$, let $\pi(i, (\omega_0, F, \omega_2)) = \omega_2(i)$, and $g(i, \omega) = \alpha^0(\pi(i, \omega), \omega)$. Denote the corresponding Loeb probability spaces of the internal probability spaces $(\Omega, \mathcal{F}_0, P_0)$ and $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes P_0)$ respectively by $(\Omega, \mathcal{F}, P)$ and $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. Since π is an internal function from $I \times \Omega$ to $I \cup \{J\}$, it is $\mathcal{I} \boxtimes \mathcal{F}$-measurable.

Denote the internal set $\left\{(\omega_0, F, \omega_2) \in \Omega : \omega_0 \in \Omega_0, F \in C^{\omega_0}, \omega_2 \in \Omega_2^{\omega_0, F}\right\}$ by $\hat{\Omega}$. By the construction of P_0, it is clear that $P_0\left(\hat{\Omega}\right) = 1$. By its construction, it is clear that π is a random matching as in Definition 1 (iii) and satisfies part (i) of the lemma. It remains to prove parts (ii) and (iii) of the lemma.

Define an internal process f from $I \times \Omega$ to $S \cup \{J\}$ such that for any $(i, \omega) \in I \times \Omega$,

$$f(i, \omega) = \begin{cases} \omega_0(i) & \text{if } \pi^0(i) = J \\ \alpha^0(\pi^0(i)) & \text{if } \pi^0(i) \neq J. \end{cases}$$

It is clear that if $\alpha^0(i) = k$ and $\pi^0(i) = J$, then $P(f_i = J) = {}^\circ\eta_k$ and $P(f_i = l) = {}^\circ q_{kl}$, where ${}^\circ x$ is the standard part of a bounded hyperreal number $x \in {}^*\mathbb{R}$. It is also obvious that for $i \neq j$ in I, f_i and f_j are independent random variables on the sample space $(\Omega, \mathcal{F}, P)$. The exact law of large number as in Theorem 2.8 in Sun (2006)) implies that for P-almost all $\omega = (\omega_0, F, \omega_2) \in \Omega$, $\lambda(\{\alpha^0(i) = k, \pi^0(i) = J, \omega_0(i) = l\}) = {}^\circ\hat{\rho}_{kJ} \cdot {}^\circ q_{kl}$ holds for any $k, l \in S$, and $\lambda(\{\alpha^0(i) = k, \pi^0(i) = J, \omega_0(i) = J\}) = {}^\circ\hat{\rho}_{kJ} \cdot {}^\circ\eta_k$, which means that

$$\frac{|\bar{A}^{\omega_0}_{kl}|}{\hat{M}} \simeq \hat{\rho}_{kJ} q_{kl} \simeq \hat{\rho}_{lJ} q_{lk} \simeq \frac{|\bar{A}^{\omega_0}_{lk}|}{\hat{M}} \text{ and } \frac{|\bar{B}^{\omega_0}_{k}|}{\hat{M}} \simeq \hat{\rho}_{kJ} \eta_k. \tag{30}$$

Let $\tilde{\Omega}$ be the set of $\omega = (\omega_0, F, \omega_2) \in \Omega$ such that Equation (30) holds. Then, $P\left(\tilde{\Omega}\right) = 1$, and hence $P\left(\hat{\Omega} \cap \tilde{\Omega}\right) = 1$.

Fix any $\omega = (\omega_0, F, \omega_2) \in \hat{\Omega} \cap \tilde{\Omega}$; then $F = A^{\omega_0}$ for some $A^{\omega_0} \in C^{\omega_0}$ and $\omega_2 \in \Omega_2^{\omega_0, A^{\omega_0}}$. For any $k \neq l \in S$, we have

$$\frac{|A^{\omega_0}_{kl}|}{\hat{M}} = \min\left(\frac{|\bar{A}^{\omega_0}_{kl}|}{\hat{M}}, \frac{|\bar{A}^{\omega_0}_{lk}|}{\hat{M}}\right) \simeq \hat{\rho}_{kJ} q_{kl} \simeq \frac{|\bar{A}^{\omega_0}_{kl}|}{\hat{M}} \text{ and } \frac{|A^{\omega_0}_{kk}|}{\hat{M}} \simeq \frac{|\bar{A}^{\omega_0}_{kk}|}{\hat{M}} \simeq \hat{\rho}_{kJ} q_{kk}, \tag{31}$$

which also implies $\frac{|B_k^{\omega_0}|}{\hat{M}} \simeq \hat{\rho}_{kJ}\eta_k \simeq \frac{|\bar{B}_k^{\omega_0}|}{\hat{M}}$. For any $i \in I_k$, $i \in A_{kl}^{\omega_0}$ if and only if $\pi(\omega_0, A^{\omega_0}, \omega_2) = \omega_2(i) \in A_{lk}^{\omega_0}$; and $i \in B_k^{\omega_0}$ if and only if $\pi(\omega_0, A^{\omega_0}, \omega_2) = \omega_2(i) = J$. Hence, for the fixed $\omega = (\omega_0, A^{\omega_0}, \omega_2)$, and for any $k, l \in S$, we can obtain that if $i \in A_{kl}^{\omega_0} \subseteq \bar{A}_{kl}^{\omega_0}$, $f(i,\omega) = \omega_0(i) = l = \alpha^0(\omega_2(i)) = g(i,\omega)$; if $i \in \bar{B}_k^{\omega_0} \subseteq B_k^{\omega_0}$, $f(i,\omega) = \omega_0(i) = J = \alpha^0(\omega_2(i)) = g(i,\omega)$. For any $i \in I \backslash (\cup_{k\in S} I_k)$ which means $\pi^0(i) \neq J$, we can obtain that $f(i,\omega) = \alpha^0(\pi^0(i)) = \alpha^0(\pi(i,\omega)) = g(i,\omega)$. It is clear that the set $\{i \in I : f(i,\omega) \neq g(i,\omega)\}$ is a subset of $\bigcup_{l\in S}\left(\bar{A}_{kl}^{\omega_0} \setminus A_{kl}^{\omega_0}\right)$, which has λ-measure zero by Equation (31).

By the fact that $P\left(\hat{\Omega} \cap \tilde{\Omega}\right) = 1$, we know that for P-almost all $\omega \in \Omega$,

$$\lambda\left(i \in I : f(i,\omega) = g(i,\omega)\right) = 1.$$

Since $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a Fubini extension, the Fubini property implies that for λ-almost all $i \in I$, $g(i,\omega)$ is equal to $f(i,\omega)$ for P-almost all $\omega \in \Omega$. Hence g satisfies part (ii) of the lemma. Let $\tilde{I}$ be an $\mathcal{I}$-measurable set with $\lambda(\tilde{I}) = 1$ such that for any $i \in \tilde{I}$, $g_i(\omega) = f_i(\omega)$ for P-almost all $\omega \in \Omega$. Therefore, by the construction of f, we know that the collection of random variables $\{g_i\}_{i\in\tilde{I}}$ is mutually independent in the sense that any finitely many random variables from that collection are mutually independent. This also implies part (iii) of the lemma. ■

Proof of Theorem 1: We follow Lemma 1. Let α^0 be an internal type function from I to S such that $\lambda_0\left(\{\alpha^0(i) = k\}\right) \simeq p_k$ for any $k \in S$.[22] Let $\pi^0(i) = J$ for any $i \in I$. Given that matching probability function q from $S \times S$ to $\mathbb{R}_+$ with $\sum_{r\in S} q_{kr} \leq 1$ and $p_k q_{kl} = p_l q_{lk}$ for all $k, l \in S$, the condition $\hat{\rho}_{kJ} q_{kl} \simeq \hat{\rho}_{lJ} q_{lk}$ in the statement of Lemma 1 is obviously satisfied. It is clear that the random matching π and the probability measure P constructed in Lemma 1 satisfies all the conditions in Theorem 1. Let α be α^0. Then, α and π, which are defined on a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, are a type function and an independent directed random matching with respective parameters p and q.

B.2 Proofs of Theorem 4 and Proposition 3

Before proving Theorem 4, we need to prove a few lemmas. To prove that the agents' extended type processes are essentially pairwise independent in Lemma 3 below, we need the following elementary lemma, which is Lemma 5 in Duffie and Sun (2012).

Lemma 2 *Let ϕ_m be a random variable from $(\Omega, \mathcal{F}, P)$ to a finite space A_m, for $m = 1, 2, 3, 4$. If the random variables ϕ_3 and ϕ_4 are independent, and if, for all $a_1 \in A_1$ and $a_2 \in A_2$,*

$$P(\phi_1 = a_1, \phi_2 = a_2 \mid \phi_3, \phi_4) = P(\phi_1 = a_1 \mid \phi_3) P(\phi_2 = a_2 \mid \phi_4), \tag{32}$$

then the two pairs of random variables (ϕ_1, ϕ_3) and (ϕ_2, ϕ_4) are independent.

[22] For any given $p \in \Delta$, the atomless property of λ_0 implies the existence of such an α^0.

The following lemma is useful for applying the exact law of large numbers for discrete time processes in Theorem 2.16 of Sun (2006) to our setting.

Lemma 3 *Assume that the dynamical system $\mathbb{D}$ is Markov conditionally independent. Then, the discrete time processes $\{\beta_i^n\}_{n=0}^{\infty}, i \in I$, are essentially pairwise independent. In addition, for each fixed $n \geq 1$, the random variables $\bar{\beta}_i^n, i \in I$ ($\bar{\bar{\beta}}_i^n, i \in I$) are also essentially pairwise independent.*

Proof. Let E be the set of all $(i,j) \in I \times I$ such that Equations (21), (22) and (23) hold for all $n \geq 1$. Then, by grouping countably many null sets together, we obtain that for λ-almost all $i \in I$, λ-almost all $j \in I$, $(i,j) \in E$, i.e., for λ-almost all $i \in I$, $\lambda(E_i) = \lambda(\{j \in I : (i,j) \in E\}) = 1$.

We can use induction to prove that for any fixed $(i,j) \in E$, $(\beta_i^0, \ldots, \beta_i^n)$ and $(\beta_j^0, \ldots, \beta_j^n)$ are independent for $n \geq 0$, so are the pairs $\bar{\beta}_i^n$ and $\bar{\beta}_j^n$, $\bar{\bar{\beta}}_i^n$ and $\bar{\bar{\beta}}_j^n$ for $n \geq 1$. The case of $n = 0$ is simply the assumption of initial independence in Subsection A.2. Suppose that it is true for the case $n-1$, i.e., $(\beta_i^0, \ldots, \beta_i^{n-1})$ and $(\beta_j^0, \ldots, \beta_j^{n-1})$ are independent, so are the pairs $\bar{\beta}_i^{n-1}$ and $\bar{\beta}_j^{n-1}$, $\bar{\bar{\beta}}_i^{n-1}$ and $\bar{\bar{\beta}}_j^{n-1}$. Then, the Markov conditional independence condition and Lemma 2 imply that $(\beta_i^0, \ldots, \beta_i^{n-1}, \bar{\beta}_i^n)$ and $(\beta_j^0, \ldots, \beta_j^{n-1}, \bar{\beta}_j^n)$ are independent, so are the pairs $(\beta_i^0, \ldots, \beta_i^{n-1}, \bar{\beta}_i^n, \bar{\bar{\beta}}_i^n)$ and $(\beta_j^0, \ldots, \beta_j^{n-1}, \bar{\beta}_j^n, \bar{\bar{\beta}}_j^n)$, and $(\beta_i^0, \ldots, \beta_i^{n-1}, \bar{\beta}_i^n, \bar{\bar{\beta}}_i^n, \beta_i^n)$ and $(\beta_j^0, \ldots, \beta_j^{n-1}, \bar{\beta}_j^n, \bar{\bar{\beta}}_j^n, \beta_j^n)$. Hence, the random vectors $(\beta_i^0, \ldots, \beta_i^n)$ and $(\beta_j^0, \ldots, \beta_j^n)$ are independent for all $n \geq 0$, which means that $\{\beta_i^n\}_{n=0}^{\infty}$ and $\{\beta_j^n\}_{n=0}^{\infty}$ are independent. It is also clear that for each $n \geq 1$, the random variables $\bar{\beta}_i^n$ and $\bar{\beta}_j^n$ are independent, so are $\bar{\bar{\beta}}_i^n$ and $\bar{\bar{\beta}}_j^n$. The desired result follows. ■

The following lemma shows how to compute the expected cross-sectional type distributions $\mathbb{E}(\hat{p}^n)$ and $\mathbb{E}(\check{p}^n)$.

Lemma 4 *The following hold:*

1. *For each $n \geq 1$, $\mathbb{E}(\hat{p}^n) = \Gamma^n(\mathbb{E}(\hat{p}^{n-1}))$.*

2. *For each $n \geq 1$, the expected cross-sectional type distribution $\tilde{p}^n = \mathbb{E}(\check{p}^n)$ immediately after random mutation at time n, satisfies $\mathbb{E}(\check{p}_{kl}^n) = \sum_{k_1, l_1 \in S} \mathbb{E}(\hat{p}_{k_1 l_1}^{n-1}) b_{k_1 k}^n b_{l_1 l}^n$ and $\mathbb{E}(\check{p}_{kJ}^n) = \sum_{l \in S} \mathbb{E}(\hat{p}_{lJ}^{n-1}) b_{lk}^n$.*

Proof. Fix any $k, l \in S$. Equations (11) and (12) imply respectively that for any $k_1, l_1 \in S$,

$$P\left(\bar{\beta}_i^n = (k, J) \,|\, \beta_i^{n-1} = (k_1, l_1)\right) = 0, \text{ and } P\left(\bar{\beta}_i^n = (k, l) \,|\, \beta_i^{n-1} = (k_1, J)\right) = 0. \tag{33}$$

The Fubini property will be used extensively in the computations below. We shall illustrate its usage in Equation (34). It then follows from the Fubini property, and Equations (11) and (33) that

$$\begin{aligned}\tilde{p}^n_{kl} &= \int_\Omega \lambda\left(i \in I : \bar{\beta}^n_\omega(i) = (k,l)\right) dP(\omega) = \int_I P\left(\bar{\beta}^n_i = (k,l)\right) d\lambda(i) \\ &= \int_I \sum_{k_1,l_1 \in S} P\left(\bar{\beta}^n_i = (k,l), \beta^{n-1}_i = (k_1,l_1)\right) d\lambda(i) \\ &= \int_I \sum_{k_1,l_1 \in S} P\left(\bar{\beta}^n_i = (k,l) \,|\, \beta^{n-1}_i = (k_1,l_1)\right) P\left(\beta^{n-1}_i = (k_1,l_1)\right) d\lambda(i) \\ &= \sum_{k_1,l_1 \in S} \mathbb{E}(\hat{p}^{n-1}_{k_1 l_1}) b^n_{k_1 k} b^n_{l_1 l}. \end{aligned} \tag{34}$$

By Equations (12) and (33), we obtain that

$$\begin{aligned}\tilde{p}^n_{kJ} &= \int_I P\left(\bar{\beta}^n_i = (k,J)\right) d\lambda(i) = \int_I \sum_{k_1 \in S} P\left(\bar{\beta}^n_i = (k,J), \beta^{n-1}_i = (k_1,J)\right) d\lambda(i) \\ &= \int_I \sum_{k_1 \in S} P\left(\bar{\beta}^n_i = (k,J) \,|\, \beta^{n-1}_i = (k_1,J)\right) P\left(\beta^{n-1}_i = (k_1,J)\right) d\lambda(i) \\ &= \sum_{k_1 \in S} \int_I b^n_{k_1 k} P\left(\beta^{n-1}_i = (k_1,J)\right) d\lambda(i) \\ &= \sum_{k_1 \in S} \mathbb{E}(\hat{p}^{n-1}_{k_1 J}) b^n_{k_1 k}. \end{aligned} \tag{35}$$

By Lemma 3, $\bar{\beta}^n$ is essentially pairwise independent process. The exact law of large numbers in Corollary 2.9 of Sun (2006) implies that $\check{p}^n(\omega) = \mathbb{E}(\check{p}^n) = \tilde{p}^n$ for P-almost all $\omega \in \Omega$. Combining with Equations (15) and (16), we can obtain that for any $l \in S$,

$$P(\bar{\bar{g}}^n_i = l \,|\, \bar{\alpha}^n_i = k, \bar{g}^n_i = J) = q^n_{kl}\left(\tilde{p}^n\right), \text{ and } P(\bar{\bar{g}}^n_i = J \,|\, \bar{\alpha}^n_i = k, \bar{g}^n_i = J) = \eta^n_k\left(\tilde{p}^n\right). \tag{36}$$

It follows from Equations (18) and (19) that

$$\begin{aligned}\mathbb{E}(\hat{p}^n_{kl}) &= \int_I P\left(\beta^n_i = (k,l)\right) d\lambda(i) = \int_I \sum_{k_1,l_1 \in S} P\left(\beta^n_i = (k,l), \bar{\bar{\beta}}^n_i = (k_1,l_1)\right) d\lambda(i) \\ &= \int_I \sum_{k_1,l_1 \in S} P\left(\beta^n_i = (k,l) \,|\, \bar{\bar{\beta}}^n_i = (k_1,l_1)\right) P\left(\bar{\bar{\beta}}^n_i = (k_1,l_1)\right) d\lambda(i) \\ &= \int_I \sum_{k_1,l_1 \in S} (1-\theta^n_{k_1 l_1}) \sigma^n_{k_1 l_1}(k,l) P\left(\bar{\bar{\beta}}^n_i = (k_1,l_1)\right) d\lambda(i) \\ &= \sum_{k_1,l_1 \in S} (1-\theta^n_{k_1 l_1}) \sigma^n_{k_1 l_1}(k,l) \int_I P\left(\bar{\bar{\beta}}^n_i = (k_1,l_1)\right) d\lambda(i). \end{aligned} \tag{37}$$

Equations (14) and (36) imply that

$$\int_I P\left(\bar{\bar{\beta}}_i^n = (k,l)\right) d\lambda(i) = \int_I \sum_{k_1,l_1\in S} P\left(\bar{\bar{\beta}}_i^n = (k,l) \,|\, \bar{\beta}_i^n = (k_1,l_1)\right) P\left(\bar{\beta}_i^n = (k_1,l_1)\right) d\lambda(i)$$
$$+ \int_I \sum_{k_1\in S} P\left(\bar{\bar{\beta}}_i^n = (k,l) \,|\, \bar{\beta}_i^n = (k_1,J)\right) P\left(\bar{\beta}_i^n = (k_1,J)\right) d\lambda(i)$$
$$= \int_I P\left(\bar{\bar{\beta}}_i^n = (k,l) \,|\, \bar{\beta}_i^n = (k,l)\right) P\left(\bar{\beta}_i^n = (k,l)\right) d\lambda(i)$$
$$+ \int_I P\left(\bar{\bar{\beta}}_i^n = (k,l) \,|\, \bar{\beta}_i^n = (k,J)\right) P\left(\bar{\beta}_i^n = (k,J)\right) d\lambda(i)$$
$$= \tilde{p}_{kl}^n + q_{kl}^n\left(\tilde{p}^n\right)\tilde{p}_{kJ}^n. \tag{38}$$

By substituting Equation (38) into Equation (37), we can express $\mathbb{E}(\hat{p}_{kl}^n)$ in terms of $\mathbb{E}(\check{p}^n)$:

$$\mathbb{E}(\hat{p}_{kl}^n) = \sum_{k_1,l_1\in S} \tilde{p}_{k_1l_1}^n(1-\theta_{k_1l_1}^n)\sigma_{k_1l_1}^n(k,l) + \sum_{k_1,l_1\in S} \tilde{p}_{k_1J}^n q_{k_1l_1}^n\left(\tilde{p}^n\right)(1-\theta_{k_1l_1}^n)\sigma_{k_1l_1}^n(k,l). \tag{39}$$

Similarly, Equations (18) and (20) imply the second and third identities while Equations (36) and (38) imply the last identity in the following equation:

$$\mathbb{E}(\hat{p}_{kJ}^n) = \int_I P\left(\beta_i^n = (k,J)\right) d\lambda(i)$$
$$= \int_I P\left(\beta_i^n = (k,J), \bar{\bar{\beta}}_i^n = (k,J)\right) d\lambda(i) + \int_I \sum_{k_1,l_1\in S} P\left(\beta_i^n = (k,J), \bar{\bar{\beta}}_i^n = (k_1,l_1)\right) d\lambda(i)$$
$$= \int_I P\left(\bar{\bar{\beta}}_i^n = (k,J)\right) d\lambda(i) + \int_I \sum_{k_1,l_1\in S} \theta_{k_1l_1}^n \varsigma_{k_1l_1}^n(k) P\left(\bar{\bar{\beta}}_i^n = (k_1,l_1)\right) d\lambda(i)$$
$$= \int_I P\left(\bar{\bar{\beta}}_i^n = (k,J) \,|\, \bar{\beta}_i^n = (k,J)\right) P\left(\bar{\beta}_i^n = (k,J)\right) d\lambda(i)$$
$$+ \sum_{k_1,l_1\in S} \theta_{k_1l_1}^n \varsigma_{k_1l_1}(k) \int_I P\left(\bar{\bar{\beta}}_i^n = (k_1,l_1)\right) d\lambda(i)$$
$$= \tilde{p}_{kJ}^n \eta_k^n(\tilde{p}^n) + \sum_{k_1,l_1\in S} \tilde{p}_{k_1l_1}^n \theta_{k_1l_1}^n \varsigma_{k_1l_1}^n(k) + \sum_{k_1,l_1\in S} \tilde{p}_{k_1J}^n q_{k_1l_1}^n(\tilde{p}^n)\theta_{k_1l_1}^n \varsigma_{k_1l_1}^n(k) \tag{40}$$

By combining Equations (34), (35),(39) and (40), we obtain that $\mathbb{E}(\hat{p}^n) = \Gamma^n(\mathbb{E}(\hat{p}^{n-1}))$. ■

The following lemma shows the Markov property of the agents' extended type processes.

Lemma 5 *Suppose the dynamical system $\mathbb{D}$ is Markov conditional independent. Then, for λ-almost all $i \in I$, the extended type process for agent i, $\{\beta_i^n\}_{n=0}^{\infty}$, is a Markov chain with transition matrix z^n at time $n-1$.*

Proof. Fix $n \geq 1$, by summing over all the $(k_2,l_2) \in \tilde{S}$ in Equation (21), we obtain that for λ-almost all $i \in I$,

$$P\left(\bar{\beta}_i^n = (k_1,l_1) \mid (\beta_i^t)_{t=0}^{n-1}\right) = P\left(\bar{\beta}_i^n = (k_1,l_1) \mid \beta_i^{n-1}\right). \tag{41}$$

By grouping countably many null sets together, we know that for λ-almost all $i \in I$, Equation (41) holds for all $n \geq 1$.

Similarly, Equations (22) and (23) imply that for λ-almost all $i \in I$,

$$P\left(\bar{\bar{\beta}}_i^n = (k_1, l_1) \,|\, \bar{\beta}_i^n, (\beta_i^t)_{t=0}^{n-1}\right) = P\left(\bar{\bar{\beta}}_i^n = (k_1, l_1) \,|\, \bar{\beta}_i^n\right),$$
$$P\left(\beta_i^n = (k_1, l_1) \,|\, \bar{\bar{\beta}}_i^n, (\beta_i^t)_{t=0}^{n-1}\right) = P\left(\beta_i^n = (k_1, l_1) \,|\, \bar{\bar{\beta}}_i^n\right)$$

hold for all $n \geq 1$. A simple computation shows that for λ-almost all $i \in I$,

$$P(\beta_i^n = (k_1, l_1) \mid \beta_i^0, \ldots, \beta_i^{n-1}) = P(\beta_i^n = (k_1, l_1) \mid \beta_i^{n-1})$$

for all $a_1 \in S$, $r_1 \in S \cup \{J\}$ and $n \geq 1$. Hence, for λ-almost all $i \in I$, agent i's extended type process $\{\beta_i^n\}_{n=0}^{\infty}$ is a Markov chain.

By combining Equations (34), (35) and (39), we can obtain that

$$\begin{aligned}
\mathbb{E}(\hat{p}_{kl}^n) \;&=\; \sum_{k_1, l_1, k' \in S} b_{k'k_1} q_{k_1 l_1}^n(\tilde{p}^n)(1 - \theta_{k_1 l_1}^n)\sigma_{k_1 l_1}^n(k, l)\mathbb{E}(\hat{p}_{k'J}^{n-1}) \\
&\quad + \sum_{k_1, l_1, k', l' \in S} b_{k'k_1}^n b_{l'l_1}^n (1 - \theta_{k_1 l_1}^n)\sigma_{k_1 l_1}^n(k, l)\mathbb{E}(\hat{p}_{k'l'}^{n-1}),
\end{aligned}$$

Since the transition probabilities $z_{(k'l')(kl)}^n$ and $z_{(k'J)(kl)}^n$ from time $n-1$ to time n are the respective coefficients of $\mathbb{E}(\hat{p}_{k'l'}^{n-1})$ and $\mathbb{E}(\hat{p}_{k'J}^{n-1})$ for any $k, l, k', l' \in S$, we can obtain that

$$\begin{aligned}
z_{(k'l')(kl)}^n \;&=\; \sum_{k_1, l_1 \in S} b_{k'k_1}^n b_{l'l_1}^n (1 - \theta_{k_1 l_1}^n)\sigma_{k_1 l_1}^n(k, l), \\
z_{(k'J)(kl)}^n \;&=\; \sum_{k_1, l_1 \in S} b_{k'k_1} q_{k_1 l_1}^n(\tilde{p}^n)(1 - \theta_{k_1 l_1}^n)\sigma_{k_1 l_1}^n(k, l),
\end{aligned}$$

which follow the corresponding formulas in Equation (24). Similarly, by combining Equations (34), (35) and (40), we can obtain that

$$\begin{aligned}
\mathbb{E}(\hat{p}_{kJ}^n) \;&=\; \sum_{k' \in S} b_{k'k}^n \eta_k^n(\tilde{p}^n)\mathbb{E}(\hat{p}_{k'J}^{n-1}) + \sum_{k_1, l_1, k', l' \in S} b_{k'k_1}^n b_{l'l_1}^n \theta_{k_1 l_1}^n \varsigma_{k_1 l_1}^n(k)\mathbb{E}(\hat{p}_{k'l'}^{n-1}) \\
&\quad + \sum_{k_1, l_1, k' \in S} b_{k'k_1}^n q_{k_1 l_1}^n(\tilde{p}^n)\theta_{k_1 l_1}^n \varsigma_{k_1 l_1}^n(k)\mathbb{E}(\hat{p}_{k'J}^{n-1})
\end{aligned}$$

Since the transition probabilities $z_{(k'l')(kJ)}^n$ and $z_{(k'J)(kJ)}^n$ from time $n-1$ to time n are the respective coefficients of $\mathbb{E}(\hat{p}_{k'l'}^{n-1})$ and $\mathbb{E}(\hat{p}_{k'J}^{n-1})$ for any $k, k', l' \in S$, we can obtain that

$$\begin{aligned}
z_{(k'l')(kJ)}^n \;&=\; \sum_{k_1, l_1 \in S} b_{k'k_1}^n b_{l'l_1}^n \theta_{k_1 l_1}^n \varsigma_{k_1 l_1}^n(k), \\
z_{(k'J)(kJ)}^n \;&=\; b_{k'k}^n \eta_k^n(\tilde{p}^n) + \sum_{k_1, l_1 \in S} b_{k'k_1}^n q_{k_1 l_1}^n(\tilde{p}^n)\theta_{k_1 l_1}^n \varsigma_{k_1 l_1}^n(k),
\end{aligned}$$

which follow the corresponding formulas in Equation (24). ∎

Now, for each $n \geq 1$, we view each β^n as a random variable on $I \times \Omega$. Then $\{\beta^n\}_{n=0}^{\infty}$ is a discrete-time stochastic process.

Lemma 6 *Assume that the dynamical system $\mathbb{D}$ is Markov conditionally independent. Then, $\{\beta^n\}_{n=0}^{\infty}$ is also a Markov chain with transition matrix z^n at time $n-1$.*

Proof. We can compute the transition matrix of $\{\beta^n\}_{n=0}^{\infty}$ at time $n-1$ by using Lemma 5 and the Fubini property. Fix any $k_1, k_2 \in S$ and any $l_1, l_2 \in S \cup \{J\}$. We have

$$\begin{aligned}
&(\lambda \boxtimes P)(\beta^n = (k_2, l_2), \beta^{n-1} = (k_1, l_1)) \\
= \;& \int_I P(\beta_i^n = (k_2, l_2) \mid \beta^{n-1} = (k_1, l_1)) P(\beta_i^{n-1} = (k_1, l_1)) \, d\lambda(i) \\
= \;& \int_I z^n_{(k_1 l_1)(k_2 l_2)} P(\beta^{n-1} = (k_1, l_1)) \, d\lambda(i) \\
= \;& z^n_{(k_1 l_1)(k_2 l_2)} \cdot (\lambda \boxtimes P)(\beta^{n-1} = (k_1, l_1)),
\end{aligned} \tag{42}$$

which implies that $(\lambda \boxtimes P)(\beta^n = (k_2, l_2) \mid \beta^{n-1} = (k_1, l_1)) = z^n_{(k_1 l_1)(k_2 l_2)}$.

Next, for any $n \geq 1$, and for any $(a^0, \ldots, a^{n-2}) \in (S \times (S \cup \{J\}))^{n-1}$, we have

$$\begin{aligned}
&(\lambda \boxtimes P)\left((\beta^0, \ldots, \beta^{n-2}) = (a^0, \ldots, a^{n-2}), \beta^{n-1} = (k_1, l_1), \beta^n = (k_2, l_2)\right) \\
= \;& \int_I P\left((\beta_i^0, \ldots, \beta_i^{n-2}) = (a^0, \ldots, a^{n-2}), \beta_i^{n-1} = (k_1, l_1), \beta_i^n = (k_2, l_2)\right) \, d\lambda(i) \\
= \;& \int_I P\left(\beta_i^n = (k_2, l_2) \mid \beta_i^{n-1} = (k_1, l_1)\right) P\left((\beta_i^0, \ldots, \beta_i^{n-2}) = (a^0, \ldots, a^{n-2}), \beta_i^{n-1} = (k_1, l_1)\right) \, d\lambda(i) \\
= \;& z^n_{(k_1 l_1)(k_2 l_2)} \cdot (\lambda \boxtimes P)((\beta^0, \ldots, \beta^{n-2}) = (a^0, \ldots, a^{n-2}), \beta^{n-1} = (k_1, l_1)),
\end{aligned} \tag{43}$$

which implies that

$$(\lambda \boxtimes P)(\beta^n = (k_2, l_2) \mid (\beta^0, \ldots, \beta^{n-2}) = (a^0, \ldots, a^{n-2}), \beta^{n-1} = (k_1, l_1)) = z^n_{(k_1 l_1)(k_2 l_2)}.$$

Hence the discrete-time process $\{\beta^n\}_{n=0}^{\infty}$ is indeed a Markov chain with transition matrix z^n at time $n-1$. ∎

Proof of Theorem 4: Properties (1), (2), and (3) of the theorem are shown in Lemmas 4, 5, and 3 respectively.

By the exact law of large numbers for discrete time processes in Theorem 2.16 of Sun (2006), we know that for P-almost all $\omega \in \Omega$, $(\beta_\omega^0, \ldots, \beta_\omega^n)$ and $(\beta^0, \ldots, \beta^n)$ (viewed as random vectors) have the same distribution for all $n \geq 1$. Since, as noted in Lemma 6, $\{\beta^n\}_{n=0}^{\infty}$ is a Markov chain with transition matrix z^n at time $n-1$, so is $\{\beta_\omega^n\}_{n=0}^{\infty}$ for P-almost all $\omega \in \Omega$. Thus property (4) is shown.

Since the processes $\bar{\beta}^n$ and β^n are essentially pairwise independent as shown in Lemma 3, the exact law of large numbers in Corollary 2.9 of Sun (2006) implies that at time period n, for P-almost all $\omega \in \Omega$, the realized cross-sectional distribution after the random mutation, $\check{p}^n(\omega) = \lambda(\bar{\beta}^n_\omega)^{-1}$ is the expected cross-sectional distribution $\mathbb{E}(\check{p}^n)$, and the realized cross-sectional distribution at the end of period n, $\hat{p}^n(\omega) = \lambda(\beta^n_\omega)^{-1}$ is the expected cross-sectional distribution $\mathbb{E}(\hat{p}^n)$. Thus, property (5) is shown.

Assume that there exists $\ddot{p}^0 \in \hat{\Delta}$ such that $P\left(\beta^0_i\right)^{-1} = \ddot{p}^0$ holds for λ-almost every $i \in I$. The exact law of large numbers in Corollary 2.9 of Sun (2006) implies that $\ddot{p}^0 = \mathbb{E}(\hat{p}^0)$. For λ-almost all $i \in I$, since the transition matrix of $\{\beta^n_i\}_{n=1}^\infty$ is $\{z^n\}_{n=1}^\infty$, the Markov chains $\{\beta^n_i\}_{n=0}^\infty$ induce the same distribution on $\hat{S}^\infty$ as ξ. For P-almost all $\omega \in \Omega$, the Markov chains $\{\beta^n_\omega\}_{n=0}^\infty$ induce the same distribution on $\hat{S}^\infty$ as ξ. Thus, property (6) is shown. ■

Proof of Proposition 3: Given that the parameters $(b, q, \theta, \sigma, \varsigma)$ are time independent, the mapping Γ^n from $\hat{\Delta}$ to $\hat{\Delta}$ in Subsection A.3 is time independent, and will simply be denoted by Γ. By the continuity assumption in the sentence above Equation (7), $\hat{p}_{kJ} q^n_{kl}(\hat{p})$ is continuous in $\hat{p} \in \hat{\Delta}$ for any $k, l \in S$. For any $k_1, l_1 \in S$, since $\tilde{p}_{k_1 J} = \sum_{r \in S} \hat{p}_{rJ} b^n_{rk_1}$ is continuous in $\hat{p} \in \hat{\Delta}$, we can also obtain that $\tilde{p}_{k_1 J}\, q^n_{k_1 l_1}(\tilde{p})$ is continuous in $\hat{p} \in \hat{\Delta}$. Therefore, Γ is a continuous function from $\hat{\Delta}$ to itself. By Brower's Fixed Point Theorem, Γ has a fixed point $\hat{p}^*$. In this case, $\mathbb{E}(\hat{p}^n) = \hat{p}^*$, $z^n = z^1$ for all $n \geq 1$. Hence the Markov chains $\{\beta^n_i\}_{n=0}^\infty$ for λ-almost all $i \in I$, $\{\beta^n\}_{n=0}^\infty$, $\{\beta^n_\omega\}_{n=0}^\infty$ for P-almost all $\omega \in \Omega$ are time-homogeneous.

If the initial extended type process β^0 is i.i.d., then the extended type distribution of agent i at time $n = 0$ is $P\left(\beta^0_i\right)^{-1} = \hat{p}^*$ for λ-almost every $i \in I$. By (6) of Theorem 4, for any $n \geq 1$, β^n_i induce the same distribution on $\hat{S}$ for λ-almost all $i \in I$. Therefore, for any $n \geq 1$, $P\left(\beta^n_i\right)^{-1} = \hat{p}^*$ for λ-almost all $i \in I$.

B.3 Proof of Theorem 5

What we need to do is to construct sequences of internal transition probabilities, internal type functions, and internal random matchings. Since we need to consider random mutation, random matching and random type changing with break-up at each time period, three internal measurable spaces with internal transition probabilities will be constructed at each time period.

Let T_0 be the hyperfinite discrete time line $\{n\}_{n=0}^M$ and $(I, \mathcal{I}_0, \lambda_0)$ be the agent space, where $I = \{1, \ldots, \hat{M}\}$, $\mathcal{I}_0$ is the internal power set on I, λ_0 is the internal counting probability measure on $\mathcal{I}_0$, M and $\hat{M}$ are unlimited hyperfinite numbers in ${}^*\mathbb{N}_\infty$. We transfer the sequences of numbers $b^n, \theta^n, \sigma^n, \varsigma^n, n \in \mathbb{N}$ to the nonstandard universe to obtain $b^n, \theta^n, \sigma^n, \varsigma^n, n \in {}^*\mathbb{N}$. The transfer of the sequence of functions $q^n, n \in \mathbb{N}$ to the nonstandard universe is denoted by ${}^*q^n, n \in {}^*\mathbb{N}$. Then, for any $k, l \in S$, ${}^*q^n_{kl}$ is an internal function from ${}^*\hat{\Delta}$ to ${}^*[0,1]$. Let

$\hat{q}^n_{kl}(\hat{\rho}) = ({}^*q^n_{kl})(\hat{\rho})$ and $\hat{\eta}^n_k = 1 - \sum_{l\in S} \hat{q}^n_{kl}(\hat{\rho})$ for any $k, l \in S$ and $\hat{\rho} \in {}^*\hat{\Delta}$. Note that an object with an upper left star means the transfer of a standard object to the nonstandard universe.

We shall first consider the case of an initial condition Π^0 that is deterministic. Let $\{A_{kl}\}_{(k,l)\in\tilde{S}}$ be an internal partition of I such that $\frac{|A_{kl}|}{\hat{M}} \simeq \ddot{p}_{kl}$ for any $k \in S$ and $l \in S \cup \{J\}$, and $|A_{kl}| = |A_{lk}|$ and $|A_{kk}|$ are even for any $k, l \in S$. Let α^0 be an internal function from $(I, \mathcal{I}_0, \lambda_0)$ to S such that $\alpha^0(i) = k$ if $i \in \bigcup_{l\in S\cup\{J\}} A_{kl}$. Let π^0 be an internal partial matching from I to $I \cup \{J\}$ such that $\pi^0(i) = J$ on $\bigcup_{k\in S} A_{kJ}$, and the restriction $\pi^0|_{A_{kl}}$ is an internal bijection from A_{kl} to A_{lk} for any $k, l \in S$. Let $g^0(i) = \alpha^0(\pi^0(i))$. It is clear that $\lambda_0(\{i : \alpha^0(i) = k, g^0(i) = l\}) \simeq \ddot{p}^0_{kl}$ for any $k \in S$ and $l \in S \cup \{J\}$.

Suppose that the construction for the dynamical system $\mathbb{D}$ has been done up to time period $n-1 \in {}^*\mathbb{N}$. That is, $\{(\Omega_m, \mathcal{F}_m, Q_m)\}_{m=1}^{3n-3}$ and $\{\alpha^l, \pi^l\}_{l=0}^{n-1}$ have been constructed, where each Ω_m is a hyperfinite internal set with its internal power set $\mathcal{F}_m$, Q_m an internal transition probability from Ω^{m-1} to $(\Omega_m, \mathcal{F}_m)$, α^l an internal type function from $I \times \Omega^{3l-1}$ to the type space S, and π^l an internal random matching from $I \times \Omega^{3l}$ to $I \cup \{J\}$.[23] Here, $\Omega^m = \prod_{j=1}^m \Omega_j$, and $\{\omega_j\}_{j=1}^m$ will also be denoted by ω^m when there is no confusion. Denote the internal product transition probability $Q_1 \otimes Q_2 \otimes \cdots \otimes Q_m$ by Q^m, and $\otimes_{j=1}^m \mathcal{F}_j$ by $\mathcal{F}^m$ (which is simply the internal power set on Ω^m). Then, Q^m is the internal product of the internal transition probability Q_m with the internal probability measure Q^{m-1}.

We shall now consider the constructions for time n. We first work with the random mutation step. Let $\Omega_{3n-2} = S^I$ (the space of all internal functions from I to S) with its internal power set $\mathcal{F}_{3n-2}$. For each $i \in I$, $\omega^{3n-3} \in \Omega^{3n-3}$, if $\alpha^{n-1}(i, \omega^{3n-3}) = k$, define a probability measure $\gamma_i^{\omega^{3n-3}}$ on S by letting $\gamma_i^{\omega^{3n-3}}(l) = b^n_{kl}$ for each $l \in S$. Define an internal probability measure $Q_{3n-2}^{\omega^{3n-3}}$ on $(S^I, \mathcal{F}_{3n-2})$ to be the internal product measure $\prod_{i\in I} \gamma_i^{\omega^{3n-3}}$. Let $\bar{\alpha}^n : \left(I \times \prod_{m=1}^{3n-2} \Omega_m\right) \to S$ be such that $\bar{\alpha}^n\left(i, \omega^{3n-2}\right) = \omega_{3n-2}(i)$. Let $\bar{g}^n : \left(I \times \prod_{m=1}^{3n-2} \Omega_m\right) \to S \cup \{J\}$ be such that

$$\bar{g}^n\left(i, \omega^{3n-2}\right) = \bar{\alpha}^n(\pi^{n-1}(i, \omega^{3n-3}), \omega^{3n-2}).$$

Let $\check{\rho}^n_{\omega^{3n-2}} = \lambda_0\left(\bar{\alpha}^n_{\omega^{3n-2}}, \bar{g}^n_{\omega^{3n-2}}\right)^{-1}$ be the internal cross-sectional extended type distribution after random mutation.

Next, we consider the step of directed random matching. Let $(\Omega_{3n-1}, \mathcal{F}_{3n-1}) = (\bar{\Omega}, \bar{\mathcal{F}})$, where $(\bar{\Omega}, \bar{\mathcal{F}})$ is the measurable space constructed in the proof of Lemma 1. For any given $\omega^{3n-2} \in \Omega^{3n-2}$, the type function is $\bar{\alpha}^n_{\omega^{3n-2}}(\,\cdot\,)$ while the partial matching function is $\pi^{n-1}_{\omega^{3n-3}}(\,\cdot\,)$. We can construct an internal probability measure $Q_{3n-1}^{\omega^{3n-2}} = P_{\bar{\alpha}^n_{\omega^{3n-2}}, \pi^{n-1}_{\omega^{3n-3}}, \hat{q}^n(\check{\rho}^n_{\omega^{3n-2}})}$ and a directed random matching $\pi_{\bar{\alpha}^n_{\omega^{3n-2}}, \pi^{n-1}_{\omega^{3n-3}}, \hat{q}^n(\check{\rho}^n_{\omega^{3n-2}})}$ by Lemma 1. Let $\bar{\pi}^n : \left(I \times \prod_{m=1}^{3n-1} \Omega_m\right) \to$

[23] To handle the deterministic case at the initial step with $l = 0$ ($3l-1 = -1$ and $3l = 0$), one can let $\Omega^0 = \Omega^{-1}$ be a singleton set.

$I \cup \{J\}$ be such that

$$\bar{\pi}^n\left(i, \omega^{3n-1}\right) = \pi_{\bar{\alpha}^n_{\omega^{3n-2}}, \pi^{n-1}_{\omega^{3n-3}}, \hat{q}^n(\check{\rho}^n_{\omega^{3n-2}})}(i, \omega_{3n-1}),$$

$$\bar{\bar{g}}^n(i, \omega^{3n-1}) = \bar{\alpha}^n(\bar{\pi}^n(i, \omega^{3n-1}), \omega^{3n-2}).$$

Now, we consider the final step of random type changing with break-up for matched agents. Let $\Omega_{3n} = (S \times \{0,1\})^I$ with its internal power set $\mathcal{F}_{3n}$, where 0 represents "unmatched" and 1 represents "paired"; each point $\omega_{3n} = (\omega^1_{3n}, \omega^2_{3n}) \in \Omega_{3n}$ is an internal function from I to $S \times \{0,1\}$. Define a new type function $\alpha^n : (I \times \Omega^{3n}) \to S$ by letting $\alpha^n(i, \omega^{3n}) = \omega^1_{3n}(i)$. Fix $\omega^{3n-1} \in \Omega^{3n-1}$. For each $i \in I$, (1) if $\bar{\pi}^n(i, \omega^{3n-1}) = J$ (i is not paired after the matching step at time n), let $\tau_i^{\omega^{3n-1}}$ be the probability measure on the type space $S \times \{0,1\}$ that gives probability one to the type $\left(\bar{\alpha}^n(i, \omega^{3n-2}), 0\right)$ and zero for the rest; (2) if $\bar{\pi}^n(i, \omega^{3n-1}) \neq J$ (i is paired after the matching step at time n), $\bar{\alpha}^n(i, \omega^{3n-2}) = k$, $\bar{\pi}^n(i, \omega^{3n-1}) = j$ and $\bar{\alpha}^n(j, \omega^{3n-2}) = l$, define a probability measure $\tau_{ij}^{\omega^{3n-1}}$ on $(S \times \{0,1\}) \times (S \times \{0,1\})$ such that $\tau_{ij}^{\omega^{3n-1}}((k',1),(l',1)) = (1 - \theta^n_{kl})\sigma^n_{kl}(k',l')$ and $\tau_{ij}^{\omega^{3n-1}}((k',0),(l',0)) = \theta^n_{kl}\varsigma^n_{kl}(k')\varsigma^n_{lk}(l')$ for $k', l' \in S$, and zero for the rest. Let $A^n_{\omega^{3n-1}} = \{(i,j) \in I \times I : i < j,\ \bar{\pi}^n(i, \omega^{3n-1}) = j\}$ and $B^n_{\omega^{3n-1}} = \{i \in I : \bar{\pi}^n(i, \omega^{3n-1}) = J\}$. Define an internal probability measure $Q_{3n}^{\omega^{3n-1}}$ on $(S \times \{0,1\})^I$ to be the internal product measure

$$\prod_{i \in B^n_{\omega^{3n-1}}} \tau_i^{\omega^{3n-1}} \otimes \prod_{(i,j) \in A^n_{\omega^{3n-1}}} \tau_{ij}^{\omega^{3n-1}}.$$

Let

$$\pi^n(i, \omega^{3n}) = \begin{cases} J & \text{if } \bar{\pi}^n(i, \omega^{3n-1}) = J \text{ or } \omega^2_{3n}(i) = 0 \text{ or } \omega^2_{3n}(\bar{\pi}^n(i, \omega^{3n-1})) = 0 \\ \bar{\pi}^n(i, \omega^{3n-1}) & \text{otherwise.} \end{cases}$$

and $g^n(i, \omega^{3n}) = \alpha^n(\pi^n(i, \omega^{3n}), \omega^{3n})$. It is clear that π^n is a random matching and Equation (17) holds.

Keep repeating the construction. We can then construct a hyperfinite sequence of internal transition probabilities $\{(\Omega_m, \mathcal{F}_m, Q_m)\}_{m=1}^{3M}$ and a hyperfinite sequence of internal type functions and internal random matchings $\{(\alpha^n, \pi^n)\}_{n=0}^{M}$.

Let $(I \times \Omega^{3M}, \mathcal{I}_0 \otimes \mathcal{F}^{3M}, \lambda_0 \otimes Q^{3M})$ be the internal product probability space of $(I, \mathcal{I}_0, \lambda_0)$ and $(\Omega^{3M}, \mathcal{F}^{3M}, Q^{3M})$. Denote the Loeb spaces of $(\Omega^{3M}, \mathcal{F}^{3M}, Q^{3M})$ and the internal product $(I \times \Omega^{3M}, \mathcal{I}_0 \otimes \mathcal{F}^{3M}, \lambda_0 \otimes Q^{3M})$ by $(\Omega^{3M}, \mathcal{F}, P)$ and $(I \times \Omega^{3M}, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ respectively. For simplicity, let Ω^{3M} be denoted by Ω, Q^{3M} be denoted by P_0.

In the following, we will often work with functions or sets that are measurable in $(\Omega^m, \mathcal{F}^m, Q^m)$ or its Loeb space for some $m \leq 3M$, which may be viewed as functions or sets based on $(\Omega^{3M}, \mathcal{F}^{3M}, Q^{3M})$ or its Loeb space by allowing for dummy components for the tail part. We can thus continue to use P to denote the Loeb measure generated by Q^m for convenience. Since all the type functions, random matchings and the partners' type functions are internal in the relevant hyperfinite settings, they are all $\mathcal{I} \boxtimes \mathcal{F}$-measurable when viewed as functions on $I \times \Omega$.

For $n = 0$, the initial independence condition in the definition of Markov conditional independence in Subsection A.2 is trivially satisfied. Suppose that the Markov conditional independence are satisfied up to period $n-1 \in \mathbb{N}$. It remains to check the Markov conditional independence for period n.

For the mutation step in period n, fix any $(a_1, r_1), (a_2, r_2)$ and $k_1^t, l_1^t), (k_2^t, l_2^t)$, $t = 1, \ldots, n-1$ in $\tilde{S}$. For any agents i and j with $i \neq j$, we can obtain that

$$\begin{aligned}
&P\left(\bar{\beta}_i^n = (a_1, r_1), \bar{\beta}_j^n = (a_2, r_2), \beta_i^t = (k_1^t, l_1^t), \beta_j^t = (k_2^t, l_2^t), t = 1, \ldots, n-1\right) \\
&\simeq \int_{D_{ij}^{3n-3}} Q_{3n-2}^{\omega^{3n-3}} \left(\bar{\beta}^n(i, \omega^{3n-2}) = (a_1, r_1), \bar{\beta}^n(j, \omega^{3n-2}) = (a_2, r_2)\right) dQ^{3n-3}(\omega^{3n-3}) \\
&= \int_{\underline{D}_{ij}^{3n-3}} Q_{3n-2}^{\omega^{3n-3}} \left(\bar{\beta}^n(i, \omega^{3n-2}) = (a_1, r_1), \bar{\beta}^n(j, \omega^{3n-2}) = (a_2, r_2)\right) dQ^{3n-3}(\omega^{3n-3}) \\
&\quad + \int_{\overline{D}_{ij}^{3n-3}} Q_{3n-2}^{\omega^{3n-3}} \left(\bar{\beta}^n(i, \omega^{3n-2}) = (a_1, r_1), \bar{\beta}^n(j, \omega^{3n-2}) = (a_2, r_2)\right) dQ^{3n-3}(\omega^{3n-3}),
\end{aligned}$$

where

$$D_{ij}^{3n-3} = \{\omega^{3n-3} : \beta^t(i, \omega^{3t}) = (k_1^t, l_1^t), \beta^t(j, \omega^{3t}) = (k_2^t, l_2^t), t = 1, \ldots, n-1\},$$

$$\underline{D}_{ij}^{3n-3} = \{\omega^{3n-3} : \pi^{n-1}(i, \omega^{3n-3}) \neq j, \beta^t(i, \omega^{3t}) = (k_1^t, l_1^t), \beta^t(j, \omega^{3t}) = (k_2^t, l_2^t), t = 1, \ldots, n-1\},$$

$$\overline{D}_{ij}^{3n-3} = \{\omega^{3n-3} : \pi^{n-1}(i, \omega^{3n-3}) = j, \beta^t(i, \omega^{3t}) = (k_1^t, l_1^t), \beta^t(j, \omega^{3t}) = (k_2^t, l_2^t), t = 1, \ldots, n-1\}.$$

Fix any agent $i \in I$. It is clear that $\overline{D}_{ij}^{3n-3} \cap \overline{D}_{ij'}^{3n-3} = \emptyset$ for different j and j'. Then there are at most countably many $j \in I$ such that $P(\overline{D}_{ij}^{3n-3}) > 0$. Let $F_i^{3n-3} = \{j \in I : j \neq i, P(\overline{D}_{ij}^{3n-3}) = 0\}$; then $\lambda(F_i^{3n-3}) = 1$. Fix any $j \in F_i^{3n-3}$. The probability for agents i and j to be partners is zero at the end of period $n-1$. When agents i and j are not partners, their random extended types will be independent by the construction of $Q_{3n-2}^{\omega^{3n-3}}$. Hence, we can obtain that

$$\begin{aligned}
&P\left(\bar{\beta}_i^n = (a_1, r_1), \bar{\beta}_j^n = (a_2, r_2), \beta_i^t = (k_1^t, l_1^t), \beta_j^t = (k_2^t, l_2^t), t = 1, \ldots, n-1\right) \\
&\simeq \int_{\underline{D}_{ij}^{3n-3}} Q_{3n-2}^{\omega^{3n-3}} \left(\bar{\beta}^n(i, \omega^{3n-2}) = (a_1, r_1), \bar{\beta}^n(j, \omega^{3n-2}) = (a_2, r_2)\right) dQ^{3n-3}(\omega^{3n-3}) \\
&= \int_{\underline{D}_{ij}^{3n-3}} Q_{3n-2}^{\omega^{3n-3}} \left(\bar{\beta}^n(i, \omega^{3n-2}) = (a_1, r_1)\right) Q_{3n-2}^{\omega^{3n-3}} \left(\bar{\beta}^n(j, \omega^{3n-2}) = (a_2, r_2)\right) dQ^{3n-3}(\omega^{3n-3}) \\
&= \int_{\underline{D}_{ij}^{3n-3}} B_{k_1^{n-1} l_1^{n-1}}^{3n-2}(a_1, r_1) B_{k_2^{n-1} l_2^{n-1}}^{3n-2}(a_2, r_2) dQ^{3n-3}(\omega^{3n-3}) \\
&\simeq P(D_{ij}^{3n-3}) B_{k_1^{n-1} l_1^{n-1}}^{3n-2}(a_1, r_1) B_{k_2^{n-1} l_2^{n-1}}^{3n-2}(a_2, r_2),
\end{aligned}$$

where

$$B^{3n-2}_{kl}(r,s) = \begin{cases} b^n_{kr} b^n_{ls} & \text{if } l,s \in S \\ b^n_{kr} & \text{if } l = s = J \\ 0 & \text{otherwise.} \end{cases}$$

Thus, for λ-almost all agent $j \in I$,

$$\begin{aligned} & P\left(\bar{\beta}^n_i = (a_1, r_1), \bar{\beta}^n_j = (a_2, r_2) \,|\, \beta^t_i = (k^t_1, l^t_1), \beta^t_j = (k^t_2, l^t_2), t = 1, \ldots, n-1\right) \\ & = B^{3n-2}_{k^{n-1}_1 l^{n-1}_1}(a_1, r_1) B^{3n-2}_{k^{n-1}_2 l^{n-1}_2}(a_2, r_2). \end{aligned} \tag{44}$$

Note that for any $i \in I$,

$$\begin{aligned} & P(\bar{\beta}^n_i = (a_1, r_1), \beta^{n-1}_i = (k^{n-1}_1, l^{n-1}_1)) \simeq \int_{E^{3n-3}_i} Q^{\omega^{3n-3}}_{3n-2} \left(\bar{\beta}^n(i, \omega^{3n-2}) = (a_1, r_1)\right) dQ^{3n-3}(\omega^{3n-3}) \\ & = \int_{E^{3n-3}_i} B^{3n-2}_{k^{n-1}_1 l^{n-1}_1}(a_1, r_1) dQ^{3n-3}(\omega^{3n-3}) \simeq P(E^{3n-3}_i) B^{3n-2}_{k^{n-1}_1 l^{n-1}_1}(a_1, r_1), \end{aligned}$$

where $E^{3n-3}_i = \{\omega^{3n-3} : \beta^{n-1}(i, \omega^{3n-3}) = (k^{n-1}_1, l^{n-1}_1)\}$. Then, we have

$$P(\bar{\beta}^n_i = (a_1, r_1) \,|\, \beta^{n-1}_i = (k^{n-1}_1, l^{n-1}_1)) = B^{3n-2}_{k^{n-1}_1 l^{n-1}_1}(a_1, r_1). \tag{45}$$

Hence, Equations (11) and (12) in the definition of dynamical system are satisfied. By Equation (44), we can obtain for each $i \in I$, and for λ-almost all $j \in I$,

$$\begin{aligned} & P\left(\bar{\beta}^n_i = (a_1, r_1), \bar{\beta}^n_j = (a_2, r_2) \,|\, \beta^t_i = (k^t_1, l^t_1), \beta^t_j = (k^t_2, l^t_2), t = 1, \ldots, n-1\right) \\ & = P\left(\bar{\beta}^n_i = (a_1, r_1) \,|\, \beta^{n-1}_i = (k^{n-1}_1, l^{n-1}_1)\right) P\left(\bar{\beta}^n_j = (a_2, r_2) \,|\, \beta^{n-1}_j = (k^{n-1}_2, l^{n-1}_2)\right). \end{aligned} \tag{46}$$

Hence, Equation (21) in the definition of Markov conditional independence is satisfied.

For the random matching step in period n, fix any $(a_1, r_1), (a_2, r_2)$ in $S \times S$ and any $k^t_1, l^t_1), (k^t_2, l^t_2)$ in $\tilde{S}$ for $t = 1, \ldots, n-1$. Fix any $\omega^{3n-2} \in \Omega^{3n-2}$. Let $A^{\omega^{3n-3}} = \{i \in I : \pi^{n-1}_{\omega^{3n-3}}(i) \neq J\}$. By Lemma 1 (i), we know that

$$Q^{\omega^{3n-2}}_{3n-1}\left(\omega_{3n-1} \in \Omega_{3n-1} : \bar{\pi}^n\left(i, (\omega^{3n-2}, \omega_{3n-1})\right) = \pi^{n-1}\left(i, \omega^{3n-3}\right) \text{ for any } i \in A^{\omega^{3n-3}}\right) = 1,$$

which implies that Equation (13) holds.

Lemma 2 and Equation (46) imply that the extended type process $\bar{\beta}^n$ is essentially pairwise independent. It follows from the exact law of large numbers in Corollary 2.9 of Sun (2006) that for P-almost all $\omega^{3n-2} \in \Omega^{3n-2}$,

$$\check{\rho}^n(\omega^{3n-2}) \simeq \check{p}^n(\omega^{3n-2}) = \lambda\left(\bar{\beta}^n_{\omega^{3n-2}}\right)^{-1} = \mathbb{E}\left(\check{p}^n\left(\omega^{3n-2}\right)\right) = \tilde{p}^n \simeq \mathbb{E}\left(\check{\rho}^n\right). \tag{47}$$

Then Equation (15) is equivalent to

$$P(\bar{g}^n_i = l \,|\, \bar{\alpha}^n_i = k, \bar{g}^n_i = J) = q^n_{kl}\left(\tilde{p}^n\right).$$

Since paired agents do not match in this step, their extended types will not change. Thus, to verify Equation (22), we only need to prove

$$\begin{aligned}
&P\Big(\bar{\bar{\beta}}_i^n = (a_1, r_1), \bar{\bar{\beta}}_j^n = (a_2, r_2) \,|\, \bar{\beta}_i^n = (a_1, J), \bar{\beta}_j^n = (a_2, J),\\
&\qquad \beta_i^t = (k_1^t, l_1^t), \beta_j^t = (k_2^t, l_2^t), t = 1, \ldots, n-1\Big)\\
&= P\left(\bar{\bar{\beta}}_i^n = (a_1, r_1) \,|\, \bar{\beta}_i^n = (a_1, J)\right) P\left(\bar{\bar{\beta}}_j^n = (a_2, r_2) \,|\, \bar{\beta}_j^n = (a_2, J)\right).
\end{aligned}$$

Fix any $k \in S$. If $\tilde{p}_{kJ}^n = \int_I P(\bar{\beta}_i^n = (k, J)) d\lambda(i) = 0$, then $P(\bar{\beta}_i^n = (k, J)) = 0$ for λ-almost all agent $i \in I$, which means that Equation (22) automatically holds. It follows from the continuity requirement above Equation (47) that

$$\check{\rho}_{a_1J}^n \hat{q}_{a_1r_1}^n(\check{\rho}^n) \simeq \tilde{p}_{a_1J}^n q_{a_1r_1}^n(\tilde{p}^n)$$

for P-almost all $\omega^{3n-2} \in \Omega^{3n-2}$. Suppose $\tilde{p}_{a_1J}^n > 0$ and $\tilde{p}_{a_2J}^n > 0$. Hence, we can obtain that for P-almost all $\omega^{3n-2} \in \Omega^{3n-2}$, $\hat{q}_{a_1r_1}^n(\check{\rho}^n) \simeq q_{a_1r_1}^n(\tilde{p}^n)$, and $\hat{q}_{a_2r_2}^n(\check{\rho}^n) \simeq q_{a_2r_2}^n(\tilde{p}^n)$.

We can now derive

$$\begin{aligned}
&\int_I \int_I \Big| P\Big(\bar{\bar{\beta}}_i^n = (a_1, r_1), \bar{\bar{\beta}}_j^n = (a_2, r_2), \bar{\beta}_i^n = (a_1, J), \bar{\beta}_j^n = (a_2, J),\\
&\qquad\qquad \beta_i^t = (k_1^t, l_1^t), \beta_j^t = (k_2^t, l_2^t), t = 1, \ldots, n-1\Big)\\
&\quad - q_{a_1r_1}^n(\tilde{p}^n)\, q_{a_2r_2}^n(\tilde{p}^n)\, P(D_{ij}^{3n-2})\Big| d\lambda(j) d\lambda(i)\\
&\simeq \int_I \int_I \Big| \int_{D_{ij}^{3n-2}} \Big(Q_{3n-1}^{\omega^{3n-2}}\left(\bar{\bar{g}}^n(i, \omega^{3n-1}) = r_1, \bar{\bar{g}}^n(j, \omega^{3n-1}) = r_2\right)\\
&\quad - \hat{q}_{a_1r_1}^n\left(\check{\rho}^n(\omega^{3n-2})\right) \hat{q}_{a_2r_2}^n\left(\check{\rho}^n(\omega^{3n-2})\right)\Big) dQ^{3n-2}(\omega^{3n-2})\Big| d\lambda_0(j) d\lambda_0(i)\\
&\le \int_I \int_I \int_{\Omega^{3n-2}} \mathbf{1}_{D_{ij}^{3n-2}}(\omega^{3n-2}) \Big| Q_{3n-1}^{\omega^{3n-2}}\left(\bar{\bar{g}}^n(i, \omega^{3n-1}) = r_1, \bar{\bar{g}}^n(j, \omega^{3n-1}) = r_2\right)\\
&\quad - \hat{q}_{a_1r_1}^n\left(\check{\rho}^n(\omega^{3n-2})\right) \hat{q}_{a_2r_2}^n\left(\check{\rho}^n(\omega^{3n-2})\right)\Big| dQ^{3n-2}(\omega^{3n-2}) d\lambda_0(j) d\lambda_0(i)\\
&= \int_{\Omega^{3n-2}} \int_I \int_I \mathbf{1}_{D_{ij}^{3n-2}}(\omega^{3n-2}) \Big| Q_{3n-1}^{\omega^{3n-2}}\left(\bar{\bar{g}}^n(i, \omega^{3n-1}) = r_1, \bar{\bar{g}}^n(j, \omega^{3n-1}) = r_2\right)\\
&\quad - \hat{q}_{a_1r_1}^n\left(\check{\rho}^n(\omega^{3n-2})\right) \hat{q}_{a_2r_2}^n\left(\check{\rho}^n(\omega^{3n-2})\right)\Big| d\lambda_0(j) d\lambda_0(i) dQ^{3n-2}(\omega^{3n-2}),
\end{aligned} \tag{48}$$

where

$$\begin{aligned}
D^{3n-2} = \{(\omega^{3n-2}, i, j) : \quad & \bar{\beta}^n(i, \omega^{3n-2}) = (a_1, J), \bar{\beta}^n(j, \omega^{3n-2}) = (a_2, J),\\
& \beta^t(i, \omega^{3t}) = (k_1^t, l_1^t), \beta^t(j, \omega^{3t}) = (k_2^t, l_2^t), t = 1, \ldots, n-1\},
\end{aligned}$$

D_{ij}^{3n-2} is the (i, j)-section of D^{3n-2}, and $\mathbf{1}_{D_{ij}^{3n-2}}$ is the indicator function of the set $\mathbf{1}_{D_{ij}^{3n-2}}$ in Ω^{3n-2}. By Lemma 1 (iii), it is clear that for λ-almost all $i \in I$, for λ-almost all $j \in I$, and for any $\omega^{3n-2} \in D_{ij}^{3n-2}$, we have

$$Q_{3n-1}^{\omega^{3n-2}}\left(\bar{\bar{g}}^n(i, \omega^{3n-1}) = r_1, \bar{\bar{g}}^n(j, \omega^{3n-1}) = r_2\right) \simeq \hat{q}_{a_1r_1}^n\left(\check{\rho}^n(\omega^{3n-2})\right) \hat{q}_{a_2r_2}^n\left(\check{\rho}^n(\omega^{3n-2})\right).$$

Hence, the last term of Equation (48) is equal to an infinitesimal. Therefore, the first term of Equation (48) is equal to zero, which implies that for λ-almost all $i \in I$

$$\begin{aligned} & P\Big(\bar{\bar{\beta}}_i^n = (a_1, r_1), \bar{\bar{\beta}}_j^n = (a_2, r_2) \,|\, \bar{\beta}_i^n = (a_1, J), \bar{\beta}_j^n = (a_2, J), \\ & \qquad \beta_i^t = (k_1^t, l_1^t), \beta_j^t = (k_2^t, l_2^t), t = 1, \ldots, n-1\Big) \\ & = q_{a_1 r_1}^n \left(\tilde{p}^n\right) q_{a_2 r_2}^n \left(\tilde{p}^n\right) \end{aligned} \tag{49}$$

for λ-almost all $j \in I$.

For $i \in I$, let $E_i^{3n-2} = \{\omega^{3n-2} : \bar{\beta}^n(i, \omega^{3n-2}) = (a_1, J)\}$. We can obtain that for λ-almost all $i \in I$, and for any $\omega^{3n-2} \in E_i^{3n-2}$,

$$P\left(\bar{\bar{\beta}}_i^n = (a_1, r_1), \bar{\beta}_i^n = (a_1, J)\right) \simeq \int_{E_i^{3n-2}} Q_{3n-1}^{\omega^{3n-2}} \left(\bar{\bar{g}}^n(i, \omega^{3n-1}) = r_1\right) dQ^{3n-2}(\omega^{3n-2}),$$

and $Q_{3n-1}^{\omega^{3n-2}} \left(\bar{\bar{g}}^n(i, \omega^{3n-1}) = r_1\right) \simeq \hat{q}_{a_1 r_1}^n \left(\check{\rho}^n(\omega^{3n-2})\right)$. Hence, we can obtain that for λ-almost all $i \in I$,

$$\begin{aligned} & P\left(\bar{\bar{\beta}}_i^n = (a_1, r_1), \bar{\beta}_i^n = (a_1, J)\right) \\ \simeq & \int_{E_i^{3n-2}} Q_{3n-1}^{\omega^{3n-2}} \left(\bar{\bar{g}}^n(i, \omega^{3n-1}) = r_1\right) dQ^{3n-2}(\omega^{3n-2}) \\ \simeq & \int_{E_i^{3n-2}} \hat{q}_{a_1 r_1}^n \left(\check{\rho}^n(\omega^{3n-2})\right) dQ^{3n-2}(\omega^{3n-2}) \simeq P(E_i^{3n-2}) q_{a_1 r_1}^n \left(\tilde{p}^n\right). \end{aligned}$$

Therefore, we have for λ-almost all $i \in I$,

$$P\left(\bar{\bar{\beta}}_i^n = (a_1, r_1) \,|\, \bar{\beta}_i^n = (a_1, J)\right) = q_{a_1 r_1}^n \left(\tilde{p}^n\right). \tag{50}$$

Since $\check{p}^n(\omega^{3n-2}) \simeq \tilde{p}^n$ for P-almost all $\omega^{3n-2} \in \Omega^{3n-2}$, Equation (50) implies Equation (15). Combining Equations (49) and (50) together, we have

$$\begin{aligned} & P\Big(\bar{\bar{\beta}}_i^n = (a_1, r_1), \bar{\bar{\beta}}_j^n = (a_2, r_2) \,|\, \bar{\beta}_i^n = (a_1, J), \bar{\beta}_j^n = (a_2, J), \\ & \qquad \beta_i^t = (k_1^t, l_1^t), \beta_j^t = (k_2^t, l_2^t), t = 1, \ldots, n-1\Big) \\ & = q_{a_1 r_1}^n \left(\tilde{p}^n\right) q_{a_2 r_2}^n \left(\tilde{p}^n\right) \\ & = P\left(\bar{\bar{\beta}}_i^n = (a_1, r_1) \,|\, \bar{\beta}_i^n = (a_1, J)\right) P\left(\bar{\bar{\beta}}_j^n = (a_2, r_2) \,|\, \bar{\beta}_j^n = (a_2, J)\right). \end{aligned}$$

Hence, Equation (22) in the definition of Markov conditional independence is satisfied.

For the step of type changing with break-up in period n, fix any $(a_1, r_1), (a_2, r_2), (x_1, y_1), (x_2, y_2)$, and $k_1^t, l_1^t), (k_2^t, l_2^t)$, $t = 1, \ldots, n-1$ in $\tilde{S}$. For any agents i and j with $i \neq j$, we can obtain that

$$\begin{aligned}
&P\Big(\beta_i^n = (a_1, r_1), \beta_j^n = (a_2, r_2), \bar{\bar{\beta}}_i^n = (x_1, y_1), \bar{\bar{\beta}}_j^n = (x_2, y_2), \\
&\qquad \beta_i^t = (k_1^t, l_1^t), \beta_j^t = (k_2^t, l_2^t), t = 1, \ldots, n-1\Big) \\
&\simeq \int_{D_{ij}^{3n-1}} Q_{3n}^{\omega^{3n-1}} \left(\beta^n(i, \omega^{3n}) = (a_1, r_1), \beta^n(j, \omega^{3n}) = (a_2, r_2)\right) dQ^{3n-1}(\omega^{3n-1}) \\
&= \int_{\underline{D}_{ij}^{3n-1}} Q_{3n}^{\omega^{3n-1}} \left(\beta^n(i, \omega^{3n}) = (a_1, r_1), \beta^n(j, \omega^{3n}) = (a_2, r_2)\right) dQ^{3n-1}(\omega^{3n-1}) \\
&\quad + \int_{\overline{D}_{ij}^{3n-1}} Q_{3n}^{\omega^{3n-1}} \left(\beta^n(i, \omega^{3n}) = (a_1, r_1), \beta^n(j, \omega^{3n}) = (a_2, r_2)\right) dQ^{3n-1}(\omega^{3n-1}),
\end{aligned}$$

where

$$\begin{aligned}
D_{ij}^{3n-1} = \{\omega^{3n-1} : \quad &\bar{\bar{\beta}}_i^n = (x_1, y_1), \bar{\bar{\beta}}_j^n = (x_2, y_2), \\
&\beta^t(i, \omega^{3t}) = (k_1^t, l_1^t), \beta^t(j, \omega^{3t}) = (k_2^t, l_2^t), t = 1, \ldots, n-1\},
\end{aligned}$$

$$\begin{aligned}
\underline{D}_{ij}^{3n-1} = \{\omega^{3n-1} : \bar{\pi}^n(i, \omega^{3n-1}) \neq j, \quad &\bar{\bar{\beta}}_i^n = (x_1, y_1), \bar{\bar{\beta}}_j^n = (x_2, y_2), \\
&\beta^t(i, \omega^{3t}) = (k_1^t, l_1^t), \beta^t(j, \omega^{3t}) = (k_2^t, l_2^t), t = 1, \ldots, n-1\},
\end{aligned}$$

$$\begin{aligned}
\overline{D}_{ij}^{3n-1} = \{\omega^{3n-1} : \bar{\pi}^n(i, \omega^{3n-1}) = j, \quad &\bar{\bar{\beta}}_i^n = (x_1, y_1), \bar{\bar{\beta}}_j^n = (x_2, y_2), \\
&\beta^t(i, \omega^{3t}) = (k_1^t, l_1^t), \beta^t(j, \omega^{3t}) = (k_2^t, l_2^t), t = 1, \ldots, n-1\}.
\end{aligned}$$

Fix any agent $i \in I$. It is clear that $\overline{D}_{ij}^{3n-1} \cap \overline{D}_{ij'}^{3n-1} = \emptyset$ for different j and j'. Then there are at most countably many $j \in I$ such that $P(\overline{D}_{ij}^{3n-1}) > 0$. Let $F_i^{3n-1} = \{j \in I : j \neq i, P(\overline{D}_{ij}^{3n-1}) = 0\}$; then $\lambda(F_i^{3n-1}) = 1$. Next, fix any $j \in F_i^{3n-1}$. The probability for agents i and j to be partners is zero at the matching step in period n. When agents i and j are not partners, their random extended types will be independent by the construction of $Q_{3n}^{\omega^{3n-1}}$. Hence, we can obtain that

$$\begin{aligned}
&P\Big(\beta_i^n = (a_1, r_1), \beta_j^n = (a_2, r_2), \bar{\bar{\beta}}_i^n = (x_1, y_1), \bar{\bar{\beta}}_j^n = (x_2, y_2), \\
&\qquad \beta_i^t = (k_1^t, l_1^t), \beta_j^t = (k_2^t, l_2^t), t = 1, \ldots, n-1\Big) \\
&\simeq \int_{\underline{D}_{ij}^{3n-1}} Q_{3n}^{\omega^{3n-1}} \left(\beta^n(i, \omega^{3n}) = (a_1, r_1), \beta^n(j, \omega^{3n}) = (a_2, r_2)\right) dQ^{3n-1}(\omega^{3n-1}) \\
&= \int_{\underline{D}_{ij}^{3n-1}} Q_{3n}^{\omega^{3n-1}} \left(\beta^n(i, \omega^{3n}) = (a_1, r_1)\right) Q_{3n}^{\omega^{3n-1}} \left(\beta^n(j, \omega^{3n}) = (a_2, r_2)\right) dQ^{3n-1}(\omega^{3n-1}) \\
&= \int_{\underline{D}_{ij}^{3n-1}} B_{x_1 y_1}^{3n}(a_1, r_1) B_{x_2 y_2}^{3n}(a_2, r_2) dQ^{3n-1}(\omega^{3n-1}) \\
&\simeq P(D_{ij}^{3n-1}) B_{x_1 y_1}^{3n}(a_1, r_1) B_{x_2 y_2}^{3n}(a_2, r_2),
\end{aligned}$$

where

$$B_{kl}^{3n}(r,s) = \begin{cases} (1-\theta_{kl}^n)\sigma_{kl}^n(r,s) & \text{if } l,s \in S \\ \theta_{kl}^n \varsigma_{kl}^n(r) & \text{if } l \in S \text{ and } s = J \\ \delta_k(r)\delta_J(s) & \text{if } l = J. \end{cases}$$

Therefore, for any $i \in I$, and for λ-almost all $j \in I$,

$$P\Big(\beta_i^n = (a_1,r_1), \beta_j^n = (a_2,r_2) \,|\, \bar{\bar{\beta}}_i^n = (x_1,y_1), \bar{\bar{\beta}}_j^n = (x_2,y_2), \tag{51}$$
$$\beta_i^t = (k_1^t, l_1^t), \beta_j^t = (k_2^t, l_2^t), t = 1,\ldots,n-1\Big)$$
$$= B_{x_1y_1}^{3n}(a_1,r_1) B_{x_2y_2}^{3n}(a_2,r_2). \tag{52}$$

Note that for any agent $i \in I$,

$$P(\beta_i^n = (a_1,r_1), \bar{\bar{\beta}}_i^n = (x_1,y_1)) \simeq \int_{E_i^{3n-1}} Q_{3n}^{\omega^{3n-1}} \left(\beta^n(i,\omega^{3n}) = (a_1,r_1)\right) dQ^{3n-1}(\omega^{3n-1})$$
$$= \int_{E_i^{3n-1}} B_{x_1y_1}^{3n}(a_1,r_1) dQ^{3n-3}(\omega^{3n-3}) \simeq P(E_j^{3n-1}) B_{x_1y_1}^{3n}(a_1,r_1),$$

where $E_i^{3n-1} = \{\omega^{3n-1} : \bar{\bar{\beta}}^n(i,\omega^{3n-1}) = (x_1,y_1)\}$, which implies that $P(\beta_i^n = (a_1,r_1) \,|\, \bar{\bar{\beta}}_i^n = (x_1,y_1)) = B_{x_1y_1}^{3n}(a_1,r_1)$. Hence, Equations (18), (19) and (20) in the definition of the dynamical system $\mathbb{D}$ are satisfied. By Equation (51), we can obtain for each $i \in I$, and for λ-almost all $j \in I$,

$$P\Big(\beta_i^n = (a_1,r_1), \beta_j^n = (a_2,r_2) \,|\, \bar{\bar{\beta}}_i^n = (x_1,y_1), \bar{\bar{\beta}}_j^n = (x_2,y_2),$$
$$\beta_i^t = (k_1^t, l_1^t), \beta_j^t = (k_2^t, l_2^t), t = 1,\ldots,n-1\Big)$$
$$= P\left(\beta_i^n = (a_1,r_1) \,|\, \bar{\bar{\beta}}_i^n = (x_1,y_1)\right) P\left(\beta_j^n = (a_2,r_2) \,|\, \bar{\bar{\beta}}_j^n = (x_2,y_2)\right).$$

Hence, Equation (23) in the definition of Markov conditional independence is satisfied.

In summary, we have shown the validity of Equations (11) to (20), and (21) to (23). Hence $\mathbb{D}$ is a dynamical system with the Markov conditional independence property, where the initial condition Π^0 is deterministic.

Finally, we consider the case that the initial extended type process β^0 is i.i.d. across agents. We shall use the construction for the case of deterministic initial condition. We choose $n = -1$ to be the initial period so that we can have some flexibility in choosing the parameters in period 0. Assume that at $n = -1$, all agents have type 1, and no agents are matched. Namely, the initial type function is $\alpha^{-1} \equiv 1$ while the initial matching is $\pi^{-1} \equiv J$.

Denote $\sum_{r \in S \cup \{J\}} \ddot{p}_{kr}^0$ by $\ddot{p}_k^0$. For the parameters in period 0, let

$$b_{kr}^0 = \begin{cases} \ddot{p}_r^0 & \text{if } k = 1 \\ \delta_k(r) & \text{if } k \neq 1, \end{cases}$$

$$q^0_{kl}(\hat{p}) = \begin{cases} \frac{\min(\frac{\ddot{p}^0_{kl}\hat{p}_{kJ}}{\ddot{p}^0_k}, \frac{\ddot{p}^0_{kl}\hat{p}_{lJ}}{\ddot{p}^0_l})}{\hat{p}_{kJ}} & \text{if } \hat{p}_{kJ} \neq 0, \ddot{p}^0_k \neq 0 \text{ and } \ddot{p}^0_l \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$\sigma^0_{kl}(k', l') = \delta_k(k')\delta_l(l')$, $\varsigma^0_{kl}(k') = \delta_k(k')$ and $\theta^0_{kl} = 0$ for any $k, k', l, l' \in S$. Following the construction for the case of deterministic initial condition, there exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ on which is defined a dynamical system $\mathbb{D} = (\Pi^n)^\infty_{n=-1}$ that is Markov conditionally independent with the parameters $(b^n, q^n, \sigma^n, \theta^n)^\infty_{n=0}$.

By Lemma 4, $\tilde{p}^0_{kl} = \delta_J(l)\ddot{p}^0_k$. It follows from part (2) of Theorem 4 that,

$$z^0_{(1J)(kl)} = \ddot{p}^0_k \frac{\ddot{p}^0_{kl}}{\ddot{p}^0_k} = \ddot{p}^0_{kl},$$

$$z^0_{(1J)(kJ)} = 1 - \sum_{l \in S} z^0_{(1J)(kl)} = 1 - \sum_{l \in S} \ddot{p}^0_{kl} = \ddot{p}^0_{kJ}.$$

Therefore, for λ-almost all $i \in I$,

$$P(\beta^0_i = (k, l)) = P(\beta^0_i = (k, l) \,|\, \beta^{-1}_i = (1, J))P(\beta^{-1}_i = (1, J)) = z^0_{(1J)(kl)} = \ddot{p}^0_{kl}$$

for any $k \in S$, $l \in S \cup \{J\}$. Part (3) of Theorem 4 implies the essential pairwise independence of β^0. Thus, we can simply start the dynamical system $\mathbb{D}$ from time zero instead of time -1 so that we can have an i.i.d. initial extended type process β^0.

B.4 Proofs of Propositions 2 and 4

In this subsection, the unit interval $[0, 1]$ will have a different notation in a different context. Recall that $(L, \mathcal{L}, \chi)$ is the Lebesgue unit interval, where χ is the Lebesgue measure defined on the Lebesgue σ-algebra $\mathcal{L}$. We shall prove Proposition 4 first. The proof of Proposition 2 then follows easily.

Note that the agent space used in the proof of Theorem 5 is a hyperfinite Loeb counting probability space. Using the usual ultrapower construction as in Loeb and Wolff (2015), the hyperfinite index set of agents can be viewed as an equivalence class of a sequence of finite sets with elements in natural numbers, and thus this index set of agents has the external cardinality of the continuum. The purpose of Proposition 4 is to show that one can find some extension of the Lebesgue unit interval as the agent space so that the associated version of Theorem 5 still holds.

Fix a Fubini extension $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$ as constructed in the proof of Theorem 5. Following Appendix A of Sun and Zhang (2009) and Appendix B in Duffie and Sun (2012), we can state the following lemma.[24]

[24] Parts (2) and (3) of Lemma 7 are taken from Lemma 11 in Duffie and Sun (2012).

Lemma 7 *There exists a Fubini extension* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ *such that:*

(1) The agent space $(I, \mathcal{I}, \lambda)$ *is an extension of the Lebesgue unit interval* $(L, \mathcal{L}, \chi)$*.*

(2) There exists a surjective mapping φ *from* I *to* $\hat{I}$ *such that* $\varphi^{-1}(\hat{i})$ *has the cardinality of the continuum for any* $\hat{i} \in \hat{I}$ *and* φ *is measure preserving, in the sense that for any* $A \in \hat{\mathcal{I}}$*,* $\varphi^{-1}(A)$ *is measurable in* $\mathcal{I}$ *with* $\lambda[\varphi^{-1}(A)] = \hat{\lambda}(A)$*.*

(3) Let Φ *be the mapping* $(\varphi, \mathrm{Id}_\Omega)$ *from* $I \times \Omega$ *to* $\hat{I} \times \Omega$*, that is,* $\Phi(i, \omega) = (\varphi, \mathrm{Id}_\Omega)(i, \omega) = (\varphi(i), \omega)$ *for any* $(i, \omega) \in I \times \Omega$*. Then* Φ *is measure preserving from* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ *to* $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$ *in the sense that for any* $V \in \hat{\mathcal{I}} \boxtimes \mathcal{F}$*,* $\Phi^{-1}(V)$ *is measurable in* $\mathcal{I} \boxtimes \mathcal{F}$ *with* $(\lambda \boxtimes P)[\Phi^{-1}(V)] = (\hat{\lambda} \boxtimes P)(V)$*.*

Denote the MCI dynamical system with parameters $(b, q, \sigma, \varsigma, \theta)$ and a deterministic initial condition, as constructed in proof of Theorem 5 by $\hat{\mathbb{D}}$. For that dynamical system, we add a hat to the relevant type processes, matching functions, and partners' type processes. We shall follow the proof of Theorem 4 in Duffie and Sun (2012).

Proof of Proposition 4: Based on the dynamical system $\hat{\mathbb{D}}$ on the Fubini extension $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$, we shall now define, inductively, a new dynamical system $\mathbb{D}$ on the Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

For any $\hat{i}, \hat{i}' \in \hat{I}$ with $\hat{i} \neq \hat{i}'$, let $\Theta^{\hat{i}, \hat{i}'}$ be a bijection from $\varphi^{-1}(\hat{i})$ to $\varphi^{-1}(\hat{i}')$, and $\Theta^{\hat{i}', \hat{i}}$ be the inverse mapping of $\Theta^{\hat{i}, \hat{i}'}$. This is possible since both $\varphi^{-1}(\hat{i})$ and $\varphi^{-1}(\hat{i}')$ have cardinality of the continuum.

Let α^0 be the mapping $\hat{\alpha}^0(\varphi)$ from I to S,

$$\pi^0(i) = \begin{cases} J & \text{if } \hat{\pi}^0(\varphi(i)) = J \\ \Theta^{\varphi(i),\, \hat{\pi}^0(\varphi(i))}(i) & \text{if } \hat{\pi}^0(\varphi(i)) \neq J, \end{cases}$$

and $g^0(i) = \alpha^0\left(\pi^0(i)\right) = \hat{g}^0(\varphi(i))$. By the measure preserving property of φ in Lemma 7, we know that $\beta^0 = (\alpha^0, g^0)$ is $\mathcal{I}$-measurable type function with distribution $\hat{p}^0$ on $S \times (S \cup \{J\})$.

For each time period $n \geq 1$, let $\bar{\alpha}^n$ and α^n be the respective mappings $\hat{\bar{\alpha}}^n(\Phi)$ and $\hat{\alpha}^n(\Phi)$ from $I \times \Omega$ to S. Define mappings $\bar{\pi}^n$, and π^n from $I \times \Omega$ to I such that for each $(i, \omega) \in I \times \Omega$,

$$\bar{\pi}^n(i, \omega) = \begin{cases} J & \text{if } \hat{\bar{\pi}}^n_\omega(\varphi(i)) = J \\ \Theta^{\varphi(i),\, \hat{\bar{\pi}}^n_\omega(\varphi(i))}(i) & \text{if } \hat{\bar{\pi}}^n_\omega(\varphi(i)) \neq J, \end{cases}$$

$$\pi^n(i, \omega) = \begin{cases} J & \text{if } \hat{\pi}^n_\omega(\varphi(i)) = J \\ \Theta^{\varphi(i),\, \hat{\pi}^n_\omega(\varphi(i))}(i) & \text{if } \hat{\pi}^n_\omega(\varphi(i)) \neq J. \end{cases}$$

When $\pi^n_\omega(\varphi(i)) \neq J$, π^n_ω defines a full matching on $\varphi^{-1}(\hat{H}^n_\omega)$, where $\hat{H}^n_\omega = \hat{I} - \{i : \hat{\pi}_\omega(i)^n = J\}$, which implies that $\pi^n_\omega(i) \neq i$. Hence, π^n is a well-defined mapping from $I \times \Omega$ to $I \cup \{J\}$. $\bar{\pi}^n$ is also well-defined for the same reason.

Since Φ is measure-preserving and $\hat{\bar{\alpha}}^n$ and $\hat{\alpha}^n$ are measurable mappings from $(\hat{I}\times\Omega, \hat{\mathcal{I}}\boxtimes\mathcal{F}, \hat{\lambda}\boxtimes P)$ to S. By the definitions of $\bar{\alpha}^n$ and α^n, it is obvious that for each $i \in I$,

$$\bar{\alpha}_i^n = \hat{\bar{\alpha}}_{\varphi(i)}^n \text{ and } \alpha_i^n = \hat{\alpha}_{\varphi(i)}^n. \tag{53}$$

Next, we consider the property of $\bar{\pi}^n$ and π^n. Fix any $\omega \in \Omega$. Let $H_\omega^n = I - \{i : \pi_i^n = J\}$; then $H_\omega^n = \varphi^{-1}(\hat{H}_\omega^n)$. Pick any $i \in H_\omega^n$ and denote $\pi_\omega^n(i)$ by j. Then, $\varphi(i) \in \hat{H}_\omega^n$. The definition of π^n implies that $j = \Theta^{\varphi(i),\, \hat{\pi}_\omega^n(\varphi(i))}(i)$. Since $\Theta^{\varphi(i),\, \hat{\pi}_\omega^n(\varphi(i))}$ is a bijection between $C_{\varphi(i)}$ and $C_{\hat{\pi}_\omega^n(\varphi(i))}$, it follows that $\varphi(j) = \varphi(\pi_\omega^n(i)) = \hat{\pi}_\omega^n(\varphi(i))$ by the definition of φ. Thus, $j = \Theta^{\varphi(i),\, \varphi(j)}(i)$. Since the inverse of $\Theta^{\varphi(i),\, \varphi(j)}$ is $\Theta^{\varphi(j),\, \varphi(i)}$, we know that $\Theta^{\varphi(j),\, \varphi(i)}(j) = i$. By the full matching property of $\hat{\pi}_\omega^n$, $\varphi(j) \neq \varphi(i)$, $\varphi(j) \in \hat{H}_\omega^n$ and $\hat{\pi}_\omega^n(\varphi(j)) = \varphi(i)$. Hence, we have $j \neq i$, and

$$\pi_\omega^n(j) = \Theta^{\varphi(j),\, \hat{\pi}_\omega^n(\varphi(j))}(j) = \Theta^{\varphi(j),\, \varphi(i)}(j) = i.$$

This means that the composition of π_ω^n with itself on H_ω^n is the identity mapping on H_ω^n, which also implies that π_ω^n is a bijection on H_ω^n. Therefore π_ω^n is a full matching on $H_\omega^n = I - \{i : \pi_i^n = i\}$.

We define $g^n : I \times \Omega \to S \cup \{J\}$ by $g^n(i,\omega) = \alpha^n(\pi^n(i,\omega),\omega)$. As noted in the above paragraph, for any fixed $\omega \in \Omega$, $\varphi(\pi_\omega^n(i)) = \hat{\pi}_\omega^n(\varphi(i))$ for $i \in H_\omega^n$. When $i \notin H_\omega^n$, we have $\varphi(i) \notin \hat{H}_\omega^n$, and $\pi_\omega^n(i) = J$, $\hat{\pi}_\omega^n(\varphi(i)) = J$. Therefore, $\varphi(\pi_\omega^n(i)) = \hat{\pi}_\omega^n(\varphi(i))$ for any $i \in I$. Then,

$$g^n(i,\omega) = \hat{\alpha}^n(\varphi(\pi^n(i,\omega)),\omega) = \hat{\alpha}^n(\hat{\pi}^n(\varphi(i),\omega),\omega) = \hat{g}^n(\varphi(i),\omega) = \hat{g}^n(\Phi)(i,\omega).$$

We can prove that $\bar{g}^n(i,\omega) = \hat{\bar{g}}^n(\Phi)(i,\omega)$ and $\bar{\bar{g}}^n(i,\omega) = \hat{\bar{\bar{g}}}^n(\Phi)(i,\omega)$ in the same way. Hence, the measure-preserving property of Φ implies that g^n is $\mathcal{I}\boxtimes\mathcal{F}$-measurable. The previous three identities on the partners' processes also mean that for any $i \in I$,

$$g_i^n(\cdot) = \hat{g}_{\varphi(i)}^n(\cdot),\ \bar{g}_i^n(\cdot) = \hat{\bar{g}}_{\varphi(i)}^n(\cdot),\ \bar{\bar{g}}_i^n(\cdot) = \hat{\bar{\bar{g}}}_{\varphi(i)}^n(\cdot).$$

Since $\bar{\alpha}^n = \hat{\bar{\alpha}}^n(\Phi)$ and $\bar{g}^n(i,\omega) = \hat{\bar{g}}^n(\Phi)(i,\omega)$, Equation (11) implies that for λ-almost all $i \in I$,

$$\begin{aligned}
&P(\bar{\alpha}_i^n = k_2, \bar{g}_i^n = l_2 \,|\, \alpha_i^{n-1} = k_1, g_i^{n-1} = l_1) \\
&= P(\hat{\bar{\alpha}}_{\varphi(i)}^n = k_2, \hat{\bar{g}}_{\varphi(i)}^n = l_2 |\, \hat{\alpha}_{\varphi(i)}^{n-1} = k_1, \hat{g}_{\varphi(i)}^{n-1} = l_1) \\
&= b_{k_1k_2}^n b_{l_1l_2}^n.
\end{aligned}$$

Similarly, we can obtain that for λ-almost all $i \in I$,

$$P(\bar{\alpha}_i^n = k_2, \bar{g}_i^n = r \,|\, \alpha_i^{n-1} = k_1, g_i^{n-1} = J) = b_{k_1k_2}^n \delta_J(r),$$

$$P(\bar{\bar{g}}_i^n = l \,|\, \bar{\alpha}_i^n = k, \bar{g}_i^n = J, \check{p}^n) = q_{kl}^n(\check{p}^n(\omega)),$$

$$P(\alpha_i^n = l_1, g_i^n = r \,|\, \bar{\alpha}_i^n = k_1, \bar{\bar{g}}_i^n = J) = \delta_{k_1}(l_1)\, \delta_J(r),$$

$$P(\alpha_i^n = l_1, g_i^n = J \,|\, \bar{\alpha}_i^n = k_1, \bar{\bar{g}}_i^n = k_2) = \theta^n_{k_1k_2} \varsigma^n_{k_1k_2}(l_1),$$

$$P(\alpha_i^n = l_1, g_i^n = l_2 \,|\, \bar{\alpha}_i^n = k_1, \bar{\bar{g}}_i^n = k_2) = (1 - \theta^n_{k_1k_2}) \sigma^n_{k_1k_2}(l_1, l_2).$$

Therefore, $\mathbb{D}$ is a dynamical system with random mutation, directed random matching and type changing with break-up and with the parameters $(p^0, b, q, \sigma, \varsigma, \theta)$.

It remains to check the Markov conditional independence for $\mathbb{D}$. Since the dynamical system $\hat{\mathbb{D}}$ is Markov conditionally independent, for each $n \geq 1$, there is a set $\hat{I}' \in \hat{\mathcal{I}}$ with $\hat{\lambda}(\hat{I}') = 1$, and for each $\hat{i} \in \hat{I}'$, there exists a set $\hat{E}_{\hat{i}} \in \hat{\mathcal{I}}$ with $\hat{\lambda}(\hat{E}_{\hat{i}}) = 1$, with Equations (21) to (23) being satisfied for any $\hat{i} \in \hat{I}'$ and any $\hat{j} \in \hat{E}_{\hat{i}}$. Let $I' = \varphi^{-1}(\hat{I}')$. For any $i \in I'$, let $E_i = \varphi^{-1}(\hat{E}_{\varphi(i)})$. Since φ is measure-preserving, $\lambda(I') = \lambda(E_i) = 1$. Fix any $i \in I'$, and any $j \in E_i$. Denote $\varphi(i)$ by $\hat{i}$ and $\varphi(j)$ by $\hat{j}$. Then, it is obvious that $\hat{i} \in \hat{I}'$ and $\hat{j} \in \hat{E}_{\hat{i}}$. Therefore Equations (21) to (23) are satisfied for any $i' \in I'$ and any $j' \in E_{i'}$. Therefore the dynamical system $\mathbb{D}$ is Markov conditionally independent.

By using exactly the same proof as in the end of the proof of Theorem 5, we can have an i.i.d. (instead of deterministic) initial extended type process β^0 in the statement of this proposition. ■

Proof of Proposition 2: In the proof of Proposition 4, take the initial type distribution $\hat{p}^0_{kl} = p_k \delta_J(l)$. Assume that there is no genuine random mutation. Then, it is clear that $\tilde{p}^0_{kl} = p_k \delta_J(l)$ for any $k \in S$. Consider the random matching π^1 in period one.

Fix an agent i with $\alpha^0(i) = k$, then $P(\bar{\alpha}_i^1 = k) = 1$, $P\left(\bar{\bar{g}}_i^1 = l\right) = q_{kl}$ and $P\left(\bar{\bar{g}}_i^1 = J\right) = \eta_k$. Similarly, Equation (22) implies that the process $\bar{\bar{g}}^1$ is essentially pairwise independent. By taking the type function α to be α^0, the matching function π to be $\bar{\pi}^1$, and the associated process g to be $\bar{\bar{g}}^1$, the proposition holds. ■