The Exact Law of Large Numbers for Independent Random Matching*

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Abstract

This paper provides a mathematical foundation for independent random matching of a large population, as widely used in the economics literature. We consider both static and dynamic systems with random mutation, partial matching arising from search, and type changes induced by matching. Under independence assumptions at each randomization step, we show that there is an almost-sure constant cross-sectional distribution of types in a large population, and moreover that the multi-period cross-sectional distribution of types is deterministic and evolves according to the transition matrices of the type process of a given agent. We also show the existence of a joint agent-probability space, and randomized mutation, partial matching and match-induced type-changing functions that satisfy appropriate independence conditions, where the agent space is an extension of the classical Lebesgue unit interval.

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1 Introduction

A deterministic (almost surely) cross-sectional distribution of types in independent random matching models for continuum populations had been widely used in several literatures, without a foundation. Economists and geneticists, among others, have implicitly or explicitly assumed the law of large numbers for independent random matching in a continuum population, by which we mean an atomless measure space of agents. This result is relied upon in large literatures within general equilibrium theory (e.g. [24], [25], [42], [56]), game theory (e.g. [6], [8], [11], [23], [30]), monetary theory (e.g. [14], [28], [31], [36], [37], [49], [53]), labor economics (e.g. [13], [32], [45], [46], [48]), illiquid financial markets (e.g. [17], [18], [38], [54], [55]), and biology (e.g. [9], [29], [41]). Mathematical foundations, however, have been lacking, as has been noted by Green and Zhou [28].

We provide an exact law of large numbers for independent random matching, under which there is an almost-sure constant cross-sectional distribution of types in a large population. We address both static and dynamic systems with random mutation, partial matching arising from search, and type changes induced by matching. Based on a suitable measure-theoretic framework, an exact law of large numbers is proved for each case under an independence assumption[1] on each of the randomization steps: matching, mutation, and matching-induced type changes. We also show the existence of a joint agent-probability space, and randomized mutation, partial matching and match-induced type-changing functions that satisfy appropriate independence conditions, where the agent space is an extension of the classical Lebesgue unit interval.[2]

The mathematical abstraction of an atomless measure space of agents not only provides a convenient idealization of an economy with a large but finite number of agents, but is often relied upon for tractability, especially in dynamic settings. It is intractable, at best, to propagate finite-agent approximations in every time step, given the many underlying state variables that would be required to capture the payoff relevant states of the economy. This may partially explain why a plethora of papers in economics have been based on independent random matching of a continuum of agents, even without a mathematical foundation. In our setting, the continuum model allows us to show that the time evolution of the cross-sectional distribution of types is completely determined by the agent-level Markov chain for type, with

[1] The independence condition we propose is natural, but may not be obvious. For example, a random matching in a finite population may not allow independence among agents since the matching of agent i to agent j implies of course that j is also matched to i, implying some correlation among agents. The effect of this correlation is reduced to zero in a continuum population. A new concept, "Markov conditional independence in types," is proposed for dynamic matching, under which the transition law at each randomization step depends on only the previous one or two steps of randomization.

[2] A rich measure-theoretic extension of the Lebesgue unit interval was already considered by Kakutani in [34].

explicitly calculated transition matrices. This convenient property is not even considered for models with a large but finite number of agents.

For a simple illustration of our results, suppose that each agent within a fraction p of a continuum population has an item for sale, and that the agents in the remaining fraction $q = 1 - p$ are in need of the item. If the agents "pair off independently," a notion that we formalize shortly, then each would-be seller meets some would-be buyer with probability q. At such a meeting, a trade occurs. One presumes that, almost surely, in a natural model, exactly a fraction q of the seller population would trade, implying that a fraction qp of the total population are sellers who trade, that the same fraction pq of the total population are buyers who trade, and that the fraction of the population that would not trade is $1 - 2pq$. Among other results, we show that this presumption is correct in a suitable mathematical framework. Moreover, we prove in Section 5 below that such a model exists.

Hellwig [31] is the first, to our knowledge, to have relied on the effect of the exact law of large numbers for random pairwise matching in a market, in a study of a monetary exchange economy.[3] Much earlier reliance can be found in genetics. In 1908, G.H. Hardy [29] and W. Weinberg (see [9]) independently proposed that with random mating in a large population, one could determine the constant fractions of each allele in the population. Hardy wrote: "suppose that the numbers are fairly large, so that the mating may be regarded as random," and then used, in effect, an exact law of large numbers for random matching to deduce his results.[4] For a simple illustration, consider a continuum population of gametes consisting of two alleles, A and B, in initial proportions p and $q = 1 - p$. Then, following the Hardy-Weinberg approach, the new population would have a fraction p^2 whose parents are both of type A, a fraction q^2 whose parents are both of type B, and a fraction $2pq$ whose parents are of mixed type (heterozygotes). These genotypic proportions asserted by Hardy and Weinberg are already, implicitly, based on an exact law of large numbers for random matching in a large population. In order to consider the implications for the steady-state distribution of alleles, suppose that, with both parents of allele A, the offspring are of allele A, and with both parents of allele B, the offspring are of allele B. Suppose that the offspring of parents of different alleles are, say, equally likely to be of allele A or allele B. The Hardy-Weinberg equilibrium for this special case is a population with steady-state constant proportions $p = 60\%$ of allele A and $q = 40\%$ of allele B. Provided that

[3] Diamond [12] had earlier treated random matching of a large population with, in effect, finitely many employers, but not pairwise matching within a large population. The matching of a large population with a finite population can be treated directly by the exact law of large numbers for a continuum of independent random variables. For example, let $N(i)$ be the event that worker i is matched with an employer of a given type, and suppose this event is pairwise independent and of the same probability p, in a continuum population of such workers. Then, under the conditions of [50], the fraction of the population that is matched to this type of employer is p, almost surely.

[4] Later in his article, Hardy did go on to consider the effect of "casual deviations," and the issue of stability.

the law of large numbers for random matching indeed applies, this is verified by checking that, if generation k has this cross-sectional distribution, then the fraction of allele A in generation $k+1$ is almost surely $0.6^2 + 0.5 \times (2 \times 0.6 \times 0.4) = 0.6$. This Hardy-Weinberg Law, governing steady-state allelic and genotypic frequencies, is a special case of our results treating dynamics and steady-state behavior.

In applications, random-matching models have also allowed for random mutation of agents, obviously in genetics, and in economics via random changes in preferences, productivity, or endowments. Typical models are also based on "random search," meaning that the time at which a given agent is matched is also uncertain. With random search, during each given time period, some fraction of the agents are not matched. Finally, in some cases, it is important that the impact of a match between two agents on their post-match types is itself random, as in [37] and [46]. For instance, trade sometimes depends on a favorable outcome of a productivity shock to the buyer, allowing the buyer to produce, or not, the output necessary to pay the seller. In some models, once paired by matching, agents use mixed strategies for their actions, causing another stage of random type changes. It is also often the case that one wishes not only an (almost surely) deterministic cross-sectional distribution of types as a result of each round of matching, but also a cross-sectional distribution of types that is constant over time, as in the Hardy-Weinberg Equilibrium. It may also help if one knows the cross-sectional distribution of the type process almost surely. We provide a collection of results treating all of these cases.

Our results include the potential for random birth and death, because we allow for random mutation of types, which can include "alive" or "dead." We do not, however, consider population growth, which could be handled by relatively straightforward extensions of the results here, that we leave for future work. It would also be straightforward to extend our results in order to consider the effect of "aggregate shocks," for example common adjustments to the parameters determining speed of matching, according to a Markov chain, as in the business-cycle effects on employer-worker matching studied by[5] Mortensen and Pissarides [46]. When we treat dynamic models, we take only the discrete-time case, although continuous-time models of random matching are also popular (e.g. [17], [46], [53], [55]). Using different methods, we are in the process of extending our results to continuous-time settings.

Because there are fundamental measurability problems associated with a continuum of independent random variables,[6] there has up to now been no theoretical treatment of the exact law of large numbers for independent random matching among a continuum population. In [50], various versions of the exact law of large numbers and their converses are proved by direct

[5]Ljungqvist and Sargent [39] present a discrete-time version of the Mortensen-Pissarides model, which is further treated in discrete time by Cole and Rogerson [10] and by Merz [43].

[6]See, for example, [4], [15], [16], [20], [33] and discussions in [50].

application of simple measure-theoretic methods in the framework of an extension of the usual product probability space that retains the Fubini property.[7] This paper adopts the measure-theoretic framework of [50] to provide the first theoretical treatment of the exact law of large numbers for a general independent random matching among a continuum population. The existence of such an independent random matching is shown in [19] for the case of a hyperfinite number of agents via the method of nonstandard analysis. Since the unit interval and the class of Lebesgue measurable sets with the Lebesgue measure provide the archetype for models of economies with a continuum of agents, we show in this paper that one can take an extension of the classical Lebesgue unit interval as the agent space for the construction of an independent random matching in both static and dynamic settings.

In comparison, earlier papers on random matching, such as [7], [26], and [42], consider either the non-existence of random matching with certain desired properties, or provide for an approximate law of large numbers for some particular random matching with a countable population (and with a purely finitely additive sample measure space in [26]). Section 6 provides additional discussion of the literature.[8] A continuum of agents with independent random types is never measurable with respect to the completion of the usual product σ-algebra, except in the trivial case that almost all the random types in the process are constants.[9] Instead, we work with extensions of the usual product measure spaces (of agents and states of the world) that retain the Fubini property, allowing us to resolve the measurability problem in this more general measure-theoretic framework.

The remainder of the paper is organized as follows. Section 2 is a user's guide, going immediately to the form and implications of the main results, and putting off most of the underlying mathematical developments. In Section 3, we consider random full and partial matchings in the static case. Section 3.1 includes a brief introduction of the measure-theoretic framework (a Fubini extension). A random full matching is formally defined in Section 3.2 and its properties shown in Theorem 1. Random partial matching (the case of search models) is considered in Section 3.3.

[7] While it is relatively straightforward to construct examples of a continuum of independent random variables whose sample means or distributions are constant (see, for example, Anderson [4], Green [27], or Judd [33]), one can also construct other pathological examples of a continuum of independent random variables whose sample functions may not be measurable, or may behave in "strange" ways. (For example, the sample function can be made equal to any given function on the continuum almost surely, as in [33] and [50].) By working with a Fubini extension of the usual product probability space, however, one is able to obtain general results on the exact law of large numbers, as in [50], while at the same time ruling out such pathologies.

[8] To prove our results, we cannot use the particular example of an iid process constructed from the coordinate functions of the Kolmogorov continuum product, as in [33]. While the Kolmogorov continuum product space gives a product measure on the continuum product easily, there is no simple way to define a useful measure on the space of matching functions (which are special one-to-one and onto mappings on the agent space) that will lead to an independent random matching.

[9] See, for example, Proposition 2.1 in [50].

Section 4 considers a dynamical system for agent types, allowing at each time period for random mutation, partial matching, and match-induced random type changes. We introduce the condition of Markov conditional independence to model these three stages of uncertainty. Markov conditional independence allows us to show that the individual type processes of almost all agents are essentially pairwise independent Markov chains. Using this last result, we can then show that there is an almost-sure constant cross-sectional distribution of types in a large population (including stationarity of the cross-sectional distribution of agent types), and moreover, that the time evolution of the cross-sectional distribution of types is (almost surely) completely determined as that of a Markov chain with known transition matrices. All of these results are included in Theorem 3.

Existence results for random matching, in static settings and in dynamic settings that are (Markov conditionally) independent in types, are stated in Section 5. Although the Lebesgue unit interval fails to be an agent space suitable for modeling a continuum of agents with independent random matching, we show that an extension of the Lebesgue unit interval does work well in our setting.

A brief discussion of the relevant literature is given in Section 6. Proofs of Theorems 1, 2 and 3 (as stated in Sections 3 and 4) are given in Appendix 1 (Section 7). Proofs of Theorem 4 and Corollaries 1 and 2 (as stated in Section 5) are given in Appendix 2 (Section 8).

2 User's Guide

This section gives a simple understanding of some of the key results, without detailing most of the definitions and arguments that we later use to formalize and prove these results.

We fix a probability space $(\Omega, \mathcal{F}, P)$ representing uncertainty, an atomless probability space $(I, \mathcal{I}, \lambda)$ representing the set of agents,[10] and a finite agent-type space $S = \{1, \ldots, K\}$.[11] As shown in Section 5 below, one may take the agent space $(I, \mathcal{I}, \lambda)$ to be an extension of the Lebesgue unit interval $(L, \mathcal{L}, \eta)$ in the sense that $I = L = [0, 1]$, the σ-algebra $\mathcal{I}$ contains the Lebesgue σ-algebra $\mathcal{L}$, and the restriction of λ to $\mathcal{L}$ is the Lebesgue measure η.

In order to discuss independent random matching, we consider a product probability space $(I \times \Omega, \mathcal{W}, Q)$ such that $\mathcal{W}$ contains the product σ-algebra $\mathcal{I} \otimes \mathcal{F}$, and such that the marginals of Q on $(I, \mathcal{I})$ and $(\Omega, \mathcal{F})$ are λ and P respectively. This extension $(I \times \Omega, \mathcal{W}, Q)$ of

[10] A probability space $(I, \mathcal{I}, \lambda)$ is atomless if there does not exist $A \in \mathcal{I}$ such that $\lambda(A) > 0$, and for any $\mathcal{I}$-measurable subset C of A, $\lambda(C) = 0$ or $\lambda(C) = \lambda(A)$.

[11] In order to study independent random partial matching systematically in the static and dynamic settings, we focus on the finite type case here. It is pointed out in Remarks 1 – 4 below that for the case of independent random full matching, some results in [19] and this paper can be readily restated to the setting of a complete separable metric type space. However, an independent random partial matching with random mutation and match-induced random type changes in such a type space would require transition probabilities in a general setup. Some new tools will be needed to handle that case, which is beyond the scope of this paper.

the product of the two underlying spaces must have the basic Fubini property in order for the following law-of-large numbers results to make sense.

A cross-sectional or probability distribution of types is an element of $\Delta = \{p \in \mathbb{R}_+^K : p_1 + \cdots + p_K = 1\}$.

For each time period $n \geq 1$, we first have a random mutation, and then a random partial matching, followed by a random type changing for matched agents. The random mutation is modeled by some $\mathcal{W}$-measurable function $h^n : I \times \Omega \to S$ that specifies a mutated type for agent i at state of nature ω.

As for the random partial matching at time n, there is some random matching function $\pi^n : I \times \Omega \to I \cup \{J\}$, where $\{J\}$ is a singleton representing 'unmatched,' that specifies either an agent $j = \pi^n(i,\omega) \neq i$ in I to whom i is matched in state ω, or specifies the outcome $\pi^n(i,\omega) = J$ that i is not matched. It must be the case that if i is matched to j, then j is matched to i. Specifically, for all ω, i, and j, $\pi^n(i,\omega) = j$ if and only if $\pi^n(j,\omega) = i$. Let g^n be a $\mathcal{W}$-measurable matching type function on $I \times \Omega$ into $S \cup \{J\}$, such that $g^n(i,\omega)$ is the type of the agent $j = \pi^n(i,\omega)$ who is matched with agent i in state of nature ω, or $g^n(i,\omega) = J$ if $\pi^n(i,\omega) = J$.

When agents are not matched, they keep their types. Otherwise, the types of two matched are randomly changed, with a distribution that depends on their pre-match types. Some $\mathcal{W}$-measurable $\alpha^n : I \times \Omega \to S$ specifies the type $\alpha^n(i,\omega)$ of agent $i \in I$ in state of the world ω after the type changing. The associated cross-sectional distribution of types at time n is the Δ-valued random variable $p^n(\omega)$ (also denoted by p^n_ω) defined by

$$p^n_k(\omega) = \lambda(\{i \in I : \alpha^n(i,\omega) = k\}).$$

The initial type function $\alpha^0 : I \to S$ is non-random. As usual, we let g^n_i and α^n_i denote the random variables whose outcomes in state ω are $g^n(i,\omega)$ and $\alpha^n(i,\omega)$, respectively. For a realized state of nature ω, h^n_ω, π^n_ω, g^n_ω and α^n_ω denote, respectively, the realized mutation, matching, matching type and type functions on I.

The parameters of a random matching model with type space S are

1. Some initial cross-sectional distribution $p^0 \in \Delta$ of types (the type distribution induced by α^0).

2. A $K \times K$ transition matrix b fixing the probability b_{kl} that an agent of type k mutates to an agent of type l in a given period, before matching.

3. Some $q \in [0,1]^S$ specifying, for each type k, the probability q_k that an agent of type k is not matched within one period. An agent who is not matched keeps her type in a given period, but may mutate to another type at the beginning of next period.

4. Some $\nu : S \times S \to \Delta$ specifying the probability $\nu_{kl}(r)$ that an agent of type k who is matched with an agent of type l will become, after matching, an agent of type r.

Fixing the parameters (p^0, b, q, ν) of some random matching model, under a natural definition of "Markov conditional independence for mutation, matching, and type changing" which we provide later in this paper, one conjectures the following results.[12]

- At each time $n \geq 1$, the realized cross-sectional type distribution $p^n(\omega)$ is P-almost surely (henceforth, "*a.s.*") equal to the expected cross-sectional type distribution $\overline{p}^n = \int_\Omega p^n(\omega)\, dP(\omega)$.

- After the random mutation step at time n, the fraction $p_l^n(\omega)$ of the population of a given type l is almost surely $\sum_{k=1}^K \overline{p}\,_k^{n-1} b_{kl}$, denoted by $\tilde{p}_l^n$.

- At each time $n \geq 1$ and for any type k, the fraction of the population of type k that are not matched[13] at period n is

$$\lambda(\{i \in I : h_\omega^n(i) = k, g_\omega^n(i) = J\}) = \tilde{p}_k^n q_k \quad a.s. \tag{1}$$

For any types $k, l \in S$, the fraction of the population who are agents of type k that are matched with agents of type l is

$$\lambda(\{i : h_\omega^n(i) = k, g_\omega^n(i) = l\}) = \frac{\tilde{p}_k^n(1-q_k)\tilde{p}_l^n(1-q_l)}{\sum_{r=1}^K \tilde{p}_r^n(1-q_r)} \quad a.s. \tag{2}$$

- At the end of each time period $n \geq 1$ (after match-induced type changing), for each type r, the new fraction of agents of type r is

$$p_r^n(\omega) = \overline{p}\,_r^n = \tilde{p}_r^n q_r + \sum_{k,l=1}^K \frac{\nu_{kl}(r)\tilde{p}_k^n(1-q_k)\tilde{p}_l^n(1-q_l)}{\sum_{t=1}^K \tilde{p}_t^n(1-q_t)} \quad a.s. \tag{3}$$

Using the fact that $\tilde{p}_l^n = \sum_{k=1}^K \overline{p}\,_k^{n-1} b_{kl}$, one has a recursive formula for $\overline{p}\,^n$ in terms of $\overline{p}\,^{n-1}$, and thus $\overline{p}\,^n$ (and also p^n) can be computed directly from p^0.

- For λ-almost every agent $i \in I$, the type process $\alpha_i^0, \alpha_i^1, \alpha_i^2, \ldots$ is an S-valued Markov chain,[14] with a $K \times K$ transition matrix z^n specifying the probability of transition from

[12] Models with random full matching, or with deterministic match-induced type changing, or without random mutation, are special cases of our model. To avoid random mutation, one can simply take b_{kk} to be one for all $k \in S$. If $q_k = 0$ for all $k \in S$, then an agent will be matched with probability one. For $k, l \in S$, if $\nu_{kl}(r)$ is one for some r, then the match-induced type change is deterministic.

[13] We note that $\sum_{r=1}^K \tilde{p}_r^n(1-q_r)$ is the fraction of population who are matched, while $(\tilde{p}_l^n(1-q_l)) / \left(\sum_{r=1}^K \tilde{p}_r^n(1-q_r)\right)$ is the relative fraction of the population who are matched agents of type l among all matched agents.

[14] For a complete statement of what constitutes a Markov process, one must fix a filtration $\{\mathcal{F}_0, \mathcal{F}_1, \ldots\}$ of sub-σ-algebras of $\mathcal{F}$. For our purposes, it is natural, and suffices for this result, to take $\mathcal{F}_t$ to be the σ-algebra generated by $\{\alpha_i^s : s \leq t\}$.

type k at time $n-1$ to type l at time n (for $n \geq 1$), given by

$$z_{kl}^n = P(\alpha_i^n = l \mid \alpha_i^{n-1} = k) = q_l b_{kl} + \sum_{r,t=1}^{K} \nu_{rt}(l) b_{kr} \frac{(1-q_r)(1-q_t)\tilde{p}_t^n}{\sum_{r'=1}^{K}(1-q_{r'})\tilde{p}_{r'}^n}, \tag{4}$$

provided the event $\{\alpha_i^{n-1} = k\}$ has positive probability.

- For P-almost every state of nature $\omega \in \Omega$, the cross-sectional type process $\alpha^0, \alpha_\omega^1, \alpha_\omega^2, \ldots$ is an S-valued Markov chain with the transition matrix z^n at time $n-1$ and initial type distribution p^0. Thus, almost surely, the evolution of the fractions of each type is deterministic and coincides with the evolution of the probability distribution of type for a given agent, except for the initial distributions.[15]

Given the mutation, search and match-induced type changing parameters (b, q, ν), one also conjectures that, under the assumption of "Markov conditional independence," there is some steady-state constant cross-sectional type distribution p^* in Δ, in the sense that, for the parameters (p^*, b, q, ν) we have, almost surely, for all $n \geq 0$, $p_\omega^n = p^*$. Moreover, the Markov chains for the type process $\alpha_i^0, \alpha_i^1, \alpha_i^2, \ldots$ (for λ-almost every agent $i \in I$) and for the cross-sectional type process $\alpha^0, \alpha_\omega^1, \alpha_\omega^2, \ldots$ (for P-almost every state of nature $\omega \in \Omega$) are time-homogeneous, and the latter has p^* as a stationary distribution. That is, for some fixed transition matrix z, we have $z^n = z$ for all $n \geq 1$, and we have, for all ℓ in S,

$$\sum_{k=1}^{K} z_{kl}\, p_k^* = p_\ell^*.$$

We will demonstrate all of the results stated above, based on the following version of the exact law of large numbers, proved in Sun [50]. Given some $\mathcal{W}$-measurable $f : I \times \Omega \to X$, where X is a finite set (we state the result for general X in Section 7.1), the random variables $\{f_i : i \in I\}$, defined by $f_i(\omega) = f(i, \omega)$, are said to be *essentially pairwise independent*[16] if for λ-almost all $i \in I$, the random variables f_i and f_j are independent for λ-almost all $j \in I$. For brevity, in this case we say that f itself is essentially pairwise independent. With the assumption of the Fubini property on $(I \times \Omega, \mathcal{W}, Q)$, the exact law of large numbers in [50] (which is stated as Lemma 1 in Section 7.1, for the convenience of the reader) says that if f is essentially pairwise independent, then the sample functions f_ω have essentially constant

[15] We do not take the initial probability distribution of agent i's type to be p^0, but rather the Dirac measure at the type $\alpha^0(i)$. See Footnote 30 for a generalization.

[16] This condition is weaker than pairwise/mutual independence since each agent is allowed to have correlation with a null set of agents (including finitely many agents since a finite set is null under an atomless measure). For example, the agent space I is divided into a continuum of cohorts, with each cohort containing a fixed number L of agents ($L \in \mathbb{N}$). If the agents across cohorts act independently (correlation may be allowed for agents in the same cohort), then the essential pairwise independence condition is satisfied.

distributions. Then, the notion of Markov conditional independence is used to derive the essential pairwise independence of the n-th period mutation, matching and type processes h^n, g^n and α^n, as well as the essential pairwise independence of the Markov chains $\alpha_i^0, \alpha_i^1, \alpha_i^2, \ldots$, which imply all the results stated above. In addition, we show the existence of an independent random matching satisfying all the above properties when the agent space is an extension of the classical Lebesgue unit interval.

3 Exact law of large numbers for independent random matchings

In this section, we consider independent random matchings, full or partial, in a static setting. Some background definitions are given in Section 3.1. Exact laws of large numbers for random full and partial matchings are presented, respectively, in Sections 3.2 and 3.3, and their proofs are given in Section 7.2 of Appendix 1.

3.1 Some background definitions

Let $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ be two probability spaces that represent the index and sample spaces respectively.[17] In our applications, $(I, \mathcal{I}, \lambda)$ is an atomless probability space that is used to index the agents. If one prefers, I can be taken to be the unit interval $[0,1]$, $\mathcal{I}$ an extension of the Lebesgue σ-algebra $\mathcal{L}$, and λ an extension of the Lebesgue measure η. Let $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ be the usual product probability space. For a function f on $I \times \Omega$ (not necessarily $\mathcal{I} \otimes \mathcal{F}$-measurable), and for $(i, \omega) \in I \times \Omega$, f_i represents the function $f(i, \cdot\,)$ on Ω, and f_ω the function $f(\,\cdot\,, \omega)$ on I.

In order to work with independent type processes arising from random matching, we need to work with an extension of the usual measure-theoretic product that retains the Fubini property. A formal definition, as in [50], is as follows.

Definition 1 *(Fubini extension)* *A probability space $(I \times \Omega, \mathcal{W}, Q)$ extending the usual product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ is said to be a* Fubini extension *of $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ if for any real-valued Q-integrable function g on $(I \times \Omega, \mathcal{W})$, the functions $g_i = g(i, \cdot\,)$ and $g_\omega = g(\,\cdot\,, \omega)$ are integrable respectively on $(\Omega, \mathcal{F}, P)$ for λ-almost all $i \in I$ and on $(I, \mathcal{I}, \lambda)$ for P-almost all $\omega \in \Omega$; and if, moreover, $\int_\Omega g_i \, dP$ and $\int_I g_\omega \, d\lambda$ are integrable, respectively, on $(I, \mathcal{I}, \lambda)$ and on $(\Omega, \mathcal{F}, P)$, with $\int_{I \times \Omega} g \, dQ = \int_I \left(\int_\Omega g_i \, dP \right) d\lambda = \int_\Omega \left(\int_I g_\omega \, d\lambda \right) dP$. To reflect the fact that the probability space $(I \times \Omega, \mathcal{W}, Q)$ has $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ as its marginal spaces, as required by the Fubini property, it will be denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.*

[17] All measures in this paper are complete, countably additive set functions defined on σ-algebras.

An $\mathcal{I} \boxtimes \mathcal{F}$-measurable function f will also be called a process, each f_i will be called a random variable of this process, and each f_ω will be called a sample function of the process.

We now introduce the following crucial independence condition. We state the definition using a complete separable metric space X for the sake of generality; in particular, a finite space or an Euclidean space is a complete separable metric space.

Definition 2 *(Essential pairwise independence)* *An $\mathcal{I} \boxtimes \mathcal{F}$-measurable process f from $I \times \Omega$ to a complete separable metric space X is said to be essentially pairwise independent if for λ-almost all $i \in I$, the random variables f_i and f_j are independent for λ-almost all $j \in I$.*[18]

3.2 An exact law of large numbers for independent random full matchings

We follow the notation in Section 3.1. Below is a formal definition of random full matching.

Definition 3 *(Full matching)*

1. *Let $S = \{1, 2, \ldots, K\}$ be a finite set of types, $\alpha : I \to S$ an $\mathcal{I}$-measurable type function of agents. Let p denote the distribution on S. That is, for $1 \le k \le K$ and $I_k = \{i \in I : \alpha(i) = k\}$, let $p_k = \lambda(I_k)$ for each $1 \le k \le K$.*

2. *A full matching ϕ is a one-to-one mapping from I onto I such that, for each $i \in I$, $\phi(i) \neq i$ and $\phi(\phi(i)) = i$.*

3. *A random full matching π is a mapping from $I \times \Omega$ to I such that (i) π_ω is a full matching for each $\omega \in \Omega$; (ii) the type process $g = \alpha(\pi)$ is a measurable map from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to S;*[19] *(iii) for λ-almost all $i \in I$, g_i has distribution p.*

4. *A random full matching π is said to be independent in types if the type process g is essentially pairwise independent.*[20]

Condition (1) of this definition says that a fraction p_k of the population is of type k. Condition (2) says that all individuals are matched, there is no self-matching, and that if i is matched to $j = \phi(i)$, then j is matched to i. Condition (3) *(iii)* means that for almost every agent i, the probability that i is matched to a type-k agent is p_k, the fraction of type-k agents in the population. Condition (4) says that for almost all agents i and $j \in I$, the event that

[18] Two random variables ϕ and ψ from $(\Omega, \mathcal{F}, P)$ to X are said to be independent, if the σ-algebras $\sigma(\phi)$ and $\sigma(\psi)$ generated respectively by ϕ and ψ are independent.

[19] In general, we only require the measurability condition on the type process g rather than on the random full matching π; the latter implies the former. This allows one to work with a more general class of random full matchings.

[20] This weaker condition of independence (see Footnote 16) allows one to work with a more general class of independent random matchings.

agent i matched to a type-k agent is independent of the event that agent j matched to a type-l agent, for any k and l in S.

Because agents of type k have a common probability p_l of being matched to type-l agents, Condition (4) allows the application of the exact law of large numbers in [50] (which is stated as Lemma 1 in Section 7.1 below) in order to claim that the relative fraction of agents matched to type-l agents among the type-k population is almost surely p_l (or, intuitively, frequency coincides with probability). This means that the fraction of the total population consisting of type-k agents that are matched to type-l is almost surely $p_k \cdot p_l$. This result is formally stated in the following theorem, whose proof is given in Section 7.2.

Theorem 1 *Let $\alpha : I \to S$ be an $\mathcal{I}$-measurable type function with type distribution $p = (p_1, \ldots, p_K)$ on S. Let π be a random full matching from $I \times \Omega$ to I. If π is independent in types, then for any given types $(k, l) \in S \times S$,*

$$\lambda(\{i : \alpha(i) = k, \alpha(\pi_\omega(i)) = l\}) = p_k \cdot p_l \tag{5}$$

holds for P-almost all $\omega \in \Omega$.[21]

3.3 An exact law of large numbers for independent random partial matchings

We shall now consider the case of random partial matchings, starting with the formal definition.

Definition 4 *Let $\alpha : I \to S$ be an $\mathcal{I}$-measurable type function with type distribution $p = (p_1, \ldots, p_K)$ on S. Let π be a mapping from $I \times \Omega$ to $I \cup \{J\}$, where J denotes "no match."*

1. *We say that π is a random partial matching with no-match probabilities $q_1, \ldots, q_K$ in $[0, 1]$ if (i) for each $\omega \in \Omega$, the restriction of π_ω to $I - \pi_\omega^{-1}(\{J\})$ is a full matching on $I - \pi_\omega^{-1}(\{J\})$;*[22] *(ii) after extending the type function α to $I \cup \{J\}$ so that $\alpha(J) = J$, and letting $g = \alpha(\pi)$, we have g measurable from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to $S \cup \{J\}$; (iii) for λ-almost all $i \in I_k$, $P(g_i = J) = q_k$ and*[23]
$$P(g_i = l) = \frac{(1 - q_k) p_l (1 - q_l)}{\sum_{r=1}^{K} p_r (1 - q_r)}.$$

[21] It means that the joint distribution of α and g_ω is the product distribution $p \otimes p$ on $S \times S$. As noted in Remarks 1 and 3, one can readily generalize the finite type case to the case of a complete separable metric type space for the case of independent random full matching.

[22] This means that an agent i with $\pi_\omega(i) = J$ is not matched, while any agent in $I - \pi_\omega^{-1}(\{J\})$ is matched. This produces a partial matching on I.

[23] Note that if an agent of type k is matched, its probability of being matched to a type-l agent should be proportional to the type distribution of matched agents. The fraction of the population of matched agents among the total population is $\sum_{r=1}^{K} p_r(1 - q_r)$. Thus, the relative fraction of type l matched agents to that of all the matched agents is $(p_l(1 - q_l))/\sum_{r=1}^{K} p_r(1 - q_r)$. This implies that the probability that a type-k agent is matched to a type-l agent is $(1 - q_k)(p_l(1 - q_l))/\sum_{r=1}^{K} p_r(1 - q_r)$. When $\sum_{r=1}^{K} p_r(1 - q_r) = 0$, we have $p_k(1 - q_k) = 0$ for all $1 \le k \le K$, in which case almost no agents are matched, and we can interpret the ratio $((1 - q_k)p_l(1 - q_l))/\sum_{r=1}^{K} p_r(1 - q_r)$ as zero.

2. *A random partial matching π is said to be independent in types if the process g (taking values in $S \cup \{J\}$) is essentially pairwise independent.*[24]

The following result, proved in Section 7.2, generalizes Theorem 1 to the case of random partial matchings.

Theorem 2 *If π is an independent-in-types random partial matching from $I \times \Omega$ to $I \cup \{J\}$ with no-match probabilities $q_1, \ldots, q_K$ then, for P-almost all $\omega \in \Omega$:*

1. *The fraction of the total population consisting of unmatched agents of type k is*

$$\lambda(\{i \in I : \alpha(i) = k, g_\omega(i) = J\}) = p_k q_k. \tag{6}$$

2. *For any types $(k, l) \in S^2$, the fraction of the total population consisting of type-k agents that are matched to type-l agents is*

$$\lambda(\{i : \alpha(i) = k, g_\omega(i) = l\}) = \frac{p_k(1 - q_k)p_l(1 - q_l)}{\sum_{r=1}^{K} p_r(1 - q_r)}. \tag{7}$$

4 A dynamical system with random mutation, partial matching, and type changing that is Markov conditionally independent in types

In this section, we consider a dynamical system with random mutation, partial matching and type changing that is Markov conditionally independent in types. We first define such a dynamical system in Section 4.1. Then, we formulate in Section 4.2 the key condition of Markov conditional independence in types, and finally present in Theorem 3 of Section 4.3 an exact law of large numbers and stationarity for the dynamical system.

4.1 Definition of a dynamical system with random mutation, partial matching and type changing

Let $S = \{1, 2, \ldots, K\}$ be a finite set of types. A discrete-time dynamical system $\mathbb{D}$ with random mutation, partial matching and type changing in each period can be defined intuitively as follows. The initial distribution of types is p^0. That is, $p^0(k)$ (denoted by p^0_k) is the initial fraction of agents of type k. In each time period, each agent of type k first goes through a stage of random mutation, becoming an agent of type l with probability b_{kl}. In models such as [17], for example, this mutation generates new motives for trade. Then, each agent of type k is either not matched, with probability q_k, or is matched to a type-l agent with a probability proportional to the fraction of type-l agents in the population immediately after the random

[24] This means that for almost all agents $i, j \in I$, whether agent i is unmatched or matched to a type-k agent is independent of a similar event for agent j.

mutation step. When an agent is not matched, she keeps her type. Otherwise, when a pair of agents with respective types k and l are matched, each of the two agents changes types; the type-k agent becomes type r with probability $\nu_{kl}(r)$, where ν_{kl} is a probability distribution on S, and similarly for the type-l agent. Under appropriate independence conditions, one would like to have an almost-surely deterministic cross-sectional type distribution at each time period.

We shall now define formally a dynamical system $\mathbb{D}$ with random mutation, partial matching and type changing. As in Section 3, let $(I, \mathcal{I}, \lambda)$ be an atomless probability space representing the space of agents, $(\Omega, \mathcal{F}, P)$ a sample probability space, and $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ a Fubini extension of the usual product probability space.

Let $\alpha^0 : I \to S = \{1, \ldots, K\}$ be an initial $\mathcal{I}$-measurable type function with distribution p^0 on S. For each time period $n \geq 1$, we first have a random mutation that is modeled by a process h^n from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to S, then a random partial matching that is described by a function π^n from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to $I \cup \{J\}$ (where J represents no matching), followed by a random assignment of types for the matched agents, given by α^n from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to S.

For the random mutation step at time n, given a $K \times K$ probability transition matrix[25] b, we require that, for each agent $i \in I$,

$$P\left(h_i^n = l \mid \alpha_i^{n-1} = k\right) = b_{kl}, \tag{8}$$

the specified probability with which an agent i of type k at the end of time period $n-1$ mutates to type l.

For the random partial matching step at time n, we let $\tilde{p}^n$ be the expected cross-sectional type distribution immediately after random mutation. That is,

$$\tilde{p}_k^n = \tilde{p}^n(k) = \int_\Omega \lambda(\{i \in I : h_\omega^n(i) = k\})\, dP(\omega). \tag{9}$$

The random partial matching function π^n at time n is defined by:

1. For any $\omega \in \Omega$, $\pi_\omega^n(\,\cdot\,)$ is a full matching on $I - (\pi_\omega^n)^{-1}(\{J\})$, as defined in Section 3.3.

2. Extending h^n so that $h^n(J, \omega) = J$ for any $\omega \in \Omega$, we define $g^n : I \times \Omega \to S \cup \{J\}$ by

$$g^n(i, \omega) = h^n(\pi^n(i, \omega), \omega),$$

and assume that g^n is $\mathcal{I} \boxtimes \mathcal{F}$-measurable.

[25] Here, b_{kl} is in $[0, 1]$, with $\sum_{l=1}^K b_{kl} = 1$ for each k. We do not require that the mutation probability b_{kl} is strictly positive. Thus, as noted in Footnote 12, the degenerate case of no random mutation is allowed.

3. Let $q \in [0,1]^S$. For each agent $i \in I$,

$$
\begin{aligned}
P\left(g_i^n = J \mid h_i^n = k\right) &= q_k, \\
P\left(g_i^n = l \mid h_i^n = k\right) &= \frac{(1-q_k)(1-q_l)\tilde{p}_l^n}{\sum_{r=1}^K (1-q_r)\tilde{p}_r^n}.
\end{aligned} \tag{10}
$$

Equation (10) means that, for any agent whose type before the matching is k, the probability of being unmatched is q_k, and the probability of being matched to a type-l agent is proportional to the expected cross-sectional type distribution for matched agents. When g^n is essentially pairwise independent (as under the Markov conditional independence condition used in Section 4.3 below), the exact law of large numbers in [50] (see Lemma 1 below) implies that the realized cross-sectional type distribution $\lambda(h_\omega^n)^{-1}$ after random mutation at time n is indeed the expected distribution $\tilde{p}^n$, P-almost surely.[26]

Finally, for the step of random type changing for matched agents at time n, a given $\nu : S \times S \to \Delta$ specifies the probability distribution $\nu_{kl} = \nu(k,l)$ of the new type of a type-k agent who has met a type-l agent. When agent i is not matched at time n, she keeps her type h_i^n with probability one. We thus require that the type function α^n after matching satisfies, for each agent $i \in I$,

$$
\begin{aligned}
P\left(\alpha_i^n = r \mid h_i^n = k, g_i^n = J\right) &= \delta_k^r, \\
P\left(\alpha_i^n = r \mid h_i^n = k, g_i^n = l\right) &= \nu_{kl}(r),
\end{aligned} \tag{11}
$$

where δ_k^r is one if $r = k$, and zero otherwise.

Thus, we have inductively defined a dynamical system $\mathbb{D}$ with random mutation, partial matching, and match-induced type changing with parameters (p^0, b, q, ν).

4.2 Markov conditional independence in types

In this section, we consider a suitable independence condition on the dynamical system $\mathbb{D}$. In order to formalize the intuitive idea that, given their type function α^{n-1}, the agents randomly mutate to other types independently at time n, in such a way that their types in earlier periods have no effect on this mutation, we say that the random mutation is **Markov conditionally independent in types** if, for λ-almost all $i \in I$ and λ-almost all $j \in I$,

$$
P(h_i^n = k, h_j^n = l \mid \alpha_i^0, \ldots, \alpha_i^{n-1}; \alpha_j^0, \ldots, \alpha_j^{n-1}) = P(h_i^n = k \mid \alpha_i^{n-1}) P(h_j^n = l \mid \alpha_j^{n-1}) \tag{12}
$$

holds for all types $k, l \in S$.[27]

[26] As noted in Footnote 23, if the denominator in equation (10) is zero, then almost no agents will be matched and we can simply interpret the ratio as zero.

[27] We could include the functions h^m and g^m for $1 \le m \le n-1$ as well. However, it is not necessary to do so since we only care about the dependence structure across time for the type functions at the end of each time period.

Intuitively, the random partial matching at time n should depend only on agents' types immediately after the latest random mutation step. One may also want the random partial matching to be independent across agents, given events that occurred in the first $n-1$ time periods and the random mutation at time n. We say that the random partial matching π^n is **Markov conditionally independent in types** if, for λ-almost all $i \in I$ and λ-almost all $j \in I$,

$$P(g_i^n = c, g_j^n = d \mid \alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n; \alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n) = P(g_i^n = c \mid h_i^n) P(g_j^n = d \mid h_j^n) \quad (13)$$

holds for all types $c, d \in S \cup \{J\}$.

The agents' types at the end of time period n should depend on the agents' types immediately after the random mutation stage at time n, as well as the results of random partial matching at time n, but not otherwise on events that occurred in previous periods. This motivates the following definition. The random type changing after partial matching at time n is said to be **Markov conditionally independent in types** if for λ-almost all $i \in I$ and λ-almost all $j \in I$, and for each $n \geq 1$,

$$\begin{aligned} & P(\alpha_i^n = k, \alpha_j^n = l \mid \alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n, g_i^n; \alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n, g_j^n) \\ = \; & P(\alpha_i^n = k \mid h_i^n, g_i^n) P(\alpha_j^n = l \mid h_j^n, g_j^n) \end{aligned} \quad (14)$$

holds for all types $k, l \in S$.

The dynamical system $\mathbb{D}$ is said to be **Markov conditionally independent in types** if, in each time period n, each random step (random mutation, partial matching, and type changing) is so.[28]

4.3 Exact law of large numbers and stationarity

With the goal of a stationarity result for the cross-sectional type distribution, we now define a mapping Γ from Δ to Δ such that, for each $p = (p_1, \ldots, p_K) \in \Delta$, the r-th component of Γ is

$$\Gamma_r(p_1, \ldots, p_K) = q_r \sum_{m=1}^{K} p_m b_{mr} + \sum_{k,l=1}^{K} \frac{\nu_{kl}(r)(1-q_k)(1-q_l) \sum_{m=1}^{K} p_m b_{mk} \sum_{j=1}^{K} p_j b_{jl}}{\sum_{t=1}^{K} (1-q_t) \sum_{j=1}^{K} p_j b_{jt}}. \quad (15)$$

We note that the second term of this expression for $\Gamma_r(p_1, \ldots, p_K)$ can be written as

$$\sum_{k=1}^{K} (1-q_k) \sum_{m=1}^{K} p_m b_{mk} \frac{\sum_{l=1}^{K} \nu_{kl}(r)(1-q_l) \sum_{j=1}^{K} p_j b_{jl}}{\sum_{l=1}^{K} (1-q_l) \sum_{j=1}^{K} p_j b_{jl}},$$

[28] For the conditions of Markov conditional independence in equations (12), (13) and (14), one could state the Markov type property and the independence condition separately. However, this would double the number of equations.

which is less than or equal to $\sum_{l=1}^{K}(1-q_l)\sum_{j=1}^{K}p_j b_{jl}$. This means that one can define $\Gamma_r(p_1,\ldots,p_K)$ to be $q_r\sum_{m=1}^{K}p_m b_{mr}$ when $\sum_{l=1}^{K}(1-q_l)\sum_{j=1}^{K}p_j b_{jl}=0$, in order to have continuity of Γ on all of Δ.

We let $p^n(\omega)_k=\lambda(\{i\in I:\alpha_\omega^n(i)=k\})$ be the fraction of the population of type k at the end of time period n in state of nature ω, and let $\overline{p}\,_k^n$ be it's expectation. That is,

$$\overline{p}\,_k^n=\int_\Omega p^n(\omega)_k\,dP(\omega)=\int_I P(\alpha_i^n=k)\,d\lambda(i), \tag{16}$$

where the last equality follows from the Fubini property.

The following theorem provides an exact law of large numbers and shows stationarity for a dynamical system $\mathbb{D}$ with random mutation, partial matching, and type changing that is Markov conditionally independent in types. Its proof is given in Section 7.3 of Appendix 1.

Theorem 3 *Let $\mathbb{D}$ be a dynamical system with random mutation, partial matching and type changing whose parameters are (p^0,b,q,ν).*[29] *If $\mathbb{D}$ is Markov conditionally independent in types, then:*

1. *For each time $n\geq 1$, the expected cross-sectional type distribution is given by $\overline{p}\,^n=\Gamma(\overline{p}\,^{n-1})=\Gamma^n(p^0)$, and $\tilde{p}_k^n=\sum_{l=1}^{K}b_{lk}\overline{p}\,_l^{n-1}$, where Γ^n is the composition of Γ with itself n times, and where $\tilde{p}^n$ is the expected cross-sectional type distribution after the random mutation (see equation (9)).*

2. *For λ-almost all $i\in I$, $\{\alpha_i^n\}_{n=0}^\infty$ is a Markov chain with transition matrix z^n at time $n-1$ defined by*
$$z_{kl}^n=q_l b_{kl}+\sum_{r,j=1}^{K}\nu_{rj}(l)b_{kr}\frac{(1-q_r)(1-q_j)\tilde{p}_j^n}{\sum_{r'=1}^{K}(1-q_{r'})\tilde{p}_{r'}^n}. \tag{17}$$

3. *For λ-almost all $i\in I$ and λ-almost all $j\in I$, the Markov chains $\{\alpha_i^n\}_{n=0}^\infty$ and $\{\alpha_j^n\}_{n=0}^\infty$ are independent (which means that the random vectors $(\alpha_i^0,\ldots,\alpha_i^n)$ and $(\alpha_j^0,\ldots,\alpha_j^n)$ are independent for all $n\geq 0$).*

4. *For P-almost all $\omega\in\Omega$, the cross-sectional type process $\{\alpha_\omega^n\}_{n=0}^\infty$ is a Markov chain with transition matrix z^n at time $n-1$.*

5. *For P-almost all $\omega\in\Omega$, at each time period $n\geq 1$, the realized cross-sectional type distribution after the random mutation $\lambda(h_\omega^n)^{-1}$ is its expectation $\tilde{p}^n$, and the realized*

[29] It is straightforward to restate the results in this theorem to the case of a dynamical system of random full matching with a complete separable metric type space S and deterministic match induced type changes, and without random mutation; see Remark 2. As noted in Footnote 11, the study of independent random partial matching with random mutation and match induced random type changes in such a general type space is beyond the scope of this paper.

cross-sectional type distribution at the end of period n, $p^n(\omega) = \lambda(\alpha^n_\omega)^{-1}$*, is equal to its expectation* $\overline{p}^{\,n}$*, and thus, P-almost surely,* $p^n = \Gamma^n(p^0)$*.*

6. *There is a stationary distribution* p^**. That is, with initial cross-sectional type distribution* $p^0 = p^*$*, for every* $n \geq 1$*, the realized cross-sectional type distribution* p^n *at time n is* p^* *P-almost surely, and* $z^n = z^1$*. In particular, all of the relevant Markov chains are time-homogeneous with a constant transition matrix having* p^* *as a fixed point.*[30]

5 Existence of random matching models that are independent in types

We now show the existence of a joint agent-probability space, and randomized mutation, partial matching and match-induced type-changing functions that satisfy Markov conditional independence, where the agent space $(I, \mathcal{I}, \lambda)$ is an extension of the classical Lebesgue unit interval $(L, \mathcal{L}, \eta)$ in the sense that $I = L = [0, 1]$, the σ-algebra $\mathcal{I}$ contains the Lebesgue σ-algebra $\mathcal{L}$, and the restriction of λ to $\mathcal{L}$ is the Lebesgue measure η. We also obtain as corollaries existence results for random partial and full matchings in the static case.

First, we present the existence result for the dynamic case. Its proof is given in Appendix 2 (Section 8).

Theorem 4 *Fixing any parameters* p^0 *for the initial cross-sectional type distribution, b for mutation probabilities,* $q \in [0, 1]^S$ *for no-match probabilities, and* ν *for match-induced type-change probabilities, there exists a Fubini extension* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ *such that*

1. *The agent space* $(I, \mathcal{I}, \lambda)$ *is an extension of the Lebesgue unit interval* $(L, \mathcal{L}, \eta)$*.*

2. *There is defined on the Fubini extension a dynamical system* $\mathbb{D}$ *with random mutation, partial matching and type changing that is Markov conditionally independent in types with the parameters* (p^0, b, q, ν)*.*[31]

Next, by restricting the dynamic model of Theorem 4 to the first period without random mutation, we obtain the following existence result for independent random partial matching.

Corollary 1 *For any type distribution* $p = (p_1, \ldots, p_K)$ *on S, and any* $q = (q_1, \ldots, q_K)$ *as no-match probabilities, there exists a Fubini extension* $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ *such that*

[30] Our initial type function α^0 is assumed to be non-random. It is easy to generalize to the case in which α^0 is a function from $I \times \Omega$ to S such that for λ-almost all $i \in I$, $j \in I$, α^0_i and α^0_j are independent. Let $\overline{p}^{\,0}$ be the expected cross-sectional type distribution. Then, all the results in Theorem 3 remain valid. In the case that for λ-almost all $i \in I$, α^0_i has distribution $\overline{p}^{\,0}$, the evolution of the fractions of each type is essentially deterministic and coincides exactly with the evolution of the probability distribution of type for almost every given agent (in comparison with Footnote 15).

[31] The existence of a dynamical system of random full matching with a complete separable metric type space S and deterministic match induced type changes is noted in Remark 4 below.

1. *The agent space $(I, \mathcal{I}, \lambda)$ is an extension of the Lebesgue unit interval $(L, \mathcal{L}, \eta)$.*

2. *There is defined on the Fubini extension an independent-in-types random partial matching π from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to I with type distribution p and with q as the no-match probabilities.*

By taking the no-match probabilities to be zero, we can obtain an existence result for independent random matching where almost all agents are matched. We cannot, however, claim the existence for an independent random full matching, for which all agents are matched. On the other hand, the general constructions in the proof of Theorem 4 can also be used to prove the following corollary.

Corollary 2 *There exists a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ such that*

1. *The agent space $(I, \mathcal{I}, \lambda)$ is an extension of the Lebesgue unit interval $(L, \mathcal{L}, \eta)$.*

2. *For any type distribution $p = (p_1, \ldots, p_K)$ on S, there exists an independent-in-types random full matching π from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to I with type distribution p.*

The proofs for Corollaries 1 and 2 are given in Appendix 2 (Section 8). The dynamic and static matching models described in Theorem 4 and its Corollaries 1 and 2 satisfy the respective conditions of Theorems 3, 2, and 1. Thus, the respective conclusions in Theorems 1, 2, and 3 also hold for these matching models.

6 Discussion

As noted in the introduction, this is the first theoretical treatment of the exact law of large numbers for independent random matching among a continuum population (modeled by an atomless, countably additive probability measure space). All three basic issues concerning independent random matching for a continuum population, namely, mathematical formulation of the analytic framework, proof of general results on the exact law of large numbers for independent random matching, and existence of independent random matching with an extension of the Lebesgue unit interval as the agent space, are addressed for both static and dynamic systems. Our results on dynamical systems with random mutation, random partial matching, and random type changing provide an understanding of the time evolution of the cross-sectional type process, identifying it as a Markov chain with known transition matrices.

Based on the classical asymptotic law of large numbers, Boylan constructed an example of random full matching for a countable population in [7, Proposition 2] with the properties that an individual's probability of matching a type-k agent is the fraction p_k of type-k agents in the total population, and that the asymptotic fraction of type-k agents matching type-l agents in a realized matching approximates $p_k p_l$ almost surely.[32] A repeated matching scheme is then considered in [7] for the dynamic setting.

A special example of random full matching is constructed in [3, Theorem 4.2] for a given type function on the population space $[0, 1]$ by rearranging intervals in $[0, 1]$ through measure-preserving mappings. For repeated matching schemes with an infinite number of time periods, it is recognized in [3, p. 262] that one may run into problems when the matching in the next period follows from the type function in a previous period.[33] It is then proposed to arbitrarily rearrange agents with the same types into half-open intervals. Aside from the question of a natural interpretation of this rearrangement of agents' names using intervals, the random full matching considered in [3] does not satisfy the intuitive idea that agents are matched independently in types. That is, this example is not a model for independent random matching. It is made clear in [3, p. 266] that "This paper should not be viewed as a justification for the informal use of a law of large numbers in random matching with a continuum of agents."

Gilboa and Matsui [26] constructed a particular example for a matching model of two countable populations with a countable number of encounters in the time interval $[0, 1)$, where the space $\mathbb{N}$ of agents is endowed with a purely finitely additive measure μ extending the usual density. They showed that their matching model satisfies a few desired matching properties in their setting, including the fact that an agent is matched exactly once with probability one. Their matching model for a countable population with a countable number of encounters within a continuous time framework is quite different from our static or dynamic matching models for a continuum population. As they also point out, a disadvantage of their approach is that the underlying state of the world is "drawn" according to a purely finitely-additive measure.

In comparison with the particular examples of a non-independent random matching with some matching properties in [7], [3], and [26],[34] we prove the exact law of large numbers for general independent random matchings,[35] which can be applied to different matching schemes. The papers [1] and [2] also formalize a link between matching and informational constraints,

[32]It is not clear whether this example satisfies the kind of condition, independence in types, considered by us.

[33]In the dynamic random matching model defined in the proof of Theorem 3.1 in [19] (and of Theorem 4 here), every step of randomization uses the realized type function generated in the step of randomization immediately before.

[34]As noted in Subsection 3.2 of [44], there are many non-independent random matchings with some matching properties even for finitely many agents.

[35]Independence is in general viewed as a behavioral assumption. When agents make their random choices without explicit coordinations among themselves, it is reasonable to assume independence. It is important to distinguish an ad hoc example with some particular correlation structure from a general result in the setting of law of large numbers. For example, one can take a sequence of bounded random variables $\{\phi_n\}_{n=1}^{\infty}$ with mean zero. When all the odd terms in the sequence equal ϕ_1 and even terms equal $-\phi_1$, then $(\sum_{k=1}^{n} \phi_k)/n$ converges to zero almost surely. Such kind of result will not be useful at all in situations that require the use of the law of large numbers.

which do not consider random matching under the independence assumption as in the models considered here. Random mutation, random partial matching and random type changing induced by matching are not considered in any of those earlier papers.

In addition, our results in Section 5 also provide the first existence results for independent random matching where an extension of the Lebesgue unit interval is used as the agent space. This goes beyond the existence results for independent random matching with a hyperfinite number of agents as studied in [19].

7 Appendix 1 – Proof of Theorems 1, 2 and 3

7.1 Exact law of large numbers for a continuum of independent random variables

The following general version of the exact law of large numbers is shown by Sun in [50], and is stated as a lemma here for the convenience of the reader.[36]

Lemma 1 *Let f be a measurable process from a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ of the usual product probability space to a complete separable metric space X. Assume that the random variables f_i are essentially pairwise independent in the sense that for λ-almost all $i \in I$, the random variables f_i and f_j are independent for λ-almost all $j \in I$.*

1. *For P-almost all $\omega \in \Omega$, the sample distribution λf_ω^{-1} of the sample function f_ω is the same as the distribution $(\lambda \boxtimes P) f^{-1}$ of the process.*[37]

2. *For any $A \in \mathcal{I}$ with $\lambda(A) > 0$, let f^A be the restriction of f to $A \times \Omega$, λ^A and $\lambda^A \boxtimes P$ the probability measures rescaled from the restrictions λ and $\lambda \boxtimes P$ to $\{D \in \mathcal{I} : D \subseteq A\}$ and $\{C \in \mathcal{I} \boxtimes \mathcal{F} : C \subseteq A \times \Omega\}$ respectively. Then for P-almost all $\omega \in \Omega$, the sample distribution $\lambda^A (f^A)_\omega^{-1}$ of the sample function $(f^A)_\omega$ is the same as the distribution of $(\lambda^A \boxtimes P)(f^A)^{-1}$ of the process f^A.*

3. *If there is a distribution μ on X such that for λ-almost all $i \in I$, the random variable f_i has distribution μ, then the sample function f_ω (or $(f^A)_\omega$) also has distribution μ for P-almost all $\omega \in \Omega$.*

[36] Part (2) of the lemma is part of Theorem 2.8 in [50]. That theorem actually shows that the statement in Part (2) here is equivalent to the condition of essential pairwise independence. While Parts (1) and (3) of the lemma are special cases of Part (2), they are stated respectively in Corollary 2.9 and Theorem 2.12 of [50]. In addition, it is noted in [51] that under the condition of essential pairwise independence on the process f, the statement in Part (2) here is equivalent to the existence of a Fubini extension in which f is measurable. Thus, in a certain sense, a Fubini extension provides the only right measure-theoretic framework for working with independent processes (and independent random matchings).

[37] Here, $(\lambda \boxtimes P) f^{-1}$ is the distribution ν on X such that $\nu(B) = (\lambda \boxtimes P)(f^{-1}(B))$ for any Borel set B in X; λf_ω^{-1} is defined similarly.

By viewing a discrete-time stochastic process taking values in X as a random variable taking values in X^∞, Lemma 1 implies the following exact law of large numbers for a continuum of discrete-time stochastic processes, which is formally stated in Theorem 2.16 in [50].

Corollary 3 *Let f be a mapping from $I \times \Omega \times \mathbb{N}$ to a complete separable metric space X such that for each $n \geq 0$, $f^n = f(\cdot, \cdot, n)$ is an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process. Then, for λ-almost all $i \in I$, $\{f_i^n\}_{n=0}^\infty$ is a discrete-time stochastic process. Assume that the stochastic processes $\{f_i^n\}_{n=0}^\infty$, $i \in I$ are essentially pairwise independent, i.e., for λ-almost all $i \in I$, λ-almost all $j \in I$, the random vectors $(f_i^0, \ldots, f_i^n)$ and $(f_j^0, \ldots, f_j^n)$ are independent for all $n \geq 0$. Then, for P-almost all $\omega \in \Omega$, the empirical process $f_\omega = \{f_\omega^n\}_{n=0}^\infty$ has the same finite-dimensional distributions as that of $f = \{f^n\}_{n=0}^\infty$, i.e. $(f_\omega^0, \ldots, f_\omega^n)$ and $(f^0, \ldots, f^n)$ have the same distribution for any $n \geq 0$.*

7.2 Proofs of Theorems 1 and 2

Proof of Theorem 1: If $p_k = 0$, equation (5) is automatically satisfied. Consider $p_k > 0$. Let $I_k = \{i \in I : \alpha(i) = k\}$ and $g = \alpha(\pi)$. Since the random variables g_i are essentially pairwise independent, Lemma 1 (3) implies that the sample function $(g^{I_k})_\omega$ on I_k has distribution p on S for P-almost all $\omega \in \Omega$. This means that $\lambda(\{i \in I_k : g_\omega(i) = l\})/p_k = p_l$ for P-almost all $\omega \in \Omega$. Hence equation (5) follows. ■

Remark 1 *One can restate Definition 3 for the case in which the type space S is a complete separable metric space. It is clear that Theorem 1 still holds in the setting that α is a $\mathcal{I}$-measurable type function from I to such a general type space S. Let p be the induced probability distribution of α on the type space S. Then, if π is independent in types, the joint distribution $\lambda(\alpha, g_\omega)^{-1}$ is simply the product distribution $p \otimes p$ on $S \times S$ for P-almost all $\omega \in \Omega$. This is also a direct consequence of the exact law of large numbers in Lemma 1. One can simply consider the process $G(i, \omega) = (\alpha(i), g(i, \omega))$, which still has essentially pairwise independent random variables. By Lemma 1 (1), we have for P-almost all $\omega \in \Omega$, $\lambda(\alpha, g_\omega)^{-1} = (\lambda \boxtimes P)G^{-1}$, which is simply $p \otimes p$.*

Proof of Theorem 2: The proof is similar to that of Theorem 1; we adopt the same notation and consider only $p_k > 0$. Lemma 1 says that for P-almost all $\omega \in \Omega$, the sample function $g_\omega^{I_k}$ on I_k has the same distribution as g^{I_k} on $I_k \times \Omega$. Hence for P-almost all $\omega \in \Omega$,

$$\lambda^{I_k}\left((g_\omega^{I_k})^{-1}(\{J\})\right) = \left(\lambda^{I_k} \boxtimes P\right)\left((g^{I_k})^{-1}(\{J\})\right),$$

which means that

$$\lambda(\{i \in I : \alpha(i) = k, g_\omega(i) = J\}) = \int_{I_k} \int_\Omega 1_{(g_i = J)}\, dP\, d\lambda = \int_{I_k} q_k\, d\lambda = p_k q_k;^{38}$$

and also for any $1 \leq l \leq K$,

$$\begin{aligned}
\lambda(I_k \cap g_\omega^{-1}(\{l\})) &= (\lambda \boxtimes P)((I_k \times \Omega) \cap g^{-1}(\{l\})) = \int_{I_k} \int_\Omega 1_{(g_i = l)}\, dP\, d\lambda \\
&= \int_{I_k} \frac{(1 - q_k) p_l (1 - q_l)}{\sum_{r=1}^K p_r (1 - q_r)} d\lambda = \frac{p_k (1 - q_k) p_l (1 - q_l)}{\sum_{r=1}^K p_r (1 - q_r)}.
\end{aligned}$$

Thus, equations (6) and (7) follow. ■

7.3 Proof of Theorem 3

Before proving Theorem 3, we need to prove a few lemmas. The first lemma shows how to compute the expected cross-sectional type distributions $\overline{p}^{\,n}$ and $\tilde{p}^n$.

Lemma 2 *(1) For each $n \geq 1$, $\overline{p}^{\,n} = \Gamma(\overline{p}^{\,n-1})$, and hence $\overline{p}^{\,n} = \Gamma^n(p^0)$, where Γ^n is the composition of Γ with itself n times.*

(2) For each $n \geq 1$, the expected cross-sectional type distribution $\tilde{p}^n$ immediately after random mutation at time n, as defined in equation (9), satisfies $\tilde{p}^n_k = \sum_{l=1}^K b_{lk} \overline{p}^{\,n-1}_l = \sum_{l=1}^K b_{lk} \Gamma^{n-1}_l (p^0)$.

Proof. Equations (8) and (9) and the Fubini property imply that

$$\begin{aligned}
\tilde{p}^n_k &= \int_I P(h^n_i = k)\, d\lambda(i) = \int_I \sum_{l=1}^K P(h^n_i = k, \alpha^{n-1}_i = l)\, d\lambda(i) \\
&= \int_I \sum_{l=1}^K P(h^n_i = k \mid \alpha^{n-1}_i = l) P(\alpha^{n-1}_i = l)\, d\lambda(i) \\
&= \sum_{l=1}^K \int_I b_{lk} P(\alpha^{n-1}_i = l)\, d\lambda(i) = \sum_{l=1}^K b_{lk} \overline{p}^{\,n-1}_l. \qquad (18)
\end{aligned}$$

Then, we can express $\overline{p}^{\,n}$ in terms of $\tilde{p}^n$ by equations (10) and (11) as

$$\begin{aligned}
\overline{p}^{\,n}_r &= \int_I P(\alpha^n_i = r)\, d\lambda(i) \\
&= \int_I \sum_{k=1}^K \Big[P(\alpha^n_i = r, h^n_i = k, g^n_i = J) + \sum_{l=1}^K P(\alpha^n_i = r, h^n_i = k, g^n_i = l) \Big] d\lambda(i) \\
&= \int_I \sum_{k=1}^K \Big[P(\alpha^n_i = r \,|\, h^n_i = k, g^n_i = J) P(g^n_i = J \,|\, h^n_i = k) P(h^n_i = k)
\end{aligned}$$

[38] For a set C in a space, 1_C denotes its indicator function.

$$+\sum_{l=1}^{K} P(\alpha_i^n = r \,|\, h_i^n = k, g_i^n = l) P(g_i^n = l \,|\, h_i^n = k) P(h_i^n = k)\Big] d\lambda(i)$$
$$= \tilde{p}_r^n q_r + \sum_{k,l=1}^{K} \frac{\nu_{kl}(r) \tilde{p}_k^n (1 - q_k) \tilde{p}_l^n (1 - q_l)}{\sum_{t=1}^{K} \tilde{p}_t^n (1 - q_t)}. \tag{19}$$

By combining equations (18) and (19), it is easy to see that $\overline{p}^{\,n} = \Gamma(\overline{p}^{\,n-1})$, and hence that $\overline{p}^{\,n} = \Gamma^n(p^0)$, where Γ^n is the composition of Γ with itself n times. Hence, part (1) of the lemma is shown. Part (2) of the lemma follows from part (1) and equation (18). ∎

The following lemma shows the Markov property of the agents' type processes.

Lemma 3 *Suppose the dynamical system $\mathbb{D}$ is Markov conditionally independent in types. Then, for λ-almost all $i \in I$, the type process for agent i, $\{\alpha_i^n\}_{n=0}^{\infty}$, is a Markov chain with transition matrix z^n at time $n-1$, where z_{kl}^n is defined in equation (17).*

Proof. Fix $n \geq 1$. Equation (12) implies that for λ-almost all $i \in I$, λ-almost all $j \in I$,

$$P(h_i^n = k_n, h_j^n \in S \mid \alpha_i^0 = k_0, \ldots, \alpha_i^{n-1} = k_{n-1}; \alpha_j^0 \in S, \ldots, \alpha_j^{n-1} \in S)$$
$$= P(h_i^n = k_n \mid \alpha_i^{n-1} = k_{n-1}) P(h_j^n \in S \mid \alpha_j^{n-1}), \tag{20}$$

holds for any $(k_0, \ldots, k_n) \in S^{n+1}$. Thus, for λ-almost all $i \in I$,

$$P(h_i^n = k \mid \alpha_i^0, \ldots, \alpha_i^{n-1}) = P(h_i^n = k \mid \alpha_i^{n-1}) \tag{21}$$

holds for any $k \in S$. By grouping countably many null sets together, we know that for λ-almost all $i \in I$, equation (21) holds for all $k \in S$ and $n \geq 1$.

Similarly, equations (13) and (14) imply that for λ-almost all $i \in I$,

$$P(g_i^n = c \mid \alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n) = P(g_i^n = c \mid h_i^n)$$
$$P(\alpha_i^n = k \mid \alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n, g_i^n) = P(\alpha_i^n = k \mid h_i^n, g_i^n) \tag{22}$$

hold for all $k \in S$, $c \in S \cup \{J\}$ and $n \geq 1$. A simple computation shows that for λ-almost all $i \in I$, $P(\alpha_i^n = k \mid \alpha_i^0, \ldots, \alpha_i^{n-1}) = P(\alpha_i^n = k \mid \alpha_i^{n-1})$ for all $k \in S$ and $n \geq 1$. Hence, for λ-almost all $i \in I$, agent i's type process $\{\alpha_i^n\}_{n=0}^{\infty}$ is a Markov chain; it is also easy to see that the transition matrix z^n from time $n-1$ to time n is

$$z_{kl}^n = P(\alpha_i^n = l \mid \alpha_i^{n-1} = k)$$
$$= \sum_{r=1}^{K} \sum_{c \in S \cup \{J\}} P(\alpha_i^n = l \mid h_i^n = r, g_i^n = c) P(g_i^n = c \mid h_i^n = r) P(h_i^n = r \mid \alpha_i^{n-1} = k). \tag{23}$$

Then, equations (8), (10) and (11) imply that the formula for z_{kl}^n in equation (17) holds. ∎

Now, for each $n \geq 1$, we view each α^n as a random variable on $I \times \Omega$. Then $\{\alpha^n\}_{n=0}^{\infty}$ is a discrete-time stochastic process.

Lemma 4 *Assume that the dynamical system $\mathbb{D}$ is Markov conditionally independent in types. Then, $\{\alpha^n\}_{n=0}^{\infty}$ is also a Markov chain with transition matrix z^n at time $n-1$ given by equation (17).*

Proof. We can compute the transition matrix of $\{\alpha^n\}_{n=0}^{\infty}$ at time $n-1$ as follows. For any $k, l \in S$, we have

$$
\begin{aligned}
(\lambda \boxtimes P)(\alpha^n = l, \alpha^{n-1} = k) &= \int_I P(\alpha_i^n = l \mid \alpha_i^{n-1} = k) P(\alpha_i^{n-1} = k)\, d\lambda(i) \\
&= \int_I z_{kl}^n P(\alpha_i^{n-1} = k)\, d\lambda(i) \\
&= z_{kl}^n \cdot (\lambda \boxtimes P)(\alpha^{n-1} = k),
\end{aligned} \tag{24}
$$

which implies that $(\lambda \boxtimes P)(\alpha^n = l \mid \alpha^{n-1} = k) = z_{kl}^n$.

Next, for any $n \geq 1$, and for any $(a^0, \ldots, a^{n-2}) \in S^{n-1}$, we have

$$
\begin{aligned}
& (\lambda \boxtimes P)((\alpha^0, \ldots, \alpha^{n-2}) = (a^0, \ldots, a^{n-2}), \alpha^{n-1} = k, \alpha^n = l) \\
= & \int_I P((\alpha_i^0, \ldots, \alpha_i^{n-2}) = (a^0, \ldots, a^{n-2}), \alpha_i^{n-1} = k, \alpha_i^n = l)\, d\lambda(i) \\
= & \int_I P(\alpha_i^n = l \mid \alpha_i^{n-1} = k) P((\alpha_i^0, \ldots, \alpha_i^{n-2}) = (a^0, \ldots, a^{n-2}), \alpha_i^{n-1} = k)\, d\lambda(i) \\
= & z_{kl}^n \cdot (\lambda \boxtimes P)((\alpha^0, \ldots, \alpha^{n-2}) = (a^0, \ldots, a^{n-2}), \alpha^{n-1} = k),
\end{aligned} \tag{25}
$$

which implies that $(\lambda \boxtimes P)(\alpha^n = l \mid (\alpha^0, \ldots, \alpha^{n-2}) = (a^0, \ldots, a^{n-2}), \alpha^{n-1} = k) = z_{kl}^n$. Hence the discrete-time process $\{\alpha^n\}_{n=0}^{\infty}$ is indeed a Markov chain with transition matrix z^n at time $n-1$. ■

To prove that the agents' type processes are essentially pairwise independent in Lemma 6 below, we need the following elementary lemma.

Lemma 5 *Let ϕ_m be a random variable from $(\Omega, \mathcal{F}, P)$ to a finite space A_m, for $m = 1, 2, 3, 4$. If the random variables ϕ_3 and ϕ_4 are independent, and if, for all $a_1 \in A_1$ and $a_2 \in A_2$,*

$$
P(\phi_1 = a_1, \phi_2 = a_2 \mid \phi_3, \phi_4) = P(\phi_1 = a_1 \mid \phi_3) P(\phi_2 = a_2 \mid \phi_4), \tag{26}
$$

then the two pairs of random variables (ϕ_1, ϕ_3) and (ϕ_2, ϕ_4) are independent.

Proof. For any $a_m \in A_m$, $m = 1, 2, 3, 4$, we have

$$
\begin{aligned}
& P(\phi_1 = a_1, \phi_2 = a_2, \phi_3 = a_3, \phi_4 = a_4) \\
= & P(\phi_1 = a_1, \phi_2 = a_2 \mid \phi_3 = a_3, \phi_4 = a_4) P(\phi_3 = a_3, \phi_4 = a_4) \\
= & P(\phi_1 = a_1 \mid \phi_3 = a_3) P(\phi_2 = a_2 \mid \phi_4 = a_4) P(\phi_3 = a_3) P(\phi_4 = a_4) \\
= & P(\phi_1 = a_1, \phi_3 = a_3) P(\phi_2 = a_2, \phi_4 = a_4).
\end{aligned} \tag{27}
$$

Hence, the pairs (ϕ_1, ϕ_3) and (ϕ_2, ϕ_4) are independent. ■

The following lemma is useful for applying the exact law of large numbers in Corollary 3 to Markov chains.

Lemma 6 *Assume that the dynamical system $\mathbb{D}$ is Markov conditionally independent in types. Then, the Markov chains $\{\alpha_i^n\}_{n=0}^{\infty}, i \in I$, are essentially pairwise independent. In addition, the processes h^n and g^n are also essentially pairwise independent for each $n \geq 1$.*

Proof. Let E be the set of all $(i, j) \in I \times I$ such that equations (12), (13) and (14) hold for all $n \geq 1$. Then, by grouping countably many null sets together, we obtain that for λ-almost all $i \in I$, λ-almost all $j \in I$, $(i, j) \in E$, i.e., for λ-almost all $i \in I$, $\lambda(E_i) = \lambda(\{j \in I : (i, j) \in E\}) = 1$.

We can use induction to prove that for any fixed $(i, j) \in E$, $(\alpha_i^0, \ldots, \alpha_i^n)$ and $(\alpha_j^0, \ldots, \alpha_j^n)$ are independent, so are the pairs h_i^n and h_j^n, and g_i^n and g_j^n. This is obvious for $n = 0$. Suppose that it is true for the case $n - 1$, i.e., $(\alpha_i^0, \ldots, \alpha_i^{n-1})$ and $(\alpha_j^0, \ldots, \alpha_j^{n-1})$ are independent, so are the pairs h_i^{n-1} and h_j^{n-1}, and g_i^{n-1} and g_j^{n-1}. Then, the Markov conditional independence condition and Lemma 5 imply that $(\alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n)$ and $(\alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n)$ are independent, so are the pairs $(\alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n, g_i^n)$ and $(\alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n, g_j^n)$, and $(\alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n, g_i^n, \alpha_i^n)$ and $(\alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n, g_j^n, \alpha_j^n)$. Hence, the random vectors $(\alpha_i^0, \ldots, \alpha_i^n)$ and $(\alpha_j^0, \ldots, \alpha_j^n)$ are independent for all $n \geq 0$, which means that the Markov chains $\{\alpha_i^n\}_{n=0}^{\infty}$ and $\{\alpha_j^n\}_{n=0}^{\infty}$ are independent. It is also clear that for each $n \geq 1$, the random variables h_i^n and h_j^n are independent, so are g_i^n and g_j^n. The desired result follows. ■

Proof of Theorem 3: Properties (1), (2), and (3) of the theorem are shown in Lemmas 2, 3, and 6 respectively.

By the exact law of large numbers in Corollary 3, we know that for P-almost all $\omega \in \Omega$, $(\alpha_\omega^0, \ldots, \alpha_\omega^n)$ and $(\alpha^0, \ldots, \alpha^n)$ (viewed as random vectors) have the same distribution for all $n \geq 1$. Since, as noted in Lemma 4, $\{\alpha^n\}_{n=0}^{\infty}$ is a Markov chain with transition matrix z^n at time $n - 1$, so is $\{\alpha_\omega^n\}_{n=0}^{\infty}$ for P-almost all $\omega \in \Omega$. Thus (4) is shown.

Since the processes h^n and g^n are essentially pairwise independent as shown in Lemma 6, the exact law of large numbers in Lemma 1 implies that at time period n, for P-almost all $\omega \in \Omega$, the realized cross-sectional distribution after the random mutation, $p^n(\omega) = \lambda(h_\omega^n)^{-1}$ is the expected cross-sectional distribution $\tilde{p}^n$, and the realized cross-sectional distribution at the end of period n, $p^n(\omega) = \lambda(\alpha_\omega^n)^{-1}$ is the expected cross-sectional distribution $\overline{p}^{\,n}$. Thus, (5) is shown.

To prove (6), note that Γ is a continuous function from Δ to itself. Hence, Brower's Fixed Point Theorem implies that Γ has a fixed point p^*. In this case, $\overline{p}^{\,n} = \Gamma^n(p^*) = p^*$,

$z_{kl}^n = z_{kl}^1$ for all $n \geq 1$. Hence the Markov chains $\{\alpha_i^n\}_{n=0}^\infty$ for λ-almost all $i \in I$, $\{\alpha^n\}_{n=0}^\infty$, $\{\alpha_\omega^n\}_{n=0}^\infty$ for P-almost all $\omega \in \Omega$ are time-homogeneous. ∎

Remark 2 *It is simple to define inductively a dynamical system of random full matching with a complete separable metric type space S and deterministic match-induced type changes, and without random mutation. Let α^0 be an initial measurable type function from I to S with distribution p^0 on S, and ν a deterministic assignment of types for the matched agents that is given by a Borel measurable function from $S \times S$ to S. For each time $n \geq 1$, we first have a random full matching that is described by a function π^n from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to I, and then a new type function α^n from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to S.*

Let $\bar{p}^{n-1}$ be the expected cross-sectional type distribution on S at time $n-1$, i.e., $\bar{p}^{n-1}(\cdot) = \int_\Omega \lambda(\alpha_\omega^{n-1})^{-1}(\cdot)\, dP(\omega)$. The random full matching function π^n at time n is defined by:

1. *For any $\omega \in \Omega$, $\pi_\omega^n(\,\cdot\,)$ is a full matching on I.*

2. *Define $g^n : I \times \Omega \to S$ by $g^n(i,\omega) = \alpha^{n-1}(\pi^n(i,\omega),\omega)$, and assume that g^n is $\mathcal{I} \boxtimes \mathcal{F}$-measurable.*

3. *For λ-almost all agent $i \in I$, the conditional distribution*

$$P\left((g_i^n)^{-1} \mid \alpha_i^{n-1}\right) = \bar{p}^{n-1}. \tag{28}$$

Item 3 means that given agent i's type at time $n-1$, the conditional probability of agent i being matched in time n to an agent with type in a measurable subset B of S is simply the expected proportion of agents with types in B at time $n-1$. Thus, for λ-almost all agent $i \in I$, the random variables α_i^{n-1} and g_i^n are independent with distributions $P(\alpha_i^{n-1})^{-1}$ and $\bar{p}^{n-1}$. We also have $\alpha^n = \nu(\alpha^{n-1}, g^n)$.

We assume that the random full matching π^n is **Markov conditionally independent in types** *in the sense that for λ-almost all $i \in I$ and λ-almost all $j \in I$,*

$$P(g_i^n \in C, g_j^n \in D \mid \alpha_i^0, \ldots, \alpha_i^{n-1}; \alpha_j^0, \ldots, \alpha_j^{n-1}) = P(g_i^n \in C \mid \alpha_i^{n-1}) P(g_j^n \in D \mid \alpha_j^{n-1}) \tag{29}$$

holds for measurable subsets C and D of S.

Define a mapping Γ from the space $\Delta(S)$ of Borel probability measures on S by letting $\Gamma(p) = (p \otimes p)\nu^{-1}$ for each $p \in \Delta(S)$.

For the expected cross-sectional type distribution $\bar{p}^n$ on S at time $n \geq 1$, we have

$$\begin{aligned}
\bar{p}^n(\,\cdot\,) &= \int_\Omega \lambda(\alpha_\omega^n)^{-1}(\,\cdot\,)\, dP = \int_I P(\alpha_i^n)^{-1}(\,\cdot\,)\, d\lambda \\
&= \int_I \left(P(\alpha_i^{n-1})^{-1} \otimes \bar{p}^{n-1}\right) \nu^{-1}(\,\cdot\,)\, d\lambda = \left(\bar{p}^{n-1} \otimes \bar{p}^{n-1}\right) \nu^{-1}(\,\cdot\,),
\end{aligned} \tag{30}$$

where the second equality is obtained by the Fubini property and the third equality follows from the definition of α^n *and equation (28). Thus, we have* $\bar{p}^n = \Gamma(\bar{p}^{n-1}) = \Gamma^n(p^0)$.

For $s \in S$, *let* ν_s *be the function* $\nu(s, \cdot)$ *from* S *to* S. *Define a transition probability* z^n *by letting* z^n_s *be the probability distribution* $\bar{p}^{n-1}\nu_s^{-1}$ *for* $s \in S$. *Then, the same proof as above allows us to claim that (1) for* λ*-almost all* $i \in I$, $\{\alpha^n_i\}_{n=0}^{\infty}$ *is a Markov chain with transition probability* z^n *at time* $n-1$*; (2) for* λ*-almost all* $i \in I$ *and* λ*-almost all* $j \in I$*, the Markov chains* $\{\alpha^n_i\}_{n=0}^{\infty}$ *and* $\{\alpha^n_j\}_{n=0}^{\infty}$ *are independent; (3) for* P*-almost all* $\omega \in \Omega$*, the cross-sectional type process* $\{\alpha^n_\omega\}_{n=0}^{\infty}$ *is a Markov chain with transition matrix* z^n *at time* $n-1$*; (4) for* P*-almost all* $\omega \in \Omega$*, the realized cross-sectional type distribution at each period* $n \geq 1$, $p^n(\omega) = \lambda(\alpha^n_\omega)^{-1}$*, is equal to its expectation* $\overline{p}^{\,n}$*, and thus,* P*-almost surely,* $p^n = \Gamma^n(p^0)$.

In addition, if S *is compact and* ν *is continuous, then* Γ *is a continuous mapping from the the compact and convex space* $\Delta(S)$ *to itself. The classical Tychonoff Fixed Point Theorem implies the existence of a stationary distribution* p^* *with* $\Gamma(p^*) = p^*$. *That is, with initial cross-sectional type distribution* $p^0 = p^*$*, for every* $n \geq 1$*, the realized cross-sectional type distribution* p^n *at time* n *is* p^* P*-almost surely, and* $z^n = z^1$. *In particular, all of the relevant Markov chains are time-homogeneous with transition probability* z^1 *having* p^* *as a fixed point.*

8 Appendix 2 – Proofs of results in Section 5

In this appendix, the unit interval $[0, 1]$ will have a different notation in a different context. Recall that $(L, \mathcal{L}, \eta)$ is the Lebesgue unit interval, where η is the Lebesgue measure defined on the Lebesgue σ-algebra $\mathcal{L}$.

The following result is Theorem 3.1 in [19]. Note that the agent space used in the proof of Theorem 3.1 in [19] is a hyperfinite Loeb counting probability space. Using the usual ultrapower construction as in [40], the hyperfinite index set of agents can be viewed as an equivalence class of a sequence of finite sets with elements in natural numbers, and thus this index set of agents has the cardinality of the continuum.[39]

Proposition 1 *Fixing any parameters* p^0 *for the initial cross-sectional type distribution,* b *for mutation probabilities,* $q \in [0, 1]^S$ *for no-match probabilities, and* ν *for match-induced type-change probabilities, there exist (1) an atomless probability space* $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$ *of agents, where the index space* $\hat{I}$ *has cardinality of the continuum; (2) a sample probability space* $(\Omega, \mathcal{F}, P)$*; and (3) a Fubini extension* $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$ *on which is defined a dynamical system* $\hat{\mathbb{D}}$ *with random mutation, partial matching and type changing that is Markov conditionally independent in types with these parameters* (p^0, b, q, ν).

[39] The notation for the agent space in the proof of Theorem 3.1 in [19] is $(I, \mathcal{I}, \lambda)$, which is replaced by the notation $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$ in Proposition 1 here. The notation $(I, \mathcal{I}, \lambda)$ will be used below for a different purpose.

The purpose of Theorem 4 in this paper is to show that one can find some extension of the Lebesgue unit interval as the agent space so that the associated version of Proposition 1 still holds.

Fix a set $\hat{I}$ with cardinality of the continuum as in Proposition 1.[40] The following lemma is a strengthened version of Lemma 2 in [34]; see also Lemma 419I of Fremlin [21] and Lemma 3 in [52].[41] The proof given below is a slight modification of the proof of Lemma 2 in [34].

Lemma 7 *There is a disjoint family $\mathcal{C} = \{C_{\hat{i}} : \hat{i} \in \hat{I}\}$ of subsets of $L = [0,1]$ such that $\bigcup_{\hat{i}\in\hat{I}} C_{\hat{i}} = L$, and for each $\hat{i} \in \hat{I}$, $C_{\hat{i}}$ has the cardinality of the continuum, $\eta_*(C_{\hat{i}}) = 0$ and $\eta^*(C_{\hat{i}}) = 1$, where η_* and η^* are, respectively, the inner and outer measures of the Lebesgue measure η.*

Proof. Let c be the cardinality of the continuum. As usual in set theory, c can be viewed as the set of all ordinals below the cardinality of the continuum. Let $\mathcal{H}$ be the family of closed subsets of $L = [0,1]$ with positive Lebesgue measure. Then, the cardinality of $\mathcal{H}$ is c, and hence the cardinality of $\mathcal{H} \times c$ is c as well. Enumerate the elements in $\mathcal{H} \times c$ as a transfinite sequence $\{(F_\xi, \alpha_\xi)\}_{\xi<c}$, where ξ is an ordinal.

Define a transfinite sequence $\{x_\xi\}_{\xi<c}$ by transfinite induction as follows. Suppose that for an ordinal $\xi < c$, $\{x_\beta\}_{\beta<\xi}$ is defined. Note that the set of elements $\{x_\beta\}_{\beta<\xi}$ has cardinality strictly less than the continuum. Since F_ξ has the cardinality of the continuum, one can take any x_ξ from the nonempty set $F_\xi \setminus \{x_\beta\}_{\beta<\xi}$. One can continue this procedure to define the whole transfinite sequence $\{x_\xi\}_{\xi<c}$. Note that the elements in the transfinite sequence $\{x_\xi\}_{\xi<c}$ are all distinct.

For each ordinal $\alpha < c$, let A_α be the set of all the x_ξ with $\xi < c$ and $\alpha_\xi = \alpha$, that is, $A_\alpha = \{x_\xi : \xi < c, \alpha_\xi = \alpha\}$. It is clear that the sets A_α, $\alpha < c$ are disjoint.

Next, fix an ordinal $\alpha < c$. Since $\{(F_\xi, \alpha_\xi)\}_{\xi<c}$ enumerates the elements in $\mathcal{H} \times c$, the set $\{F_\xi : \xi < c, \alpha_\xi = \alpha\}$ equals $\mathcal{H}$, which has cardinality of the continuum. Thus, $\{\xi : \xi < c, \alpha_\xi = \alpha\}$ has the cardinality of the continuum. Since the elements in $\{x_\xi\}_{\xi<c}$ are all distinct, the set $A_\alpha = \{x_\xi : \xi < c, \alpha_\xi = \alpha\}$ has the cardinality of the continuum as well.

Suppose that the inner measure $\eta_*(L \setminus A_\alpha)$ is positive. Then there is a Lebesgue measurable subset E of $L \setminus A_\alpha$ with $\eta(E) > 0$, which implies the existence of $F \in \mathcal{H}$ with $F \subseteq E \subseteq L \setminus A_\alpha$. Let $\xi < c$ be the unique ordinal such that $F_\xi = F$ and $\alpha_\xi = \alpha$. By the definitions of x_ξ and A_α, we have $x_\xi \in F_\xi$ and $x_\xi \in A_\alpha$, and hence $x_\xi \in F \cap A_\alpha$. However,

[40] Note that we replace the corresponding notation $(K, \mathcal{K}, \kappa)$ used in the Appendix of [52] by $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$ in this paper. The reason is that the notation K has been used earlier as the number of agent types.

[41] The original version of Lemma 2 of [34] and Lemma 419I of Fremlin [21] requires neither that each $C_{\hat{i}}$ has cardinality of the continuum nor that $\bigcup_{\hat{i}\in\hat{I}} C_{\hat{i}} = L$.

$F \subseteq L \setminus A_\alpha$, which means that $F \cap A_\alpha = \emptyset$. This is a contradiction. Hence, $\eta_*(L \setminus A_\alpha) = 0$, which means that the outer measure $\eta^*(A_\alpha) = 1$. It is clear that $0 \leq \eta_*(A_\alpha) \leq \eta_*(L \setminus A_{\alpha+1}) = 0$. Therefore $\eta_*(A_\alpha) = 0$.

Finally, since $\hat{I}$ has the cardinality of the continuum, there is a bijection $\hat{\xi}$ between $\hat{I}$ and c. For each $\hat{i} \in \hat{I}$, let $C_{\hat{i}} = A_{\hat{\xi}(\hat{i})}$. Let $B = L \setminus \bigcup_{\hat{i} \in \hat{I}} C_{\hat{i}}$. Since the cardinality of B is at most the cardinality of the continuum, we can redistribute at most one point of B into each $C_{\hat{i}}$ in the family $\mathcal{C} = \{C_{\hat{i}} : \hat{i} \in \hat{I}\}$. The rest is clear. ∎

Kakutani [34] provided a non-separable extension of the Lebesgue unit interval by adding subsets of the unit interval directly. As in the Appendix of [52], we follow some constructions used in the proof of Lemma 521P(b) of [22], which allows one to work with Fubini extensions in a more transparent way. The spirit of the Lebesgue extension itself is similar in the constructions used in [34] and here. Define a subset C of $L \times \hat{I}$ by letting $C = \{(l, \hat{i}) \in L \times \hat{I} : l \in C_{\hat{i}}, \hat{i} \in \hat{I}\}$. Let $(L \times \hat{I}, \mathcal{L} \otimes \hat{\mathcal{I}}, \eta \otimes \hat{\lambda})$ be the usual product probability space. For any $\mathcal{L} \otimes \hat{\mathcal{I}}$-measurable set U that contains C, $C_{\hat{i}} \subseteq U_{\hat{i}}$ for each $\hat{i} \in \hat{I}$, where $U_{\hat{i}} = \{l \in L : (l, \hat{i}) \in U\}$ is the $\hat{i}$-section of U. The Fubini property of $\eta \otimes \hat{\lambda}$ implies that for $\hat{\lambda}$-almost all $\hat{i} \in \hat{I}$, $U_{\hat{i}}$ is $\mathcal{L}$-measurable, which means that $\eta(U_{\hat{i}}) = 1$ (since $\eta^*(C_{\hat{i}}) = 1$). Since $\eta \otimes \hat{\lambda}(U) = \int_{\hat{I}} \eta(U_{\hat{i}})\, d\hat{\lambda}$, we have $\eta \otimes \hat{\lambda}(U) = 1$. Therefore, the $\eta \otimes \hat{\lambda}$-outer measure of C is one.

Since the $\eta \otimes \hat{\lambda}$-outer measure of C is one, the method in [16] (see p. 69) can be used to extend $\eta \otimes \hat{\lambda}$ to a measure γ on the σ-algebra $\mathcal{U}$ generated by the set C and the sets in $\mathcal{L} \otimes \hat{\mathcal{I}}$ with $\gamma(C) = 1$. It is easy to see that $\mathcal{U} = \left\{(U^1 \cap C) \cup (U^2 \setminus C) : U^1, U^2 \in \mathcal{L} \otimes \hat{\mathcal{I}}\right\}$, and that $\gamma[(U^1 \cap C) \cup (U^2 \setminus C)] = \eta \otimes \hat{\lambda}(U^1)$ for any measurable sets $U^1, U^2 \in \mathcal{L} \otimes \hat{\mathcal{I}}$. Let $\mathcal{T}$ be the σ-algebra $\{U \cap C : U \in \mathcal{L} \otimes \hat{\mathcal{I}}\}$, which is the collection of all the measurable subsets of C in $\mathcal{U}$. The restriction of γ to $(C, \mathcal{T})$ is still denoted by γ. Then, $\gamma(U \cap C) = \eta \otimes \hat{\lambda}(U)$, for every measurable set $U \in \mathcal{L} \otimes \hat{\mathcal{I}}$. Note that $(L \times \hat{I}, \mathcal{U}, \gamma)$ is an extension of $(L \times \hat{I}, \mathcal{L} \otimes \hat{\mathcal{I}}, \eta \otimes \hat{\lambda})$.

Consider the projection mapping $p^L : L \times \hat{I} \to L$ with $p^L(l, \hat{i}) = l$. Let ψ be the restriction of p^L to C. Since the family $\mathcal{C}$ is a partition of $L = [0, 1]$, ψ is a bijection between C and L. It is obvious that p^L is a measure-preserving mapping from $(L \times \hat{I}, \mathcal{L} \otimes \hat{\mathcal{I}}, \eta \otimes \hat{\lambda})$ to $(L, \mathcal{L}, \eta)$ in the sense that for any $B \in \mathcal{L}$, $(p^L)^{-1}(B) \in \mathcal{L} \otimes \hat{\mathcal{I}}$ and $\eta \otimes \hat{\lambda}[(p^L)^{-1}(B)] = \eta(B)$; and thus p^L is a measure-preserving mapping from $(L \times \hat{I}, \mathcal{U}, \gamma)$ to $(L, \mathcal{L}, \eta)$. Since $\gamma(C) = 1$, ψ is a measure-preserving mapping from $(C, \mathcal{T}, \gamma)$ to $(L, \mathcal{L}, \eta)$, that is, $\gamma[\psi^{-1}(B)] = \eta(B)$ for any $B \in \mathcal{L}$.

To introduce one more measure structure on the unit interval $[0, 1]$, we shall also denote it by I. Let $\mathcal{I}$ be the σ-algebra $\{S \subseteq I : \psi^{-1}(S) \in \mathcal{T}\}$. Define a set function λ on $\mathcal{I}$ by letting $\lambda(S) = \gamma[\psi^{-1}(S)]$ for each $S \in \mathcal{I}$. Since ψ is a bijection, λ is a well-defined probability measure on $(I, \mathcal{I})$. Moreover, ψ is also an isomorphism from $(C, \mathcal{T}, \gamma)$ to $(I, \mathcal{I}, \lambda)$. Since ψ

is a measure-preserving mapping from $(C, \mathcal{T}, \gamma)$ to $(L, \mathcal{L}, \eta)$, it is obvious that $(I, \mathcal{I}, \lambda)$ is an extension of the Lebesgue unit interval $(L, \mathcal{L}, \eta)$.

We shall now follow the procedure used in the proof of Proposition 2 in [52] to construct a Fubini extension based on the probability spaces $(I, \mathcal{I}, \lambda)$ as defined above, and $(\Omega, \mathcal{F}, P)$ as in our Proposition 1 here.

First, consider the usual product space $(L \times \hat{I} \times \Omega, \mathcal{L} \otimes (\hat{\mathcal{I}} \boxtimes \mathcal{F}), \eta \otimes (\hat{\lambda} \boxtimes P))$ of the Lebesgue unit interval $(L, \mathcal{L}, \eta)$ with the Fubini extension $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$. The following lemma is shown in Step 1 of the proof of Proposition 2 in [52].

Lemma 8 *The probability space $(L \times \hat{I} \times \Omega, \mathcal{L} \otimes (\hat{\mathcal{I}} \boxtimes \mathcal{F}), \eta \otimes (\hat{\lambda} \boxtimes P))$ is a Fubini extension of the usual triple product space $((L \times \hat{I}) \times \Omega, (\mathcal{L} \otimes \hat{\mathcal{I}}) \otimes \mathcal{F}, (\eta \otimes \hat{\lambda}) \otimes P)$.*

Next, as shown in Step 2 of the proof of Proposition 2 in [52], the set $C \times \Omega$ has $\eta \otimes (\hat{\lambda} \boxtimes P)$-outer measure one. Based on the Fubini extension $(L \times \hat{I} \times \Omega, \mathcal{L} \otimes (\hat{\mathcal{I}} \boxtimes \mathcal{F}), \eta \otimes (\hat{\lambda} \boxtimes P))$, we can construct a measure structure on $C \times \Omega$ as follows. Let $\mathcal{E} = \{D \cap (C \times \Omega) : D \in \mathcal{L} \otimes (\hat{\mathcal{I}} \boxtimes \mathcal{F})\}$ (which is a σ-algebra on $C \times \Omega$), and τ be the set function on $\mathcal{E}$ defined by $\tau(D \cap (C \times \Omega)) = \eta \otimes (\hat{\lambda} \boxtimes P)(D)$ for any measurable set D in $\mathcal{L} \otimes (\hat{\mathcal{I}} \boxtimes \mathcal{F})$.[42] Then, τ is a well-defined probability measure on $(C \times \Omega, \mathcal{E})$ since the $\eta \otimes (\hat{\lambda} \boxtimes P)$-outer measure of $C \times \Omega$ is one. The result in the following lemma is shown in Step 2 of the proof of Proposition 2 in [52].

Lemma 9 *The probability space $(C \times \Omega, \mathcal{E}, \tau)$ is a Fubini extension of the usual product probability space $(C \times \Omega, \mathcal{T} \otimes \mathcal{F}, \gamma \otimes P)$.*

Let Ψ be the mapping $(\psi, \mathrm{Id}_\Omega)$ from $C \times \Omega$ to $I \times \Omega$, where Id_Ω is the identity map on Ω. That is, for each $(l, \hat{i}) \in C$, $\omega \in \Omega$, $\Psi((l, \hat{i}), \omega) = (\psi, \mathrm{Id}_\Omega)((l, \hat{i}), \omega) = (\psi(l, \hat{i}), \omega)$. Since ψ is a bijection from C to I, Ψ is a bijection from $C \times \Omega$ to $I \times \Omega$. Let $\mathcal{W} = \{H \subseteq I \times \Omega : \Psi^{-1}(H) \in \mathcal{E}\}$; then $\mathcal{W}$ is a σ-algebra of subsets of $I \times \Omega$. Define a probability measure ρ on $\mathcal{W}$ by letting $\rho(H) = \tau[\Psi^{-1}(H)]$ for any $H \in \mathcal{W}$. Therefore, Ψ is an isomorphism from the probability space $(C \times \Omega, \mathcal{E}, \tau)$ to the probability space $(I \times \Omega, \mathcal{W}, \rho)$. The following lemma is shown in Step 3 of the proof of Proposition 2 in [52].

Lemma 10 *The probability space $(I \times \Omega, \mathcal{W}, \rho)$ is a Fubini extension of the usual product probability space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$.*

Since $(I \times \Omega, \mathcal{W}, \rho)$ is a Fubini extension, we shall follow Definition 1 to denote $(I \times \Omega, \mathcal{W}, \rho)$ by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

[42] We replace here the notation "ν" used in the Appendix of [52] with "τ" here, because "ν" has been used earlier here for match-induced type-change probabilities.

Now, define a mapping φ from I to $\hat{I}$ by letting $\varphi(i) = \hat{i}$ if $i \in C_{\hat{i}}$. Since the family $\mathcal{C} = \{C_{\hat{i}} : \hat{i} \in \hat{I}\}$ is a partition of $I = [0,1]$, φ is well-defined.

Lemma 11 *The following properties of φ hold.*

1. *The mapping φ is measure preserving from $(I, \mathcal{I}, \lambda)$ to $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$, in the sense that for any $A \in \hat{\mathcal{I}}$, $\varphi^{-1}(A)$ is measurable in $\mathcal{I}$ with $\lambda[\varphi^{-1}(A)] = \hat{\lambda}(A)$.*

2. *Let Φ be the mapping $(\varphi, \mathrm{Id}_\Omega)$ from $I \times \Omega$ to $\hat{I} \times \Omega$, that is, $\Phi(i,\omega) = (\varphi, \mathrm{Id}_\Omega)(i,\omega) = (\varphi(i), \omega)$ for any $(i,\omega) \in I \times \Omega$. Then Φ is measure preserving from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$ in the sense that for any $V \in \hat{\mathcal{I}} \boxtimes \mathcal{F}$, $\Phi^{-1}(V)$ is measurable in $\mathcal{I} \boxtimes \mathcal{F}$ with $(\lambda \boxtimes P)[\Phi^{-1}(V)] = (\hat{\lambda} \boxtimes P)(V)$.*

Proof. Property (1) obviously follows from (2) by considering those sets V in the form of $A \times \Omega$ for $A \in \hat{\mathcal{I}}$. Thus, we only need to prove (2). Consider the projection mapping $p^{\hat{I} \times \Omega} : L \times \hat{I} \times \Omega \to \hat{I} \times \Omega$ with $p^{\hat{I} \times \Omega}(l, \hat{i}, \omega) = (\hat{i}, \omega)$. Let Ψ_1 be the restriction of $p^{\hat{I} \times \Omega}$ to $C \times \Omega$.

Fix any $(i,\omega) \in I \times \Omega$. There is a unique $\hat{i} \in \hat{I}$ such that $i \in C_{\hat{i}}$. Thus, $\varphi(i) = \hat{i}$, and $(i, \hat{i}) \in C$ by the definition of C. We also have $\psi(i, \hat{i}) = i$, $\psi^{-1}(i) = (i, \hat{i})$, and $\Psi^{-1}(i,\omega) = ((i,\hat{i}), \omega)$. Note that Ψ^{-1} is a well-defined mapping from $I \times \Omega$ to $C \times \Omega$ since Ψ is a bijection from $C \times \Omega$ to $I \times \Omega$. Hence, we have

$$\Psi_1[\Psi^{-1}(i,\omega)] = (\hat{i}, \omega) = (\varphi(i), \omega) = \Phi(i,\omega).$$

Therefore Φ is the composition mapping $\Psi_1[\Psi^{-1}]$.

Fix any $V \in \hat{\mathcal{I}} \boxtimes \mathcal{F}$. We have $\Phi^{-1}(V) = \Psi[\Psi_1^{-1}(V)]$. By the definition of Ψ_1, we obtain that $\Psi_1^{-1}(V) = (L \times V) \cap (C \times \Omega)$, which is obviously measurable in $\mathcal{E}$. For simplicity, we denote the set $\Psi_1^{-1}(V)$ by E . It follows from the definition of τ that $\tau(E) = \eta \otimes (\hat{\lambda} \boxtimes P)(L \times V) = (\hat{\lambda} \boxtimes P)(V)$. Since Ψ is an isomorphism from the probability space $(C \times \Omega, \mathcal{E}, \tau)$ to the probability space $(I \times \Omega, \mathcal{W}, \rho)$, we know that $\Psi(E)$ is measurable in $\mathcal{W}$ and $\rho[\Psi(E)] = \tau(E) = (\hat{\lambda} \boxtimes P)(V)$. It is clear that $\Psi(E) = \Phi^{-1}(V)$. Therefore, $\Phi^{-1}(V)$ is measurable in $\mathcal{W}$ with $\rho[\Phi^{-1}(V)] = (\hat{\lambda} \boxtimes P)(V)$. The rest follows from the fact that $(I \times \Omega, \mathcal{W}, \rho)$ is denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$. ■

For notational convenience, we let $\hat{\mathbb{D}}$ denote the dynamical system with random mutation, partial matching and type changing that is Markov conditionally independent in types with parameters (p^0, b, q, ν), as presented in Proposition 1 here and Theorem 3.1 in [19]. For $\hat{\mathbb{D}}$, we add a hat to the relevant type functions, random mutation functions, and random assignments of types for the matched agents. Let $\hat{\alpha}^0 : \hat{I} \to S = \{1, \ldots, K\}$ be an initial $\hat{\mathcal{I}}$-measurable type function with distribution p^0 on S.

For each time period $n \geq 1$, $\hat{h}^n$ is a random mutation function from $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$ to S such that for each agent $\hat{i} \in \hat{I}$, and for any types $k, l \in S$,

$$P\left(\hat{h}^n_{\hat{i}} = l \mid \hat{\alpha}^{n-1}_{\hat{i}} = k\right) = b_{kl}. \tag{31}$$

The expected cross-sectional type distribution immediately after random mutation $\tilde{p}^n$ follows from the recursive formula in part (1) of Theorem 3.

The random partial matching at time n is described by a function $\hat{\pi}^n$ from $\hat{I} \times \Omega$ to $\hat{I} \cup \{J\}$ such that

1. For any $\omega \in \Omega$, $\hat{\pi}^n_\omega(\cdot)$ is a full matching on $\hat{I} - (\hat{\pi}^n_\omega)^{-1}(\{J\})$. For simplicity, the set $\hat{I} - (\hat{\pi}^n_\omega)^{-1}(\{J\})$ will be denoted by $\hat{H}^n_\omega$.

2. $\hat{g}^n$ is a $\hat{\mathcal{I}} \boxtimes \mathcal{F}$-measurable mapping from $\hat{I} \times \Omega$ to $S \cup \{J\}$ with $\hat{g}^n(\hat{i}, \omega) = \hat{h}^n(\hat{\pi}^n(\hat{i}, \omega), \omega)$, where we assume that $\hat{h}^n(J, \omega) = J$ for any $\omega \in \Omega$.

3. For each agent $i \in I$ and for any types $k, l \in S$,

$$\begin{aligned} P\left(\hat{g}^n_{\hat{i}} = J \mid \hat{h}^n_{\hat{i}} = k\right) &= q_k, \\ P\left(\hat{g}^n_{\hat{i}} = l \mid \hat{h}^n_{\hat{i}} = k\right) &= \frac{(1-q_k)(1-q_l)\tilde{p}^n_l}{\sum_{r=1}^K (1-q_r)\tilde{p}^n_r}. \end{aligned} \tag{32}$$

A random assignment of types for the matched agents at time n is a function $\hat{\alpha}^n$ from $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$ to S such that for each agent $\hat{i} \in \hat{I}$,

$$\begin{aligned} P\left(\hat{\alpha}^n_{\hat{i}} = r \mid \hat{h}^n_{\hat{i}} = k, \hat{g}^n_{\hat{i}} = J\right) &= \delta^r_k, \\ P\left(\hat{\alpha}^n_{\hat{i}} = r \mid \hat{h}^n_{\hat{i}} = k, \hat{g}^n_{\hat{i}} = l\right) &= \nu_{kl}(r). \end{aligned} \tag{33}$$

Proof of Theorem 4: Based on the dynamical system $\hat{\mathbb{D}}$ on the Fubini extension $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$, we shall now define, inductively, a new dynamical system $\mathbb{D}$ on the Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

We first fix some bijections between the $\hat{i}$-sections of the set C. For any $\hat{i}, \hat{i}' \in \hat{I}$ with $\hat{i} \neq \hat{i}'$, let $\Theta^{\hat{i},\hat{i}'}$ be a bijection from $C_{\hat{i}}$ to $C_{\hat{i}'}$, and $\Theta^{\hat{i}',\hat{i}}$ be the inverse mapping of $\Theta^{\hat{i},\hat{i}'}$. This is possible since both $C_{\hat{i}}$ and $C_{\hat{i}'}$ have cardinality of the continuum, as noted in Lemma 7.

Let α^0 be the mapping $\hat{\alpha}^0(\varphi)$ from I to S. By the measure preserving property of φ in Lemma 11, we know that α^0 is $\mathcal{I}$-measurable type function with distribution p^0 on S.

For each time period $n \geq 1$, let h^n and α^n be the respective mappings $\hat{h}^n(\Phi)$ and $\hat{\alpha}^n(\Phi)$ from $I \times \Omega$ to S. Define a mapping π^n from $I \times \Omega$ to $I \cup \{J\}$ such that for each $(i, \omega) \in I \times \Omega$,

$$\pi^n(i, \omega) = \begin{cases} J & \text{if } \hat{\pi}^n_\omega(\varphi(i)) = J, \\ \Theta^{\varphi(i),\, \hat{\pi}^n_\omega(\varphi(i))}(i) & \text{if } \hat{\pi}^n_\omega(\varphi(i)) \neq J. \end{cases}$$

When $\hat{\pi}^n_\omega(\varphi(i)) \neq J$, $\hat{\pi}^n_\omega$ defines a full matching on $\hat{H}^n_\omega = \hat{I} - (\hat{\pi}^n_\omega)^{-1}(\{J\})$, which implies that $\hat{\pi}^n_\omega(\varphi(i)) \neq \varphi(i)$. Hence, π^n is a well-defined mapping from $I \times \Omega$ to $I \cup \{J\}$.

Since Φ is measure-preserving and $\hat{h}^n$ is a measurable mapping from $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$ to S, h^n is $\mathcal{I} \boxtimes \mathcal{F}$-measurable. By the definitions of h^n and α^n, it is obvious that for each $i \in I$,

$$h^n_i = \hat{h}^n_{\varphi(i)} \text{ and } \alpha^n_i = \hat{\alpha}^n_{\varphi(i)}, \tag{34}$$

which, together with equation (31), implies that

$$P\left(h^n_i = l \mid \alpha^{n-1}_i = k\right) = P\left(\hat{h}^n_{\varphi(i)} = l \mid \hat{\alpha}^{n-1}_{\varphi(i)} = k\right) = b_{kl}. \tag{35}$$

Next, we consider the partial matching property of π^n.

1. Fix any $\omega \in \Omega$. Let $H^n_\omega = I - (\pi^n_\omega)^{-1}(\{J\})$; then $H^n_\omega = \varphi^{-1}(\hat{H}^n_\omega)$. Pick any $i \in H^n_\omega$ and denote $\pi^n_\omega(i)$ by j. Then, $\varphi(i) \in \hat{H}^n_\omega$. The definition of π^n implies that $j = \Theta^{\varphi(i),\, \hat{\pi}^n_\omega(\varphi(i))}(i)$. Since $\Theta^{\varphi(i),\, \hat{\pi}^n_\omega(\varphi(i))}$ is a bijection between $C_{\varphi(i)}$ and $C_{\hat{\pi}^n_\omega(\varphi(i))}$, it follows that $\varphi(j) = \varphi(\pi^n_\omega(i)) = \hat{\pi}^n_\omega(\varphi(i))$ by the definition of φ. Thus, $j = \Theta^{\varphi(i),\, \varphi(j)}(i)$. Since the inverse of $\Theta^{\varphi(i),\, \varphi(j)}$ is $\Theta^{\varphi(j),\, \varphi(i)}$, we know that $\Theta^{\varphi(j),\, \varphi(i)}(j) = i$. By the full matching property of $\hat{\pi}^n_\omega$, $\varphi(j) \neq \varphi(i)$, $\varphi(j) \in \hat{H}^n_\omega$ and $\hat{\pi}^n_\omega(\varphi(j)) = \varphi(i)$. Hence, we have $j \neq i$, and

   $$\pi^n_\omega(j) = \Theta^{\varphi(j),\, \hat{\pi}^n_\omega(\varphi(j))}(j) = \Theta^{\varphi(j),\, \varphi(i)}(j) = i.$$

   This means that the composition of π^n_ω with itself on H^n_ω is the identity mapping on H^n_ω, which also implies that π^n_ω is a bijection on H^n_ω. Therefore π^n_ω is a full matching on $H^n_\omega = I - (\pi^n_\omega)^{-1}(\{J\})$.

2. Extending h^n so that $h^n(J, \omega) = J$ for any $\omega \in \Omega$, we define $g^n : I \times \Omega \to S \cup \{J\}$ by $g^n(i, \omega) = h^n(\pi^n(i, \omega), \omega)$. Denote $\varphi(J) = J$. As noted in the above paragraph, for any fixed $\omega \in \Omega$, $\varphi(\pi^n_\omega(i)) = \hat{\pi}^n_\omega(\varphi(i))$ for $i \in H^n_\omega$. When $i \notin H^n_\omega$, we have $\varphi(i) \notin \hat{H}^n_\omega$, and $\pi^n_\omega(i) = J$, $\hat{\pi}^n_\omega(\varphi(i)) = J$. Therefore, $\varphi(\pi^n_\omega(i)) = \hat{\pi}^n_\omega(\varphi(i))$ for any $i \in I$. Then,

   $$g^n(i, \omega) = \hat{h}^n(\varphi(\pi^n(i, \omega)), \omega) = \hat{h}^n(\hat{\pi}^n(\varphi(i), \omega), \omega) = \hat{g}^n(\varphi(i), \omega) = \hat{g}^n(\Phi)(i, \omega).$$

   Hence, the measure-preserving property of Φ implies that g^n is $\mathcal{I} \boxtimes \mathcal{F}$-measurable. The above equation also means that

   $$g^n_i(\cdot) = \hat{g}^n_{\varphi(i)}(\cdot), \qquad i \in I. \tag{36}$$

3. Equations (32), (34) and (36) imply that for each agent $i \in I$,

   $$\begin{aligned} P\left(g^n_i = J \mid h^n_i = k\right) &= P\left(\hat{g}^n_{\varphi(i)} = J \mid \hat{h}^n_{\varphi(i)} = k\right) = q_k, \\ P\left(g^n_i = l \mid h^n_i = k\right) &= P\left(\hat{g}^n_{\varphi(i)} = l \mid \hat{h}^n_{\varphi(i)} = k\right) = \frac{(1-q_k)(1-q_l)\tilde{p}^n_l}{\sum_{r=1}^K (1-q_r)\tilde{p}^n_r}. \end{aligned} \tag{37}$$

Now, we consider the type-changing function α^n for the matched agents. Since Φ is measure-preserving and $\hat{\alpha}^n$ is a measurable mapping from $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$ to S, α^n is $\mathcal{I} \boxtimes \mathcal{F}$-measurable. Equations (33), (34) and (36) imply that for each agent $i \in I$,

$$\begin{aligned} P\left(\alpha_i^n = r \mid h_i^n = k, g_i^n = J\right) &= P\left(\hat{\alpha}_{\varphi(i)}^n = r \mid \hat{h}_{\varphi(i)}^n = k, \hat{g}_{\varphi(i)}^n = J\right) = \delta_k^r, \\ P\left(\alpha_i^n = r \mid h_i^n = k, g_i^n = l\right) &= P\left(\hat{\alpha}_{\varphi(i)}^n = r \mid \hat{h}_{\varphi(i)}^n = k, \hat{g}_{\varphi(i)}^n = l\right) = \nu_{kl}(r). \end{aligned} \tag{38}$$

Therefore, $\mathbb{D}$ is a dynamical system with random mutation, partial matching and type changing and with the parameters (p^0, b, q, ν).

It remains to check the Markov conditional independence for $\mathbb{D}$. Since the dynamical system $\hat{\mathbb{D}}$ is Markov conditionally independent in types, for each $n \geq 1$, there is a set $\hat{I}' \in \hat{\mathcal{I}}$ with $\hat{\lambda}(\hat{I}') = 1$, and for each $\hat{i} \in \hat{I}'$, there exists a set $\hat{E}_{\hat{i}} \in \hat{\mathcal{I}}$ with $\hat{\lambda}(\hat{E}_{\hat{i}}) = 1$, with the following properties being satisfied for any $\hat{i} \in \hat{I}'$ and any $\hat{j} \in \hat{E}_{\hat{i}}$:

1. For all types $k, l \in S$,

$$P(\hat{h}_{\hat{i}}^n = k, \hat{h}_{\hat{j}}^n = l \mid \hat{\alpha}_{\hat{i}}^0, \ldots, \hat{\alpha}_{\hat{i}}^{n-1};\ \hat{\alpha}_{\hat{j}}^0, \ldots, \hat{\alpha}_{\hat{j}}^{n-1}) = P(\hat{h}_{\hat{i}}^n = k \mid \hat{\alpha}_{\hat{i}}^{n-1}) P(\hat{h}_{\hat{j}}^n = l \mid \hat{\alpha}_{\hat{j}}^{n-1}). \tag{39}$$

2. For all types $c, d \in S \cup \{J\}$,

$$P(\hat{g}_{\hat{i}}^n = c, \hat{g}_{\hat{j}}^n = d \mid \hat{\alpha}_{\hat{i}}^0, \ldots, \hat{\alpha}_{\hat{i}}^{n-1}, \hat{h}_{\hat{i}}^n;\ \hat{\alpha}_{\hat{j}}^0, \ldots, \hat{\alpha}_{\hat{j}}^{n-1}, \hat{h}_{\hat{j}}^n) = P(\hat{g}_{\hat{i}}^n = c \mid \hat{h}_{\hat{i}}^n) P(\hat{g}_{\hat{j}}^n = d \mid \hat{h}_{\hat{j}}^n). \tag{40}$$

3. For all types $k, l \in S$,

$$\begin{aligned} & P(\hat{\alpha}_{\hat{i}}^n = k, \hat{\alpha}_{\hat{j}}^n = l \mid \hat{\alpha}_{\hat{i}}^0, \ldots, \hat{\alpha}_{\hat{i}}^{n-1}, \hat{h}_{\hat{i}}^n, \hat{g}_{\hat{i}}^n;\ \hat{\alpha}_{\hat{j}}^0, \ldots, \hat{\alpha}_{\hat{j}}^{n-1}, \hat{h}_{\hat{j}}^n, \hat{g}_{\hat{j}}^n) \\ = \;& P(\hat{\alpha}_{\hat{i}}^n = k \mid \hat{h}_{\hat{i}}^n, \hat{g}_{\hat{i}}^n) P(\hat{\alpha}_{\hat{j}}^n = l \mid \hat{h}_{\hat{j}}^n, \hat{g}_{\hat{j}}^n). \end{aligned} \tag{41}$$

Let $I' = \varphi^{-1}(\hat{I}')$. For any $i \in I'$, let $E_i = \varphi^{-1}(\hat{E}_{\varphi(i)})$. Since φ is measure-preserving, $\lambda(I') = \lambda(E_i) = 1$. Fix any $i \in I'$, and any $j \in E_i$. Denote $\varphi(i)$ by $\hat{i}$ and $\varphi(j)$ by $\hat{j}$. Then, it is obvious that $\hat{i} \in \hat{I}'$ and $\hat{j} \in \hat{E}_{\hat{i}}$.

By equations (34) and (36), we can rewrite equations (39), (40) and (41) as follows. For all types $k, l \in S$,

$$\begin{aligned} & P(h_i^n = k, h_j^n = l \mid \alpha_i^0, \ldots, \alpha_i^{n-1}; \alpha_j^0, \ldots, \alpha_j^{n-1}) \\ = \;& P(\hat{h}_{\hat{i}}^n = k, \hat{h}_{\hat{j}}^n = l \mid \hat{\alpha}_{\hat{i}}^0, \ldots, \hat{\alpha}_{\hat{i}}^{n-1};\ \hat{\alpha}_{\hat{j}}^0, \ldots, \hat{\alpha}_{\hat{j}}^{n-1}) \\ = \;& P(\hat{h}_{\hat{i}}^n = k \mid \hat{\alpha}_{\hat{i}}^{n-1}) P(\hat{h}_{\hat{j}}^n = l \mid \hat{\alpha}_{\hat{j}}^{n-1}) \\ = \;& P(h_i^n = k \mid \alpha_i^{n-1}) P(h_j^n = l \mid \alpha_j^{n-1}). \end{aligned} \tag{42}$$

For all types $c, d \in S \cup \{J\}$,

$$
\begin{aligned}
& P(g_i^n = c, g_j^n = d \mid \alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n; \alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n) \\
= \;& P(\hat{g}_{\hat{i}}^n = c, \hat{g}_{\hat{j}}^n = d \mid \hat{\alpha}_{\hat{i}}^0, \ldots, \hat{\alpha}_{\hat{i}}^{n-1}, \hat{h}_{\hat{i}}^n;\ \hat{\alpha}_{\hat{j}}^0, \ldots, \hat{\alpha}_{\hat{j}}^{n-1}, \hat{h}_{\hat{j}}^n) \\
= \;& P(\hat{g}_{\hat{i}}^n = c \mid \hat{h}_{\hat{i}}^n) P(\hat{g}_{\hat{j}}^n = d \mid \hat{h}_{\hat{j}}^n) \\
= \;& P(g_i^n = c \mid h_i^n) P(g_j^n = d \mid h_j^n).
\end{aligned} \tag{43}
$$

For all types $k, l \in S$,

$$
\begin{aligned}
& P(\alpha_i^n = k, \alpha_j^n = l \mid \alpha_i^0, \ldots, \alpha_i^{n-1}, h_i^n, g_i^n; \alpha_j^0, \ldots, \alpha_j^{n-1}, h_j^n, g_j^n) \\
= \;& P(\hat{\alpha}_{\hat{i}}^n = k, \hat{\alpha}_{\hat{j}}^n = l \mid \hat{\alpha}_{\hat{i}}^0, \ldots, \hat{\alpha}_{\hat{i}}^{n-1}, \hat{h}_{\hat{i}}^n, \hat{g}_{\hat{i}}^n;\ \hat{\alpha}_{\hat{j}}^0, \ldots, \hat{\alpha}_{\hat{j}}^{n-1}, \hat{h}_{\hat{j}}^n, \hat{g}_{\hat{j}}^n) \\
= \;& P(\hat{\alpha}_{\hat{i}}^n = k \mid \hat{h}_{\hat{i}}^n, \hat{g}_{\hat{i}}^n) P(\hat{\alpha}_{\hat{j}}^n = l \mid \hat{h}_{\hat{j}}^n, \hat{g}_{\hat{j}}^n) \\
= \;& P(\alpha_i^n = k \mid h_i^n, g_i^n) P(\alpha_j^n = l \mid h_j^n, g_j^n).
\end{aligned} \tag{44}
$$

Therefore the dynamical system $\mathbb{D}$ is Markov conditionally independent in types. ■

Proof of Corollary 1: In the proof of Theorem 4, take the initial type distribution p^0 to be p. Assume that there is no genuine random mutation in the sense that $b_{kl} = \delta_{kl}$ for all $k, l \in S$. Then, it is clear that $\tilde{p}_k^1 = p_k$ for any $k \in S$. Consider the random partial matching π^1 in period one.

Fix an agent i with $\alpha^0(i) = k$. Then equation (35) implies that $P(h_i^1 = k) = 1$. By equation (37),

$$
P\left(g_i^1 = J\right) = q_k, \;\; P\left(g_i^1 = l\right) = \frac{(1-q_k)(1-q_l)p_l}{\sum_{r=1}^{K}(1-q_r)p_r}. \tag{45}
$$

Similarly, equation (43) implies that the process g^1 is essentially pairwise independent. By taking the type function α to be α^0, the partial matching function π to be π^1, and the associated process g to be g^1, the corollary holds. ■

Remark 3 *For the proof of Corollary 2, we state Theorem 2.4 of [19] here using the notation $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$ for the agent space, instead of the notation $(I, \mathcal{I}, \lambda)$ from [19]. In particular, Theorem 2.4 of [19] shows the existence of an atomless probability space $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$ of agents with $\hat{I}$ having the cardinality of the continuum, a sample probability space $(\Omega, \mathcal{F}, P)$, a Fubini extension $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$, and a random full matching $\hat{\pi}$ from $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$ to $\hat{I}$ such that (i) for each $\omega \in \Omega$, $\hat{\lambda}(\hat{\pi}_\omega^{-1}(A)) = \hat{\lambda}(A)$ for any $A \in \hat{\mathcal{I}}$; (ii) for each $\hat{i} \in \hat{I}$, $P(\hat{\pi}_{\hat{i}}^{-1}(A)) = \hat{\lambda}(A)$ for any $A \in \hat{\mathcal{I}}$; (iii) for any $A_1, A_2 \in \hat{\mathcal{I}}$, $\hat{\lambda}(A_1 \cap \hat{\pi}_\omega^{-1}(A_2)) = \hat{\lambda}(A_1)\hat{\lambda}(A_2)$ holds for P-almost all $\omega \in \Omega$. Since the random matching considered here does not depend on type functions, it is universal in the sense that it can be applied to any type functions.*

When $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$ is taken to be the unit interval with the Borel algebra and Lebesgue measure, Footnote 4 of McLennan and Sonnenschein [42] shows the non-existence of a random full matching $\hat{\pi}$ that satisfies (i)-(iii). Theorem 2.4 of [19] resolves this issue posed by McLennan and Sonnenschein by working with a suitable agent space; see the main theorem of [47] for another proof of such a result.[43]

To be consistent with the general terminology in the paper, the statement of Theorem 2.4 in [19] stated the independence condition in terms of independence in types. However, it is shown in the proof of Theorem 2.4 of [19, p. 399] that for $\hat{i} \neq \hat{j}$ in $\hat{I}$, $(\hat{\pi}_{\hat{i}}, \hat{\pi}_{\hat{j}})$ is a measure-preserving mapping from $(\Omega, \mathcal{F}, P)$ to $(\hat{I} \times \hat{I}, \hat{\mathcal{I}} \boxtimes \mathcal{I}, \hat{\lambda} \boxtimes \hat{\lambda})$, which implies that $\hat{\pi}_{\hat{i}}$ and $\hat{\pi}_{\hat{j}}$ are independent as measurable mappings.[44] *The idea of the proof of [19] can be used to show that for finitely many different agents, the mappings of their random partners are independent. This stronger independence property is shown explicitly in [47] for the particular universal random matching considered there.*

As noted in Remark 1, the exact law of large numbers for an independent random full matching of Theorem 1 generalizes trivially to a setting in which $\hat{\alpha}$ is a $\hat{\mathcal{I}}$-measurable type function from $\hat{I}$ to a complete separable metric type space S. The existence of such an independent random full matching follows immediately from Theorem 2.4 of [19] by working with the type process $\hat{g} = \hat{\alpha}(\hat{\pi})$.

Since the finiteness of S is not used in the proof of Corollary 2 below, Corollary 2 also holds in the setting of a complete separable metric type space S.

Proof of Corollary 2: We follow the notation in Remark 3. For any given type distribution p on S, take a $\hat{\mathcal{I}}$-measurable type function $\hat{\alpha}$ from $\hat{I}$ to S with type distribution p.

By following the same constructions used before the proof of Theorem 4, we can obtain (1) an atomless probability space $(I, \mathcal{I}, \lambda)$ which is an extension of the Lebesgue unit interval $(L, \mathcal{L}, \eta)$; (2) a Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$; and (3) a measure preserving mapping φ from $(I, \mathcal{I}, \lambda)$ to $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$.

As in the proof of Theorem 4, for any $\hat{i}, \hat{i}' \in \hat{I}$ with $\hat{i} \neq \hat{i}'$, let $\Theta^{\hat{i},\hat{i}'}$ be a bijection from $C_{\hat{i}}$ to $C_{\hat{i}'}$, and $\Theta^{\hat{i}',\hat{i}}$ be the inverse mapping of $\Theta^{\hat{i},\hat{i}'}$.

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[43] The agent space in Theorem 2.4 of [19] is a hyperfinite probability space. It is a well-known property that hyperfinite probability spaces capture the asymptotic properties of large but finite probability spaces; see [40]. So the use of such a probability space does provide some advantages. In contrast, the agent space as considered in [47] is the space of all transfinite sequences of 0 or 1 whose length is the first uncountable ordinal and whose terms are constant 1 except for countably many terms; it is not known how such a probability space can be linked to large but finite probability spaces. Unlike [19], random partial matching and dynamic random matching are not considered in [47]. In addition, we note that as indicated in the last paragraph of the proof of Theorem 2.4 of [19, p. 400], property (iii) above is simply a special case of the exact law of large numbers (as stated in Lemma 1) for an i.i.d. process in a Fubini extension with a common distribution on the two-pint space $\{0, 1\}$; Proposition 3 of [47] provided another proof for such a special case.

[44] We add the hat notation here.

Define a mapping π from $I \times \Omega$ to I such that for each $(i, \omega) \in I \times \Omega$,

$$\pi(i, \omega) = \Theta^{\varphi(i),\, \hat{\pi}_\omega(\varphi(i))}(i).$$

Fix any $\omega \in \Omega$. Pick any $i \in I$ and denote $\pi_\omega(i)$ by j. The definition of π implies that $j = \Theta^{\varphi(i),\, \hat{\pi}_\omega(\varphi(i))}(i)$. Since $\Theta^{\varphi(i),\, \hat{\pi}_\omega(\varphi(i))}$ is a bijection between $C_{\varphi(i)}$ and $C_{\hat{\pi}_\omega(\varphi(i))}$, it follows that $\varphi(j) = \varphi(\pi_\omega(i)) = \hat{\pi}_\omega(\varphi(i))$ by the definition of φ. Thus, $j = \Theta^{\varphi(i),\, \varphi(j)}(i)$. Since the inverse of $\Theta^{\varphi(i),\, \varphi(j)}$ is $\Theta^{\varphi(j),\, \varphi(i)}$, we know that $\Theta^{\varphi(j),\, \varphi(i)}(j) = i$. By the full matching property of $\hat{\pi}_\omega$, $\varphi(j) \neq \varphi(i)$ (and thus $j \neq i$), and $\hat{\pi}_\omega(\varphi(j)) = \varphi(i)$. Hence,

$$\pi_\omega(j) = \Theta^{\varphi(j),\, \hat{\pi}_\omega(\varphi(j))}(j) = \Theta^{\varphi(j),\, \varphi(i)}(j) = i.$$

This means that the composition of π_ω with itself is the identity mapping on I, which also implies that π_ω is a bijection on I. Hence, π_ω is a full matching on I.

Let the type function α on I be defined as the composition $\hat{\alpha}(\varphi)$. Since φ is measure preserving, the distribution of α is still p. Let $\hat{g} = \hat{\alpha}(\hat{\pi})$ and $g = \alpha(\pi)$. Then, it is easy to see that for any $(i, \omega) \in I \times \Omega$,

$$g(i, \omega) = \hat{\alpha}(\varphi(\pi(i, \omega))) = \hat{\alpha}(\hat{\pi}_\omega(\varphi(i))) = \hat{g}(\varphi(i), \omega).$$

Hence, the essential pairwise independence of g follows immediately from that of $\hat{g}$ and the measure-preserving property of φ. ■

Remark 4 *To obtain the existence of a dynamical system $\hat{\mathbb{D}}$ with random full matching for a complete separable metric type space S and deterministic match induced type changing function ν that is Markov conditionally independent in types, we can follow the proof of Theorem 3.1 in [19].*

Let M be a fixed unlimited hyperfinite natural number in ${}^*\mathbb{N}_\infty$, $\hat{I} = \{1, 2, ..., M\}$, $\hat{\mathcal{I}}_0$ be the internal power set on $\hat{I}$, and $\hat{\lambda}_0$ be the internal counting probability measure on $\hat{\mathcal{I}}_0$. Let $(\hat{I}, \hat{\mathcal{I}}, \hat{\lambda})$ be the Loeb space of the internal probability space $(\hat{I}, \hat{\mathcal{I}}_0, \hat{\lambda}_0)$.

For $n \geq 1$, let $(\Omega_n, \mathcal{F}_n, Q_n)$ be the internal sample measurable space $(\Omega, \mathcal{F}_0, P_0)$ as in the proof of Theorem 2.4 of [19]. Let $\hat{\pi}_n$ be the random full matching defined by $\hat{\pi}_n(\hat{i}, \omega_n) = \omega_n(\hat{i})$ for $\hat{i} \in \hat{I}$ and $\omega_n \in \Omega_n$. Let P_n be the corresponding Loeb measure of Q_n on $(\Omega_n, \sigma(\mathcal{F}_n))$.

Follow the notation in Subsection 5.2 of [19]. We can construct generalized infinite product spaces $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$ and $(\hat{I} \times \Omega^\infty, \hat{\mathcal{I}} \boxtimes \mathcal{A}^\infty, \hat{\lambda} \boxtimes P^\infty)$, where $\Omega^\infty = \prod_{n=1}^\infty \Omega_n$. Let $\hat{\pi}^n$ be defined by $\hat{\pi}^n(\hat{i}, \{\omega_m\}_{m=1}^\infty) = \omega_n(\hat{i})$ for $\hat{i} \in \hat{I}$ and $\{\omega_m\}_{m=1}^\infty \in \Omega^\infty$. For simplicity, we shall also use $(\Omega, \mathcal{F}, P)$ and $(\hat{I} \times \Omega, \hat{\mathcal{I}} \boxtimes \mathcal{F}, \hat{\lambda} \boxtimes P)$ to denote $(\Omega^\infty, \mathcal{A}^\infty, P^\infty)$ and $(\hat{I} \times \Omega^\infty, \hat{\mathcal{I}} \boxtimes \mathcal{A}^\infty, \hat{\lambda} \boxtimes P^\infty)$ respectively.

Let α^0 be any initial measurable type function from $\hat{I}$ to S with distribution p^0 on S. The type function $\hat{\alpha}^n$ from $(\hat{I}\times\Omega, \hat{\mathcal{I}}\boxtimes\mathcal{F}, \hat{\lambda}\boxtimes P)$ to S can be defined inductively by letting $\hat{\alpha}^n(\hat{i},\omega) = \nu(\hat{\alpha}^{n-1}(\hat{i},\omega), \hat{\alpha}^{n-1}(\hat{\pi}^n(\hat{i},\omega),\omega))$. Define $\hat{g}^n : \hat{I}\times\Omega \to S$ by $\hat{g}^n(\hat{i},\omega) = \hat{\alpha}^{n-1}(\hat{\pi}^n(\hat{i},\omega),\omega)$; then we have $\hat{\alpha}^n = \nu(\hat{\alpha}^{n-1}, \hat{g}^n)$. It can be checked that equations (28) and (29) are satisfied (with the hat notation). Thus, the desired existence result for $\hat{\mathbb{D}}$ follows.

As in the proof of Theorem 4, we can use the dynamical system $\hat{\mathbb{D}}$ on the Fubini extension $(\hat{I}\times\Omega, \hat{\mathcal{I}}\boxtimes\mathcal{F}, \hat{\lambda}\boxtimes P)$ to construct a new dynamical system $\mathbb{D}$ on the Fubini extension $(I\times\Omega, \mathcal{I}\boxtimes\mathcal{F}, \lambda\boxtimes P)$.

For the dynamical system $\mathbb{D}$, define mappings π^n from $I\times\Omega$ to I, α^n from $I\times\Omega$ to S such that for each $(i,\omega)\in I\times\Omega$,

$$\pi^n(i,\omega) = \Theta^{\varphi(i),\, \hat{\pi}^n_\omega(\varphi(i))}(i),\ \alpha^n(i,\omega) = \hat{\alpha}^n(\varphi(i),\omega).$$

Define $g^n : I\times\Omega \to S$ by $g^n(i,\omega) = \alpha^{n-1}(\pi^n(i,\omega),\omega)$. As shown above, π^n is a random full matching, and $g^n = \hat{g}^n(\Phi)$. It is then easy to see that $\alpha^n = \hat{\alpha}^n(\Phi) = \nu(\hat{\alpha}^{n-1}, \hat{g}^n)(\Phi) = \nu(\alpha^{n-1}, g^n)$. It is also easy to check that equations (28) and (29) are satisfied. This shows the existence of a dynamical system $\mathbb{D}$ of random full matching with a complete separable metric type space S and with initial type distribution p^0 and deterministic match induced type changing function ν that is Markov conditionally independent in types, where the agent space is an extension of the Lebesgue unit interval.

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