

Publ 11-B-12 32

Research Paper Number 915

EQUILIBRIUM AND THE ROLE OF THE FIRM IN INCOMPLETE MARKETS

Darrell Duffie and Wayne Shafer

First Draft: August, 1986 This Draft: November, 1986

ABSTRACT

This paper studies the role of the firm in incomplete markets. Stock market equilibria are shown to exist generically in economies with "smooth" preferences and production sets. The set of equilibrium allocations is generically infinite. The stochastic setting is described by an arbitrary event tree. At each state and date agents trade on markets for spot commodities, common stocks, and other general securities. The goal of share value maximization by firms is shown to be generically strictly sub-optimal in equilibrium for all but (at most) a single shareholder. The Modigliani-Miller Invariance Principle, showing the irrelevance of the financial policy of the firm, is re-examined in the light of incomplete markets.*

* Shafer is with the Department of Economics, University of Southern California, Los Angeles CA 90089. Duffie is with the Graduate School of Business, Stanford University, Stanford CA 94305. We are pleased to acknowledge conversations with David Cass, Andreu Mas-Colell, and Peter DeMarzo. This work was supported by the National Science Foundation under grants SES-851335, SES-8420114, and MCS-8120790, and was undertaken while Duffie visited the Mathematical Sciences Research Institute, Berkeley, California. The typing assistance of Ann Bush and Jill Fukuhara is much appreciated.

87p

§1. Introduction

The firm plays several fundamental roles in a stock market economy:

- (i) by virtue of its production possibilities, the firm augments the goods available for consumption,
- (ii) through common stock valuation, the firm affects the distribution of wealth among agents, and
- (iii) via the dividend processes marketed as a firm's equity, debt, and other corporate issues, the firm augments the set of consumption processes that agents can finance by trading over time on security and spot markets.

The last role, typically called *spanning*, is absent in a standard complete markets competitive model, and is the focal point of this paper. Our model of an economy comprises agents with smooth preferences, firms with smooth production sets, spot commodity markets, and security markets for common stocks and other assets. A stock market equilibrium is defined roughly as follows. Taking prices as given, firms make production choices and security trades maximizing the market value of their common shares; agents choose security trading strategies and spot consumption strategies maximizing utility. The system is in equilibrium if all spot commodity and security markets clear. Excluding from the set of economies a subset whose parameters form a closed set of measure zero, we reach the following conclusions.

- (1) Equilibria exist.
- (2) If the number of securities is large enough to provide full spanning at some prices, there is only a finite number of equilibrium allocations, each being Pareto optimal, and shareholders unanimously support the production goal of share value maximization.
- (3) Without a sufficient number of securities for spanning, (i) no equilibrium allocation is Pareto optimal, (ii) the number of equilibrium allocations is infinite if and only if at least one of the securities is a firm's common share, and (iii) the production goal of share value maximization is strictly sub-optimal for all shareholders, except perhaps one.
- (4) Regardless of spanning, the issuing or trading of securities by a firm has no effect on the firm's share value. This includes the indirect effect of dividends and price changes from securities held by the firm which themselves hold shares of the firm, and so on.

DeMarzo (1986) has subsequently shown, in the context of our security valuation model, that the trading of securities by firms has no effect on equilibrium shareholder utility, yielding a full version of the Modigliani–Miller Theorem in this general incomplete markets stochastic framework. It remains the case, of course, that shareholders are not generally indifferent to the issuance of new securities by the firm.

The remainder of this section is a discussion of these results and their antecedents in the literature. Sections 2 through 8 formulate the model and state the results more carefully. Proofs of theorems are collected in Section 9.

Existence of Equilibria

We first consider the question of existence of competitive equilibria. Since the span of markets changes both with spot prices and with the production choices of the firm (not to mention its financial policy), standard fixed point analysis has been applied with limited success. Even with a short sales limitation on portfolios, Radner (1972) did not resolve the existence issue with production in his model of a sequence of markets. Grossman and Hart (1979) show existence in the Radner setting with limited short sales and the assumption of a single commodity. Burke (1986) shows existence with a short sales restriction and multiple commodities under additional regularity assumptions on production sets. Without a short sales restriction, Hart (1975) has shown that equilibria do not generally exist, even under the smooth preference assumptions of Debreu (1972) that we adopt in this paper. Hart's counterexample is based on a collapse in the span of security markets at certain "bad" spot prices. After Hart's paper, attention was focused on models with purely financial securities [Werner (1984), Cass (1984), Duffie (1985)] or purely numeraire securities [Geanakoplos and Polemarchakis (1985), Chae (1985)], or toward showing that bad spot prices are relevant only for an exceptional set of economies, a program of *generic existence*. In the pure exchange model, McManus (1984), Repullo (1984), as well as Magill and Shafer (1984, 1985) show generic existence provided there is a sufficient number of securities to potentially span complete markets. Duffie and Shafer (1985a, 1985b) extend generic existence to the case of an insufficient number of securities for full spanning, still in a pure exchange setting. This paper extends our work to include production, exploiting the fact that a smooth production economy and its pure exchange version have homotopic excess demand

functions. [Homotopic functions have the same fixed point index.] We gradually extend the generality of our basic model, finally showing generic existence for smooth stochastic stock market economies with incomplete markets, security trading by firms, unrestricted or linearly restricted portfolio formation, purely financial securities (such as bonds), real securities (such as commodity futures contracts), and mixtures of these security types, including mutual funds. We have not extended existence to securities whose dividends are general non-linear functions of spot commodity prices, such as stock options, commodity futures options, defaultable corporate debt issues, and so on. [See Polemarchakis and Ku (1986) for a counterexample in a special sense.] The proofs, located in Section 9, make extensive use of differential topology, introduced to the study of general economic equilibrium by Debreu (1970,1972,1976). We use the approach of Balasko (1975), studying the properties of the projection map on the space of economies and their equilibria (the "equilibrium manifold") into the space of economies. Existence follows from an application of mod 2 degree theory along the lines of Dierker (1972), whereby the projection map on a closely related "pseudo-equilibrium" manifold is onto.

Multiplicity and Optimality of Equilibria

Debreu's (1970, 1972, 1976) demonstration that smooth exchange economies with complete markets generically have a finite set of equilibria has been extended to production economies [Smale (1974), Fuchs (1974), Kehoe (1983)]. With incomplete markets and purely financial securities, Geanakoplos and Mas-Colell (1985) as well as Cass (1985) showed that the set of equilibrium allocations is generically infinite. For illustration, with S states of the world (in the second period of a two period model) and n purely financial assets, Geanakoplos and Mas-Colell showed that the set of equilibrium allocations generically has a subset equivalent (topologically) to an $(S - 1)$ -dimensional ball! Speaking generically, we show that stock markets play a special role in the multiplicity of equilibria. With n real (rather than purely financial) assets, the set of equilibrium allocations is finite if and only if none of these assets is a common stock. If even one of these n assets is a common stock, the set of equilibrium allocations again has a subset homeomorphic to a ball in \mathbb{R}^{S-n} .

In one of our examples, the entire (one-dimensional) set of equilibrium allocations is strictly Pareto ordered. Even before Hart's (1975) examples, it was known that incom-

plete stock market equilibria can be Pareto inefficient. For a sample of early studies of this problem, some of which give conditions for optimality or constrained optimality, we cite Diamond (1967), Jensen and Long (1972), Stiglitz (1972), Leland (1973), Drèze (1974), Ekern and Wilson (1974), Gevers (1974), and Merton and Subrahmanyam (1974). Even a cursory examination of this literature or of the first order conditions for optimality in incomplete markets will leave the reader unsurprised at our proof of generic Pareto inefficiency of incomplete markets equilibria. Although Grossman and Hart (1979) showed that Grossman's (1977) Social Nash Optimality property carries over to stock market equilibria if firms have appropriate objectives, they make it quite clear that this optimality property is not especially normative, and that firms must collect each shareholder's marginal rates of substitution in order to implement Social Nash Optimal equilibria.

Beginning with Arrow (1953), the fact that repeated trade of a sufficient number of securities can dynamically span the entire consumption space and thus allow fully Pareto optimal equilibrium allocations has been shown in various settings by Guesnerie and Jaffray (1974), Friesen (1974), Kreps (1982), McManus (1984), Repullo (1984), Magill and Shafer (1985), Nermuth (1985), and Duffie and Huang (1986). The basic premise in this paper is that the securities are too few in number to provide complete spanning. Of course it is trivial that if re-allocations and production changes are constrained to the subspace M of consumption bundles that can be traded via security and spot markets, then an equilibrium is Pareto optimal in this constrained sense. This follows from the observation that the allocation is an Arrow-Debreu equilibrium allocation for the economy restricted to the marketed consumption space M . Given strong linearity restrictions on production sets, this is the idea behind Diamond's (1967) demonstration that stock market equilibria are constrained efficient (in an appropriate sense). With a slightly stronger sense of constrained optimality, however, Geanakoplos and Polemarchakis (1985) show that pure exchange economies have generically inefficient equilibrium allocations in incomplete markets. This result (probably) carries over easily to production economies. Frankly, we are unsure about where next to look in a study of constrained optimality of stock markets.

Shareholder Agreement

We work with the basic premise that a shareholder takes prices as given and agrees

with the production choice of a firm if, given unilateral control of the firm, the shareholder would maximize his or her own utility with the same production choice. The classical decentralization properties of the competitive market mechanism in complete markets are known to include unanimous shareholder agreement with market value maximizing production choices. Unanimity follows from the simple observation that, if prices are taken as given (the competitive assumption), then a firm affects a shareholder only to the extent of the shareholder's wealth, which is a strictly monotone function of the firm's value. The firm thus maximizes the optimal utility of any locally non-satiated shareholder if and only if the firm maximizes its own market value. This neat coincidence of goals via the price mechanism does not carry over to incomplete markets. Indeed, in a smooth economy, market value maximization is strictly sub-optimal for any shareholder restricted by the span of markets. Again, the reasoning is simple. The first order conditions for maximization of a firm's market value are precisely that the marginal effect of any change in production on the value of the firm is zero. A value maximizing firm can therefore move the span of markets in a direction strictly favorable to a shareholder with zero marginal effect on the shareholder's budget. The shareholder will thus prefer that the firm change its production choice. Using Hart's (1979b, 1979a) terminology, the "wealth effect" of a production change is always locally dominated by the "consumption effect" at a production choice maximizing the firm's value. This depends on smoothness, of course, and the wealth effect may dominate if the firm's value-maximizing production choice occurs at a sufficiently sharp kink in the production frontier. We provide the details in Section 4, using the fact that shareholders are generically restricted by the span of markets in equilibrium to conclude that market value maximization is generically strictly sub-optimal for every shareholder, except perhaps one.

Our conclusion must be reconciled with a long history of literature showing unanimity among shareholders even in incomplete markets. In some of the literature [Diamond (1967), Ekern and Wilson (1974), Radner (1974), Leland (1974)] the firm cannot affect the span of markets, and the reasoning of the complete markets case can be applied to deduce unanimity. Other papers base unanimity on some particular version of the "competitive assumption," taking something more than prices, or other than prices, as given by agents. For example, the "perfect competition" assumption of Ostroy (1980) and Makowski (1980,

1983) has the effect that shareholders take the span of markets as given independently of the firm's production choice. Again, unanimity follows from the complete markets reasoning. Diamond (1967), Drèze (1974), as well as Grossman and Hart (1979) derive unanimity not for value maximization, but for "pseudo-value" maximization," where the "pseudo-prices" are share-weighted sums of agents marginal rates of substitution. Here, a "utility taking" competitive assumption is invoked. Several papers exploit a "large number" of firms, meaning either a sequence of economies with a growing number of firms or a measure space of firms. The spirit of these models is that a small firm can affect market clearing prices or allocations only negligibly, so that the consumption effect of a production shift can be dominated by the wealth effect. A measure space of negligible firms in our model would not overturn our rejection of unanimity for value maximization. Although the impact of a small firm on equilibrium prices or allocations is negligible, its effect on the span of incomplete markets is not. For illustration, a shareholder unable to hedge a random endowment risk may wish that a firm alter its production choice purely for purposes of providing a better hedge, and the effect on the span of security markets is independent of the size of the firm. Indeed, rejection of value maximization holds *a fortiori* with negligibly sized firms, since the contribution of a small firm to the wealth of a shareholder is negligible, even without smoothness, and the consumption effect of production changes will dominate the wealth effect. [One should also see Rubinstein (1977) on a related point.] Our approach is not comparable with Hart's (1979b), however, for Hart compares sequences of equilibrium allocations as the numbers of agents and active firms diverge, rather than examining a particular agent's attitude to a particular firm's choice in a particular economy. Under conditions, Hart reaches the opposite conclusion that the wealth effect dominates, and thus that unanimity prevails. The unanimity results of Bester (1982) and Haller (1984), also relying on a "large number" of firms, are quite special. They assume mean-variance utility, an absence of future random endowments, and exogenously distributed future spot prices (independent of production choices.) In that setting, unanimity follows from the absence of any incentive for an agent to change the span of the markets. Leland (1978) and Satterthwaite (1981) derive unanimity under special conditions and partial observation of the state by shareholders. Kreps (1983) has surveyed the spanning and unanimity issue more carefully.

The Dividends and Arbitrage—Valuation of Inter-Dependent Securities

We will be considering the financial policy of the firm. This issue is traditionally framed in terms of the debt-equity decision and the corresponding dividend policy. Of course, a firm may implement a given financial policy by trading in security markets. For example, a firm may borrow or lend in bond markets to accelerate or smooth dividends. More generally, firms trade the shares of other firms; in fact, they occasionally buy their own common shares. Similarly, certain securities such as mutual funds are set up purely to generate dividends by trading in other securities. This poses an obvious simultaneity question: “How does one determine the dividends and prices of securities that invest in one another”? In a static setting, for example, suppose n securities generate internal cash flows corresponding to a vector $\delta \in \mathbb{R}^n$. For example, δ_k could be the market value of the commodities produced by firm k . Further, suppose that the portfolios of securities held by securities are given by an $n \times n$ matrix γ , where γ_{jk} is the number of shares of security k held by security j . [Firm k has repurchased some of its shares from shareholders, for example, if and only if $\gamma_{kk} \neq 0$.] Then the total vector $\Delta \in \mathbb{R}^n$ of dividends paid by the n securities must solve the equation $\Delta = \delta + \gamma\Delta$. A unique solution exists if and only if $I - \gamma$ is non-singular, in which case $\Delta = (I - \gamma)^{-1}\delta$. The story is much richer in an incomplete markets stochastic setting. Our result is Lemma 2 of Section 6. Then, given a stochastic process Δ for dividends, we study restrictions on a security price process π , such that (π, Δ) is arbitrage-free, meaning that one cannot generate a positive cash flow from security trading without a positive investment. Proposition 3 is a finite-dimensional extension of results by Rubinstein (1976), Ross (1978), and Harrison and Kreps (1979), showing that (π, Δ) is arbitrage-free if and only if every security’s price is some fixed strictly positive weighted sum of its future dividends.

The Modigliani–Miller Theorem

Broadly stated, there are two major implications of the Modigliani–Miller theory in competitive linear markets, by which we mean markets with price taking and free portfolio formation (no short sales restrictions, transactions costs, or taxes). First, the current market value of a firm’s share is independent of its financial policy. Second, if the span of markets is fixed, the shareholders of a firm are indifferent to the firm’s financial policy. In Section 7 we re-examine and re-affirm these implications, with emphasis on a formal model of the

financial policy of the firm and on the role of the fixed-market-span assumption.

We use our construction of arbitrage-free dividends and prices for interdependent securities. Allowing the firm to adopt a general security trading strategy, we then confirm the intuition that a firm cannot change its market value by trading securities. This is the case even in incomplete markets and even if one accounts for the impact of changes in the future dividends and prices of the firm on the dividends and prices of other securities that hold shares of the firm, and the feedback effect on the firm itself through its holdings of other securities, and so on. [This includes re-purchases by the firm of its own stock.] Of course, a firm can also implement a financial policy by issuing new securities, the policy form originally considered by Modigliani and Miller (1958) when they showed that the issuance of debt has no effect on the total value of the firm. Whether the value received from the sale of (defaultable) bonds is paid immediately to the original shareholders as a dividend, or is re-invested (financially) elsewhere, this has no effect on the initial *cum* dividend equity value of the firm. This argument holds all other prices fixed and assumes that the issuance of debt does not affect the range of feasible production choices. Of course, with credit limitations or limited liability restrictions, which we do not model, the issuance of debt or trading of securities could allow a firm to undertake a production project of higher present market value than would otherwise be possible. [This bald observation hardly constitutes a satisfactory model of financial policy, which is still considered to have a relatively unexplained role.] Our arguments also implicitly use perfect foresight on the part of investors as to the dividends of defaultable securities. That is, a corporate bond that defaults in some states of the world is taken as such, and not as riskless. Further, bondholders understand the entire financial policy of the firm, including any sale of "new debt" in some future states of the world, which would generally reduce the market value of previously issued defaultable debt. [The alternative to this perfect foresight assumption is a so-called "me-first" rule, as shown by Fama and Miller (1972) and Fama (1978).]

As to the effect of financial policy on shareholders, we point out that, generically, shareholders find the span of incomplete markets a binding constraint. This yields the obvious conclusion that shareholders are not indifferent to the financial policy of the firm if it can change the span of markets (which is typically the case in incomplete markets). We provide a trivial example of the impact of financial innovation by the firm. DeMarzo (1986)

has gone beyond this and such earlier work as Stiglitz (1974), however, in showing that shareholders are indifferent to the trading of existing securities by firms. Anything the firm can do by trading securities, agents can undo by trading securities on their own account. Indeed, any change of security trading strategy by the firm can be accomodated within a new equilibrium that preserves consumption allocations. Hellwig (1981) distinguishes situations in which this is not the case, such as limited short sales.

§2. The Basic Equilibrium Problem in Incomplete Markets.

This section includes the definition of an economy and its equilibria in a two period model with uncertainty over the state of nature in the second period. We will later extend the basic problem in several directions.

There are ℓ commodities consumed at time zero and in each of S states at time one, making for the consumption space $\mathcal{L} = \mathbb{R}^{\ell(S+1)}$. For any $x \in \mathcal{L}$, let $x_0 \in \mathbb{R}^\ell$ denote the time zero consumption bundle and $x_1(s) \in \mathbb{R}^\ell$ denote the time one consumption bundle in state s , for $1 \leq s \leq S$. For any spot price vector $p \in \mathcal{L}$ and any consumption plan $x \in \mathcal{L}$, let $p_1 \square x_1 \in \mathbb{R}^S$ denote the vector of state-contingent spot market values of x_1 , with s -th element $p_1(s) \cdot x_1(s)$. Let $\mathcal{L}_{++} = \mathbb{R}_{++}^{\ell(S+1)}$. An economy is defined by a collection

$$((u_i, \omega^i), (Y_j), (\theta_{ij})), \quad 1 \leq i \leq m, \quad 1 \leq j \leq n,$$

where $u_i : \mathcal{L}_{++} \rightarrow \mathbb{R}$ is a utility function, $\omega^i \in \mathcal{L}_{++}$ is an endowment vector, $Y_j \subset \mathcal{L}$ is a production set, and $\theta_{ij} \geq 0$ is the share of firm j endowed to agent i , for firms $1 \leq j \leq n$ and agents $1 \leq i \leq m$. By convention, $\sum_i \theta_{ij} = 1$ for all j .

For the basic model, we allow agents to trade only in markets for spot commodities and the shares of firms. Later we extend the model to allow trade in additional securities in zero net supply, such as bonds, forward contracts for commodities, and so on. We will also extend to a stochastic setting with sequential trading. These generalizations do not affect the basic results.

Firms take as given a market valuation function $v : \mathcal{L} \rightarrow \mathbb{R}$ mapping each production choice $y \in Y_j$ to the initial *cum* dividend market value $v(y)$ of firm j . We assume the free formation of portfolios by agents, complete spot markets, and ex-dividend trading of firm shares at time zero. Given a spot price vector $p \in \mathcal{L}$, the absence of arbitrage in security

markets then implies (as shown in the more general stochastic setting of Section 6) the existence of a state-price vector $q \in \mathbb{R}_{++}^S$ such that

$$v(y) = v_{qp}(y) \equiv p_0 \cdot y_0 + q \cdot (p_1 \square y_1), \quad y \in Y_j, \quad 1 \leq j \leq n. \quad (1)$$

For production choices $y = (y^1, \dots, y^n) \in \mathcal{L}^n$ of the n firms and a spot vector $p \in \mathcal{L}$, let $V(p, y_1)$ denote the $S \times n$ matrix whose j -th column is $p_1 \square y_1^j$. A portfolio $\gamma \in \mathbb{R}^n$ of firms' shares then yields the state contingent dividend vector $V(p, y_1)\gamma \in \mathbb{R}^S$. Given a state price vector q , a spot price vector p , and production choices $y = (y^1, \dots, y^n)$, agent i thus faces the problem:

$$\max_{x \in \mathcal{L}_{++}, \gamma \in \mathbb{R}^n} u_i(x) \quad (2)$$

subject to:

$$\begin{aligned} p_0 \cdot (x_0 - \omega_0^i) + \sum_j (\gamma_j - \theta_{ij}) v_{qp}(y^j) - \gamma_j p_0 \cdot y_0^j &\leq 0 \\ p_1 \square (x_1 - \omega_1^i) &\leq V(p, y_1)\gamma. \end{aligned}$$

Firm j takes p and q as given and solves, if possible, the market value maximization problem

$$\max_{y \in Y_j} v_{qp}(y). \quad (3)$$

A collection $((x^i, \gamma^i), (y^j), p, q)$ is an *equilibrium* for the given economy provided

- (a) (x^i, γ^i) solves problem (2) for each agent i ,
- (b) y^j solves problem (3) for each firm j ,
- (c) $\sum_i x^i - \omega^i = \sum_j y^j$, and
- (d) $\sum_i \gamma_j^i = 1$ for each firm j .

§3. Basic Equilibrium Theorems

This section states sufficient conditions for the generic existence of equilibria. The conditions are mainly directed toward smooth well-behaved demand and supply functions. We use the word *smooth* to describe functions with as many continuous derivatives as required for our proofs, and adopt the following "smooth preference assumptions" of Debreu (1972). For all i ,

- (U.1) u_i is smooth,

- (U.2) $Du_i(x) \in \mathcal{L}_{++}$ for all x in \mathcal{L}_{++} (strict monotonicity),
 (U.3) for all x in \mathcal{L}_{++} , $h^\top D^2 u_i(x) h < 0$ for all $h \neq 0$ satisfying $Du_i(x)h = 0$ (differentiably strictly convex preferences), and
 (U.4) $\{x \in \mathcal{L}_{++} : u_i(x) \geq u_i(\bar{x})\}$ is closed in \mathcal{L} for all \bar{x} in \mathcal{L}_{++} (a boundary condition).

For convenience, we define a *state-complete subspace* to be a vector subspace of \mathcal{L} equivalent to $\Pi_{s=0}^S \mathbb{R}^{\ell(s)}$ for some subset of $\ell(s) > 0$ different commodities used in production out of the ℓ available for consumption in state s or at time $s = 0$, $0 \leq s \leq S$. Substituting a state-complete subspace for \mathcal{L} in condition (P.3) below makes for a weakening of the strong assumption of Gaussian curvature on production sets.

- (P.1) Y_j is closed, convex, and intersects \mathcal{L}_+ ,
 (P.2) $p \cdot Y_j$ is bounded above for all p in \mathcal{L}_{++} , and
 (P.3) as a subset of some state-complete subspace $\mathcal{L}(j)$, the boundary of Y_j is a smooth manifold with non-zero Gaussian curvature.

As Hart (1975) has shown by counterexample, we must avoid the singularities induced in security demands when the rank of the dividend matrix $V(p, y_1)$ changes. As p and y approach such a singularity, two securities become closer and closer substitutes, and only short sales restrictions can guarantee the existence of an equilibrium. In order to guarantee generic existence, we will perturb Y_j by a translation $t^j \in \mathcal{L}(j)$ in order to generate the production sets

$$Y_j^t = Y_j + \{t^j\}, \quad 1 \leq j \leq n,$$

for $t = (t^1, \dots, t^n) \in \mathcal{T} \equiv \prod_j \mathcal{L}(j)_+$. This perturbation preserves the existence of smooth supply functions satisfying (P.1)–(P.3) and is continuous in any of the usual topologies placed on production sets, such as the topology of uniform C^r convergence on compacta of the associated distance function (Mas-Colell (1985)).

The word *generic* is taken throughout to mean: for all parameters in the stated set, except for a closed subset of Lebesgue measure zero.

Theorem 1. *Suppose utility assumptions (U.1)–(U.4) and production assumptions (P.1)–(P.3) apply. Then, for generic $(\omega, t) \in \mathcal{L}_+^m \times \mathcal{T}$, there exists an equilibrium for the incomplete markets economy $((u_i, \omega^i), (Y_j^t), (\theta_{ij}))$.*

Theorem 1 is a special case of Theorem 2, which is stated after the following simple example.

Example. To illustrate our results we consider an example with $m = 2$ agents, $\ell = 1$ commodity, $S = 2$ states, and $n = 1$ firm. The utility functions are

$$u_i(x) = \ell n(x_0) + \beta_1^i \ell n(x_1(1)) + \beta_2^i \ell n(x_1(2)), \quad x \in \mathcal{L}_{++},$$

for some $\beta^i \in \mathbb{R}_{++}^2$, $i \in \{1, 2\}$. The production set is

$$Y = \{y \in \mathcal{L} : y_0 \leq 0, y_1 \geq 0, y_1 \cdot y_1 \leq -y_0\} - \mathcal{L}_+.$$

All of our assumptions (P.1)–(P.4) and (U.1)–(U.3) are satisfied. Taking the state price vector $\bar{q} = (1, 1)$ and normalizing the spot price vector $p \in \mathcal{L}_{++}$ by choosing $p_0 = 1$, the market value maximizing production choices are

$$\begin{aligned} y_0 &= -p_1 \cdot p_1 / 4 \\ y_1(s) &= p_1(s) / 2, \quad s \in \{1, 2\}. \end{aligned}$$

The initial market value of the firm is thus $p \cdot y = p_1 \cdot p_1 / 4$.

Problem (2) of agent i is reduced to choosing the fraction γ^i of the firm to hold. The first order condition for optimal γ^i is

$$\frac{-p \cdot y - y_0}{\omega_0^i + \gamma^i(y_0 - p \cdot y) + \theta_i p \cdot y} + \sum_s \frac{\beta_s^i y_1(s)}{\omega_1^i(s) + y_1(s) \gamma^i} = 0.$$

Substituting the market value maximizing y yields

$$\frac{-p_1 \cdot p_1}{\omega_0^i + \left(\frac{\theta_i}{4} - \frac{\gamma^i}{2}\right) p_1 \cdot p_1} + \sum_s \frac{\beta_s^i p_1(s)}{\omega_1^i(s) + p_1(s) \gamma^i / 2} = 0, \quad i \in \{1, 2\}. \quad (*)$$

We note that $\gamma^1 + \gamma^2 = 1$ implies spot market clearing, and that the set of equilibria is therefore equivalent to the set of $(p_1, \gamma^1, \gamma^2)$ solving the two equations (*) and $\gamma^2 = 1 - \gamma^1$. By the usual transversality argument, the set of solutions is generically a one-dimensional manifold.

Distinct $(p_1, \gamma^1, \gamma^2)$ and $(\bar{p}_1, \bar{\gamma}^1, \bar{\gamma}^2)$ solving (*) and $\gamma^1 + \gamma^2 = \bar{\gamma}^1 + \bar{\gamma}^2 = 1$ correspond to distinct allocations for the agents. To see this, suppose not. Then $p_1 = \bar{p}_1$, for otherwise the corresponding production choices y and \bar{y} differ. Furthermore $p_1 \gamma^1 = \bar{p}_1 \bar{\gamma}^1$ for otherwise the corresponding consumption choices x_1 and \bar{x}_1 differ. Thus $\gamma^1 = \bar{\gamma}^1$.

Special Case: Let $\omega^i = (1, 0, 0)$, $\beta_1^i + \beta_2^i = 1$, and $\theta_i = 1/2$ for $i \in \{1, 2\}$. Then (*) reduces to

$$\frac{p_1 \cdot p_1 / 2}{1 + \left(\frac{1}{8} - \frac{\gamma^i}{2}\right) p_1 \cdot p_1} - \frac{\beta_1^i + \beta_2^i}{\gamma^i} = 0, \quad i \in \{1, 2\}.$$

Taking $\gamma^2 = 1 - \gamma^1$ and solving for $p_1 \cdot p_1$ and γ^1 , we have $\gamma^1 = 1/2$ and $p_1 \cdot p_1 = 8/3$. Thus the set of spot price vectors $p_1 \gg 0$ on the circle of radius $\sqrt{8/3}$ is in one-to-one correspondence with the set of equilibrium consumption allocations.

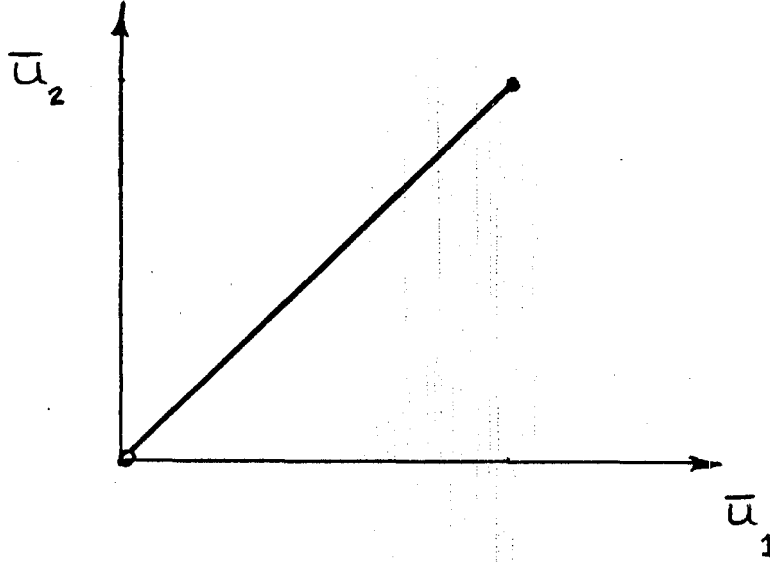


Figure 1. Equilibrium Utilities for $\beta^1 = \beta^2$.

We graph the equilibria in terms of the monotonic transformation of utility

$$\bar{u}_i(x) = e^{u_i(x)}, \quad i \in \{1, 2\}.$$

The graph of equilibrium utilities for $\beta^1 = \beta^2$ is shown in Figure 1. In this case, the equilibria are strictly Pareto ordered! The graph of equilibrium utilities for $\beta^1 = (\frac{1}{3}, \frac{2}{3})$ and $\beta^2 = (\frac{2}{3}, \frac{1}{3})$ is shown in Figure 2 as the set of non-zero solutions to the cubic equation $\bar{u}_1^6 + \bar{u}_2^6 = k \bar{u}_1^2 \bar{u}_2^2$ for a scalar k . Only at point A in Figure 2, does agent 2 agree with the production choice of the firm. [It also happens that A is the point of highest utility for agent 2 on the graph.] Similarly, agent 1 is in agreement with the firm's choice only at point B .

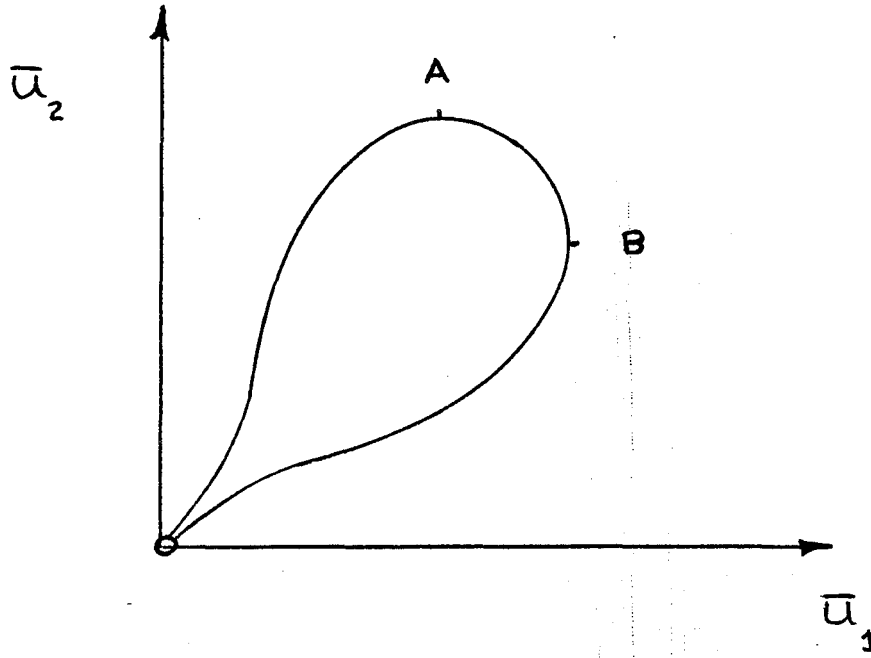


Figure 2. Equilibrium Utilities for $\beta^1 \neq \beta^2$

Multiplicity and Optimality of Equilibria

In order to study the determinacy of equilibrium allocations in an appropriate setting, we introduce assets $a^k \in \mathbb{R}^{S^t}$, $0 \leq k \leq n_2 \leq n$, in zero net supply, where $a^k(s) \in \mathbb{R}^t$ is the commodity bundle paid by asset k in state s . The remaining $n_1 = n - n_2 \geq 0$ securities are firms' shares, as before. We let

$$V(p, (y_1, a)) = (p_1 \square y_1^1 | \dots | p_1 \square y_1^{n_1} | p_1 \square a^1 | \dots | p_1 \square a^{n_2})$$

denote the corresponding $S \times n$ dividend matrix. Given a state price vector $q \in \mathbb{R}_{++}^S$ and a spot price vector $p \in \mathcal{L}_{++}$, the initial market value of the asset a^k is $q \cdot (p \square a^k)$. Problem (2) is amended to read

$$\max_{x \in \mathcal{L}_{++}, \gamma \in \mathbb{R}^n} u_i(x) \quad (2')$$

subject to

$$p_0 \cdot (x_0 - \omega_0^i) + \sum_{j=1}^{n_1} \left[(\gamma_j - \theta_{ij}) v_{qp}(y^j) - \gamma_j p_0 \cdot y_0^j \right] + \sum_{k=n_1+1}^{n_2} \gamma_k q \cdot (p \square a^k) \leq 0$$

$$p_1 \square (x_1 - \omega_1^i) \leq V(p, (y_1, a)) \gamma.$$

The conditions (a)–(d) defining a collection $((x^i, \gamma^i), (y^j), p, q)$ to be an equilibrium for an economy $((u_i, \omega^i), (Y_j), (\theta_{ij}), a)$ are otherwise changed only by adding the condition:

(e) $\sum_i \gamma_k^i = 0$ for $n_1 + 1 \leq k \leq n$.

We allow the case $n_1 = 0$ of pure exchange or $n_1 = n$ (of pure stock markets) for comparison, asking the reader to make the obvious notational adjustments for these cases.

Theorem 2. Suppose utility assumptions (U.1)–(U.4) and production assumptions (P.1)–(P.3) apply. Then there is an open subset E of $\mathcal{L}_+^m \times T \times \mathbb{R}^{Stn_2}$ whose complement has Lebesgue measure zero such that, for each $(\omega, t, a) \in E$,

- (1) there exists an equilibrium for the economy $((u_i, \omega^i), (Y_j^t), (\theta_{ij}), a)$,
- (2) if $n \geq S$, the set of equilibrium allocations is finite,
- (3) if $n_1 = 0$ (pure exchange), the set of equilibrium allocations is finite,
- (4) if $n_1 \geq 1$ (production) and $n < S$, the set of equilibrium allocations contains a set homeomorphic to a ball in \mathbb{R}^{S-n} ,
- (5) if $n \geq S$, every equilibrium allocation is Pareto optimal, and
- (6) if $n < S$, every equilibrium allocation is not Pareto optimal.

The proof of Theorem 2 is given in the final section of the paper. In the following section we define and prove an additional generic property:

- (7) If $n < S$, for any equilibrium, all shareholders (except possibly one) of any firm disagree with maximization of the firm's market value.

In Sections 7 and 8 we extend this existence result to general stochastic economies.

§4. The Production Goals of the Firm

We adopt the usual competitive assumption that agents take prices as given. With incomplete markets, this includes both the spot commodity price vector p and the state price vector q , both of which we fix for this discussion. We also fix the production choices y^k , $k \neq j$, of all firms other than a particular firm j , with $v_{qp}(y^k) \geq 0$. The value of problem (2') can then be written as the function $U_{ij} : Y_j \rightarrow \mathbb{R}$ defined by $U_{ij}(y^j) = u_i(x^i)$, where x^i solves problem (2'). [We define $U_{ij}(y^j)$ to be $-\infty$ if the market value of y^j leaves an empty

budget feasible set for i .] A production choice $y^j \in Y_j$ is defined to be optimal for agent i if y^j solves the problem

$$\max_{y \in Y_j} U_{ij}(y). \quad (4)$$

We first recall the optimality of market value maximization in complete markets, using none of the regularity assumptions (U.1)–(U.4) or (P.1)–(P.3) for this result, whose proof is obvious.

Proposition 1. *Suppose $\text{span}(V(p, (y_1, a))) = \mathbb{R}^S$ (complete markets). For any firm j and any agent i , if y^j solves the market value maximization problem (3), then y^j is optimal for agent i . Suppose, moreover, that $\theta_{ij} > 0$ and u_i is locally non-satiated. Then y^j is optimal for agent i if and only if y^j solves (3).*

In other words, assuming only locally non-satiated preferences, the unique production objective supported by any shareholder in complete markets is market value maximization, a case of *unanimity*. This result is completely overturned (generically) in incomplete markets equilibria. To show this, we begin by demonstrating that a firm and an agent agree on market value maximization only when the incompleteness of markets is not a binding constraint on the agent.

Proposition 2. *Suppose u_i satisfies (U.1)–(U.4) and Y_j satisfies (P.1)–(P.3). If (x^i, γ^i) solves (2') with $\gamma_j^i \neq 0$ and y^j solves (3) and (4), then x^i also solves the complete markets problem*

$$\max_{x \in \mathcal{L}_{++}} u_i(x) \quad \text{subject to} \quad v_{qp}(x - \omega^i - \sum_k \theta_{ik} y^k) \leq 0. \quad (2'')$$

PROOF: Let $\mathcal{N} \subset \mathcal{L}$ denote a neighborhood of y^j small enough that $U : \mathcal{N} \rightarrow \mathbb{R}$ is a smooth function when defined by

$$U(y^j) = \max_{x, \gamma_k, k \neq j} u_i(x)$$

subject to the constraints of problem (2'), holding γ_j fixed at γ_j^i . We can assume without loss of generality that $q = (1, 1, \dots, 1)$. We calculate $\hat{p} = D_y U(y^j)^\top$ to be $\hat{p}_0 = \nu \theta_{ij} p_0$ and

$$\hat{p}_1(s) = (-\nu(\gamma_j^i - \theta_{ij}) + \lambda_s \gamma_j^i) p_1(s), \quad 1 \leq s \leq S,$$

where ν and $\lambda \in \mathbb{R}^S$ are the Lagrange multipliers for problem (2'). By Lemma B.1 of the final section, we can write the market value maximizing production choice of firm j as a smooth function $y^j : \mathcal{L}_{++} \rightarrow Y_j$ of spot prices. Since $y^j(p)$ solves (4), we have

$$U(y^j(p')) \leq U(y^j(p))$$

for all p' in a neighborhood of p . It follows that $U \circ y^j : \mathcal{L}_{++} \rightarrow \mathbb{R}$ has a local maximum at p , implying that $D_y U(y^j(p)) D_p y^j(p) = 0$. Then $\hat{p}^\top D_p y^j(p) \hat{p} = 0$, but by the second order conditions for market value maximization (Lemma B.4 (4)), this can only be true if $\gamma_j^i \nu = \gamma_j^i \lambda_s$ for all s . Since $\gamma_j^i \neq 0$, we have $\nu = \lambda_s$, $1 \leq s \leq S$. An examination of the first order conditions for problem (2') then shows that x^i solves the first order conditions for the complete markets problem (2''). ■

We say that *shareholder i disagrees with market value maximization by firm j at $((x^i, \gamma^i), (y^j), p, q)$* if $\gamma_j^i \neq 0$ and if y^j does not solve both (3) and (4).

Corollary. *Theorem 2 holds with appended property (7).*

PROOF: This follows from Proposition B.4 (2), which shows that, generically in equilibrium, the market subspace constraint is binding, so that the equilibrium solution x^i to problem (2') does not solve the corresponding complete markets problem (2''). ■

Given an absence of unanimity for maximization of market value, it may be prudent to extend our existence results to a general class of objective functions for the firm. Aside from the fact that perturbing a production set induces a full rank perturbation of dividends (Lemma B.3), the only properties of the firm's objective that we actually use in our existence proof are the smoothness of supply functions, homogeneity in prices, and positive market value. Thus an extension of our existence results to a general class of production objectives that includes market value maximization is quite conceivable.

§5. Stochastic Equilibria with General Securities

In this section we extend our existence result for incomplete markets to the general stochastic setting of Debreu (1959), Chapter 7, incorporating zero net supply securities of various types. The extension is along the lines of the pure exchange model of Duffie and Shafer (1985b).

The Event Tree

The model of uncertainty is an *event tree*, a directed graph (Ξ, \mathcal{A}) consisting of a set $\Xi = \{\xi^1, \dots, \xi^H\}$ of H vertices (or "nodes") and a set $\mathcal{A} \subset \Xi \times \Xi$ of arcs (or "branches"). If (ξ, η) is an arc, we may think of ξ as the "state-date pair" that uniquely precedes η . We denote this precedence by writing $\xi = \eta_-$. The root vertex ξ^1 is distinguished as

the unique vertex without a predecessor. A walk is a sequence η_1, \dots, η_k of vertices in Ξ with the property that $(\eta_i, \eta_{i+1}) \in \mathcal{A}$, $1 \leq i \leq k-1$. A cycle is a walk η_1, \dots, η_k with $\eta_k = \eta_1$, $k \neq 1$. An event tree is such a directed graph without a cycle. The successors of a vertex $\xi \in \Xi$, denoted $\Xi(\xi)$, is the set of vertices with the property that there is a walk from ξ to η . In other words, $\Xi(\xi)$ is the sub-tree with root vertex ξ . The vertices with the same unique predecessor ξ can be ordered and denoted ξ_1, \dots, ξ_{+k} . The integer k is the outdegree of ξ . We refer to Figure 3 for illustration.

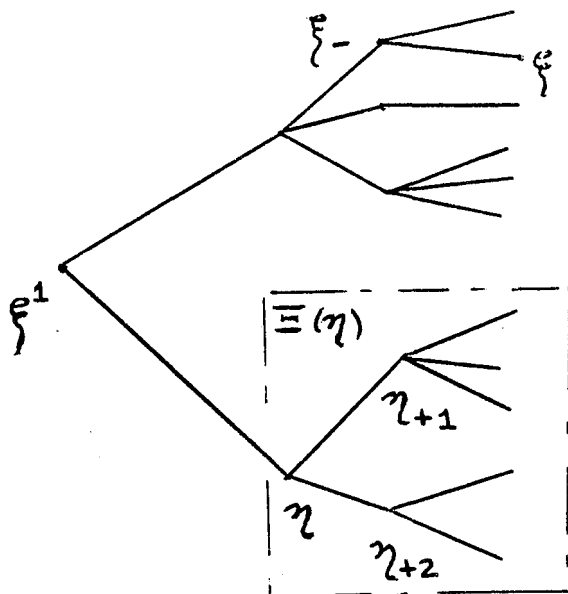


Figure 3. Event Tree Notation

The Markets

There are complete spot markets for ℓ commodities at each vertex ξ in Ξ . For any integer $k \geq 1$, let D_k denote the space of \mathbb{R}^k -valued functions on Ξ . Our consumption space is thus $\mathcal{L} = D_\ell$, treated equivalently as $\mathbb{R}^{H\ell}$ with the obvious co-ordinate identifications. We refer to any function on Ξ as a “process”, and cite Section 7 of Duffie and Shafer

(1985b) for the equivalence with a traditional probabilistic model of stochastic processes and information filtrations.

For any spot price process p in D_t and any consumption process x in D_t , let $p \square x \in D_1$ denote the real-valued process defined by

$$[p \square x](\xi) = p(\xi) \cdot x(\xi), \quad \xi \in \Xi.$$

That is, $p \square x$ is the process of spot market values required to purchase x at each vertex.

A *purely financial security* is an element δ of D_1 , a claim paying $\delta(\xi)$ units of account (say "dollars") at vertex ξ . A *real security* is an element $d \in D_t$ paying the bundle of commodities $d(\xi) \in \mathbb{R}^t$ at vertex ξ . A *security* is a pair $(\delta, d) \in D_1 \times D_t$ with purely financial component δ and real component d .

Agents take as given a collection (δ_h, d_h) , $1 \leq h \leq k$, of k securities. For each $\xi \in \Xi$, we let $\delta(\xi) = (\delta_1(\xi), \dots, \delta_k(\xi)) \in \mathbb{R}^k$ and we let $d(\xi)$ denote the $t \times k$ matrix whose h -th column is $d_h(\xi)$. A *security price process* $\pi \in D_k$ is also taken as given by agents, with $\pi(\xi) \in \mathbb{R}^k$ denoting the vector of market values of the k securities at vertex ξ , before the securities have paid their dividends (or *cum dividend*). A *security trading strategy* is a function $\gamma : \Xi' \rightarrow \mathbb{R}^k$, where $\Xi' = \Xi \cup \{\xi^0\}$ is the event tree formed by adjoining a pre-trade vertex $\xi^0 = \xi_-^1$ to Ξ . That is, $\gamma(\xi)$ is the pre-dividend portfolio of securities held by strategy γ at vertex ξ . Let Γ denote the space of security trading strategies.

Given securities (δ_h, d_h) , $1 \leq h \leq k$, a spot price process p , and a security price process π , a trading strategy $\gamma \in \Gamma$ generates the dividend process $\delta^\gamma \in D_1$ defined by

$$\delta^\gamma(\xi) = \gamma(\xi) \cdot [\delta(\xi) + p(\xi)^\top d(\xi)] + [\gamma(\xi_-) - \gamma(\xi)] \cdot \pi(\xi), \quad \xi \in \Xi.$$

The right hand side of this expression is the sum of the spot market values of the dividends paid to γ at ξ , and the market value of the portfolio of securities sold by γ at ξ .

Equilibrium

A stochastic economy may now be summarized by a collection

$$((\Xi, \mathcal{A}), (u_i, \omega^i), (Y_j), (\theta_{ij}), (\delta_h, d_h)), \quad 1 \leq i \leq m, \quad 1 \leq j \leq n, \quad j+1 \leq h \leq k,$$

where (Ξ, \mathcal{A}) is an event tree, $((u_i, \omega^i), (Y_j), (\theta_{ij}))$ is a standard Arrow-Debreu production-exchange economy for the consumption space $\mathcal{L} = D_t$, and (δ_h, d_h) , $j+1 \leq h \leq k$, is a collection of securities held in zero net supply. Without security trading by firms, the first n securities are defined by $\delta_j = 0$ and $d_j = y^j$, $1 \leq j \leq n$, where $y^j \in Y_j$ is the production choice of firm j .

A state price process $q \in (D_1)_{++}$ and spot price process $p \in \mathcal{L}_{++}$ are taken as given by agents and firms. The price process of any purely financial security is then determined by the operator $\Lambda_q : D_1 \rightarrow D_1$ defined by

$$[\Lambda_q(\delta)](\xi) = \frac{1}{q(\xi)} \sum_{\eta \in \Xi(\xi)} q(\eta) \delta(\eta), \quad \xi \in \Xi, \delta \in D_1.$$

For example, if $q \equiv 1$, then Λ_q assigns the price of a purely financial security at any vertex ξ to be the sum of its dividends in the sub-tree $\Xi(\xi)$. We can also design a state-price process q so that $[\Lambda_q(\delta)](\xi)$ corresponds to the conditional expected sum of future dividends at ξ under a given probability measure, as shown in Section 7 of Duffie and Shafer (1985b). The given k securities are priced according to Λ_q by

$$\pi_h = \Lambda_q(\delta_h + p \square d_h), \quad 1 \leq h \leq k. \quad (6)$$

Taking q and p as given, firm j chooses $y^j \in Y_j$ to solve the problem

$$\max_{y \in Y_j} v_{qp}(y) \equiv \sum_{\xi \in \Xi} q(\xi) p(\xi) \cdot y(\xi), \quad (7)$$

maximizing its initial share value $\pi_j(\xi^1)$. By the usual principle of dynamic programming, this is equivalent to maximizing the market value $\pi_j(\xi)$ at every vertex ξ in Ξ , taking preceding production choices as irrevocable. To see this, we let $c = C(y_1, y_2, \xi)$ be defined for any y_1 and y_2 in \mathcal{L} and any vertex ξ in Ξ , by

$$c(\eta) = y_1(\eta), \quad \eta \notin \Xi(\xi)$$

$$c(\eta) = y_2(\eta), \quad \eta \in \Xi(\xi).$$

We could call $C(y_1, y_2, \xi)$ the *continuation* of y_1 by y_2 at ξ . If y^j solves (7), then at any ξ in Ξ , y^j solves the ξ -*continuation problem*

$$\max_{y \in Y_j} \sum_{\eta \in \Xi(\xi)} q(\eta) p(\eta) \cdot y(\eta) \quad \text{subject to } C(y^j, y, \xi) \in Y_j.$$

If this were not true, then there exists $\xi \in \Xi$ and $\bar{y} \in \mathcal{L}$ such that $z = C(y^j, \bar{y}, \xi) \in Y_j$ and

$$v_{qp}(z) = \sum_{\eta \in \Xi(\xi)} q(\eta)p(\eta) \cdot \bar{y}(\eta) + \sum_{\eta \notin \Xi(\xi)} q(\eta)p(\eta) \cdot y^j(\eta) > v_{qp}(y^j),$$

but this contradicts the fact that y^j solves (7). Thus a production plan that maximizes initial share price always maximizes share price.

Given a spot price process $p \in \mathcal{L}_{++}$, a state price process $q \in (D_1)_{++}$, production choices $y^j \in Y_j$, $1 \leq j \leq n$, and the pricing convention (6), a pair $(x, \gamma) \in \mathcal{L}_{++} \times \Gamma$ is a *budget feasible plan* for agent i provided $\delta^\gamma \geq p \square (x - \omega^i)$ and $\gamma(\xi^0)_j = \theta_{ij}$, $1 \leq j \leq n$; $\gamma(\xi^0)_j = 0$, $n+1 \leq j \leq k$. A budget feasible plan (x, γ) is *optimal* for i if there is no budget feasible plan (x', γ') for i such that $u_i(x') > u_i(x)$.

- An *equilibrium* is a collection $((x^i, \gamma^i), (y^j), q, p)$, $1 \leq i \leq m$, $1 \leq j \leq n$, such that:
- (a') for each agent i , (x^i, γ^i) is an optimal plan given q, p , and (y^j) ;
 - (b') for each firm j , y^j solves the market value maximization problem (7);
 - (c') $\sum_i x^i - \omega^i = \sum_j y^j$,
 - (d') $\sum_i \gamma_j^i(\xi) = 1$, $1 \leq j \leq n$, $\xi \in \Xi$, and
 - (e') $\sum_i \gamma_h^i(\xi) = 0$, $j+1 \leq h \leq k$, $\xi \in \Xi$.

Theorem 3. Suppose utility assumptions (U.1)-(U.4) and production assumptions (P.1)-(P.3) apply. Let $\bar{q} \in (D_1)_{++}$ be any given state price process. Then, for generic $(\omega, t, d) \in \mathcal{L}_+^m \times \mathcal{T} \times \mathcal{L}^{k-n}$, there exists an equilibrium of the form $((x^i, \gamma^i), (y^j), \bar{q}, p)$ for the stochastic incomplete markets economy

$$((\Xi, \mathcal{A}), (u_i, \omega^i), (Y_j^t), (\theta_{ij}), (\delta_h, d_h)), \quad 1 \leq i \leq m, 1 \leq j \leq n, j+1 \leq h \leq k.$$

Theorem 3 contains Theorem 1 as a special case. The proof is given in Appendix C, based largely on the proof of Theorem 1 and on the stochastic exchange model of Duffie and Shafer (1985b). In Section 8 we extend this result by allowing security trading by firms and by allowing for linear restrictions on portfolio formation.

§6. The Simultaneous Determination of Dividends of Interdependent Securities

To study the problem of how securities that “invest” in one another may have their dividends and prices simultaneously determined, we generalize the definition of a security

to a triple $(d, \delta, \gamma) \in \mathcal{L} \times D_1 \times \Gamma$ with $\gamma(\xi^0) = 0$. [A security is not “endowed” initially with holdings of other securities.] Given a spot price process $p \in \mathcal{L}$, a state price process $q \in (D_1)_{++}$, and k such securities, $(\delta^j, d^j, \gamma^j)$, $1 \leq j \leq k$, we will first verify the existence of security dividends and prices under a non-singularity condition. If it exists, the total dividend paid by security j is defined by $\Delta^j = \delta^j + p \square d^j + \delta^{\gamma^j}$, where of course $\delta^{\gamma^j} \in D_1$ is defined simultaneously with δ^{γ^h} for $h \neq j$. Let $\Delta \in D_k$ be defined by $\Delta(\xi)_j = \Delta^j(\xi)$; $\delta^\gamma \in D_k$ be defined by $\delta^\gamma(\xi)_j = \delta^{\gamma^j}(\xi)$; $\delta \in D_k$ be defined by $\delta(\xi)_j = \delta^j(\xi)$; and $p \square d \in D_k$ be defined by $[p \square d](\xi)_j = (p \square d^j)(\xi)$. We then have

$$\Delta = \delta + p \square d + \delta^\gamma \quad (8)$$

whenever δ^γ is well-defined. Let $\gamma \in \Gamma^k$ be defined by letting $\gamma(\xi)$ denote the $k \times k$ matrix with (j, h) -element $\gamma^j(\xi)_h$. Let \succeq denote the binary order on Ξ defined by $\eta \succeq \xi \iff \eta \in \Xi(\xi)$ (“ η follows ξ ”), and \succ denote the corresponding strict order. We let I denote the $k \times k$ identity matrix.

Lemma 2. Suppose $I - \gamma(\xi)$ is non-singular for all $\xi \in \Xi$. Then, for any $(p, q) \in D_t \times D_1$, the vector dividend process Δ of (8) and the vector price process $\pi = \Lambda_q(\Delta)$ are uniquely defined. In particular,

$$\begin{aligned} \Delta(\xi) = & ([I - \gamma(\xi_-)]^{-1} \gamma(\xi_-) + I) [(p \square d)(\xi) + \delta(\xi)] \\ & + [I - \gamma(\xi_-)]^{-1} [\gamma(\xi_-) - \gamma(\xi)] \sum_{\eta \succ \xi} \frac{q(\eta)}{q(\xi)} \Delta(\eta), \quad \xi \in \Xi. \end{aligned} \quad (9)$$

PROOF: We will define Δ inductively by (9), starting from terminal vertices in Ξ , moving from ξ to ξ_- through the tree, and using the definition of Λ_q . For terminal ξ (that is, $\text{outdegree}(\xi) = 0$), the second term of (9) is null, and the first term is well defined by the assumed nonsingularity of $I - \gamma(\xi_-)$. By the definitions of δ^γ and Λ_q ,

$$\delta^\gamma(\xi) = \gamma(\xi)[a(\xi) + \delta^\gamma(\xi)] + [\gamma(\xi_-) - \gamma(\xi)] \sum_{\eta \succeq \xi} \frac{q(\eta)}{q(\xi)} \Delta(\eta),$$

where $a(\xi) \equiv (p \square d)(\xi) + \delta(\xi)$, $\xi \in \Xi$. Thus

$$\delta^\gamma(\xi) = \gamma(\xi_-)[a(\xi) + \delta^\gamma(\xi)] + [\gamma(\xi_-) - \gamma(\xi)] \sum_{\eta \succ \xi} \frac{q(\eta)}{q(\xi)} \Delta(\eta),$$

which implies that

$$\delta^\gamma(\xi) = [I - \gamma(\xi_-)]^{-1} \left[\gamma(\xi_-)a(\xi) + [\gamma(\xi_-) - \gamma(\xi)] \sum_{\eta \succ \xi} \frac{q(\eta)}{q(\xi)} \Delta(\eta) \right].$$

This relation combined with (8) yields (9) and $\pi = \Lambda_q(\Delta)$. \blacksquare

Arbitrage Valuation

Let $\Delta \in D_k$ and $\pi \in D_k$ denote, respectively, a given dividend process and a given security price process. As formulated in Section 5, a security trading strategy $\gamma \in \Gamma$ generates the dividend process $\delta^\gamma \in D_1$ defined by

$$\delta^\gamma(\xi) = \gamma(\xi) \cdot \Delta(\xi) + [\gamma(\xi_-) - \gamma(\xi)] \cdot \pi(\xi), \quad \xi \in \Xi.$$

For generality, we restrict security portfolios at each vertex in Ξ to some given linear subspace of \mathbb{R}^k , and let $\Theta \subset \Gamma$ denote the resulting linear subspace of *admissible trading strategies*. This allows us to include, for instance, securities such as futures, options, and bonds, which are typically available for trade only during fixed intervals of time before expiry.

A pair $(\pi, \Delta) \in D_k \times D_k$ is *arbitrage-free* if $\gamma \in \Theta$, $\delta^\gamma \geq 0$, and $\delta^\gamma \neq 0$ imply that $\gamma(\xi_-^1) \cdot \pi(\xi^1) > 0$. That is, (π, Δ) is arbitrage-free if there is no admissible trading strategy generating positive dividends with non-positive initial investment. We next show that our pricing convention $\pi = \Lambda_q(\Delta)$, for some strictly positive state price process $q \in (D_1)_{++}$, is natural and without loss of generality.

Proposition 4. *If $\pi = \Lambda_q(\Delta)$ for some $q \in (D_1)_{++}$, then (π, Δ) is arbitrage-free. Conversely, if $\Theta = \Gamma$ (unrestricted security trading) and (π, Δ) is arbitrage-free, then there exists a state-price process $q \in (D_1)_{++}$ such that $\pi = \Lambda_q(\Delta)$.*

PROOF: The first implication is obvious. For the second, let $M = \{\delta^\gamma : \gamma \in \Gamma\} \subset D_1$, and let $\psi : M \rightarrow R$ be the linear functional on the “marketed subspace” M defined by $\psi(\delta^\gamma) = \gamma(\xi_-^1) \cdot \pi(\xi^1)$. By the definition of arbitrage-free, ψ is strictly positive. By Stiemke’s Lemma [Mangasarian (1969)], there exists a strictly positive linear extension $\hat{\psi} : D_1 \rightarrow R$ of ψ . Let $q \in (D_1)_{++}$ represent $\hat{\psi}$ by $\hat{\psi}(\delta) = \sum_{\xi \in \Xi} q(\xi) \delta(\xi)$. We will get a contradiction if, at some $\eta \in \Xi$ and for some security j , we have $\pi(\eta)_j > [\Lambda_q(\Delta)](\eta)_j$. In this case, let γ denote the trading strategy: $\gamma(\xi) = 0$, $\eta \succ \xi$; $\gamma(\eta)_h = 0$, $h \neq j$; $\gamma(\eta)_j = -1$; and

$\gamma(\xi) = \gamma(\eta)$, $\xi \succ \eta$. This strategy generates dividends $-\Delta(\eta)_j + \pi(\eta)_j$ at node η and $\Delta(\xi)_j$ at any $\xi \succ \eta$. Thus the initial cost of this strategy is

$$\psi(\delta^\gamma) = q(\eta)[- \Delta(\eta)_j + \pi(\eta)_j] - \sum_{\xi \succ \eta} q(\xi) \Delta(\xi)_j = \pi(\eta)_j q(\eta) - q(\eta)[\Lambda_q(\Delta)](\eta)_j > 0.$$

However $\psi(\delta^\gamma) > 0$ is impossible since $\gamma(\xi_-^1) = 0$. ■

Within an event tree context, this result is a generalization of the result by Harrison and Kreps (1979) on “equivalent martingale measures.” For related literature in other settings, we cite Rubinstein (1976) and Ross (1978).

§7. The Modigliani–Miller Theory and Incomplete Markets

In order to investigate the financial policy of the firm, as suggested in Section 1, we expand the set of decisions of firm j to include a security trading strategy $\beta \in \Gamma$, as well as a production plan $y \in Y_j$. By assumption, the initial endowment of securities to firm j is zero, or $\beta(\xi_-^1) = 0$. With a limited liability restriction, the firm must choose a trading strategy β generating a positive dividend process $\Delta^j = p \square y + \delta^\beta \geq 0$. For example, a firm may wish to finance a large capital investment by borrowing on bond markets, rather than collecting funds from shareholders via negative dividends. Given the usual assumption $0 \in Y_j$, the limited liability restriction $\Delta^j \geq 0$ can always be met by the plan $(y, \beta) = (0, 0)$. For our general purposes, we will not impose limited liability except to note that doing so would not generally affect our results.

Our first task is to show conditions under which the market value of the firm cannot be affected by changes in its financial policy, Proposition I of the “Modigliani–Miller Theorem”. We always assume that the firm takes a state price process q and a spot price process p as given, the “competitive” assumption. Barring arbitrage, the pricing of securities by a state-price process is guaranteed by Proposition 4. Since markets are not generally complete, the knowledge that securities will continue to be priced by a given state price process q as the span of markets changes is important in verifying the following result.

At a superficial level, it is trivial that the initial market value $\pi(\xi_-^1)_j$ of firm j cannot be affected by a change in the firm’s security trading strategy. Let $\psi : D_1 \rightarrow \mathbb{R}$ denote the functional assigning the initial market value $[\Lambda_q(\delta)](\xi_-^1)$ to any financial security $\delta \in D_1$. By

buying or selling other securities, a firm merely pays $\psi(\delta)$ in order to add δ to its dividend process, which adds to the market value of the firm before purchase cost by $\psi(\delta)$, leaving a net effect on its initial cum dividend share price of zero. [This includes the effects of a firm's repurchase of its own shares.] While this is indeed the case, one must also consider the impact of a change in the dividends generated by firm j on the prices and dividends of other securities that hold shares in firm j , and the resultant feedback effect on firm j itself through its holdings of the other securities, and so on. We affirm in the next proposition, however, that the superficial argument yields the correct conclusion: the firm cannot change its market value via financial policy.

Let us call a collection $\gamma = (\gamma^1, \dots, \gamma^k) \in \Gamma^k$ of k security trading strategies consistent if $I - \gamma(\xi)$ is non-singular for all non-terminal $\xi \in \Xi$. Then, given any spot price process p and state price process q , a collection $(\delta, d, \gamma) = (\delta^j, d^j, \gamma^j)$, $1 \leq j \leq k$, of securities has a joint dividend process $\Delta = \delta + p \square d + \delta^\gamma$ uniquely defined by (8) and a corresponding price process $\pi = \Lambda_q(\Delta)$ if and only if γ is consistent (Lemma 2).

Proposition 5. *Let p be a spot price process, q be a state price process, and $(\delta^j, d^j, \gamma^j)$, $1 \leq j \leq k$ be a collection of securities. If $\gamma = (\gamma^1, \dots, \gamma^k)$ is consistent, then*

$$[\Lambda_q(\Delta)](\xi^1) = [\Lambda_q(\delta + p \square d)](\xi^1).$$

To re-iterate, this states that the initial market value of a security is no more or less than the market value of its own primitive dividends, independent of the security dividends that it collects from and pays to other securities through time via purchases and sales of securities.

PROOF: We collect all terms of the sum

$$[\Lambda_q(\Delta)](\xi^1) = \sum_{\xi} \frac{q(\xi)}{q(\xi^1)} \Delta(\xi)$$

that involve the primitive dividend $a(\eta) = \delta(\eta) + (p \square d)(\eta)$ for an arbitrary vertex η . We re-label the vertices along the path from ξ^1 to η as $\eta_1 = \eta$, $\eta_2 = \eta_-$, $\eta_3 = (\eta_-)_-$, \dots , $\eta_N = \xi^1$. For notational ease, let $\gamma_n = \gamma(\eta_n)$, and let T_n denote the term in $\Delta(\eta_n)$ that involves $a(\eta_1)$, $1 \leq n \leq N$. Without loss of generality, we take $q \equiv 1$. Then the terms in $[\Lambda_q(\Delta)](\xi^1)$ that involve $a(\eta)$ are T_1, \dots, T_N , where, by Lemma 2,

$$T_1 = [(I - \gamma_2)^{-1} \gamma_2 + I]a(\eta),$$

$$T_n = (I - \gamma_{n+1})^{-1}(\gamma_{n+1} - \gamma_n)(T_1 + \dots + T_{n-1}), \quad 2 \leq n \leq N-1,$$

$$T_N = -\gamma(T_1 + \dots + T_{N-1}),$$

using $\gamma(\xi_-^1) = 0$. To complete the proof, we may show that $T_1 + \dots + T_N = a(\eta)$, but this follows from the recursion:

$$\begin{aligned} & T_N + T_{N-1} + \dots + T_1 \\ &= (I - \gamma_N)(T_{N-1} + \dots + T_1) \\ &= (I - \gamma_N)[(I - \gamma_N)^{-1}(\gamma_N - \gamma_{N-1})(T_{N-2} + \dots + T_1)] + (I - \gamma_N)(T_{N-2} + \dots + T_1) \\ &= (I - \gamma_{N-1})(T_{N-2} + \dots + T_1) \\ &= \dots \\ &= (I - \gamma_2)[(I - \gamma_2)^{-1}\gamma_2 + I]a(\eta) = a(\eta). \end{aligned}$$

This completes the proof, since η is arbitrary. ■

The effect of issuing new securities on the market value of the firm is also easily modeled. *Financial policy is value neutral*, by our definition, if for any security (δ, d, γ) , any spot price process p , any state price process q , and any financial security $\bar{\delta} \in D_1$, we have

$$\psi(\delta + p \square d + \delta^\gamma - \bar{\delta}) + \psi(\bar{\delta}) = \psi(\delta + p \square d), \quad (10)$$

where $\psi : D_1 \rightarrow \mathbb{R}$ is the initial market value function $\delta \mapsto [\Lambda_q(\delta)](\xi_-^1)$. Relation (10) states that the initial market value of the security (δ, d, γ) after issuing $\bar{\delta}$, plus the market value $\psi(\bar{\delta})$ received for the sale of $\bar{\delta}$, is merely the original value of the security, which is independent of the security trading strategy γ . But relation (10) is perfectly trivial given Proposition 5, since ψ is linear and $\gamma(\xi_-^1) = 0$.

We can summarize our progress on the Modigliani-Miller theory by asserting that financial policy, including both the issuing and trading of securities, is neutral for the initial market value of the firm in competitive linear markets, complete or incomplete.

Our second area of concern is the effect of financial policy on the welfare of shareholders. Based on the model of simultaneous arbitrage-free valuation of interdependent securities constructed in Section 6, DeMarzo (1986) has subsequently shown that any regular security trading strategy adopted by firms leaves the agents' budget feasible consumption sets invariant. Hence, so long as firms trade only the securities already available to

agents, agents are indifferent to these trades. It is simple, however, to construct examples in which the utility of any price-taking shareholder is strictly improved by issuing a new corporate security appropriately tailored to the shareholder's hedging needs. Let $\bar{x} \in \arg \max_{x \in (D_t)_{++}} u_i(x)$ s.t. $\psi(p \square (x - \omega^i)) = 0$. That is, \bar{x} is an optimal choice for i ignoring the market span constraint. If the market span constraint is actually binding, then the constrained optimal choice x^i for i is by definition strictly inferior, or $u_i(x^i) < u_i(\bar{x})$. By costlessly issuing the corporate security $\bar{\delta} = p \square (\bar{x} - \omega^i)$, a firm strictly improves the allocation of shareholder i , taking prices as given, since i can then finance the consumption choice \bar{x} by purchasing one share of $\bar{\delta}$ and holding it. This is budget feasible since $\psi(\bar{\delta}) = 0$.

Of course, we do not propose that firms issue such a tailor-made security for each agent. Even taking the equilibrium price p as fixed, the cost of issuing securities is not zero. We merely make the observation that financial policy does affect shareholders when the incompleteness of markets is binding, which is generically the case in incomplete markets equilibria (Proposition B.4 (2)). This also raises the issue of whether the firm, by virtue of its access to capital markets and ability to market new securities at low cost relative to individual agents, has a special role to play in adding new span to markets. To quote a well known textbook of corporate finance,

“Proposition I [The Modigliani-Miller irrelevance principle] is violated when financial managers find an untapped demand and satisfy it by issuing something new and different. The argument between MM and the traditionalists finally boils down to whether this is difficult or easy. We lean toward MM's view: finding unsatisfied clienteles and designing exotic securities to meet their needs is a game that's fun to play but hard to win.” [Brealey and Myers (1984), p. 372]

Most of our current knowledge of the process of financial innovation is anecdotal [Sandor (1973), Silber (1983)]. In this paper we have said little about the role of the firm as a financial innovator beyond the obvious fact that it matters.

§8. A General Existence Theorem

We expand the definition of an economy to a collection

$$((\Xi, \mathcal{A})(u_i, \omega^i), (Y_j), (\theta_{ij}), (\delta_h, d_h), \Theta, \beta),$$

$i \in \{1, \dots, m\}$, $j \in \{1, \dots, n\}$, $h \in \{n+1, \dots, k\}$, where (Ξ, \mathcal{A}) is an event tree, $((u_i, \omega^i), (Y_j), (\theta_{ij}))$ is an Arrow-Debreu production economy, $\Theta \subset \Gamma$ is an admissible security trading space, and $\beta \in \Theta$ is a vector of trading strategies with $\beta(\xi^0) = 0$. The definition of an equilibrium $((x^i, \gamma^i), (y^j), q, p)$ is as given in Section 5, with the exception that $\Delta = \delta + p \square d + \delta^\beta$, when uniquely defined by (8), is substituted for the total dividend vector $\Delta = \delta + p \square d$ throughout.

Theorem 4. *Suppose assumptions (U.1)–(U.4) and (P.1)–(P.3) apply. Then, for generic $(\omega, t, \delta, d, \beta) \in \mathcal{L}_{++}^m \times \mathcal{T} \times D_1^{k-n} \times \mathcal{L}^{k-n} \times D_k^k$, there exists an equilibrium for the economy $((\Xi, \mathcal{A}), (u_i, \omega^i), (Y_j^t), (\theta_{ij}), (\delta_h, d_h), \Theta, \beta)$.*

The proof is a straightforward extension of the proof of Theorem 3 and results in Duffie and Shafer (1985b). Generic β suffice since $I - \beta(\xi_-)$ is generically non-singular. The proof proceeds by first taking the case $\beta = 0$, and by using the arguments in Duffie and Shafer (1985b) that allow the generalization from the complete trading strategy space Γ to an admissible subspace Θ . Then the Version of the Modigliani–Miller Theorem in DeMarzo (1986) allows us to substitute any regular β and to adjust agents' security trading strategies to an equilibrium with the same real allocation. We allow perturbations in the purely financial components (δ_h) of securities, since some security (d_h, δ_h) may for structural reasons have $d_h = 0$ (e.g. a nominal bond), and we would not naturally perturb its real component.

§9. Proofs of Theorems

A. The Grassmannian Manifold

We review how one can treat the space $G_{n,S}$ of n -dimensional subspaces of \mathbb{R}^S as a compact smooth manifold without boundary of dimension $n(S-n)$. For $1 \leq n < S$, a particular subspace L in $G_{n,S}$ is induced by a full rank $(S-n) \times S$ matrix A according to $L = \{y \in \mathbb{R}^S : Ay = 0\}$. This defines an equivalence relation \sim on the space X of full rank $(S-n) \times S$ matrices by: $A \sim B$ if A and B induce the same subspace. We can then identify $G_{n,S}$ with X/\sim endowed with the quotient topology. The statement " $A \in L$ ", for a matrix A in X , means A induces $L \in G_{n,S}$. We also take the following Grassmannian differentiable structure for $G_{n,S}$. Let Σ denote the set of permutations of

$\{1, 2, \dots, S\}$. For each σ in Σ , let P_σ denote the $S \times S$ permutation matrix corresponding to σ . Let $W_\sigma^n = \{L \in G_{n,S} : \exists E \in \mathbb{R}^{(S-n)n}; [I \mid E]P_\sigma \in L\}$. For each σ in Σ , we define $\varphi_\sigma^n : W_\sigma^n \rightarrow \mathbb{R}^{(S-n)n}$ by $[I \mid \varphi_\sigma^n(L)]P_\sigma \in L$.

Lemma .1. *The collection $\{(W_\sigma^n, \varphi_\sigma^n) : \sigma \in \Sigma\}$ is an atlas for $G_{n,S}$ making $G_{n,S}$ a compact C manifold without boundary of dimension $n(S-n)$.*

This is Fact 3 of Duffie and Shafer (1985a). We will also have occasion to use the following fact from Duffie and Shafer (1985b).

Lemma .2. *Direct sum from $G_{n_1, S_1} \times G_{n, S}$ into G_{n_1+n, S_1+S} is smooth.*

The following is a trivial consequence of our definitions.

Lemma .3. *Suppose V is an $S \times n$ matrix with $[I \mid \varphi_\sigma^n(L)]P_\sigma V = 0$ for some $\sigma \in \Sigma$ and $L \in W_\sigma^n$. Then $\text{span}(V)$, the span of the columns of V , is a subspace of L . If V is of full rank, then $\text{span}(V) = L$.*

B. Proof of Theorem 2

We first claim the existence of a smooth value-maximizing supply function for each firm.

Lemma B.1. *Under assumptions (P.1), (P.2), and (P.3), there is a smooth function $y^j : \mathcal{L}_{++} \rightarrow \mathcal{L}$ with the properties, for all $p \in \mathcal{L}_{++}$,*

- (1) $\{y^j(p)\} = \arg \max p \cdot z$,
- (2) $p \cdot D_p y^j(p) = 0$,
- (3) $D_p y^j(p)$ is positive semi-definite, and
- (4) $\hat{p}^\top D_p y^j(p) \hat{p} > 0$ for all $\hat{p} \in \mathcal{L}$ of the form $\hat{p} = (\lambda_0 p_0, \lambda_1 p_1(s), \dots, \lambda_S p_1(S)) \neq 0$ with $\lambda \in \mathbb{R}^{S+1}$ having $\lambda_k \neq \lambda_j$ for some j and k in $\{0, 1, \dots, S\}$.

PROOF: This is a consequence of Propositions 3.5.3 and 3.5.4 of Mas-Colell (1985), and the surrounding discussion there. ■

Without loss of generality for our proofs, we can treat \mathcal{T} as the interior of ${}_j \mathcal{L}(j)_+$. Fixing the functions $y^1(\cdot), \dots, y^{n_1}(\cdot)$ defined by the previous lemma, we define $Q : \mathbb{R} \times \mathcal{T} \times \mathcal{L}_{++} \rightarrow \mathcal{L}^{n_1}$ by $Q(\alpha, t, p) = (\alpha y^1(p) + t^1, \dots, \alpha y^{n_1}(p) + t^{n_1})$. For $t \in \mathcal{T}$ and $\alpha \in \mathbb{R}$, let $Y_j^{\alpha t} = \alpha Y_j + \{t^j\}$, $1 \leq j \leq n_1$.

Lemma B.2.

(1) Q is smooth.

(2) For $\alpha \geq 0$, $\{Q^j(\alpha, t, p)\} = \arg \max_{z \in Y_j^\alpha} p \cdot z$.

(3) For $\alpha \geq 0$ and any $t \in T$, $p \cdot Q^j(\alpha, t, p) \geq 0$.

PROOF: Properties (1) and (3) are obvious. We note that

$$\begin{aligned} \arg \max_{z \in \alpha Y_j + \{t^j\}} p \cdot z &= \left(\arg \max_{z \in \alpha Y_j} p \cdot z \right) + \{t^j\} \\ &= \alpha \left(\arg \max_{z \in Y_j} p \cdot z \right) + \{t^j\}. \end{aligned}$$

This and Lemma B.1 imply property (2). ■

Let $\mathcal{A} = \mathbb{R}^{n_2 S t}$ denote the space of assets. The $S \times n$ dividend matrix function corresponding to the supply function Q is the map $\bar{V} : \mathcal{L}_{++} \times \mathbb{R} \times T \times \mathcal{A} \rightarrow \mathbb{R}^{S n}$ defined by $\bar{V}(p, \alpha, t, a) = V(p, (Q(\alpha, t, p)_1, a))$. Let $\mathcal{P} = \{(p, t, a, \alpha) \in \mathcal{L}_{++} \times T \times \mathcal{A} \times \mathbb{R} : \bar{V}(p, \alpha, t, a) \text{ is full rank}\}$.

Lemma B.3. \bar{V} is a submersion and \mathcal{P} is open in $\mathcal{L}_{++} \times T \times \mathcal{A} \times \mathbb{R}$ with null complement.

PROOF: Let \mathcal{V} denote the space of $S \times n$ matrices of full rank, an open subset of $\mathbb{R}^{S n}$ with null complement. We note that $\mathcal{P} = \bar{V}^{-1}(\mathcal{V})$. First, suppose that $\mathcal{L}(j) = \mathcal{L}$ for all j . Then the derivative $D_{t,a} \bar{V}(p, \alpha, t, a)$ has the block diagonal $S n \times n t$ matrix form $\text{diag}(P(1), \dots, P(S))$, where $P(s)$ is the $n \times n t$ matrix $\text{diag}(p_1(s)^\top, \dots, p_1(s)^\top)$. Since $p \gg 0$, $D_{t,a} \bar{V}$ has rank $S n$. If $\mathcal{L}(j)$ is a general state-complete subspace, the j -th row of $P(s)$ has some (but not all) of the elements of $p_1(s)^\top$ replaced by zeros. Thus the rank of the derivative is $S n$ in general, implying that \bar{V} is a submersion. Therefore \mathcal{P} is open with null complement. ■

Having the required properties of the supply function Q , we turn to the demand functions. Let $G^i : \mathcal{L}_{++} \times \mathbb{R}_{++} \rightarrow \mathcal{L}_{++}$ be defined by

$$G^i(p, w) = \arg \left[\max_x u_i(x) \text{ s.t. } p \cdot x = w \right].$$

Let $e^i : \mathcal{L}_{++} \times \mathcal{L}_{++} \times [0, \infty) \times T \rightarrow \mathcal{L}$ be the total endowment function for agent i , defined by $e^i(\omega^i, p, \alpha, t) = \omega^i + \sum_j \theta_{ij} Q^j(\alpha, t, p)$, a smooth function. For any $k \in \{0, \dots, S\}$, let $F_k^i : \mathcal{L}_{++} \times G_{k,S} \times \mathcal{L}_{++} \times [0, \infty) \times T \rightarrow \mathcal{L}_{++}$ be defined by $F_k^i(p, L, \omega^i, \alpha, t)$

$$= \arg \left[\max_x u_i(x) \text{ s.t. } p \cdot (x - e^i(\omega^i, p, \alpha, t)) = 0, \quad p_1 \square (x - e^i(\omega^i, p, \alpha, t)) \in L \right].$$

We note that F_k^i is smooth on its domain. [Since Q^j is smooth, the calculations are the same as for Fact 5 of Duffie and Shafer (1985a).] Let $\mathcal{R} = \{r \in \mathbb{R}^S : r_s > -1, 1 \leq s \leq S\}$, and for any $p \in \mathcal{L}$ and any $r \in \mathcal{R}$, let $\langle p, r \rangle \in \mathcal{L}$ be the vector p' with $p'_0 = p_0$ and $p'_1(s) = (1 + r_s)p_1(s)$, $1 \leq s \leq S$. Let $\Omega = \mathcal{L}_{++}^m$ denote the space of endowments. For any $k \in \{0, \dots, S\}$, consider the excess demand function $Z_k : \Omega \times \mathcal{T} \times \mathbb{R}_+ \times \mathcal{L}_{++} \times \mathcal{R} \times G_{k,S} \rightarrow \mathcal{L}$ defined by $Z_k(\omega, t, \alpha, p, r, L)$

$$= G^1(\langle p, r \rangle, 1 + p \cdot \sum_j \theta_{1j} Q^j(\alpha, t, p)) + \sum_{i=2}^m F_k^i(p, L, \omega^i, \alpha, t) - \sum_{i=1}^m e^i(\omega^i, p, \alpha, t).$$

Lemma B.4. *Consider the conditions:*

- (A) $Z_k(\omega, t, \alpha, p, r, L) = 0$,
- (B) $r \in L^\perp$,
- (C) $\text{span}(\bar{V}(p, \alpha, t, a)) \subset L$,
- (D) $\langle p, r \rangle \cdot \omega^1 = 1$, and
- (E)

$$G^1(\langle p, r \rangle, \langle p, r \rangle \cdot \omega^1 + p \cdot \sum_{j=1}^n \theta_{1j} Q^j(\alpha, t, p)) + \sum_{i=2}^m F_k^i(p, L, \omega^i, \alpha, t) - \sum_{i=1}^m e^i(\omega^i, p, \alpha, t) = 0.$$

Then $[(A) \text{ and } (B) \text{ and } (C)] \iff [(B) \text{ and } (C) \text{ and } (D) \text{ and } (E)]$.

PROOF: In the case $r = 0$, this follows from Walras' Law: $p \cdot \omega^1 = 1 \Rightarrow p \cdot Z_k(\omega, t, \alpha, p, r, L) = 0$. If $r \neq 0$, we also use the fact that $\langle p, r \rangle \cdot z = p \cdot z$ whenever $p_1 \square z_1 \in L$ since $r \in L^\perp$. The result then follows from "adding up". ■

Lemma B.5. *For any k , if $\{(\omega_n, t_n, \alpha_n, p_n, r_n, L_n)\}$ is a sequence converging to $(\omega, t, \alpha, p, r, L) \in \Omega \times \mathcal{T} \times \mathbb{R}_+ \times \partial(\mathcal{L}_{++} \times \mathcal{R}) \times G_{k,S}$ with $\langle p, r \rangle \neq 0$, then there exists a coordinate j such that $\limsup_n Z_k(\omega_n, t_n, \alpha_n, p_n, r_n, L_n)_j = +\infty$.*

PROOF: Property P.2 implies that the sequence $\{\sum_j Q^j(\alpha_n, t_n, p_n)\}$ is either bounded above or unbounded below. This claim therefore follows from Fact 4, part (5), of Duffie and Shafer (1985a). ■

For each $k \in \{0, \dots, S\}$ and each σ in Σ , let $K_\sigma^k : \mathcal{L}_{++} \times W_\sigma^k \times \mathcal{T} \times \mathcal{A} \times \mathbb{R} \rightarrow \mathbb{R}^{(S-k)n}$ be defined by $K_\sigma^k(p, L, t, a, \alpha) = [I \mid \varphi_\sigma^k(L)] P_\sigma \bar{V}(p, \alpha, t, a)$.

Lemma B.6.

(1) K_σ^k is smooth.

(2) $D_{t,a} K_\sigma^k$ has rank $(S - k)n$.

PROOF: Part (1) follows from the smoothness of the composition of smooth functions. Part (2) is seen by taking derivatives with respect to $t_1^j(s)$, $1 \leq j \leq n_1$ and $a^j(s)$, $1 \leq j \leq n_2$ for $1 \leq s \leq S - k$. This derivative has the block diagonal form $\text{diag}(P(1), \dots, P(S - k))$, where $P(s)$ is given in the proof of Lemma B.3. \square

We define a triple $(p, r, L) \in \mathcal{L}_{++} \times \mathcal{R} \times G_{k,S}$ to be a k -pseudo-equilibrium for $(\omega, t, a) \in \Omega \times \mathcal{T} \times \mathcal{A}$ if $r \in L^\perp$ and if there exists a permutation σ in Σ such that $K_\sigma^k(p, L, t, a, 1) = 0$ and $Z_k(\omega, t, 1, p, r, L) = 0$.

Proposition B.1. For given $(p, r, t, a, \omega) \in \mathcal{L}_{++} \times \mathcal{R} \times \mathcal{T} \times \mathcal{A} \times \Omega$, suppose that the dimension of $L = \text{span}(\bar{V}(p, 1, t, a))$ is k and that $\langle p, r \rangle \cdot \omega^1 = 1$. Consider the assertions:

(A): (p, r, L) is a k -pseudo-equilibrium for (ω, t, a) ,

(B): there exist $(x^i, \gamma^i) \in \mathcal{L}_{++} \times \mathbb{R}^n$ for $i \in \{1, \dots, m\}$ such that $((x^i, \gamma^i), Q(1, t, p), p, \bar{q})$ is an equilibrium for $((u_i, \omega^i), (Y_j^t), (\theta_{ij}), a)$, with $\bar{q} = (1, 1, \dots, 1) \in \mathbb{R}^S$, and

(C): $(\langle p, r \rangle, 0, L)$ is a k -pseudo-equilibrium for (ω, t, a) .

Then:

(1) $(A) \iff (B)$, and

(2) if $n_1 = 0$ (no production), then $(A) \iff (B) \iff (C)$.

PROOF: $(A) \implies (B)$: Let $x^1 = G^1(\langle p, r \rangle, 1 + p \cdot \sum_j \theta_{1j} Q^j(1, t, p))$ and $x^i = F_k^i(p, L, \omega^i, 1, t)$ for $2 \leq i \leq m$. We will prove conditions (a)-(e) of an equilibrium. By Lemma B.4, we have spot market clearing, or condition (c). By Lemma B.2(2), we have market value maximization, or condition (b), since $\bar{q} = (1, 1, \dots, 1)$. Since $V(p, (Q(1, t, p)_1, a)) = \bar{V}(p, 1, t, a)$, we know that $x^i = F_k^i(p, L, \omega^i, 1, t)$ if and only if there exists $\gamma^i \in \mathbb{R}^n$ such that (x^i, γ^i) solves problem (2'). Let γ^i be defined in this way for $2 \leq i \leq m$, and let γ^1 be defined by

$$\gamma_j^1 = 1 - \sum_{i=2}^m \gamma_j^i, \quad 1 \leq j \leq n_1$$

$$\gamma_{n_1+k}^1 = - \sum_{i=2}^m \gamma_{n_1+k}^i, \quad 1 \leq k \leq n_2.$$

Then conditions (d) and (e) are satisfied. Since L is a linear space, and by spot market clearing condition (c), we have $p_1 \square (x^1 - e^1(\omega^1, p, 1, t))_1 \in L$. Furthermore, since $r \in L^\perp$, we have $r \cdot [p_1 \square (x^1 - e^1(\omega^1, p, 1, t))_1] = 0$. Since $\langle p, r \rangle \cdot \omega^1 = 1$, it follows that $x^1 =$

$F_k^1(p, L, \omega^1, 1, t)$. By spot market clearing condition (c), we also know that (x^1, γ^1) satisfies the budget constraints of problem (2'). Thus (x^1, γ^1) solves problem (2'). We then have condition (a) of an equilibrium.

(B) \implies (A) : By Lemma A.3, $K_\sigma^k(p, L, t, a, 1) = 0$ for some $\sigma \in \Sigma$. Let $\nu \in \mathbb{R}$ and $\lambda \in \mathbb{R}^S$ denote the Lagrange multipliers for initial and terminal wealth, respectively, for problem (2') of agent number 1 in the given equilibrium, and let $r = \lambda/\nu$. The first order conditions for problem (2') then imply the result.

If $n_1 = 0$, (A) \Leftrightarrow (B) \Leftrightarrow (C) : It suffices to show that (A) \Leftrightarrow (C). [The notation here will include t purely as a formalism.] By Lemma A.3, we must only show that, for $n_1 = 0$,

$$Z_k(\omega, t, 1, p, r, L) = 0 \Leftrightarrow Z_k(\omega, t, 1, \langle p, r \rangle, 0, L) = 0. \quad (*)$$

First, since $r \in L^\perp$ implies that $\text{span}[\bar{V}(p, 1, t, a)] = \text{span}[\bar{V}(\langle p, r \rangle, 1, t, a)]$, we have

$$p_1 \square (x - \omega^i)_1 \in \text{span}(V(p, a_1)) \iff \langle p, r \rangle_1 \square (x - \omega^i)_1 \in \text{span}(V(\langle p, r \rangle, a_1)).$$

Furthermore, since $r \in L^\perp$ and $p_1 \square (x - \omega^i)_1 \in L$, we have

$$p \cdot (x - \omega^i) - \langle p, r \rangle \cdot (x - \omega^i) = -r \cdot (p_1 \square (x - \omega^i)_1) = 0.$$

Thus $F_k^i(p, L, \omega^i, 1, t) = F_k^i(\langle p, r \rangle, L, \omega^i, 1, t)$. Of course

$$G^1(\langle p, r \rangle, \langle p, r \rangle \cdot \omega^1) = G^1(\langle \bar{p}, 0 \rangle, \langle \bar{p}, 0 \rangle \cdot \omega^1),$$

where $\bar{p} = \langle p, r \rangle$. Thus (*) holds. ■

For each $k \in \{0, \dots, S\}$ we define $Z_0^k : \Omega \times \mathcal{T} \times \mathcal{L}_{++} \times G_{k,S} \rightarrow \mathcal{L}$ by $Z_0^k(\omega, t, p, L) = Z_k(\omega, t, 1, p, 0, L)$. Let $h : \mathbb{R} \rightarrow [0, 1]$ be smooth and satisfy $h(\alpha) = 0, \alpha \leq 0$, and $h(\alpha) = 1, \alpha \geq 1$. We also define $Z^* : \Omega \times \mathcal{T} \times \mathbb{R} \times \mathcal{L}_{++} \times G_{n,S} \rightarrow \mathcal{L}$ by $Z^*(\omega, t, \alpha, p, L) = Z_n(\omega, t, h(\alpha), p, 0, L)$.

Lemma B.7. For $Z = Z^*$ or $Z = Z_0^k$,

(1) Z is smooth and

(2) $D_{\omega^1} Z \equiv -I$.

PROOF: Since the composition of smooth functions is smooth, part (1) is a consequence of Lemma B.2(1) and of Facts 4 and 5 of Duffie and Shafer (1985a). Part (2) is a trivial calculation. ■

For each $k \in \{0, \dots, S\}$ and each $\sigma \in \Sigma$, let $J_\sigma^k = \mathcal{L}_{++} \times W_\sigma^k \times \Omega \times \mathcal{T} \times \mathcal{A}$ and $H_{k\sigma} : J_\sigma^k \rightarrow \mathcal{L} \times \mathbb{R}^{(S-k)n}$ be defined by

$$H_{k\sigma}(p, L, \omega, t, a) = (Z_0^k(\omega, t, p, L), K_\sigma^k(p, L, t, a, 1)).$$

Let $H_\sigma^* : J_\sigma^n \times \mathbb{R} \rightarrow \mathcal{L} \times \mathbb{R}^{(S-n)n}$ be defined by

$$H_\sigma^*(p, L, \omega, t, a, \alpha) = (Z^*(\omega, t, p, L, \alpha), K_\sigma^n(p, L, t, a, h(\alpha))).$$

Finally, for each agent $i \in \{2, \dots, m\}$, let $H_{k\sigma}^i : J_\sigma^k \rightarrow \mathcal{L} \times \mathbb{R}^{(S-k)n} \times \mathbb{R}^{S-k}$ be defined by

$$H_{k\sigma}^i(p, L, \omega, t, a) = \left(H_{k\sigma}(p, L, \omega, t, a), [I|\varphi_\sigma^k(L)]P_\sigma p_1 \square \left[G^i(p, p \cdot \omega^i + p \cdot \sum_j \theta_{ij} Q^j(1, t, p))_1 - \omega_1^i \right] \right).$$

For each $k \in \{0, \dots, S\}$ and $i \in \{2, \dots, m\}$, let

$$\mathcal{E}_k = \{(p, L, \omega, t, a) \in \mathcal{L}_{++} \times G_{k,S} \times \Omega \times \mathcal{T} \times \mathcal{A} : \exists \sigma \in \Sigma : H_{k\sigma}(p, L, \omega, t, a) = 0\}$$

$$\mathcal{E}_k^i = \{(p, L, \omega, t, a) \in \mathcal{L}_{++} \times G_{k,S} \times \Omega \times \mathcal{T} \times \mathcal{A} : \exists \sigma \in \Sigma : H_{k\sigma}^i(p, L, \omega, t, a) = 0\}$$

$$\mathcal{E}^* = \{(p, L, \omega, t, a, \alpha) \in \mathcal{L}_{++} \times G_{n,S} \times \Omega \times \mathcal{T} \times \mathcal{A} \times \mathbb{R} : \exists \sigma \in \Sigma : H_\sigma^*(p, L, \omega, t, a, \alpha) = 0\}.$$

Lemma B.8. For any $k \in \{0, \dots, S\}$ and $i \in \{2, \dots, m\}$, the sets \mathcal{E}_k , \mathcal{E}_k^i and \mathcal{E}^* are smooth boundaryless manifolds, with

$$\dim \mathcal{E}_k = (S - k)(k - n) + D$$

$$\dim \mathcal{E}_k^i = (S - k)(k - n - 1) + D$$

$$\dim \mathcal{E}^* = D + 1,$$

where $D = \dim(\Omega \times \mathcal{T} \times \mathcal{A})$.

PROOF: Taking \mathcal{E}_k first, we begin by showing that 0 is a regular value of $H_{k\sigma}$. We have

$$D_{(\omega^1, t)} H_{k\sigma} = \begin{pmatrix} D_{\omega^1} Z_k^0 & D_t Z_k^0 \\ 0 & D_t K_\sigma^k \end{pmatrix}.$$

The rank of $DH_{k\sigma}$ is thus at least $\ell(S + 1) + (S - k)n$ by Lemmas B.6 and B.7. But this is the dimension of the range manifold $\mathcal{L} \times \mathbb{R}^{(S-k)n}$. By the preimage theorem, $H_{k\sigma}^{-1}(0)$ is therefore a submanifold of J_σ^k of dimension $[(S + 1)\ell + (S - k)k + D] - [(S + 1)\ell + (S - k)n] =$

$(S - k)(k - n) + D$. By definition, $H_{k\sigma}^{-1}(0) = J_\sigma^k \cap \mathcal{E}_k$. Since $\{W_\sigma^k : \sigma \in \Sigma\}$ is an open cover of $G_{k,S}$, we have shown that \mathcal{E}_k is a submanifold of the stated dimension.

The calculations for \mathcal{E}^* are almost the same. For \mathcal{E}_k^i , the result follows from the same arguments once we have shown that

$$\text{rank} \left(D_{\omega^i} [I|\varphi_\sigma^k(L)] P_\sigma p_1 \square \left[G^i(p, p \cdot \omega^i + p \cdot \sum_j \theta_{ij} Q^j(1, t, p))_1 - \omega_1^i \right] \right) = S - k.$$

Without loss of generality, we take $\sigma = id$, the identity permutation. Let $E = \varphi_{id}^k(L)$, and for any $s \in \{1, \dots, S - k\}$, let $\rho_s : \mathcal{L}_{++} \rightarrow \mathbb{R}$ be defined by

$$\rho_s(\omega^i) = p_1(s) \cdot [G^i(p, w(\omega^i)) - \omega^i]_1(s) + \sum_{h=S-k+1}^S E_{sh} p_1(h) \cdot [G^i(p, w(\omega^i)) - \omega^i]_1(h),$$

where $w(\omega^i) = p \cdot \omega^i + p \cdot \sum_j \theta_{ij} Q^j(1, t, p)$. We must show that $\text{rank}(A) = S - k$, where $A = D_{\omega^i}(\rho_1(\omega^i), \dots, \rho_{S-k}(\omega^i))$. We have

$$D_{\omega_0^i} \rho_s(\omega^i) = B_s p_0^\top$$

$$D_{\omega_1^i(h)} \rho_s(\omega^i) = B_s p_1(h)^\top, \quad h \neq s,$$

$$D_{\omega_1^i(s)} \rho_s(\omega^i) = (B_s - 1) p_1(s)^\top,$$

where

$$B_s = p_1(s)^\top D_w [G^i(p, w(\omega^i))_1(s)] + \sum_{h=S-k+1}^S E_{sh} p_1(s)^\top D_w [G^i(p, w(\omega^i))_1(h)].$$

It follows that

$$A = \begin{pmatrix} B_1 p_0^\top & (B_1 - 1) p_1(1)^\top & \dots & B_1 p_1(s)^\top & \dots & B_1 p_1(S - k)^\top \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ B_s p_0^\top & B_s p_1(1)^\top & \dots & (B_s - 1) p_1(s)^\top & \dots & B_s p_1(S - k)^\top \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ B_{S-k} p_0^\top & B_{S-k} p_1(1)^\top & \dots & B_{S-k} p_1(s)^\top & \dots & (B_{S-k} - 1) p_1(S - k)^\top \end{pmatrix}.$$

Since $p \gg 0$, $\text{rank}(A) = S - k$ if and only if $\text{rank}(\hat{B}) = S - k$, where

$$\hat{B} = \begin{pmatrix} B_1 & B_1 - 1 & \dots & B_1 & \dots & B_1 \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ B_s & B_s & \dots & B_s - 1 & \dots & B_s \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ B_{S-k} & B_{S-k} & \dots & B_{S-k} & \dots & B_{S-k} - 1 \end{pmatrix}_{(S-k) \times (S-k+1)}.$$

It is easy to check that the rows of \hat{B} are linearly independent. ■

Lemma B.9. Each of the projection maps $\pi_k : \mathcal{E}_k \rightarrow \Omega \times T \times \mathcal{A}$, $\pi_k^i : \mathcal{E}_k^i \rightarrow \Omega \times T \times \mathcal{A}$, and $\pi_* : \mathcal{E}^* \rightarrow \Omega \times T \times \mathcal{A} \times \mathbb{R}$ is smooth and proper.

PROOF: Projection is a smooth operation. For properness of π_* , let X be a compact subset of $\Omega \times T \times \mathcal{A} \times \mathbb{R}$, and let $\{(p_n, L_n, \omega_n, t_n, a_n, \alpha_n)\}$ be a sequence in $\pi_*^{-1}(X)$. Since $G_{n,S}$ and X are compact, $\{(L_n, \omega_n, t_n, a_n, \alpha_n)\}$ has a subsequence converging to a point $(L, \omega, t, a, \alpha) \in W_\sigma^n \times X$ for some $\sigma \in \Sigma$. Since $p_n \cdot \omega_n^1 = 1$ for all n and since $\{\omega_n^1\}$ is bounded, $\{p_n\}$ is bounded with a non-zero cluster point $p \in \mathcal{L}_{++} \cup \partial\mathcal{L}_{++}$. But $p \notin \partial\mathcal{L}_{++}$ by Lemma B.5. Since H_σ^* is continuous, $(p, L, \omega, t, a, \alpha) \in \mathcal{E}^*$. Thus $\pi_*^{-1}(X)$ is a compact subset of \mathcal{E}^* . The same arguments show properness of π_k and π_k^i . ■

We take note of the following fact, which can be found in Hirsch (1976). Let $f : X \rightarrow Y$ denote a smooth proper map between smooth boundaryless manifolds X and Y of the same dimension, with Y connected. Then the number of points in the inverse image of a regular value $y \in Y$, modulo 2, is independent of y . This invariant, the *mod 2 degree* of f , is either zero or one, and denoted $\deg_2 f$. If the mod 2 degree of f is one, then the inverse image of every point y in Y is not empty, for if $f^{-1}(y)$ is empty, then y is a regular value, and $\#f^{-1}(y)$ is odd. By the previous results, π_n and π_* are maps satisfying these conditions. In order to demonstrate the existence of n -pseudo-equilibria for every economy $(\omega, t, a) \in \Omega \times T \times \mathcal{A}$, we thus need only to show that $\deg_2 \pi_n = 1$.

Proposition B.2. $\deg_2 \pi_n = \deg_2 \pi_* = 1$.

PROOF: We first show that $\deg_2 \pi_* = 1$. To do this, we will find some $(\bar{\omega}, \bar{t}, \bar{a}, \bar{\alpha}) \in \Omega \times T \times \mathcal{A} \times \mathbb{R}$ with a unique n -pseudo-equilibrium of the form $(\bar{p}, 0, \bar{L})$, such that $(\bar{\omega}, \bar{t}, \bar{a}, \bar{\alpha})$ is a regular value of π_* . This establishes $\#\pi_*^{-1}(\bar{\omega}, \bar{t}, \bar{a}, \bar{\alpha}) = 1$ and therefore $\deg_2 \pi_* = 1$. Let $\bar{\alpha} = 0$ and choose $\bar{p} \in \mathcal{L}_{++}$ and $(\bar{t}, \bar{a}) \in T \times \mathcal{A}$ so that the last n rows of $V(\bar{p}, (\bar{t}_1, \bar{a}))$ are linearly independent. For all i , let $\bar{\omega}^i = G^i(\bar{p}, 1 + p \cdot \sum_j \theta_{ij} \bar{t}^j) - \sum_j \theta_{ij} \bar{t}^j$. By choosing \bar{t} sufficiently small, we have $\bar{\omega}^i \in \mathcal{L}_{++}$ for all i . We also have $p \cdot \bar{\omega}^i = 1$ by strict monotonicity of u_i . For all i , let $\bar{x}^i = G^i(\bar{p}, 1 + \sum_j \theta_{ij} \bar{p} \cdot \bar{t}^j)$. It follows that $((\bar{x}^i), (\bar{t}^j), \bar{p})$ is a complete markets contingent commodity market equilibrium for the economy $((u_i, \bar{\omega}^i), (Y_j^{\bar{\omega}^i}), (\theta_{ij}))$. Let \bar{L} be the unique element of $G_{n,S}$ spanned by the columns of $V(\bar{p}, (\bar{t}_1, \bar{a}))$. Since the last n rows of $V(\bar{p}, (\bar{t}_1, \bar{a}))$ are linearly independent, there is an $(S - n) \times n$ matrix \bar{E} such that $[I \mid \bar{E}]$ induces \bar{L} and such that $[I \mid \bar{E}]V(\bar{p}, (\bar{t}_1, \bar{a})) = 0$. This implies that $\bar{L} \in W_{id}^n$. Since $\bar{x}^i - \bar{\omega}^i - \sum_j \theta_{ij} \bar{t}^j = 0$, it follows that $\bar{x}^i = F_n^i(\bar{p}, \bar{L}, \bar{\omega}^i, 1, \bar{t})$ for all i . Thus $(\bar{p}, 0, \bar{L})$ is an n -pseudo-equilibrium for $(\bar{\omega}, \bar{t}, \bar{a}, \bar{\alpha})$. Since $p'_1 \square (\bar{x}^i - e^i(\bar{\omega}^i, p', 1, \bar{t}))_1 \in L'$ for all $(p', L') \in \mathcal{L}_{++} \times G_{n,S}$, it follows that \bar{x}^i is budget feasible for all $(p', 0, L')$. Then, by

Pareto optimality and the fact that G^1 is unconstrained by the subspace L , $(\bar{p}, 0, \bar{L})$ is the unique n -pseudo-equilibrium with $r = 0$ for $(\bar{\omega}, \bar{t}, \bar{a}, \bar{\alpha})$.

It remains to show that $(\bar{\omega}, \bar{t}, \bar{a}, \bar{\alpha})$ is a regular value of π_* , which follows if

$$\text{rank } (D_{(p,L)} H_{id}^*(\bar{p}, \bar{L}, \bar{\omega}, \bar{t}, \bar{a}, \bar{\alpha})) = (S+1)\ell + (S-n)n.$$

For notational convenience, we write $E = \varphi_{id}^n(L)$ for $L \in W_{id}^n$, and define:

$$\bar{Z}(p, E, \bar{t}) = Z^*(p, (\varphi_{id}^n)^{-1}(E), \bar{\omega}, \bar{t}, \bar{\alpha})$$

$$\bar{K}(p, E) = K_{id}^n(p, (\varphi_{id}^n)^{-1}(E), \bar{t}, \bar{a}, \bar{\alpha})$$

$$\bar{H}(p, E, \bar{t}) = (\bar{Z}(p, E, \bar{t}), \bar{K}(p, E)).$$

Since $(\varphi_{id}^n, W_{id}^n)$ is a chart on $G_{n,S}$, it suffices to show that $D_{p,E} \bar{H}(\bar{p}, \bar{E}, \bar{t})$ has rank $(S+1)\ell + (S-n)n$. We have

$$D_{p,E} \bar{H}(\bar{p}, \bar{E}, \bar{t}) = \begin{bmatrix} D_p \bar{Z}(\bar{p}, \bar{E}, \bar{t}) & D_E \bar{Z}(\bar{p}, \bar{E}, \bar{t}) \\ D_p \bar{K}(\bar{p}, \bar{E}) & D_E \bar{K}(\bar{p}, \bar{E}) \end{bmatrix}.$$

We have shown that $F_n^i(\bar{p}, L', \bar{\omega}^i, 1, \bar{t}) = \bar{x}^i$ for all $L' \in G_{n,S}$. Thus $D_E \bar{Z}(\bar{p}, \bar{E}, \bar{t}) = 0$. We thus finish the proof of $\deg_2 \pi_* = 1$ by showing that $D_E \bar{K}(\bar{p}, \bar{E})$ has rank $(S-n)n$ and that $D_p \bar{Z}(\bar{p}, \bar{E}, \bar{t})$ has rank $(S+1)\ell$.

$D_E \bar{K}(\bar{p}, \bar{E})$ has rank $(S-n)n$: Let V_2 denote the $n \times n$ matrix consisting of the last n rows of $V(\bar{p}, (\bar{t}_1, \bar{a}))$. The derivative of $\bar{K}(p, E)$ with respect to any row vector of E , evaluated at (\bar{p}, \bar{E}) , is V_2 . Thus $D_E \bar{K}(\bar{p}, \bar{E})$ can be given the $(S-n)n \times (S-n)n$ matrix form $\text{diag}(V_2, V_2, \dots, V_2)$. Since $\text{rank}(V_2) = n$, we have $\text{rank}(D_E \bar{K}(\bar{p}, \bar{E})) = (S-n)n$.

$D_p \bar{Z}(\bar{p}, \bar{E}, \bar{t})$ has rank $(S+1)\ell$: We must show that $D_p \bar{Z}(\bar{p}, \bar{E}, \bar{t})$ is nonsingular. By the continuity of this derivative in \bar{t} , and since \bar{t} can be chosen arbitrarily small, it suffices to show that $D_p \bar{Z}(\bar{p}, \bar{E}, 0)$ is nonsingular. But this is the pure exchange case, and nonsingularity is demonstrated in the proof of Theorem 1 of Duffie and Shafer (1985a).

We have demonstrated that $\deg_2 \pi_* = 1$. We finish by showing that π_* and π_n have the same mod 2 degree. Their respective regular values form open sets with null complements by Sard's Theorem. Thus there exists some regular value (ω, t, a, α) of π_* with $\alpha \geq 1$ such that (ω, t, a) is a regular value for π_n . Since $h(\alpha) = 1$ for $\alpha \geq 1$, we have $\#\pi_n^{-1}(\omega, t, a) = \#\pi_*^{-1}(\omega, t, a, \alpha)$. ■

For any $(\omega, t, a) \in \Omega \times \mathcal{T} \times \mathcal{A}$, let $E^k(\omega, t, a)$ denote the set of k -pseudo-equilibria (p, r, L) with $\text{rank}(\bar{V}(p, 1, t, a)) = k$, and let $E_0^k(\omega, t, a) = \{(p, r, L) \in E^k(\omega, t, a) : r = 0\}$.

Proposition B.3. *There is an open subset \mathcal{F}^1 of $\mathcal{F} = \Omega \times T \times \mathcal{A}$ whose complement is null having the property: For each $(\bar{\omega}, \bar{t}, \bar{a}) \in \mathcal{F}^1$ there is an integer $T \geq 1$, a neighborhood $U \subset \mathcal{F}^1$ of $(\bar{\omega}, \bar{t}, \bar{a})$, and smooth functions $p_k : U \rightarrow \mathcal{L}_{++}$ and $L_k : U \rightarrow G_{n,s}$, $1 \leq k \leq T$, such that, for all $(\omega, t, a) \in U$, $E_0^n(\omega, t, a) = \{(p_k(\omega, t, a), 0, L_k(\omega, t, a)), 1 \leq k \leq T\}$.*

PROOF: Let $\bar{\mathcal{F}}$ denote the set of regular values of π_n . By Sard's Theorem and the properness of π_n , we know $\bar{\mathcal{F}}$ is open with null complement. By the "Stack of Records Theorem" [Guillemin and Pollack (1974)] and Proposition B.2, for each $(\bar{\omega}, \bar{t}, \bar{a})$ in $\bar{\mathcal{F}}$, there is a neighborhood \bar{U} of $(\bar{\omega}, \bar{t}, \bar{a})$ in $\bar{\mathcal{F}}$, an integer $T \geq 1$, and smooth functions $\bar{p}_k : \bar{U} \rightarrow \mathcal{L}_{++}$ and $\bar{L}_k : \bar{U} \rightarrow G_{n,s}$, $1 \leq k \leq T$, such that, for all (ω, t, a) in \bar{U} :

- (i) $\pi_n^{-1}(\omega, t, a) = \{(\bar{p}_k(\omega, t, a), \bar{L}_k(\omega, t, a), \omega, t, a), 1 \leq k \leq T\}$, and
- (ii) for each k , there is a $\sigma^k \in \Sigma$ such that $\bar{L}_k(\omega, t, a) \in W_{\sigma^k}$.

The proof is completed in almost the same manner as the proof of Theorem 2 in Duffie and Shafer (1985a), so we only sketch out the remaining arguments. First, noting that $H_{n\sigma^k}(\bar{p}_k(\omega, t, a), \bar{L}_k(\omega, t, a), \omega, t, a) \equiv 0$ on \bar{U} , we can differentiate with respect to ω and (t, a) only, to see with the aid of Lemma B.8 that $D_{(\omega, t, a)}\bar{p}_k(\omega, t, a)$ is of full rank on \bar{U} . It follows that $\Psi_k : \bar{U} \rightarrow \mathcal{L}_{++} \times T \times \mathcal{A}$ defined by $\Psi_k(\omega, t, a) = (\bar{p}_k(\omega, t, a), t, a)$ is a submersion, $1 \leq k \leq T$. By Lemma B.3, $\Psi_k^{-1}(\mathcal{P})$ is open in \bar{U} with null complement. Let $U = \bigcap_{1 \leq k \leq T} \Psi_k^{-1}(\mathcal{P})$, and let p_k and L_k denote the restrictions of \bar{p}_k and \bar{L}_k to U . By a standard local to global argument, the set \mathcal{F}^1 is constructed with the properties claimed. \blacksquare

We say the market subspace constraint is binding for agent i at $((\omega, t, a), p, r, L) \in \mathcal{F} \times \mathcal{L}_{++} \times \mathcal{R} \times G_{k,s}$ if

$$u_i \left(G^i(p, p \cdot \omega^i + p \cdot \sum_j \theta_{ij} Q^j(1, t, p)) \right) > u_i (F_k^i(p, L, \omega^i, 1, t)),$$

meaning that utility is strictly lowered when i is forced to keep the vector of spot values of net exchange in the sub-space L .

Proposition B.4. *There is an open subset \mathcal{F}^2 of \mathcal{F} whose complement is null such that, for all $(\omega, t, a) \in \mathcal{F}^2$:*

- (1) $E^n(\omega, t, a)$ is not empty, and
- (2) for all $(p, r, L) \in E^n(\omega, t, a)$, the market subspace constraint is binding for all except possibly one agent.

PROOF: We showed in the previous proposition that $E^n(\omega, t, a)$ is not empty for all $(\omega, t, a) \in \mathcal{F}^1$. For each $i \in \{2, \dots, m\}$ let \mathcal{F}_i^1 denote the set of regular values of π_n^i . For each $(\omega, t, a) \in \mathcal{F}_1^1$, Lemma B.8 tells us that $\dim(\pi_n^1)^{-1}(\omega, t, a) < 0$, implying that $(\pi_n^1)^{-1}(\omega, t, a)$ is empty. Let $\mathcal{F}^1(1) = \bigcap_{i \geq 2} \mathcal{F}_i^1$. By construction of $H_{k\sigma}^i$, each agent $i \geq 2$ finds the subspace constraint binding at any n -pseudo-equilibrium $(p, 0, L)$ for any $(\omega, t, a) \in \mathcal{F}^1(1)$. By relabeling, we can let $\mathcal{F}^1(i)$ denote the similarly constructed set for i taking the place of agent 1. Let $\mathcal{F}_*^1 = \bigcap_{i \geq 1} \mathcal{F}^1(i)$. Finally, let $\mathcal{F}^2 = \mathcal{F}^1 \cap \mathcal{F}_*^1$, an open subset of \mathcal{F} whose complement is null. For any $(\omega, t, a) \in \mathcal{F}^2$ and any $(p, r, L) \in E^n(\omega, t, a)$, suppose there are two agents, say 1 and 2, neither of whom find the market subspace constraint binding. Then

$$G^1(p, 1 + p \cdot \sum_j \theta_{1j} Q^j(1, t, p)) = G^1(\langle p, r \rangle, 1 + p \cdot \sum_j \theta_{1j} Q^j(1, t, p)),$$

implying that $r = 0$. But then $(p, L, \omega, t, a) \in (\pi_n^2)^{-1}(\omega, t, a)$, which contradicts the fact that $(\pi_n^i)^{-1}(\omega, t, a)$ is empty for all $(\omega, t, a) \in \mathcal{F}_i^1 \subset \mathcal{F}^2$. ■

Lemma B.10. Suppose (p, r, L) is a k -pseudo-equilibrium for some (ω, t, a) at which some agent i finds the market subspace constraint binding. Then the corresponding allocation is not Pareto optimal.

PROOF: This can be checked by comparing the first order conditions for the agents problems with the first order conditions for Pareto optimality. ■

Lemma B.11. Suppose $(\bar{\omega}, \bar{t}, \bar{a})$ is a regular value of π_n and $(\bar{p}, \bar{L}, \bar{\omega}, \bar{t}, \bar{a}) \in \pi_n^{-1}(\bar{\omega}, \bar{t}, \bar{a})$. Then there exists a ball $B \subset \mathbb{R}^{S-n}$ and a one-to-one immersion $\varphi : B \rightarrow \mathcal{L}_{++} \times \mathcal{R} \times G_{n,S}$ such that, for all $z \in B$, $\varphi(z)$ is an n -pseudo-equilibrium for $(\bar{\omega}, \bar{t}, \bar{a})$.

PROOF: The hypotheses imply that (\bar{p}, \bar{L}) is, for some $\sigma \in \Sigma$, a regular point of the function $H_{n\sigma}(\cdot, \cdot, \bar{\omega}, \bar{t}, \bar{a})$, or in other words, that $D_{p,L} H_{n\sigma}(\bar{p}, \bar{L}, \bar{\omega}, \bar{t}, \bar{a})$ has full rank. Let $\psi : G_{n,S} \rightarrow G_{S-n,S}$ be the diffeomorphism defined by $\psi(L) = L^\perp$. Let $(W_{\sigma'}, \varphi_{\sigma'})$ be a chart on $G_{S-n,S}$ containing \bar{L}^\perp and $(W_\sigma, \varphi_\sigma)$ be a chart on $G_{n,S}$ containing \bar{L} . For $L \in W_\sigma \cap \psi^{-1}(W_{\sigma'})$,

$$r \in L^\perp \iff [I|\varphi_{\sigma'}(\psi(L))]P_{\sigma'}r = 0. \quad (11)$$

Without loss of generality, suppose $\sigma' = id$. Then (11) is equivalent to $r = (-\varphi_{\sigma'}(\psi(L))r_2, r_2)$ for any $r_2 \in \mathbb{R}^{S-n}$. Let $H : \mathcal{L}_{++} \times \mathcal{R} \times G_{n,S} \rightarrow \mathcal{L} \times \mathbb{R}^{(S-n)n}$ be defined by

$$H(p, r, L) = (Z_n(\bar{\omega}, \bar{t}, 1, p, r, L), K_\sigma^n(p, L, \bar{t}, \bar{a}, 1)).$$

Let $f : \mathcal{L}_{++} \times G_{n,S} \times \mathbb{R}^{S-n} \rightarrow \mathcal{L} \times \mathbb{R}^{(S-n)n}$ be defined in a sufficiently small neighborhood of $(\bar{p}, \bar{L}, 0)$ by $f(p, L, r_2) = H(p, (-\varphi_{\sigma'} \circ \psi(L))r_2, r_2, L)$. Any solution (p, L, r_2) to the

equation $f(p, L, r_2) = 0$ corresponds to an n -pseudo equilibrium for $(\bar{\omega}, \bar{t}, \bar{a})$. One can readily verify that

$$D_{(p,L)}f(\bar{p}, \bar{L}, 0) = D_{(p,L)}H(\bar{p}, 0, \bar{L}) = D_{(p,L)}H_{n\sigma}(\bar{p}, \bar{L}, \bar{\omega}, \bar{t}, \bar{a}).$$

Earlier calculations show this derivative to be full rank, so by the Implicit Function Theorem there exists a neighborhood $\mathcal{N} \subset \mathbb{R}^{S-n}$ of zero and smooth functions $\hat{p} : \mathcal{N} \rightarrow \mathcal{L}_{++}$ and $\hat{L} : \mathcal{N} \rightarrow G_{n,S}$ such that $f(\hat{p}(r_2), \hat{L}(r_2), r_2) = 0$ for all $r_2 \in \mathcal{N}$. Thus the function $\varphi : \mathcal{N} \rightarrow \mathcal{L}_{++} \times \mathbb{R}^S \times G_{n,S}$ defined by

$$\varphi(r_2) = (\hat{p}(r_2), (-\varphi_{\sigma'} \circ \psi(\hat{L}(r_2)), r_2), \hat{L}(r_2)).$$

is an appropriate one-to-one smooth immersion. ■

Lemma B.12. Suppose $n_1 \geq 1$ (production). Let (p, r, L) and (p', r', L') be distinct k -pseudo-equilibria for a given $(\omega, t, a) \in \mathcal{F}$, with $\text{rank}(\bar{V}(p, 1, t, a)) = \text{rank}(\bar{V}(p', 1, t, a)) = k$. Then the corresponding pseudo-equilibrium consumption allocations are distinct.

PROOF:

Case 1: ($p = p'$). If $p = p'$ then $L = L'$, so $r \neq r'$, and therefore

$$G^1(\langle p, r \rangle, 1 + p \cdot \sum_j \theta_{1j} Q^j(1, t, p)) \neq G^1(\langle p', r' \rangle, 1 + p' \cdot \sum_j \theta_{1j} Q^j(1, t, p')). \quad (12)$$

Case 2: ($p \neq p'$). If (12) is not true, then $\langle p, r \rangle = \nu \langle p', r' \rangle$ for some $\nu \in (0, \infty)$. But $\langle p, r \rangle \cdot \omega^1 = \langle p', r' \rangle \cdot \omega^1 = 1$ implies that $\nu = 1$. Thus p and p' are not colinear, and for some state s , we know $r_s \neq r'_s$. It follows that

$$\frac{p_{0h}}{p_1(s)_h} \neq \frac{p'_{0h}}{p'_1(s)_h} \quad \forall h \in \{1, \dots, \ell\}.$$

Since $Q = \sum_j Q^j(1, t, p)$ solves the problem $\max\{p \cdot y : y \in \sum_j Y_j^t\}$, it follows from the first order conditions for this problem that $Q \neq Q' = \sum_j Q^j(1, t, p')$. The consumption allocations for (p, r, L) and (p', r', L') are thus distinct. ■

We complete the proof of Theorem 2 merely by collecting our results. We take the generic set E referred to in the statement of Theorem 2 to be the set \mathcal{F}^2 of Proposition B.4. Part (1) of Theorem 2 is then a consequence of Proposition B.4 (1). Part (2) follows from the fact that, for $n \geq S$, all pseudo-equilibria for $(\omega, t, a) \in \mathcal{F}^2$ have complete markets

($L = \mathbb{R}^S$), and therefore $r = 0$. We can then apply Proposition B.3. For part (3), we use the last part of Proposition B.1 as well as Proposition B.3. Part (4) follows from Lemmas B.11 and B.12. (The homeomorphism is constructed as the composition of the 1-1 immersion φ of Lemma B.11 and the 1-1 map between pseudo-equilibria and allocations of Lemma B.12.) Part (5) follows from complete markets and the usual proof of Pareto optimality with production. Part (6) follows from Lemma B.10 and Proposition B.4 (2). The amended property (7), defined in Section 4, is proved as a corollary to Proposition 2, using Proposition B.4 (2) again.

C. Proof of Theorem 3

We shall be extremely brief, as the main ideas are contained in Part B and in Duffie and Shafer (1985b). First we fix the state price process $\bar{q} \in (D_1)_{++}$ to be $\bar{q} \equiv 1$. The calculations are only slightly more complicated for arbitrary $\bar{q} \in (D_1)_{++}$. This leaves each firm with the usual problem faced in a static Arrow-Debreu economy.

Let $\hat{\Xi} = \{\eta^1, \dots, \eta^{\hat{H}}\}$ denote the subset of \hat{H} non-terminal vertices, those $\eta \in \Xi$ with $\nu(\eta) \equiv \text{outdegree}(\eta) > 0$. For any $a \in D_1$ and any $\eta \in \hat{\Xi}$, let

$$a(\eta_+) = (a(\eta_{+1}), \dots, a(\eta_{+\nu(\eta)})) \in \mathbb{R}^{\nu(\eta)}.$$

Let $\Pi : D_1 \rightarrow \mathbb{R}^{\hat{H}-1}$ be defined by $\Pi(a) = (a(\eta_+^1), \dots, a(\eta_+^{\hat{H}}))$. We can assume without loss of generality that $\hat{\Xi}$ is ordered so that, for some $\mathcal{H} \leq \hat{H}$,

$$\nu(\eta) > k, \quad \forall \eta \in \Xi^* \equiv \{\eta^1, \dots, \eta^{\mathcal{H}}\}.$$

We let $G^* = G_{k, \nu(1)} \times \dots \times G_{k, \nu(\mathcal{H})}$ and $\underline{k} = \sum_{h=1}^{\hat{H}} \min\{\nu(\eta^h), k\}$. Finally, we define $B : G^* \rightarrow G_{\underline{k}, H-1}$ by

$$B(L_1, \dots, L_{\mathcal{H}}) = \Pi \left[\Lambda_{\bar{q}}^{-1} \left(\mathbb{R} \oplus L_1 \oplus \dots \oplus L_{\mathcal{H}} \oplus \mathbb{R}^{\nu(\mathcal{H}+1)} \oplus \dots \oplus \mathbb{R}^{\nu(\hat{H})} \right) \right],$$

where \oplus denotes direct sum and $\Lambda_{\bar{q}}^{-1}$ is the inverse image map corresponding to $\Lambda_{\bar{q}}$. This construction is justified by the following intermediate result. Let $\pi(\eta_+)$ denote the $\nu(\eta) \times k$ matrix whose s -th row is $\pi(\eta_{+s})$.

Lemma C.1 *Let $p \in \mathcal{L}_{++}$ and $y^j = Q^j(1, t, p)$, $1 \leq j \leq n$. Then y^j solves the market value maximization problem (7) for Y_j^t . If $\pi(\eta_+)$ is full rank for each $\eta \in \hat{\Xi}$, then*

there exists $\gamma \in \Gamma$ such that (x, γ) is an optimal plan for agent i if and only if

$$x = F_k^i(p, B[\text{span}(\pi(\eta_+^i)), \dots, \text{span}(\pi(\eta_+^{\mathcal{H}}))], \omega^i, 1, t).$$

PROOF: The first claim of this lemma follows from Lemma B.1. The second claim follows from the proof of Lemma 1 of Duffie and Shafer (1985b). ■

Let $\{(W_\sigma^h, \varphi_\sigma^h) : \sigma \in \Sigma_{\nu(h)}\}$ denote the atlas for $G_{k, \nu(h)}$ constructed in Appendix A, $1 \leq h \leq \mathcal{H}$. For each $h \in \{1, \dots, \mathcal{H}\}$ and $\sigma \in \Sigma_{\nu(h)}$, let $K_\sigma^h : \mathcal{L}_{++} \times \mathcal{L}_{++}^n \times \mathcal{L}^{k-n} \times \mathbb{R}_+ \times W_\sigma^h \rightarrow \mathbb{R}^{(\nu(h)-k)k}$ be defined by $K_\sigma^h(p, t, d_{n+1}, \dots, d_k, \alpha, L) = [I \mid \varphi_\sigma^h(L)] P_\sigma \pi(\eta_+^h)$, where

$$\begin{aligned} \pi_j &= \Lambda_{\bar{q}}(p \square Q^j(\alpha, t, p)), \quad 1 \leq j \leq n, \\ &= \Lambda_{\bar{q}}(\delta_j + p \square d_j), \quad n+1 \leq j \leq k. \end{aligned}$$

Let $\Sigma = \Sigma_{\nu(1)} \times \dots \times \Sigma_{\nu(\mathcal{H})}$, and for each $\sigma = (\sigma_1, \dots, \sigma_{\mathcal{H}}) \in \Sigma$, let $W_\sigma = W_{\sigma_1}^1 \times \dots \times W_{\sigma_{\mathcal{H}}}^{\mathcal{H}}$. For each $\sigma \in \Sigma$, let

$$H_\sigma : \mathcal{L}_{++} \times W_\sigma \times \mathcal{L}_{++}^n \times \mathbb{R}_+ \times \mathcal{L}^{k-n} \times \Omega \rightarrow \mathcal{L} \times (\mathbb{R}^{(\nu(1)-k)k} \times \dots \times \mathbb{R}^{(\nu(\mathcal{H})-k)k})$$

be defined by

$$H_\sigma(p, L_1, \dots, L_{\mathcal{H}}, t, \alpha, d_{n+1}, \dots, d_k, \omega) = (H_A, H_B),$$

where

$$H_A = Z^*[\omega, t, \alpha, p, B(L_1, \dots, L_{\mathcal{H}})]$$

and

$$H_B = (K_{\sigma_1}^1(p, t, d_{n+1}, \dots, d_k, \alpha, L_1), \dots, K_{\sigma_{\mathcal{H}}}^{\mathcal{H}}(p, t, d_{n+1}, \dots, d_k, \alpha, L_{\mathcal{H}})),$$

with Z^* defined as in Part B.

Proposition C.1. For $(p, L) \in \mathcal{L}_{++} \times W_\sigma$, if $H_\sigma(p, L, t, 1, d_{n+1}, \dots, d_k, \omega) = 0$ and $\pi(\eta_+)$ is of full rank for all $\eta \in \hat{\Xi}$, then there exists (x^i, γ^i) , $1 \leq i \leq m$, such that $((x^i, \gamma^i), (Q^j(1, t, p)), \bar{q}, p)$ is an equilibrium for $((\Xi, \mathcal{A}), (u_i, \omega^i), (Y_j^t), (\theta_{ij}), (\delta_h, d_h))$.

PROOF: By Lemma A.3., the hypotheses imply that $\text{span}(\pi(\eta_+^h)) = L_h$, $1 \leq h \leq \mathcal{H}$. Let

$$x^i = F_k^i(p, B(L_1, \dots, L_{\mathcal{H}}), \omega^i, 1, t), \quad 1 \leq i \leq m.$$

By Lemma B.4 and linearity, $\sum_i x^i - \omega^i = \sum_j Q^j(1, t, p)$. By Lemma C.1, there exists $\gamma^i \in \Gamma$ such that (x^i, γ^i) is an optimal plan for agent i , $2 \leq i \leq m$. Let $\gamma^1 \in \Gamma$ be defined for each $\xi \in \Xi'$ by

$$\begin{aligned}\gamma_j^1(\xi) &= 1 - \sum_{i=2}^m \gamma_j^i(\xi), \quad 1 \leq j \leq n, \\ &= - \sum_{i=2}^m \gamma_j^i(\xi), \quad n+1 \leq j \leq k.\end{aligned}$$

Then by linearity and spot market clearing (x^1, γ^1) is an optimal plan for agent 1. By construction, markets clear. By Lemma B.2, firms solve the market value maximization problem (7). ■

Let $\mathcal{E} = \{(p, L, \omega, t, \alpha, d_{n+1}, \dots, d_k) \in \mathcal{L}_{++} \times G^* \times \Omega \times \mathcal{L}_{++}^n \times \mathbb{R} \times \mathcal{L}^{k-n} :$

$$H_\sigma(p, L, t, \alpha, d_{n+1}, \dots, d_k, \omega) = 0 \text{ for all } \sigma \in \Sigma \text{ such that } L \in W_\sigma\}$$

and let $\kappa : \mathcal{E} \rightarrow \Omega \times \mathcal{L}_{++}^n \times \mathbb{R} \times \mathcal{L}^{k-n}$ be the projection map defined by

$$\kappa(p, L, \omega, t, \alpha, d_{n+1}, \dots, d_k) = (\omega, t, \alpha, d_{n+1}, \dots, d_k).$$

Lemma C.2.

- (1) \mathcal{E} is a smooth boundaryless manifold of dimension $(m+k)H\ell + 1$,
- (2) κ is smooth and proper, and
- (3) κ has a regular value whose inverse image is a singleton.

PROOF: Since Z^* and $K_{\sigma_h}^h$ are smooth, and by Lemma A.2, we know that H_σ is smooth. To see that 0 is a regular value of H_σ , let $b \equiv (d_{n+1}, \dots, d_k) \in \mathcal{L}^{k-n}$ and note that

$$\text{rank}(D_{(\omega^1, t, b)} H_\sigma) = \text{rank}(\text{diag}[D_{\omega^1} Z^*, D_{(t, b)} K_{\sigma_1}^1, \dots, D_{(t, b)} K_{\sigma_n}^n]).$$

It is easy to check that each of the diagonal blocks has maximal rank, and thus that $D_{(\omega^1, t, b)} H_\sigma$ has rank $H\ell + \sum_{h=1}^n (\nu(h) - k)k$. By the preimage theorem, $H_\sigma^{-1}(0)$ is a smooth submanifold of dimension $(m+k)H\ell + 1$. Since $\{W_\sigma : \sigma \in \Sigma\}$ is an open cover of G^* , it follows that \mathcal{E} is a smooth manifold of the same dimension. The projection map κ is of course smooth, and is proper by a proof almost identical to that of Lemma B.10. The construction of a regular value of κ with a unique inverse image point is by analogy with the proofs of Proposition B.2 and Proposition 3 of Duffie and Shafer (1985b). ■

The proof of Theorem 3 is completed by direct analogy with the proofs of Proposition B.3 and B.4. That is, by the last lemma and the degree invariance result used in the proof of Proposition B.3, every point in the range space of κ has a non-empty inverse image, on which $H_\sigma = 0$ for some $\sigma \in \Sigma$. By Proposition C.1, if $H_\sigma(p, L, t, 1, d_{n+1}, \dots, d_k) = 0$ and $\pi(\eta_+)$ is of full rank for all $\eta \in \hat{\Xi}$, we are done. But $\pi(\eta_+)$ is of full rank for generic (ω, t, d) by Sard's Theorem and "perturbation" calculations analagous to those in the proof of Proposition B.4. [See also the proof of Theorem 1, Duffie and Shafer (1985b).]

References

- ARROW, K. [1953], "Le rôle des valeurs boursières pour la répartition la meilleure des risques, *Econometrie*, pp. 41 - 47; discussion, pp. 47 - 48, Colloq. Internat. Centre National de la Recherche Scientifique, no. 40 (Paris, 1952) C.N.R.S. Paris, 1953; translated in," *Review of Economic Studies*, Vol. 31 (1964), pp. 91-96.
- ARROW, K. AND G. DEBREU [1954], "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, Vol. 22, pp. 265-290.
- ARZAC, E. [1983], "A Mechanism for the Allocation of Corporate Investment," *Journal of Financial and Quantitative Analysis*, Vol. 18, pp. 175-188.
- BALASKO, Y. [1975], "The Graph of The Walras Correspondence," *Econometrica*, Vol. 43, pp. 907-912.
- BALASKO, Y. AND D. CASS [1985], "Regular Demand with Several, General Budget Constraints," Working Paper 85-20, University of Pennsylvania, CARESS.
- [1986], "The Structure of Financial Equilibrium: I. Exogenous Yields and Unrestricted Participation," Working Paper 85-23R, University of Pennsylvania, CARESS.
- BARON, D. P. [1979], "On The Relationship Between Complete and Incomplete Financial Market Models," *International Economic Review*, Vol. 20, pp. 105-117.
- BESTER, H. [1982], "On Shareholder Unanimity in the Mean-Variance Model," *Economics Letters*, Vol. 10, pp. 363-367.
- BREALEY, R. AND S. MYERS [1984], *Principles of Corporate Finance (Second Edition)*, New York: McGraw-Hill.
- BURKE, J. [1986], "Existence of Equilibrium for Incomplete Market Economies with Production and Stock Trading," Unpublished, Texas A & M University.
- CASS, D. [1984], "Competitive Equilibria in Incomplete Financial Markets," Working Paper 84-09, University of Pennsylvania, CARESS.
- [1985], "On The 'Number' of Equilibrium Allocations with Incomplete Financial Markets," Working Paper 85-16, University of Pennsylvania.
- CHAE, S. [1985], "Existence of Equilibria in Incomplete Markets," Unpublished, Rice University.
- DEBREU, G. [1959], *Theory of Value*, Cowles Foundation Monograph 17, Yale University Press, New Haven and London.
- [1970], "Economies with a Finite Set of Equilibria," *Econometrica*, Vol. 38, pp. 387-392.
- [1972], "Smooth Preferences," *Econometrica*, Vol. 40, pp. 603-615.
- [1976], "Smooth Preferences, A Corrigendum," *Econometrica*, Vol. 44, pp. 831-832.
- DEMARZO, P. [1986], "An Extension of The Modigliani-Miller Theorem to Stochastic Economies with Incomplete Markets," Unpublished, Stanford University.

- DIAMOND, P. [1967], "The Role of a Stock Market in a General Equilibrium Model with Technological Uncertainty," *American Economic Review*, Vol. 57, pp. 759-776.
- DIERKER, E. [1972], "Two Remarks on the Number of Equilibria of an Economy," *Econometrica*, Vol. 40, pp. 951-953.
- DREZE, J. [1974], "Investment under Private Ownership: Optimality, Equilibrium and Stability." In J. Drèze, *Allocation Under Uncertainty: Equilibrium and Optimality*, New York: Wiley.
- DUFFIE, D. [1985], "Stochastic Equilibria with Incomplete Financial Markets," Research Paper 811, Graduate School of Business, Stanford University, forthcoming: *Journal of Economic Theory*.
- DUFFIE, D. AND M. HUANG [1986], "Stochastic Production-Exchange Equilibria," Working Paper, Stanford University.
- DUFFIE, D. AND W. SHAFER [1985a], "Equilibrium in Incomplete Markets I: A Basic Model of Generic Existence," *Journal of Mathematical Economics*, Vol. 14, pp. 285-300.
- [1985b], "Equilibrium in Incomplete Markets II: Generic Existence in Stochastic Economies," Unpublished, Mathematical Sciences Research Institute, forthcoming: *Journal of Mathematical Economics*.
- EKERN, S. AND R. WILSON [1974], "On The Theory of the Firm in an Economy with Incomplete Markets," *Bell Journal of Economics and Management Science*, Vol. 5, pp. 171-180.
- FAMA, E. [1978], "The Effects of a Firm's Investment and Financing Decisions on the Welfare of its Security Holders," *American Economic Review*, Vol. 68, pp. 272-284.
- FAMA, E. AND M. MILLER [1972], *The Theory of Finance*, New York: Holt, Rinehart and Winston.
- P. FRIESEN [1974], "A Reinterpretation of the Equilibrium Theory of Arrow and Debreu in Terms of Financial Markets," Technical Report 126, Stanford University, IMSSS.
- FUCHS, G. [1974], "Private Ownership Economies with a Finite Number of Equilibria," *Journal of Mathematical Economics*, Vol. 1, pp. 141-158.
- GEANAKOPOLOS, J. AND A. MAS-COLELL [1985], "Real Indeterminacy with Financial Assets," Unpublished, Cowles Foundation, Yale University.
- GEANAKOPOLOS, J. AND H. POLEMARCHAKIS [1985], "Existence, Regularity, and Constrained Suboptimality of Competitive Portfolio Allocations when the Asset Market is Incomplete," Unpublished, Cowles Foundation, Yale University.
- GEVERS, L. [1974], "Competitive Equilibrium of the Stock Exchange and Pareto Efficiency." In J. Drèze, *Allocation under Uncertainty: Equilibrium and Optimality*, New York: Wiley.
- GROSSMAN, S. [1977], "A Characterization of the Optimality of Equilibrium in Incomplete Markets," *Journal of Economic Theory*, Vol. 15, pp. 1-15.
- GROSSMAN, S. AND O. HART [1979], "A Theory of Competitive Equilibrium in Stock Market Economies," *Econometrica*, Vol. 47, pp. 293-330.

- GROSSMAN, S. AND J. STIGLITZ [1976], "On Stockholder Unanimity in Making Production and Financial Decisions," Technical Report 224, IMSSS, Stanford University.
- GUESNERIE, R. AND J.-Y. JAFFRAY [1974], "Optimality of Equilibrium of Plans, Prices, and Price Expectations." In J. Drèze, *Allocation Under Uncertainty: Equilibrium and Optimality*, New York: Wiley.
- GUILLEMIN, V. AND A. POLLACK [1974], *Differential Topology*, Englewood Cliffs, New Jersey: Prentice-Hall.
- HALLER, H. [1984], "Competition in a Stock Market with Small Firms," Unpublished, University of Mannheim, West Germany.
- HARRISON, J.M. AND D. KREPS [1979], "Martingales and arbitrage in multiperiod securities markets," *Journal of Economic Theory*, Vol. 20, pp. 381-408.
- HART, O. [1975], "On The Optimality of Equilibrium When the Market Structure is Incomplete," *Journal of Economic Theory*, Vol. 11, pp. 418-443.
- [1979a], "Monopolistic Competition in a Large Economy with Differentiated Commodities," *Review of Economic Studies*, Vol. 46, pp. 1-30.
- [1979b], "On Shareholder Unanimity in Large Stock Market Economies," *Econometrica*, Vol. 47, pp. 1057-1082.
- HELLWIG, M. [1981], "Bankruptcy, Limited Liability, and The Modigliani-Miller Theorem," *American Economic Review*, Vol. 71, pp. 155-170.
- HIRSCH, M. [1976], *Differential Topology*, New York: Springer-Verlag.
- JENSEN, M. AND J. LONG [1972], "Corporate Investment under Uncertainty and Pareto Optimality in The Capital Markets," *Bell Journal of Economics and Management Science*, Vol. 3, pp. 151-174.
- KEHOE, T. [1983], "Regularity and Index Theory for Economies with Smooth Production Technologies," *Econometrica*, Vol. 51, pp. 895-919.
- KREPS, D. [1979], "Three essays on capital markets," Technical Report 298, Institute for Mathematical Studies in The Social Sciences, Stanford University.
- [1982], "Multiperiod securities and the efficient allocation of risk: a comment on the Black-Scholes option pricing model." In J. McCall, *The Economics of Uncertainty and Information*, University of Chicago Press.
- LELAND, H. [1973], "Capital Asset Markets, Production, and Optimality: A Synthesis," Technical Report 115, IMSSS, Stanford University.
- [1974], "Production Theory and The Stock Market," *Bell Journal of Economics and Management Science*, Vol. 5, pp. 125-144.
- [1977], "Quality, Choice, and Competition," *American Economic Review*, Vol. 67, pp. 127-137.
- [1978], "Information, Management Choice, and Stockholder Unanimity," *Review of Economic Studies*, Vol. 45, pp. 527-534.
- MAGILL, M.J.P. AND W. SHAFER [1984], "Allocation of Aggregate and Individual Risks Through Futures and Insurance Markets," Unpublished, University of Southern California.

- [1985], "Equilibrium and Efficiency in a Canonical Asset Trading Model," Working Paper, Department of Economics, University of Southern California.
- MAKOWSKI, L. [1980], "A Characterization of Perfectly Competitive Economies with Production," *Journal of Economic Theory*, Vol. 22, pp. 208-221.
- [1983], "Competitive Stock Markets," *Review of Economic Studies*, Vol. 50, pp. 305-330.
- MANGASARIAN, O. [1969], *Nonlinear Programming*, New York: McGraw-Hill.
- MAS-COLELL, A. [1985], *The Theory of General Economic Equilibrium - A Differentiable Approach*, Cambridge at The University Press.
- MCMANUS, D. [1984], "Incomplete Markets: Generic Existence of Equilibrium and Optimality Properties in an Economy with Futures Markets," Unpublished, University of Pennsylvania.
- MERTON, R. AND M. SUBRAHMANYAM [1974], "The Optimality of a Competitive Stock Market," *Bell Journal of Economics and Management Science*, Vol. 5, pp. 145-170.
- MODIGLIANI, F. AND M. MILLER [1958], "The Cost of Capital, Corporation Finance, and the Theory of Investment," *American Economic Review*, Vol. 48, pp. 261-297.
- NERMUTH, M. [1985], "General Equilibrium with Futures Trading," Working Paper 8502, University of Vienna.
- OSTROY, J. [1980], "The No Surplus Condition as a Characterization of Perfectly Competitive Economies," *Journal of Economic Theory*, Vol. 22, pp. 183-207.
- H. POLEMARCHAKIS AND B. KU [1986], "Options and Equilibrium," Unpublished, Columbia University.
- RADNER, R. [1972], "Existence of equilibrium of plans, prices and price expectations in a sequence of markets," *Econometrica*, Vol. 40, pp. 289-303.
- [1974], "A Note on Unanimity of Stockholders' Preferences Among Alternative Production Plans: A Reformulation of the Ekern-Wilson Model," *The Bell Journal of Economics and Management Science*, Vol. 5, pp. 181-186.
- REPULLO, R. [1984], *Equilibrium and Efficiency in a Sequence of Markets*, PhD Thesis, University of London.
- [1985], "On The Generic Existence of Radner Equilibria when there are as Many Securities as States of Nature," Unpublished, London School of Economics.
- ROSS, S.A. [1978], "A Simple Approach to the Valuation of Risky Streams," *Journal of Business*, Vol. 51, pp. 453-475.
- RUBINSTEIN, M. [1976], "The Valuation of Uncertain Income Streams and the Pricing of Options," *Bell Journal of Economics*, Vol. 7, pp. 407-425.
- [1977], "Competition and Approximation," Working Paper 54, Program in Finance, University of California, Berkeley.
- SANDOR, R. [1973], "Innovation by an Exchange: A Case Study of the Development of the Plywood Futures Contract," *Journal of Law and Economics*, Vol. 16, pp. 119-136.

- SATTERTHWAITE, M. [1981], "On the Scope of the Stockholder Unanimity Theorems," *International Economic Review*, Vol. 22, pp. 119-133.
- SILBER, W. [1983], "The Process of Financial Innovation," *American Economics Association Papers and Proceedings*, Vol. 73, pp. 89-95.
- SMALE, S. [1974], "Global Analysis and Economics IV: Finiteness and Stability of Equilibria with General Consumption Sets and Production," *Journal of Mathematical Economics*, Vol. 1, pp. 119-127.
- STIGLITZ, J. [1972], "On the Optimality of the Stock Market Allocation of Investment," *Quarterly Journal of Economics*, Vol. 86, pp. 25-60.
- [1974], "On The Irrelevance of Corporate Financial Policy," *American Economic Review*, Vol. 64, pp. 851-866.
- WERNER, J. [1985], "Equilibrium in Economies with Incomplete Financial Markets," *Journal of Economic Theory*, Vol. 36, pp. 110-119.